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RIEMANNIAN STRUCTURES AND THE CODIMENSION OF EXCEPTIONAL MINIMAL SURFACES IN H^n AND R^n

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0. Introduction

Let $N^n(c)$ denote the n-dimensional simply connected space form of constant curvature c. In particular, set $R^n=N^n(0)$, $S^n=N^n(1)$ and $H^n=N^n(-1)$. Let us consider a kind of rigidity problem to classify those minimal surfaces in $N^n(c)$ which are (locally) isometric to minimal surfaces in $N^3(c)$. Concerning this problem, several results are known (see [6], [7], [8], [9], [10], [11], [13], [14], [15], [16]). In the Euclidean case where $c=0$, Lawson [6] solved this problem completely (cf. [7, Chapter IV]). He showed that if a minimal surface in R^n is isometric to a minimal surface in R^3, then either M lies in a totally geodesic R^3, or M lies fully in a totally geodesic R^6 as a special type of minimal surfaces. Here we say that a subset in $N^n(c)$ lies fully in $N^n(c)$ if it does not lie in a totally geodesic $N^{n-1}(c)$. In particular, his result implies that if $n=4$, $n=5$ or $n\geqq 7$, then the Riemannian structures of minimal surfaces lying fully in R^n are different from those of minimal surfaces in R^3. In the previous paper [13], we showed that if a minimal surface in $N^4(c)$ is isometric to a minimal surface in $N^3(c)$, then M lies in a totally geodesic $N^3(c)$. This result says that the Riemannian structures of minimal surfaces lying fully in $N^4(c)$ are different from those of minimal surfaces in $N^3(c)$. These results suggest that there are some relations between the Riemannian structures and the codimension of minimal surfaces in $N^n(c)$.

In [4] Johnson gave a nice class of minimal surfaces in $N^n(c)$ which can be intrinsically characterized by the generalized Ricci condition. They are called exceptional minimal surfaces and are related to the theory of harmonic sequences in [1], [2] and [17] (see [15]).

In this paper we will discuss the relation between the Riemannian structures and the codimension of exceptional minimal surfaces in H^n and R^n. Our results are stated as follows:

THEOREM 1. *Suppose that an exceptional minimal surface lying fully in H^{n_1} is isometric to an exceptional minimal surface lying fully in H^{n_2}. Then $n_1=n_2$.*

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THEOREM 2. (i) *Suppose that an exceptional minimal surface lying fully in* R^{n_1} *is isometric to an exceptional minimal surface lying fully in* R^{n_2}. *Then either* (a) $n_1=n_2$, (b) n_1 *is odd and* $n_2=2n_1$, *or* (c) n_2 *is odd and* $n_1=2n_2$.

(ii) *The case* (b) *or* (c) *of* (i) *may occur: Every exceptional minimal surface lying fully in* R^{2m-1} *is locally isometric to an exceptional minimal surface lying fully in* $R^{2(2m-1)}$.

Remark. (i) Theorem 1 says that if $n_1 \neq n_2$, then the Riemannian structures of exceptional minimal surfaces lying fully in H^{n_1} are different from those of exceptional minimal surfaces lying fully in H^{n_2}.

(ii) See Corollaries 6.1 and 6.2 of [4] for the rigidity of exceptional minimal surfaces lying fully in $N^n(c)$ among all exceptional minimal surfaces lying fully in $N^n(c)$.

(iii) Theorem 1 is a generalization of Theorem 2 of [11].

(iv) The spherical case is treated in [1], [8] and [15] as pseudo-holomorphic or superconformal minimal surfaces in the sense of [1] and [2].

1. Exceptional minimal surfaces

In this section we follow [4] and recall exceptional minimal surfaces. Suppose that M is a minimal surface lying fully in $N^n(c)$. Let the integer m be given by $n=2m-1$ or $2m$, and let indices have the following ranges:

$$1 \leqq j,\ k \leqq 2, \quad 3 \leqq \alpha \leqq n, \quad 1 \leqq A,\ B \leqq n.$$

Let $\tilde{e}_A$ be a local orthonormal frame field on $N^n(c)$, and let $\tilde{\theta}_A$ be the coframe dual to $\tilde{e}_A$. Then $d\tilde{\theta}_A=\Sigma_B \tilde{\omega}_{AB} \wedge \tilde{\theta}_B$, where $\tilde{\omega}_{AB}$ are the connection forms on $N^n(c)$.

Suppose that e_j is a local orthonormal frame field on M and that the frame $\tilde{e}_A$ is chosen so that on M, $\tilde{e}_j=e_j$ and $\tilde{e}_\alpha$ are normal to M. When forms and vectors on $N^n(c)$ are restricted to M, let them be denoted by the same symbol without tilde: $\theta_A=\tilde{\theta}_A|_M$, $\omega_{AB}=\tilde{\omega}_{AB}|_M$ and $e_A=\tilde{e}_A|_M$. Then $\omega_{\alpha j}=\Sigma_k h_{\alpha jk}\theta_k$, where $h_{\alpha jk}$ are the coefficients of the second fundamental form of M.

Let T_xM and $T_xN^n(c)$ denote the tangent spaces of M and $N^n(c)$, respectively, at a point x. Curves on M through x have their first derivatives at x in T_xM, but higher order derivatives will have components normal to M. The space spanned by the derivatives of order up to r is called the r-th osculating space of M at x, denoted $T_x^{(r)}M$.

The r-th normal space of M at x, denoted $\mathrm{Nor}_x^{(r)}M$, is the orthogonal complement of $T_x^{(r)}M$ in $T_x^{(r+1)}M$. At generic points of M, the dimension of $\mathrm{Nor}_x^{(r)}M$ is 2 when $1 \leqq r \leqq m-2$, and the dimension of $\mathrm{Nor}_x^{(m-1)}M$ is 1 or 2, depending on whether n is odd or even. Those normal spaces that have dimension 2 are called the normal planes of M. Let β_n denote the number of normal planes possessed by M at generic points: $\beta_n=m-2$ if $n=2m-1$, and $\beta_n=m-1$ if $n=2m$.

Choose the normal vectors e_α so that $\mathrm{Nor}_x^{(r)}M$ is spanned by $\{e_{2r+1}, e_{2r+2}\}$, where $1 \leqq r \leqq \beta_n$. When $n=2m-1$, $\mathrm{Nor}_x^{(m-1)}M$ is spanned by $\{e_{2m-1}\}$. Set $\varphi=\theta_1+i\theta_2$. Then there are H_α such that $H_\alpha=h_{\alpha 11}+ih_{\alpha 12}$ for $\alpha=3$ and 4, for each r with $2 \leqq r \leqq \beta_n$

$$H_{2r-1}\omega_{\alpha, 2r-1}+H_{2r}\omega_{\alpha, 2r}=H_\alpha\bar{\varphi}$$

where $\alpha=2r+1$ and $2r+2$, and when $n=2m-1$

$$H_{2m-3}\omega_{2m-1, 2m-3}+H_{2m-2}\omega_{2m-1, 2m-2}=H_{2m-1}\bar{\varphi}$$

(see [4], cf. [3]), which correspond to the higher fundamental forms of M.

The r-th normal plane, $\mathrm{Nor}_x^{(r)}M$, of M is called exceptional if $H_{2r+1}^2+H_{2r+2}^2=0$. The minimal surface M is called exceptional if all of its normal planes are exceptional. Note that when $n=2m-1$, $\mathrm{Nor}_x^{(m-1)}M$ is a line, not a plane, and the notion of exceptionality does not apply. So, in particular, every minimal surface in $N^3(c)$ is exceptional.

See [15] for the relation between exceptional minimal surfaces in S^n and the theory of harmonic sequences in [1], [2] and [17].

Exceptional minimal surfaces are intrinsically characterized in Theorems A and B of [4], which is a generalization of the Ricci condition for minimal surfaces in $N^3(c)$ (see [5, Theorem 8]). So exceptional minimal surfaces lying fully in $N^n(c)$ may be seen as natural generalizations of minimal surfaces in $N^3(c)$, in particular when n is odd.

Examples of exceptional minimal surfaces can be constructed as in Remark 3 of [15].

2. A lemma

For a 2-dimensional Riemannian manifold (M, ds^2), let K and Δ denote the Gaussian curvature and the Laplacian of (M, ds^2), respectively. For each $c \leqq 0$, set

$$(1) \qquad A_0^c=1/2, \quad A_1^c=c-K,$$
$$A_{p+1}^c=\begin{cases} A_p^c[\Delta \log(A_p^c)+A_p^c/A_{p-1}^c-2(p+1)K], & \text{if } A_p^c>0, \\ 0, & \text{otherwise,} \end{cases}$$

(cf. [4]). Suppose that $A_p^c>0$ for $1 \leqq p \leqq m-1$ and the metric $(A_{m-1}^c)^{1/m}ds^2$ is flat. Then

$$\frac{K}{(A_{m-1}^c)^{1/m}}-\frac{1}{2(A_{m-1}^c)^{1/m}}\Delta \log\{(A_{m-1}^c)^{1/m}\}=0,$$

or equivalently,

$$(2) \qquad \Delta \log(A_{m-1}^c)=2mK.$$

Then we have the following:

LEMMA. *For* $0\leqq r\leqq m-1$ *and* $p\geqq 2m$,

$$(3)\qquad \begin{aligned} &A_{m-1+r}^{c}=(A_{m-1}^{c})^{2}/A_{m-1-r}^{c}, \\ &A_{2m-1}^{c}=4c(A_{m-1}^{c})^{2}, \quad A_{p}^{c}=0. \end{aligned}$$

Proof. The first identity (3) is true for $r=0$. By (1) and (2) we obtain

$$A_{m}^{c}=A_{m-1}^{c}[\Delta\log(A_{m-1}^{c})+A_{m-1}^{c}/A_{m-2}^{c}-2mK]=(A_{m-1}^{c})^{2}/A_{m-2}^{c}.$$

So, (3) holds for $r=1$. Next suppose that (3) holds for r and $r+1$ with $0\leqq r\leqq m-3$. Then by (1), (2) and the assumption,

$$\begin{aligned} A_{m+r+1}^{c}= &\{(A_{m-1}^{c})^{2}/A_{m-r-2}^{c}\}[2\Delta\log(A_{m-1}^{c})-\Delta\log(A_{m-r-2}^{c}) \\ &+A_{m-r-1}^{c}/A_{m-r-2}^{c}-2(m+r+1)K] \\ &=\{(A_{m-1}^{c})^{2}/A_{m-r-2}^{c}\}[A_{m-r-1}^{c}/A_{m-r-2}^{c}-\Delta\log(A_{m-r-2}^{c})+2(m-r-1)K] \\ &=\{(A_{m-1}^{c})^{2}/A_{m-r-2}^{c}\}(A_{m-r-2}^{c}/A_{m-r-3}^{c}) \\ &=(A_{m-1}^{c})^{2}/A_{m-r-3}^{c}. \end{aligned}$$

By induction, (3) is true from A_{m-1}^{c} to A_{2m-2}^{c}. Then by (1) and (2),

$$\begin{aligned} A_{2m-1}^{c}&=2(A_{m-1}^{c})^{2}[2\Delta\log(A_{m-1}^{c})+2A_{1}-2(2m-1)K] \\ &=4c(A_{m-1}^{c})^{2}\leqq 0. \end{aligned}$$

Thus by (1) we have $A_{p}^{c}=0$ for $p\geqq 2m$. q.e.d.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Let ds^{2} be the induced metric and let A_{p}^{-1} be as in (1). By Theorem A of [4], it is easy to see that $n_{1}=n_{2}$ if both n_{1} and n_{2} are even. So we assume that $n_{1}=2m_{1}-1$. Then by Theorem A of [4], $A_{p}^{-1}\geqq 0$ for $1\leqq p\leqq m_{1}-1$ with equality only at isolated points, and the metric $(A_{m_{1}-1}^{-1})^{1/m_{1}}ds^{2}$ is flat on

$$M^{*}=\{x\in M;\, A_{p}^{-1}>0 \text{ for } 1\leqq p\leqq m_{1}-1\},$$

which is M minus isolated points. Then Lemma is valid on M^{*} for $m=m_{1}$. By the lemma, $A_{p}^{-1}>0$ for $1\leqq p\leqq 2m_{1}-2$ and $A_{2m_{1}-1}^{-1}<0$ on M^{*}. So by Theorem A of [4], we find that $n_{2}=2m_{2}$ is impossible. If $n_{2}=2m_{2}-1$, then as above, we have $A_{p}^{-1}>0$ for $1\leqq p\leqq 2m_{2}-2$ and $A_{2m_{2}-1}^{-1}<0$ on some open dense subset, which is possible only when $m_{1}=m_{2}$. q.e.d.

Proof of Theorem 2. (i) Let ds^{2} be the induced metric and let A_{p}^{0} be as in (1). We assume that $n_{1}=2m_{1}-1$ and $n_{2}=2m_{2}$. Then, using Theorem A of [4] and Lemma as in the proof of Theorem 1, we have $A_{p}^{0}>0$ for $1\leqq p\leqq 2m_{1}-2$ and $A_{2m_{1}-1}^{0}=0$ on some open dense subset. As $n_{2}=2m_{2}$, by Theorem A of [4], we have $m_2=2m_1-1=n_1$. By the same argument as in Theorem 1, we can see that $n_1=n_2$ if both n_1 and n_2 are even, or both n_1 and n_2 are odd.

(ii) Let M be an exceptional minimal surface lying fully in R^{2m-1}. Let ds^2 be the induced metric and let A_p^0 be as in (1). Using Theorem A of [4] and Lemma as above, we have $A_p^0>0$ for $1\leqq p\leqq 2m-2$ and $A_{2m-1}^0=0$ on

$$M^{**}=\{x\in M;\ A_p^0>0 \text{ for } 1\leqq p\leqq m-1\},$$

which is M minus isolated points. By Theorem B of [4], we find that (M^{**}, ds^2) is locally isometric to an exceptional minimal surface lying fully in $R^{2(2m-1)}$. q.e.d.

4. A question

There are minimal surfaces lying fully in R^{n_1} which are isometric to minimal surfaces lying fully in R^{n_2} with $n_1\neq n_2$ (see [7, Chapter IV], cf. Theorem 2), and there are minimal surfaces lying fully in S^{n_1} which are isometric to minimal surfaces lying fully in S^{n_2} with $n_1\neq n_2$ (see [8] and [15]). But in the hyperbolic case, no such examples are known (cf. [13] and Theorem 1). So we shall ask the following:

QUESTION. Are there any minimal surfaces lying fully in H^{n_1} which are isometric to minimal surfaces lying fully in H^{n_2} with $n_1\neq n_2$?

If the answer to this question is negative, then it would mean that the difference of the dimension is (locally) essential in the hyperbolic metric, with respect to the Riemannian structures of minimal surfaces.

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