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STABLE MAPS AND LINKS IN 3-MANIFOLDS

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1. Introduction

Let $h : S^3 \to \mathbf{R}$ be the standard Morse function with exactly two critical points. It is known that, if K is an embedded circle (or a knot) in S^3 such that $h|K : K \to \mathbf{R}$ is a Morse function with exactly two critical points, then K is trivial (i.e., it bounds an embedded 2-disk in S^3). The main purpose of this paper is to study links — finite disjoint union of embedded circles — in 3-manifolds using stable maps into 2-manifolds (or surfaces) instead of Morse functions. This is a continuation of the study begun in [**S1**,§6].

Let $g : M^3 \to N^2$ be a smooth map of a closed 3-manifold into a surface. Then g can be approximated, in the sense of C^∞-topology, by a stable map $f : M^3 \to N^2$, which can be regarded as a variant of Morse functions. Thus there are plenty of stable maps on a 3-manifold. The singularities of a stable map can be written down by normal forms explicitly, as non-degenerate critical points of a Morse function can be given by explicit normal forms by the Morse Lemma ([**Mi**]). In fact, there are exactly three types of singularities for a stable map: definite fold points, indefinite fold points, and cusp points. Stable maps have been studied by many authors [**L1, L2, BdR, KLP, ML1, ML2, ML3, ML4, S1, S2, S3, MPS**] and a lot of interesting results have been obtained.

Given a link L in M^3, we can always change L by an isotopy so that $f|L : L \to N^2$ is an immersion with normal crossings. In this paper we try to obtain information on L using f and the immersion $f|L$. In [**S1**] we have considered the case where f is a simple stable map, and have given a characterization of graph links in terms of such maps. In this paper, we consider a more restricted class of maps, namely full-definite simple stable maps ([**S2**]), and show that, if $f|L$ is an embedding whose image contains no critical value for a full-definite simple stable map $f : S^3 \to \mathbf{R}^2$ and a link L in S^3, then L is trivial (i.e., L bounds disjoint embedded 2-disks in S^3).

Another important fact about stable maps is that the singular point set $S(f)$ of a stable map $f : M^3 \to N^2$ is a smooth closed 1-dimensional submanifold of M^3; i.e., it is a link in M^3. Furthermore, the regular fiber $f^{-1}(a)$ for a regular value $a \in N^2$ is also a link in M^3. Note that for every link L in S^3, there exists a stable map $f_1 : S^3 \to \mathbf{R}^2$ whose singular set $S(f_1)$ coincides with L. There also exists a stable map $f_2 : S^3 \to \mathbf{R}^2$ such that $f_2^{-1}(a) = L$ for a regular value $a \in \mathbf{R}^2$. In this paper, using the above facts, we define integer invariants of a link L in S^3, which measure a kind of complexity of such maps as f_1 and f_2 above. Although these invariants are thus defined properly, we

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have very few tools to calculate them explicitly and they are yet to be studied.

The paper is organized as follows. In §2, we recall the definition and some basic theorems about stable maps. In §3, we recall the notion of the Stein factorization of a stable map, which is a principal tool in the study of stable maps, and also recall the definition of branched surfaces and fold maps. §4 is devoted to the study of links in S^3 using full-definite simple stable maps. In §5 we define some invariants of a link in S^3 using stable maps.

Throughout the paper, all manifolds and maps are of class C^∞.

2. Preliminaries

Let M be a closed orientable 3-manifold and N a 2-manifold ($\partial N = \emptyset$). We denote by $C^\infty(M, N)$ the set of the smooth maps of M into N with the C^∞-topology (for details, see [**GG**], for example). For $f \in C^\infty(M,N)$, $S(f)$ denotes the *singular set* of f; i.e., $S(f)$ is the set of the points in M where the rank of the differential df is strictly less than 2. We call f *stable* if there exists an open neighborhood U of f in $C^\infty(M,N)$ such that every g in U is right-left equivalent to f; i.e., there exist diffeomorphisms $\Phi : M \to M$ and $\varphi : N \to N$ satisfying $g = \varphi \circ f \circ \Phi^{-1}$. Note that the stable maps are open dense in $C^\infty(M,N)$ [**Ma**].

It is known that a smooth map $f : M \to N$ is stable if and only if it satisfies the following local and global conditions (see [**L1, L2**], for example): *For all $p \in S(f)$, there exist local coordinates (u, x, y) centered at p and (X, Y) centered at $f(p)$ such that f has one of the following forms:*

L_1) $\quad X \circ f = u, \quad Y \circ f = x^2 + y^2 \quad$ (*p : definite fold point*) *or*

L_2) $\quad X \circ f = u, \quad Y \circ f = x^2 - y^2 \quad$ (*p : indefinite fold point*) *or*

L_3) $\quad X \circ f = u, \quad Y \circ f = y^2 + ux - x^3 \quad$ (*p : cusp point*);

and

G_1) *If $p \in M$ is a cusp point, then $f^{-1}(f(p)) \cap S(f) = \{p\}$,*

G_2) *$f|(S(f) - \{\text{cusp points}\})$ is an immersion with normal crossings.*

We put $S_0(f) = \{\text{definite fold points}\}$, $S_1(f) = \{\text{indefinite fold points}\}$ and $C(f) = \{\text{cusp points}\}$. Note that $S(f)$ is a smooth closed 1-dimensional submanifold of M (i.e., a link in M) and that $C(f)$ is a finite set. We call a component of $S_0(f)$ a *definite fold* and a component of $S_1(f)$ an *indefinite fold*. Note that, for a regular value $a \in N$, $f^{-1}(a)$ is also a link in M.

Next we recall some classes of stable maps. A stable map $f : M \to N$ is called *special* (or *special generic*) if $S(f)$ consists only of definite fold points. This class of stable maps has been first defined by Burlet and de Rham [**BdR**], who have shown that a closed orientable 3-manifold M admits a special stable map into $\mathbf{R}^2$ if and only if M is diffeomorphic to the connected sum of S^3 and some copies of $S^1 \times S^2$. On the other hand, Levine [**L1**] has shown that, if N is an orientable surface, then every stable map $f : M \to N$ is homotopic to a stable map having no cusp points. In particular, every closed orientable 3-manifold admits a stable map into $\mathbf{R}^2$ without cusp points. An intermediate class of stable maps can be defined as follows. A stable map $f : M \to N$ is *simple* if f has no cusp points and, for all $p \in S(f)$, the connected component of $f^{-1}(f(p))$ containing

p intersects $S(f)$ only at p. Note that, for a stable map f, a connected component of $f^{-1}(f(p))$ intersects $S(f)$ at most 2 points by the global condition G_2). In particular, if $f|S(f)$ is a smooth embedding, then f is simple. We also note that a special stable map is always simple. In [**S1**], we have shown that a closed orientable 3-manifold admits a simple stable map into a surface if and only if it is a graph manifold. Recall that a compact orientable 3-manifold M is a *graph manifold* if there exist disjointedly embedded tori $T_1, \cdots, T_r$ in $\mathrm{Int}M$ such that each component of $M - \amalg_{i=1}^r \mathrm{Int}N(T_i)$ is an S^1-bundle over a surface, where $N(T_i)$ is a tubular neighborhood of T_i in $\mathrm{Int}M$. Note that graph manifolds have been extensively studied and have been completely classified using finite coded graphs [**N, JS, Jo**]. Nevertheless the class of graph manifolds is rich enough; for example, it contains Seifert fibered spaces and the link 3-manifolds which arise around an isolated complex surface singularity.

Next we discuss the study of links in 3-manifolds using stable maps.

DEFINITION 2.1 ([**S1**]). Let $f : M \to N$ be a stable map of a closed 3-manifold into a surface. A link L in M is said to be *f-trivial* if $f|L$ is a smooth embedding and $f(L) \cap f(S(f)) = \emptyset$, after we move L by an isotopy if necessary.

For some examples, see [**S1**, §6]. Note that every regular fiber of a stable map f is f-trivial.

In our previous paper [**S1**], we have shown that every link in a 3-manifold M is f-trivial for some stable map $f : M \to \mathbf{R}^2$. Because of this fact, the definition of an f-trivial link seems nonsense. However, if we restrict the class of stable maps, we have an interesting result: in [**S1**], we have shown that a link in a 3-manifold M is a graph link if and only if it is f-trivial for some simple stable map $f : M \to N$ into a surface N. Recall that a link in a compact 3-manifold M is a *graph link* if its exterior $M - \mathrm{Int}N(L)$ is a graph manifold, where $N(L)$ is a tubular neighborhood of L in M.

In §4, we consider a more restricted class of stable maps than that of simple stable maps, and study f-trivial knots with respect to such stable maps f.

3. Stein factorization and fold maps

First we recall the notion of the Stein factorization of a stable map $f : M \to N$ of a closed 3-manifold into a surface. For $p, p' \in M$, we define $p \sim p'$ if $f(p) = f(p')$ and p, p' are in the same connected component of $f^{-1}(f(p))$. Let $W_f (= M/\sim)$ be the quotient space of M under the equivalence relation and we denote by $q_f : M \to W_f$ the quotient map. We have a unique map $\bar{f} : W_f \to N$ such that $f = \bar{f} \circ q_f$. The space W_f or the decomposition $f = \bar{f} \circ q_f$ is called the *Stein factorization* of f ([**BdR, L2, KLP**]). In general, W_f is not a manifold; however, it is homeomorphic to a 2-dimensional finite CW complex ([**KLP, L2**]).

Let $f : M \to N$ be a simple stable map. Then every point $x \in W_f$ has a neighborhood as in [**S1**, Figure 1]. We set $\Sigma_0(W_f) = q_f(S_0(f)), \Sigma_1(W_f) = q_f(S_1(f))$ and $\Sigma(W_f) = \Sigma_0(W_f) \cup \Sigma_1(W_f) (= q_f(S(f)))$. Note that, if $S(f) \neq \emptyset$, then $W_f - \Sigma(W_f)$ is an open 2-manifold, which is equipped with a smooth structure such that $\bar{f}|(W_f - \Sigma(W_f))$ is an immersion. Let C be a component of $\Sigma(W_f)$. If $C \subset \Sigma_0(W_f)$, then $N(C)$ is homeomorphic to $I \times C$ ($I = [0,1]$), where $N(C)$ is a regular neighborhood of C in W_f

and $\{0\} \times C$ is identified with C. If $C \subset \Sigma_1(W_f)$, then $N(C)$ is homeomorphic to a Y-bundle over S^1, where $Y = \{r\exp(\sqrt{-1}\theta) \in \mathbf{C}; 0 \leqq r \leqq 1, \theta = 0, \pm 2\pi/3\}$ ([**L2**, p.12]). Furthermore, near C, the map $\bar{f} : W_f \to N$ is locally C^0 right-left equivalent to $\pi \times \mathrm{id} : Y \times I \to [-1/2, 1] \times I$, where $\pi : Y \to [-1/2, 1]$ is the projection to the real line $\mathbf{C} \to \mathbf{R}$ restricted to Y. Thus the monodromy homeomorphism $\alpha : Y \to Y$ of the Y-bundle $N(C)$ over C must satisfy $\pi \circ \alpha = \pi$, and hence $\alpha = \mathrm{id}$ or τ, where $\tau : Y \to Y$ is the complex conjugation restricted to $Y \subset \mathbf{C}$. In other words, $N(C)$ is homeomorphic to $Y \times S^1$ or $Y \times_\tau S^1 = Y \times I/(y,1) \sim (\tau(y), 0)$.

In [**S2**], as a generalization of the quotient map $q_f : M \to W_f$ of a simple stable map $f : M \to N$, we have defined fold maps of 3-manifolds into branched surfaces. A branched surface is a compact Hausdorff space such that each point has a conic neighborhood as in [**S1**, Figure 1]. We can define a smooth structure on a branched surface, and a smooth map $q : M \to W$ of a closed 3-manifold M into a smooth branched surface W is a fold map if it is semi-locally right-left equivalent to the standard quotient maps of a simple stable map corresponding to the regular point, the definite fold point and the indefinite fold point. For precise definitions see [**S2**]. Note that the quotient map of a simple stable map is always a fold map and that a given fold map $q : M \to W$ arises as the quotient map of a simple stable map if and only if W can be immersed into a surface and that not every branched surface can be immersed into a surface.

The principal advantage of this generalization is that one can define surgery operations of fold maps. In [**S2**], we have defined 8 types of surgery operation: six of them consist of removing a tubular neighborhood of a regular fiber or a definite fold and inserting a certain standard map, and the other surgery operations are the connected sum operation and the operation which removes tubular neighborhoods of two definite folds and sewing the torus boundaries. The main result of [**S2**] is that every fold map of a closed orientable 3-manifold into a branched surface is constructed from some standard maps — the quotient maps of special stable maps $S^3, S^1 \times S^2 \to \mathbf{R}^2$ and the S^1-fibrations over S^2 — by a finite sequence of the above surgery operations. For details, see [**S2**].

4. Full-definite simple stable maps and links in S^3

In this section, we study links in S^3 using full-definite simple stable maps. In the following, we use the same notations as in [**S2**] for fold maps and branched surfaces. In [**S1, S2**], we have shown that, for a fold map $q : M \to W$ of a closed orientable 3-manifold into a branched surface, we have $\sharp S_0(q) - 1 \leqq \sharp S_1(q) + b_1(M)$, where $\sharp$ denotes the number of connected components and b_1 denotes the first betti number. In particular, if $f : M \to N$ is a simple stable map of a closed orientable 3-manifold into a surface, then $\sharp S_0(f) - 1 \leqq \sharp S_1(f) + b_1(M)$.

DEFINITION 4.1. Let $q : M \to W$ be a fold map. We say that q is *full-definite* if $\sharp S_0(q) - 1 = \sharp S_1(q) + b_1(M)$. A simple stable map $f : M \to N$ is *full-definite* if its quotient map $q_f : M \to W_f$ is full-definite as a fold map.

Let $f_0 : S^3 \to \mathbf{R}^2$ be the special stable map defined by $f = \pi|S^3$, where $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ is the standard projection and S^3 is the unit 3-sphere in $\mathbf{R}^4$ Define $Q_0 : S^3 \to D^2$ to be the quotient map of f_0 (note that the quotient space of f_0 is homeomorphic to the

2-disk D^2). Furthermore let $f_1 : S^1 \times S^2 \to \mathbf{R}^2$ be the special stable map defined by $f_1 = \eta \circ (\mathrm{id}_{S^1} \times h)$, where $\eta : S^1 \times \mathbf{R} \to \mathbf{R}^2$ is an embedding and $h : S^2 \to \mathbf{R}$ is the standard Morse function with exactly two critical points. Define $Q_1 : S^1 \times S^2 \to S^1 \times [-1, 1]$ to be the quotient map of f_1 (note that the quotient space of f_1 is homeomorphic to $S^1 \times [-1, 1]$). Then we can characterize full-definite fold maps as follows.

THEOREM 4.2 ([**S2**]). *Let $q : M \to W$ be a fold map of a closed orientable 3-manifold into a smooth branched surface. Then q is full-definite if and only if q is right-left equivalent to a fold map obtained from Q_0 and Q_1 by applying the operations* (I), $(\mathrm{II})_1$ *and* $(\mathrm{II})_2$ *finitely many times.*

For the precise definitions of the operations, see [**S2**, §4]. Full-definite fold maps behave very much like quotient maps of special stable maps. For example, we have the following.

PROPOSITION 4.3 ([**S2**]). *Let $q : M \to W$ be a full-definite fold map of a closed orientable 3-manifold into a smooth branched surface. If $b_1(M) = 0$, then M is diffeomorphic to S^3 and every component of $S(q)$ is the trivial knot in S^3.*

Recall that, for a special stable map $f : S^3 \to \mathbf{R}^2$, $S(f)$ is the trivial knot in S^3 ([**BdR**]). Note also that a special stable map of S^3 into a surface is always full-definite.

One of the main results of this paper is the following.

THEOREM 4.4. *Let $f : S^3 \to N$ be a full-definite simple stable map into a surface. Then a smooth knot K in S^3 is trivial if and only if it is f-trivial in the sense of Definition* 2.1.

LEMMA 4.5. *Let $f : M \to N$ be a stable map of a closed 3-manifold into a surface and L a trivial link in M. Then L is f-trivial.*

Proof. Take a regular value $a \in f(M) \subset N$ and let $\Delta \subset N$ be a small open neighborhood of a disjoint from $f(S(f))$. By the implicit function theorem, we have a local coordinate (u, x, y) centered at a point $b \in f^{-1}(a)$ and defined in an open set contained in $f^{-1}(\Delta)$ and a local coordinate (X, Y) centered at $a \in N$ and defined in an open set contained in Δ such that $X \circ f = x$ and $Y \circ f = y$. Let L_0 be disjoint embedded circles in the (x, y)-plane with respect to the above coordinate such that $\sharp L_0 = \sharp L$. Then we see that L_0 is isotopic to L, that $f|L_0$ is an embedding and that $f(L_0) \cap f(S(f)) = \emptyset$. Thus L is f-trivial. $\square$

LEMMA 4.6. *Let $f : M \to N$ be a stable map of a closed 3-manifold into a surface and L a link in M. If L is f-trivial, then $q_f|L : L \to W_f - q_f(S(f))$ is a smooth embedding.*

Proof. By our assumption, the composition

$$f|L = \bar{f} \circ q_f|L : L \xrightarrow{q_f} W_f - q_f(S(f)) \xrightarrow{\bar{f}} N$$

is an embedding. Then the result follows immediately. □

Proof of Theorem 4.4. By Lemma 4.5, the necessity is clear. On the other hand, by Lemma 4.6, the sufficiency follows from the following.

LEMMA 4.7. *Let $q : S^3 \to W$ be a full-definite fold map and K a knot in S^3. If $q(K) \cap \Sigma(W) = \emptyset$ and $q|K : K \to W - \Sigma(W)$ is an embedding, then K is trivial.*

Proof. By Theorem 4.2, q is right-left equivalent to a fold map obtained from Q_0 by applying the operations finitely many times. We prove the lemma by the induction on the number of the operations needed to construct q. Note that, in our case, Q_1 is not necessary, since the source manifold is S^3.

CASE 1. The case where q is right-left equivalent to Q_0.

We may assume $q = Q_0$. Let A be the closure of the connected component of $D^2 - q(K)$ which contains $q(S(q))$. Then A is diffeomorphic to the annulus and $q^{-1}(A)$ is diffeomorphic to the solid torus. Furthermore, $q^{-1}(A)$ is a tubular neighborhood of $S(q)$. Since $q|K$ is an embedding, $K(\subset \partial(q^{-1}(A)))$ intersects the boundary of a meridian disk of $q^{-1}(A)$ transversely in one point; hence K is isotopic to $S(q)$, which is the trivial knot ([**BdR**]).

CASE 2. The case where q is right-left equivalent to the fold map obtained by applying the operation $(\mathrm{II})_1$ or $(\mathrm{II})_2$ to a fold map $q_1 : S^3 \to W_1$.

Note that q_1 is also full-definite. We may assume that q is the fold map obtained by applying the operation $(\mathrm{II})_1$ or $(\mathrm{II})_2$ to q_1. Then W is homeomorphic to the branched surface obtained by attaching $S^1 \times Y$ to W_1 along a component C of $\Sigma_0(W_1)$. Let $W - \Sigma(W) = R_1 \cup \cdots \cup R_s \cup R'_1 \cup R'_2$ be the components, where $R_1, \cdots, R_s$ correspond to W_1 and R'_1, R'_2 to $S^1 \times Y$. Since $q(K) \cap \Sigma(W) = \emptyset$ and $q(K)$ is connected, $q(K)$ is contained in some R_i or R'_j.

CASE 2–1. $q(K) \subset R_i$.

Let $N(C)$ be a regular neighborhood of C in W_1. Then by the definition of the operation (II), $q_1 = q$ on $S^3 - q_1^{-1}(\mathrm{Int}N(C))$. Furthermore, if $N(C)$ is small enough, $K \subset S^3 - q_1^{-1}(\mathrm{Int}N(C))$. This implies $q_1(K) \cap \Sigma(q_1) = \emptyset$ and that $q_1|K$ is an embedding. Then by the induction hypothesis, we see that K is trivial.

CASE 2–2. $q(K) \subset R'_j$.

Note that R'_j is diffeomorphic to the open annulus $S^1 \times \mathbf{R}$. Since $q(K)$ is an embedded circle in R'_j, $q(K)$ either bounds a disk in R'_j or is isotopic to the core $S^1 \times \{0\}$ of the open annulus.

CASE 2–2–a. The case where $q(K)$ is isotopic to the core of the open annulus.

By an argument similar to that of Case 1, we see that K is isotopic to a component of $S(q)$. Then by Proposition 4.3, we see that K is the trivial knot.

CASE 2–2–b. The case where $q(K)$ bounds a disk Δ in R'_j.

Take an embedded arc J in $\overline{R'_j}$ such that $\{a\} = \partial J \cap \mathrm{Int}\Delta \neq \emptyset \neq \partial J \cap \Sigma_0(W)$, that J is transverse to $\Sigma_0(W)$ and that J intersects $\partial\Delta$ transversely in one point. Then we see that $q^{-1}(J)$ is a 2-disk in S^3 and that $q^{-1}(\Delta)$ is a solid torus whose core $q^{-1}(a)$ is the trivial knot. Furthermore, $K(\subset \partial(q^{-1}(\Delta)))$ intersects $q^{-1}(J)$ transversely in one point. Hence K is the trivial knot in S^3.

CASE 3. The case where q is right-left equivalent to the fold map obtained by applying the operation (I) (connected sum operation) to fold maps $q_i : S^3 \to W_i$ $(i = 1, 2)$.

Note that q_i are also full-definite. We may assume that q is the fold map obtained by applying the operation (I) to q_i. By the definition of the operation (I), there exists a properly embedded arc J in $W - \Sigma_1(W)$ such that $J \cap \Sigma_0(W) = \partial J$, that J is transverse to $\Sigma_0(W)$, that $W - J$ consists of two connected components, and that $W'_i \cup_{J=B} D = W_i$, where W'_i $(i = 1, 2)$ are the closures of the two components of $W - J$, $D = \{(x, y) \in \mathbf{R}^2; x^2 + y^2 \leqq 1, x \geqq 0\}$ and $B = \{(x, y) \in D; x^2 + y^2 = 1\}$.

CASE 3–1. The case where $q(K) \cap J = \emptyset$.

We see that $K \subset q^{-1}(W'_i)$ for $i = 1$ or 2. Since $q_i = q$ on $q^{-1}(W'_i)$, we have $q_i(K) \cap \Sigma(W_i) = \emptyset$ and $q_i|K$ is an embedding. Hence, by the induction hypothesis, K is the trivial knot.

CASE 3–2. The case where $q(K) \cap J \neq \emptyset$.

Let R be the closure of the component of $W - \Sigma(W)$ which contains J. Using Theorem 4.2, we can prove that R is planar; i.e., R is a compact orientable surface of genus 0. Let $\hat{R}$ be the compact planar surface obtained by attaching 2-disks $D_1^2, \cdots, D_l^2$ along the boundary components of R not containing ∂J. Note that $\hat{R}$ is homeomorphic to the 2-disk. Hence $q(K)$ bounds a 2-disk Δ in $\hat{R}$. Let R_i $(i = 1, 2)$ be the two components of $\hat{R} - J$. Since D_j^2 are contained in $\hat{R} - J$, we may assume that $D_1^2, \cdots, D_k^2 \subset R_1$ and $D_{k+1}^2, \cdots, D_l^2 \subset R_2$. Then there exists a properly embedded arc J' in R transverse to $\Sigma_0(W)$ such that it intersects $q(K)$ transversely at two points and that $D_1^2, \cdots, D_k^2 \subset R'_1$ and $D_{k+1}^2, \cdots, D_l^2 \subset R'_2$, where R'_i $(i = 1, 2)$ are the two components of $\hat{R} - J'$. By **[S2**, Lemma 4.1], using this arc J', we can decompose q as the connected sum of fold maps q'_1 and q'_2. Then it is not difficult to see that q'_i are right-left equivalent to q_i. Thus we may assume that J intersects $q(K)$ transversely at two points from the beginning. Thus K intersects the 2-sphere $q^{-1}(J)$ embedded in S^3 transversely at two points. Hence, there exist knots K_i $(i = 1, 2)$ in S^3 such that $K_1 \sharp K_2 = K$, $q_i(K_i) \cap \Sigma(W_i) = \emptyset$ and $q_i|K_i$ are embeddings. By the induction hypothesis, K_i are the trivial knot and hence so is K. This completes the proof of Lemma 4.7 and hence Theorem 4.4. □

Note that Theorem 4.4 does not hold for links with two or more components. For example, see [**S1**, Example 6.2 (1)].

Next we consider the link type of a regular fiber of a full-definite simple stable map. Recall that a regular fiber of a stable map f is always f-trivial.

THEOREM 4.8. *Let $f : S^3 \to N$ be a full-definite simple stable map into a surface. If $a \in N$ is a regular value of f, then $f^{-1}(a)$ is a trivial link in S^3.*

Proof. Since $f^{-1}(a) = q_f^{-1}(\bar{f}^{-1}(a))$ and $\bar{f}^{-1}(a)$ is a finite set, for the proof of Theorem 4.8, it suffices to prove the following.

LEMMA 4.9. *Let $q : S^3 \to W$ be a full-definite fold map and $a_1, \cdots, a_l$ finite points in $W - \Sigma(W)$. Then $L = q^{-1}(a_1) \cup \cdots \cup q^{-1}(a_l)$ is a trivial link in S^3.*

Proof. We prove the lemma by the induction on the number of the operations needed to construct q as in Theorem 4.2.

CASE 1. The case where q is right-left equivalent to Q_0.

We may assume that $q = Q_0$. There exist disjoint embedded arcs J_i $(i = 1, \cdots, l)$ in D^2 such that J_i connects a_i and ∂D^2 and that J_i is transverse to ∂D^2. Then $q^{-1}(J_i)$ are disjoint embedded 2-disks in S^3 whose boundary coincides with L. Hence L is trivial.

CASE 2. The case where q is right-left equivalent to the fold map obtained by applying the operation $(\mathrm{II})_1$ or $(\mathrm{II})_2$ to a fold map $q_1 : S^3 \to W_1$.

Note that q_1 is also full-definite. We may assume that q is the fold map obtained by applying the operation $(\mathrm{II})_1$ or $(\mathrm{II})_2$ to q_1. Let $W - \Sigma(W) = R_1 \cup \cdots \cup R_s \cup R_1' \cup R_2'$ be the components as in Case 2 in the proof of Lemma 4.7. We may assume that $a_1, \cdots, a_k \in R_1 \cup \cdots \cup R_s, a_{k+1}, \cdots, a_m \in R_1'$ and $a_{m+1}, \cdots, a_l \in R_2'$. Then there exist a 2-disk Δ_1 in $\overline{R_1'}$ and disjoint embedded arcs J_j $(j = k+1, \cdots, m)$ in Δ_1 such that $\Delta_1 \cap \partial \overline{R_1'} (\subset \Sigma_0(W))$ is an arc, that the closure of $\partial\Delta_1 - (\Delta_1 \cap \overline{R_1'})$ is transverse to $\partial \overline{R_1'}$, that J_j connects a_j and $\Sigma_0(W)$, and that J_j is transverse to $\Sigma_0(W)$. Then $q^{-1}(\Delta_1)$ is a 3-ball embedded in S^3 and $q^{-1}(J_j)$ are disjoint embedded 2-disks in the 3-ball such that $\partial(q^{-1}(J_j)) = q^{-1}(a_j)$. We can also take a 3-ball $q^{-1}(\Delta_2)$ and disjoint 2-disks $q^{-1}(J_j)$ $(j = m+1, \cdots, l)$ for R_2'. Thus the link L is the split sum of $q^{-1}(a_1) \cup \cdots \cup q^{-1}(a_k)$ and two trivial links. Furthermore, $q^{-1}(a_1) \cup \cdots \cup q^{-1}(a_k) = q_1^{-1}(a_1) \cup \cdots \cup q_1^{-1}(a_k)$ is a trivial link by the induction hypothesis. Hence, L is trivial.

CASE 3. The case where q is right-left equivalent to the fold map obtained by applying the operation (I) to fold maps $q_i : S^3 \to W_i$ $(i = 1, 2)$.

Note that q_i are also full-definite. We may assume that q is the fold map obtained by applying the operation (I) to q_i. Let W_i' $(i = 1, 2)$ and J be as in Case 3 of the proof of Lemma 4.7. We may assume that $a_1, \cdots, a_k \in W_1'$ and $a_{k+1}, \cdots, a_l \in W_2'$. Then $q^{-1}(J)$ is a 2-sphere in S^3 disjoint from L; hence, L is the split sum of $q^{-1}(a_1) \cup \cdots \cup q^{-1}(a_k)$

and $q^{-1}(a_{k+1}) \cup \cdots \cup q^{-1}(a_l)$. By the induction hypothesis, both of these two links are trivial, and hence so is L. This completes the proof of Lemma 4.9 and hence Theorem 4.8. $\square$

By Lemma 4.9, we have a stronger result: for a full-definite simple stable map $f : S^3 \to N$ and regular values $a_1, \cdots, a_l \in N$, $f^{-1}(a_1) \cup \cdots \cup f^{-1}(a_l)$ is a trivial link in S^3. This fact holds for a larger class of maps. In fact, Lemma 4.9 holds for fold maps obtained by applying the operations (I), $(\mathrm{II})_1, (\mathrm{II})_2$ and $(\mathrm{II})_3$ finitely many times. The same proof works also for this case.

5. Invariants of links via stable maps

PROPOSITION 5.1. *For a link L in a closed orientable 3-manifold M, there exists a stable map $f : M \to \mathbf{R}^2$ and a regular value $a \in \mathbf{R}^2$ such that $L = f^{-1}(a)$ if and only if L bounds a compact orientable surface embedded in M.*

Proof. First suppose that $f : M \to \mathbf{R}^2$ is a stable map and $a \in \mathbf{R}^2$ is a regular value. Then there exists an embedded arc J in $\mathbf{R}^2$ such that $a \in \partial J$, J is transverse to f and $(\partial J - \{a\}) \cap f(M) = \emptyset$. Set $F = f^{-1}(J)$. Then we see that F is an embedded surface in M whose boundary coincides with $f^{-1}(a)$. Furthermore, since F is 2-sided in M and M is orientable, F is orientable.

Next let L be a link in M and F a compact orientable surface embedded in M such that $\partial F = L$. We may assume that F is connected. Let V be a tubular neighborhood of F in M. We may assume that $L \subset \partial V$. Since F is 2-sided in M, $S = \partial V$ is diffeomorphic the double of F and L corresponds to the common boundary. Thus there exists a Morse function $h : S \to \mathbf{R}$ such that $L = h^{-1}(b)$ for a regular value $b \in \mathbf{R}$. Let $N(S)$ be a tubular neighborhood of S in M. Then there exists a diffeomorphism $\varphi : N(S) \to S \times [-1,1]$ such that $\varphi(p) = (p, 0)$ for all $p \in S$. Define the smooth map $g : N(S) \to \mathbf{R}^2$ by $g(r) = (h \circ p_1 \circ \varphi(r), p_2 \circ \varphi(r))$ $(r \in N(S))$, where $p_1 : S \times [-1,1] \to S$ and $p_2 : S \times [-1,1] \to [-1,1]$ are the projections. On the other hand, $M - \mathrm{Int} N(S)$ consists of two components M_+ and M_-, where $M_+ \cap \varphi^{-1}(S \times \{1\}) \neq \emptyset$. Then we can extend the map g to a smooth map $\tilde{g} : M \to \mathbf{R}^2$ so that $\tilde{g}(M_+) \subset \{(x,y) \in \mathbf{R}^2; y \geqq 1\}$ and $\tilde{g}(M_-) \subset \{(x,y) \in \mathbf{R}^2; y \leqq -1\}$. Since the stable maps are dense in $C^\infty(M, \mathbf{R}^2)$, there exists a stable map $f : M \to \mathbf{R}^2$ arbitrarily close to $\tilde{g}$. Furthermore, since $\tilde{g}|N(S) = g$ is already stable, we may assume that $f = g$ on a neighborhood U of S and that $f(M - U) \cap \{(x, 0) \in \mathbf{R}^2\} = \emptyset$. Then we see that $f^{-1}(a) = L$ for $a = (b, 0)$, and f is a desired stable map. This completes the proof. $\square$

Note that the above proposition can be used to prove that every link is f-trivial for some stable map f.

Remark 5.2. Let M be a closed orientable 3-manifold such that $H_1(M; \mathbf{Z}) = 0$. Then a link in M always bounds a compact orientable surface embedded in M.

DEFINITION 5.3. Let L be a link in S^3. Then by Proposition 5.1 and Remark 5.2, there exists a stable map $f : S^3 \to \mathbf{R}^2$ such that $L = f^{-1}(a)$ for a regular value a. Define $F(L)$ to be the minimum bifurcation number $b(f, f_0)$ over all stable maps f as above, where $f_0 : S^3 \to \mathbf{R}^2$ is the standard special stable map as in §4 and $b(f, f_0) \in \mathbf{N} \cup \{0\}$ is the bifurcation number between f and f_0 defined in [**MPS**]. Roughly speaking, $b(f, f_0)$ is the minimum number of non-stable maps in generic families of maps connecting f and f_0 in $C^\infty(S^3, \mathbf{R}^2)$.

It is not difficult to see that $F(L) = 0$ if and only if L is the trivial knot.

Using the singular set instead of the regular fiber, we obtain the following invariant of a link in S^3.

DEFINITION 5.4. Let L be a link in S^3. Then by [**S3**], there exists a stable map $f : S^3 \to \mathbf{R}^2$ such that $S(f) = L$. Define $S(L)$ to be the minimum bifurcation number $b(f, f_0)$ over all stable maps f as above.

We see easily that $S(L) = 0$ if and only if L is the trivial knot.

DEFINITION 5.5. Let L be a link in S^3. Then by [**S1**, Proposition 6.3], there exists a stable map $f : S^3 \to \mathbf{R}^2$ such that L is f-trivial. Define $T(L)$ to be the minimum bifurcation number $b(f, f_0)$ over all stable maps f as above.

Note that there exist non-trivial links L such that $T(L) = 0$ ([**S1**, §6]).

LEMMA 5.6. *Let L be an n-component link in S^3. Then we have $F(L) \geqq n - 1$ and $S(L) \geqq n - 1$.*

Proof. Let $\{f_t\}_{t\in(-\varepsilon,\varepsilon)}$ $(\varepsilon > 0)$ be a π-stable homotopy of stable maps ([**ML4, C**]) of S^3 into $\mathbf{R}^2$ which has the unique bifurcation point at $t = 0$. Then we see that $|\sharp S(f_{-\varepsilon}) - \sharp S(f_\varepsilon)| \leqq 1$ and that the maximal numbers of components of a regular fiber of $f_{-\varepsilon}$ and f_ε differ by at most 1. Furthermore, for the special stable map $f_0 : S^3 \to \mathbf{R}^2$, $\sharp S(f_0) = 1$ and every regular fiber of f_0 is connected. Hence, by the definitions of the invariants, the required inequalities follow. □

For the n-component trivial link L_n, we can construct stable maps $f_1, f_2 : S^3 \to \mathbf{R}^2$ such that $b(f_1, f_0), b(f_2, f_0) \leqq n - 1$, that $S(f_1) = L_n$ and that a regular fiber of f_2 coincides with L_n. Combining this observation with Lemma 5.6, we have the following.

PROPOSITION 5.7. *For the n-component trivial link L_n in S^3, we have $F(L_n) = S(L_n) = n - 1$.*

It is not difficult to list up all stable maps $f : S^3 \to \mathbf{R}^2$ with $b(f, f_0) = 1$ using results of Mata-Lorenzo [**ML4**] and Chíncaro [**C**]. Examining these stable maps gives the following.

PROPOSITION 5.8. *For a link L in S^3, the following three are equivalent.*
(1) *L is the 2-component trivial link.*
(2) $F(L) = 1$.
(3) $S(L) = 1$.

The proof of the above proposition is left to the reader. It is also not difficult to prove the following.

PROPOSITION 5.9. *Let L be the torus $(3,3n)$-link ($n \in \mathbf{Z}$) in S^3. Then we have* $S(L) = 2$.

It would be an interesting problem to study the properties of the invariants defined above. For example, what property do they have about the connected sum operation?

REFERENCES

[BdR] O. Burlet and G. de Rham, *Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan*, l'Enseign. Math. **20** (1974), 275–292.

[C] E.A. Chíncaro, *Bifurcações de aplicações de Whitney*, Tese, IMPA, 1978.

[GG] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts in Math., no. 14, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[JS] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc., vol. 21, no. 220, Amer. Math. Soc., Rhode Island, 1979.

[Jo] K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lect. Notes in Math., vol. 761, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[KLP] L. Kushner, H. Levine and P. Porto, *Mapping three-manifolds into the plane I*, Bol. Soc. Mat. Mexicana **29** (1984), 11–33.

[L1] H. Levine, *Elimination of cusps*, Topology **3** (suppl. 2) (1965), 263–296.

[L2] H. Levine, *Classifying immersions into $\mathbf{R}^4$ over stable maps of 3-manifolds into $\mathbf{R}^2$*, Lect. Notes in Math., vol. 1157, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.

[ML1] L.E. Mata-Lorenzo, *The Stein factorization for stable maps and pi-stable arcs of maps from 3-manifolds into the plane*, Thesis, Brandeis University, 1986.

[ML2] L.E. Mata-Lorenzo, *Stable mappings from 3-manifolds into the plane and immersed polyhedra*, 5°-Encontro de Topologia, ICMSC-USP, 1986.

[ML3] L.E. Mata-Lorenzo, *Polyhedrons and stable maps from 3-manifolds into the plane*, unpublished note.

[ML4] L.E. Mata-Lorenzo, *Polyhedrons and pi-stable homotopies from 3-manifolds into the plane*, Bol. Soc. Bras. Mat. **20** (1989), 61–85.

[Ma] J. Mather, *Stability of C^∞ mappings : VI. The nice dimensions*, Proc. Liverpool Singularities - Symposium I, Lect. Notes in Math., vol. 192, Springer-Verlag, Berlin-Heidelberg-New York, 1971, pp. 207–253.

[Mi] J. Milnor, *Morse theory*, Ann. Math. Stud., no. 51, Princeton Univ. Press, 1963.

[MPS] W. Motta, P. Porto and O. Saeki, *Stable maps of 3-manifolds into the plane and their quotient spaces*, Proc. London Math. Soc. (to appear).

[N] W. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), 299–344.

[S1] O. Saeki, *Simple stable maps of 3-manifolds into surfaces*, preprint (1992).

[S2] O. Saeki, *Simple stable maps of 3-manifolds into surfaces II*, J. Fac. Sci. Univ. Tokyo, **40** (1993), 73–124.

[S3] O. Saeki, *Singular point set of a generic smooth map as a stratified submanifold*, preprint (1994).

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