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CONFORMAL GEOMETRY OF RICCI FLAT 4-MANIFOLDS

Dedicated to Professor Shoshichi Kobayashi on his sixtieth birthday

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Introduction

Let M be a compact, connected oriented 4-manifold. Let g be a smooth Riemannian metric on X. Then the Riemannian curvature tensor $R=R_{ijkl}$ is defined naturally, and the Ricci tensor $Ric=R_{ij}$ and the scalar curvature R_g by the trace of the Riemannian curvature tensor and the trace of the Ricci tensor, respectively.

Further we define the Weyl conformal tensor $W=W_{ijkl}$ by a linear combination of $R=R_{ijkl}$, Ric and R_g in such a way that the tensor W is invariant under a conformal change of metrics.

We will investigate in this paper the moduli $\mathcal{E}(M)$ of Ricci flat metrics on certain 4-manifolds M from 4-dimensional conformal geometry. Here we mean by the moduli the space of all Ricci flat metrics of volume one modulo diffeomorphisms of M.

Since a Ricci flat metric is Einstein, the moduli is considered naturally as the moduli of Einstein metrics of $R_g=0$.

We have indeed the following premoduli theorem due to K. Koiso ([16] and [5]).

Given an Einstein metric g. Then there is a finite dimensional real analytic submanifold $\mathcal{Z}$ in a slice $\mathcal{S}$ at g such that (i) $g\in\mathcal{Z}$, (ii) $T_g\mathcal{Z}$ coincides with the space of infinitesimal Einstein deformations and (iii) the intersection $\mathcal{E}(M)\cap\mathcal{S}$, called the premoduli around g, is a real analytic subvariety of $\mathcal{Z}$.

We restrict ourself to Ricci flat 4-manifolds having topological invariant $\chi+(3/2)\tau=0$, more precisely, manifolds whose universal covering is a K3 surface. We can then apply the Torelli type theorem for K3 surfaces together with the notion of anti-self-dual conformal structure and get a complete description of manifold structure of $\mathcal{E}(M)$.

By applying the Chern-Weil theorem for characteristic classes one has the following identity which is valid for an arbitrary Riemannian 4-manifold (M, g)

$$\chi(M)+\frac{3}{2}\tau(M)=\frac{1}{48\pi^2}\int_M\{R_g^2-3|Ric(g)|^2\}dv_g+\frac{1}{4\pi^2}\int_M|W^+(g)|^2dv_g$$

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Here $\chi(M)=\sum_k(-1)^k b_k(M)$ is the Euler number and $\tau(M)$ is the index $b^+(M)-b^-(M)$ of the cup product $q_M: H^2(M, \boldsymbol{Z})\times H^2(M, \boldsymbol{Z})\to H^4(M, \boldsymbol{Z})\cong\boldsymbol{Z}$ and W^+ denotes the self-dual Weyl conformal tensor, i.e., $W^+\in\Gamma(M, \Omega^+\otimes\Omega^+)$ ($\Omega^+=\Omega_g^+$ is the self-dual 2-form bundle of M).

From this we observe the following

PROPOSITION 0.1. *Let M be of $\chi(M)+(3/2)\tau(M)=0$.*

Let g be a metric on M satisfying $W^+=0$. Then the scalar curvature $R_g=0$ if and only if g is Ricci flat.

For a 4-manifold M satisfying $(\chi+(3/2)\tau)(M)=0$ the moduli $\mathcal{E}(M)$ of Ricci flat metrics on M coincides from this proposition with the quotient space of metrics satisfying the equations $W^+(g)=0$, $R_g=0$ modulo the diffeomorphism action.

Since the equation $W^+=0$ as a section of $\Omega^+\otimes\Omega^+$ and the sign of constant scalar curvature are conformal invariant, the latter space is regarded as the moduli of anti-self-dual conformal structures of zero scalar curvature, which we denote by $\mathcal{M}^{(0)}(M)$.

Metrics g and g_1 are said to be conformally equivalent if $g_1=fg$ for a positive function f and then the conformal equivalence defines a conformal structure $[g]$ represented by g. We say a conformal structure $[g]$ to be anti-self-dual if $W^+(g)=0$ and we define the moduli $\mathcal{M}(M)$ of anti-self-dual conformal structures on M to be the space of all anti-self-dual conformal structures on M modulo the diffeomorphism action so that we have

$$\mathcal{E}(M)\cong\mathcal{M}^{(0)}(M)\subset\mathcal{M}(M).$$

From the 4-dimensional speciality, for any (M, g) having $W^+=0$ there exists an elliptic complex which provides the moduli $\mathcal{M}(M)$ a real analytic variety structure.

In fact, the following sequence enjoys the ellipticity

$$0\longrightarrow\Gamma(M, T)\longrightarrow\Gamma(M, S_0^2(T^*))\longrightarrow\Gamma(M, S_0^2(\Omega^+))\longrightarrow 0 \tag{0.1}$$

with cohomology groups $\boldsymbol{H}^0$, $\boldsymbol{H}^1$, $\boldsymbol{H}^2$ of finite dimension so that one gets a real analytic variety theorem in terms of these cohomology groups ([11]).

The following theorem gives the characterization of Ricci flat 4-manifolds of $\chi+(3/2)\tau=0$.

THEOREM 0.2 ([8]). *Let (M, g) be a compact, connected oriented Riemannian 4-manifold.*

(i) *If (M, g) is Einstein, then $\chi(M)+(3/2)\tau(M)\geqq 0$ and the equality $\chi(M)+(3/2)\tau(M)=0$ holds if and only if g is Ricci flat and anti-self-dual (i.e., $W^+=0$).*

(ii) *If (M, g) is Ricci flat and $W^+=0$ (so that $\chi+(3/2)\tau=0$), then, either* (a) *(M, g) is flat and is covered by a flat Riemannian 4-torus T^4 or* (b) *(M, g) is a Kähler Einstein $K3$ surface ($\pi_1=1$), a Kähler Einstein Enriques surface ($\pi_1=\boldsymbol{Z}_2$)*

or the quotient of a Kähler Einstein Enriques surface by a free anti-holomorphic isometric involution ($\pi_1=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$).

To state our theorems we begin with a technical preliminary in terms of groups of diffeomorphisms.

We denote by $\mathrm{Diff}^+=\mathrm{Diff}^+(M)$ the group of orientation preserving diffeomorphisms of M. We denote moreover by $\mathrm{Diff}'=\mathrm{Diff}'(M)$ and $\mathrm{Diff}^o=\mathrm{Diff}^o(M)$ the normal subgroups of Diff^+ $\{\phi\in\mathrm{Diff}^+;\ \phi^*=\mathrm{id}$ on $H^2(M, \boldsymbol{Z})\}$ and $\{\phi\in\mathrm{Diff}^+;\ \phi\cong\mathrm{id}_M$, isotopic$\}$, respectively. Here two diffeomorphisms ϕ and ϕ_1 are said to be isotopic, if there exists a path in Diff^+ joining ϕ and ϕ_1.

So $\mathrm{Diff}^o\subset\mathrm{Diff}'\subset\mathrm{Diff}^+$ and accordingly we have $\hat{\mathcal{E}}(M)$, $\tilde{\mathcal{E}}(M)$ and $\mathcal{E}(M)$, the *isotopy-Teichmüller moduli*, the *Teichmüller moduli* and the *moduli*, respectively.

There are canonical projections

$$\hat{\mathcal{E}}(M)\longrightarrow\tilde{\mathcal{E}}(M)\longrightarrow\mathcal{E}(M) \tag{0.2}$$

such that $\mathcal{E}(M)=\tilde{\mathcal{E}}(M)/\Gamma(M)$ and $\tilde{\mathcal{E}}(M)=\hat{\mathcal{E}}(M)/\Gamma'(M)$ where $\Gamma(M)=\mathrm{Diff}^+/\mathrm{Diff}'$ and $\Gamma'(M)=\mathrm{Diff}'/\mathrm{Diff}^o$ are the mapping class groups of M.

A K3 surface is the 4-manifold of which the moduli $\mathcal{E}$ is completely well studied. Actually we have the so-called period map and the Torelli type theorem (see [K, § 12 in 5] or [14] for the details and we will give a brief argument of this theorem in § 1).

THEOREM 0.3 (Torelli type Theorem). *The Teichmüller moduli $\tilde{\mathcal{E}}$ of Ricci flat metrics on a K3 surface X is isomorphic to an open dense subset T of $Gr^+_{3,22}$ $=SO(3, 19)/SO(3)\times SO(19)$. The moduli $\mathcal{E}$ is then isomorphic to the quotient of T modulo the mapping class group $\Gamma(X)=\mathrm{Diff}^+/\mathrm{Diff}'$.*

The space $SO(3, 19)/SO(3)\times SO(19)$ is an irreducible symmetric space of non-compact type whose dimension is 57.

The period map $pe:\tilde{\mathcal{E}}\to Gr^+_{3,22}$ is the assignment of the naturally oriented $H^+(g)\subset H^2(X, \boldsymbol{R})$ to each Ricci flat metric g and the Torelli type theorem asserts that pe gives the embedding of the Teichmüller moduli into the Grassmannian manifold and exactly onto T ($H^+(g)$ is the space of self-dual harmonic 2-forms on X, $\dim_{\boldsymbol{R}}H^+(g)=3$).

It follows from this theorem that the moduli $\mathcal{E}$ for a K3 surface X is a locally symmetric orbifold. In fact the moduli carries finite group quotient singularities, since the isometry group $I(g)$ of g is a finite group which is isomorphic to the isotropy at $pe(g)\in T$ in the group $\mathrm{Aut}(H^2(X, \boldsymbol{Z}), q_X)\cap SO(3, 19)$ and $I(g)$ varies when g moves (see [5]).

On the other hand, it is not difficult to show that the action of $\Gamma'(X)$ on $\hat{\mathcal{E}}(X)$ is free (see Proposition 2.5) so that the isotopy-Teichmüller moduli $\hat{\mathcal{E}}$ for a 4-manifold underlying a K3 surface is a smooth manifold (possibly having many components) of which each component is isometric to $\tilde{\mathcal{E}}$ (see Theorem

2.6). The number of components is the order of $\Gamma'(X)=\mathrm{Diff}'(X)/\mathrm{Diff}^o(X)$.

By Theorem 0.2 an Enriques surface Y and a 4-manifold Z, a $\boldsymbol{Z}_2$-quotient of Y, are written as quotients of a K3 surface X modulo the respective covering transformation groups Σ. $\Sigma\subset\mathrm{Diff}^+(X)$ induces isometric transformations of the isotopy-Teichmüller moduli for a K3 manifold X. Here isometries mean transformations of the moduli preserving the "canonically equipped L^2 metric". It follows then that the fixedpoint set of the action of Σ, which is totally geodesic as a submanifold in the moduli for X, provides the manifold structures to the isotopy-Teichmüller moduli for 4-manifolds Y and Z.

THEOREM 0.4. *Let Y be a 4-manifold underlying an Enriques surface. Then the isotopy-Teichmüller moduli $\hat{\mathcal{E}}(Y)$ is isometrically embedded in $\hat{\mathcal{E}}(X)$ as a 29 dim totally geodesic submanifold. Each component of this totally geodesic submanifold is via the period map isometric to a submanifold $T\cap F$ in T where F is the fixedpoint set of the involutive deck transformation, written in the form*

$$Gr^+_{1,10}\times Gr^+_{2,12}=(SO(1,9)/SO(1)\times SO(9))\times(SO(2,10)/SO(2)\times SO(10)),$$

embedded in $SO(3,19)/SO(3)\times SO(19)$.

For a $\boldsymbol{Z}_2$-quotient of an Enriques surface we have similarly

THEOREM 0.5. *Let Z be a $\boldsymbol{Z}_2$-quotient of an Enriques surface. Then the isotopy-Teichmüller moduli $\hat{\mathcal{E}}(Z)$ is embedded in $T\subset SO(3,19)/SO(3)\times SO(19)$ isometrically and totally geodesically. This totally geodesic submanifold is 15 dimensional.*

From Theorems 0.4 and 0.5 we have

COROLLARY 0.6. *Let Y and Z be an Enriques surface and its anti-holomorphic $\boldsymbol{Z}_2$-quotient, respectively. Then the Teichmüller moduli $\tilde{\mathcal{E}}$ and the moduli $\mathcal{E}$ of Ricci flat metrics on Y or on Z admit at least an orbifold structure.*

It is open whether $\tilde{\mathcal{E}}$ for Y and Z is smooth, or equivalently whether the mapping class group $\mathrm{Diff}'/\mathrm{Diff}^o$ acts freely on the isotopy-Teichmüller moduli $\hat{\mathcal{E}}$.

§1. The moduli of Ricci flat metrics on a K3 manifold

A K3 surface is, by definition, a compact, connected complex surface with trivial canonical line bundle and $b^1=0$ (for references of K3 surfaces see [§ VIII, 4]).

We say a compact 4-manifold which underlines a K3 surface as a K3 manifold.

Example 1. Let $\iota: T^2\to T^2$; $(z_i)\to(-z_i)$ be the involution defined on a 2-dim complex torus T^2. The quotient T^2/ι has sixteen singular points. By blowing up these points we get a smooth surface, called a Kummer surface, which is simply connected and the canonical line bundle K is trivial as a holomorphic line bundle.

Example 2. A hypersurface of degree 4 in $\boldsymbol{P}^3(C)$ is simply connected and has the first Chern class $c_1=0$. So this complex surface is a K3 surface.

The following theorem is classical.

THEOREM 1.1. (i) *Any $K3$ surfaces are diffeomorphic.*
(ii) ([18]) *Every $K3$ surface has a Kähler metric.*
(iii) ([20]) *Every $K3$ surface admits a Ricci flat Kähler metric.*

A K3 manifold X has the topological invariants; $\chi=24$, $b_2=22$ and $\tau=-16$ so that $b^+=3$, $b^-=19$ where (b^+, b^-) is the signature of the cup product of $H^2(X;\boldsymbol{Z})$.

Consider the moduli $\hat{\mathcal{E}}$, $\tilde{\mathcal{E}}$ and $\mathcal{E}$ of Ricci flat metrics on X.

THEOREM 1.2 (Torelli type Theorem). *Via the period map, the Teichmüller moduli $\tilde{\mathcal{E}}(X)$ is isomorphic to an open dense subset T of $SO(3, 19)/SO(3)\times SO(19)$.*

To define the period map and explain the domain T we need to state the cup product more precisely.

The cup product q_X of a K3 manifold X has the form $q_X\cong\oplus^2(-E_8)\oplus\oplus^3 H$ where E_8 is the Cartan matrix of type E_8 and H is $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Let g be a metric on X. Then from the harmonic theory the space $H^2(X, g)$ of g-harmonic 2-forms on X, which is isomorphic to the second cohomology group $H^2(X;\boldsymbol{R})$, decomposes as the sum of the spaces $H^{\pm}(X, g)$ of self-dual harmonic 2-forms and anti-self-dual harmonic 2-forms;

$$H^2(X, g)=H^+(X, g)\oplus H^-(X, g) \tag{1.1}$$

for which $q_X>0$ on $H^+(X, g)$ and $q_X<0$ on $H^-(X, g)$.

$H^+(g)$ and $H^-(g)$ are orthogonal with respect to q_X. Here we identify q_X on $H^2(X, \boldsymbol{R})$ with the wedge product on deRham cohomology classes of closed 2-forms.

Now suppose the metric g is Ricci flat.

Then from Proposition 0.1 this metric is anti-self-dual and of zero scalar curvature. So any harmonic self-dual 2-form must be parallel by the Weitzenböck-Bochner argument and induces up to a constant a complex structure J on X compatible with g. The metric g is Ricci flat Kähler with respect to the complex surface (X, J).

We can assign from this fact a canonical orientation to $H^+(X, g)$.

In fact, if we choose α, $\beta \in H^+(g)$, $\alpha \neq \beta$, to which complex structures J_α, J_β associate, then we have another $\gamma \in H^+(g)$ such that $J_\gamma = J_\alpha \circ J_\beta$ and hence the orientation $\{\alpha, \beta, \gamma\}$ for $H^+(g)$.

We therefore get the assignment

$$g \bmod \mathrm{Diff}' \longmapsto \text{oriented } pr(H^+(g)) \tag{1.2}$$

which yields the period map

$$pe : \tilde{\mathcal{E}}(X) \longrightarrow Gr^+_{3,22} \cong SO(3, 19)/SO(3) \times SO(19) . \tag{1.3}$$

Here we regard first $H^+(g)$ as a 3-dim linear subspace of the infinite dimensional vector space $Z^2(X)$, the space of all closed 2-forms on X and then take $pr(H^+(g))$, namely the deRham cohomology projection of $H^+(g)$ in $H^2(X, \boldsymbol{R})$, as a subspace in the 22-dim space $H^2(X, \boldsymbol{R})$.

The subset T is defined as follows.

Let $\Delta = \{\gamma \in H^2(X; \boldsymbol{Z}); q_X(\gamma, \gamma) = -2\}$ be the set of "roots" and, for any oriented positive 3-plane Π in $H^2(X; \boldsymbol{R})$ we denote by $\Pi^\perp$ the q_X-orthogonal complement of Π; $\Pi^\perp = \{\alpha \in H^2(X; \boldsymbol{R}); q_X(\alpha, \Pi) = 0\}$. Then T is the set of all oriented positive 3-planes $\Pi \subset H^2(X; \boldsymbol{R})$ such that $\Pi^\perp \cap \Delta = \emptyset$. T is the complement of countably many unions of codimension 3 submanifolds in $Gr^+_{3,22}$. So, T is connected and simply connected.

The reason why one excludes the complement of T from $Gr^+_{3,22}$ stems from the fact that q_X is even and the canonical line bundle is trivial so that an arbitrary complex curve C in a K3 surface satisfies $C \cdot C \geqq -2$ and if $C \cdot C \neq -2$, then $C \cdot C \geqq 0$ ([5], [13]).

We omit proving that the period map is isomorphic. See [5] for the references and see also references cited there.

The space $SO(3, 19)/SO(3) \times SO(19)$ is the noncompact dual of an ordinary Grassmannian manifold $Gr_{3,22}$. So it is a symmetric space of noncompact type and has the invariant metric. At the origin this invariant metric $\langle , \rangle$ is given by the restriction of the negative Killing form of the Lie algebra $\mathfrak{so}(3, 19)$. Actually the tangent space at the orgin T_o is identified with $\mathfrak{m} = \{3 \times 19 \text{ matrices}\}$ and the invariant metric is, up to constant,

$$\langle U, V \rangle = \mathrm{tr}\,(UV^t + VU^t), \qquad U, V \in \mathfrak{m} . \tag{1.4}$$

We interpret this invariant metric in terms of $\mathrm{Hom}\,(H^+, H^-)$ as follows.

Take a Ricci flat metric g corresponding to the origin o. Then $T_o \cong \mathfrak{m}$ is identified with the space of homomorphisms; $H^+(g) \to H^-(g)$. For a homomorphism $f : H^+ \to H^-$ we denote by $f^* : H^- \to H^+$ the adjoint of f, namely $q_X(f(\alpha), \beta) = q_X(\alpha, f^*(\beta))$, $\alpha \in H^+$ and $\beta \in H^-$.

Then the invariant metric $\langle , \rangle$ has the form

$$\langle f, f_1 \rangle = -\mathrm{tr}\,(f f_1^* + f_1 f^*), \qquad f, f_1 \in H^+ . \tag{1.5}$$

§ 2. Anti-self-dual conformal structures on a K3 manifold

We will look at the moduli $\hat{\mathcal{E}}(X)$ and $\tilde{\mathcal{E}}(X)$ by means of anti-self-dual conformal structures.

Before doing we prepare the notion of anti-self-dual conformal structure and define several kinds of moduli of anti-self-dual conformal structures on X.

Suppose M is a compact, connected oriented 4-manifold. Let g be a metric on M. Then the Weyl conformal tensor W of g gives a section of the bundle $S_0^2(\Omega^2)$, the tracefree symmetric product of Ω^2. This is because $W_{ijkl}=-W_{jikl}=W_{klij}$ and $\sum_{ik} g^{ik}W_{ijkl}=0$.

Decomposing Ω^2 into the sum as $\Omega^2=\Omega^+\oplus\Omega^-$ for the self-dual 2-form bundle Ω^+ and the anti-self-dual 2-form bundle Ω^-, we have the decomposition of W as $W=(W^+, W^-)$ where $W^+\in\Gamma(M, S_0^2(\Omega^+))$, $W^-\in\Gamma(M, S_0^2(\Omega^-))$ and the $\Omega^+\otimes\Omega^-$-component of W vanishes.

DEFINITION ([2]). A metric g on M is called anti-self-dual if the $S_0^2(\Omega^+)$-component $W^+=0$.

We say that a conformal structure $[g]$ represented by g is anti-self-dual if $W^+(g)=0$.

Let ϕ be an orientation preserving diffeomorphism of M. Then the pull back metric $\phi^* g$ by ϕ defines a conformal structure $[\phi^*g]$ which we denote by $\phi^*[g]$ so that ϕ gives a transformation of $\mathcal{C}(M)$ where $\mathcal{C}(M)$ is the space of all conformal structures on M.

Since the self-dual Weyl conformal tensor W^+ of $\phi^* g$ is just the pull back of W^+ of g by ϕ, the space $\mathcal{C}^-(M)$ of all anti-self-dual conformal structures on M is invariant under the action of arbitrary $\phi\in\mathrm{Diff}^+(M)$. So we can get the quotient spaces $\mathcal{M}(M)=\mathcal{C}^-(M)/\mathrm{Diff}^+$, $\tilde{\mathcal{M}}(M)=\mathcal{C}^-(M)/\mathrm{Diff}'$ and $\hat{\mathcal{M}}(M)=\mathcal{C}^-(M)/\mathrm{Diff}^0$ and we call them the moduli, the Teichmüller moduli and the isotopy-Teichmüller moduli, respectively, of anti-self-dual conformal structures on M.

Let g be an anti-self-dual metric. Then it induces the following elliptic complex;

$$(2.1) \qquad 0 \longrightarrow \Gamma(M, T) \xrightarrow{L_g} \Gamma(M, S_0^2(T^*)) \xrightarrow{D_g} \Gamma(M, S_0^2(\Omega^+)) \longrightarrow 0$$

Here T, T^* are the tangent, cotangent bundles and $L=L_g$ is the tracefree Lie derivative of vector fields on X; $(L(U))_{ij}=\nabla_i U_j+\nabla_j U_i-1/2(\sum_k \nabla_k U_k)g_{ij}$. Further $D=D_g$ is the Fréchet derivative of the self-dual Weyl conformal tensor W^+; $D(h)=d/dt|_{t=0} W^+(g_t)$ for $h=d/dt|_{t=0}\, g_t$.

The cohomology spaces are $\boldsymbol{H}^0=\{U\in\Gamma(T),\ L(U)=0\}$, $\boldsymbol{H}^1=\{h\in\Gamma(S_0^2(T^*))$; $D(h)=0,\ L^*(h)=0\}$ and $\boldsymbol{H}^2=\{B\in\Gamma(S_0^2(\Omega^+)),\ D^*B=0\}$.

The index of this elliptic complex is from the Atiyah-Singer index theorem $h^0-h^1+h^2=1/2(29\tau(M)+15\chi(M))$ ($h^i=\dim \boldsymbol{H}^i$, $i=1, 2, 3$).

Remark. Any diffeomorphism $\phi \in \mathrm{Diff}^+(M)$ induces an isomorphism of elliptic compleces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(M, T) & \xrightarrow{L_g} & \Gamma(M, S_0^2(T^*)) & \xrightarrow{D_g} & \Gamma(M, S_0^2(\Omega^+)) & \longrightarrow & 0 \\ \downarrow & & \downarrow (\phi^{-1})_* & & \downarrow \phi^* & & \downarrow \phi^* & & \downarrow \\ 0 & \longrightarrow & \Gamma(M, T) & \xrightarrow{L_{\phi^* g}} & \Gamma(M, S_0^2(T^*)) & \xrightarrow{D_{\phi^* g}} & \Gamma(M, S_0^2(\Omega^+)) & \longrightarrow & 0 \end{array}$$

so that if ϕ is an isometry of an anti-self-dual metric g, then ϕ induces an isomorphism of $\boldsymbol{H}^i$, $i=0, 1, 2$.

Now consider a Ricci flat metric g on a K3 manifold X. From Proposition 0.1 g is anti-self-dual and has zero scalar curvature.

PROPOSITION 2.1. *Let g be a Ricci flat metric on a $K3$ manifold X. Then the cohomology groups are $\boldsymbol{H}^0=0$, $\boldsymbol{H}^1 \cong \boldsymbol{R}^{57}$ and $\boldsymbol{H}^2 \cong \boldsymbol{R}^5$.*

Proof. $\boldsymbol{H}^0$ is the space of conformal Killing fields. First we show that the group of conformal transformations for the conformal structure $[g]$ coincides with the group of isometries for g. Since g is Ricci flat, it is a Yamabe metric of Yamabe invariant zero for the conformal structure $[g]$. From [9] the uniqueness of Yamabe metrics of nonpositive Yamabe invariant applies so that an arbitrary conformal transformation for $[g]$ is an isometry for g. Since conformal transformations generated by a conformal Killing field U are isometries, U is a Killing field. Because of the Ricci flatness the 1-form ξ corresponding to U must be parallel (by Theorem 2.3 in [15], for example) and hence ξ vanishes from $b_1=0$. So $\boldsymbol{H}^0=0$.

To compute $\dim \boldsymbol{H}^2$ we can apply the Weitzenböck-Bochner formula given in [12] to the elliptic operator of fourth order D^*D.

Actually, since g is Ricci flat, $D^*DB=(\nabla^*\nabla)^2B$ for $B \in \Gamma(S_0^2(\Omega^+))$, where $\nabla^*\nabla$ is the covariant Laplacian, so that $B \in \boldsymbol{H}^2$ if and only if B is a parallel section of the bundle $S_0^2(\Omega^+)$. Thus $h^2=5$. Here we used the fact that for a K3 manifold X the orthonormal basis of $H^+(g)$ makes the bundle Ω^+ trivial and then $S_0^2(\Omega^+)$ has five, linearly independent parallel sections.

The index of the complex for X is -52 so that $\dim \boldsymbol{H}^1=57$.

Now we state the real analytic variety theorem for the isotopy-Teichmüller moduli $\mathscr{M}(M)$ valid for a general 4-manifold M except for S^4.

THEOREM 2.2. *Let M be a compact, connected oriented 4-manifold and g an anti-self-dual metric on M. Let $[g]$ be the corresponding point in $\mathscr{M}(M)$ (here we identify the conformal structure $[g]$ and the point of the moduli derived from $[g]$ by the diffeomorphism quotient). Then $\mathscr{M}(M)$ is isomorphic around $[g]$ to the real analytic variety $\{h \in \boldsymbol{H}^1;\ |h|<\varepsilon,\ \Psi(h)=0\}/C_g^0$, where $\Psi: \boldsymbol{H}^1 \to \boldsymbol{H}^2$ is an analytic map associated to the Kuranishi map, $\Psi(0)=0$ and $C_g^0=C_g \cap \mathrm{Diff}^0$ denotes* *the group of conformal transformations of* $[g]$.

See (iv), Sect. 3 in [11] for the details. We explain here the action C_g^0 on the complex (2.1). We noticed in the remark that any $\phi \in \mathrm{Diff}^+(M)$ acts on (2.1) equivariantly. Since C_g^0 is compact except for the standard 4-sphere (for instance see [9]), we can choose a metric g inside the conformal structure $[g]$ in such a way that C_g^0 is the isometries of g. Therefore C_g^0 acts on the complex (2.1) so that this action induces the action on the cohomology groups.

We see from Theorems 2.1, 2.2 that $\mathcal{M}(X)$ has dimension at most 57. Moreover, we have the following proposition from which $\mathcal{M}(X)$ is at $[g]$ locally an analytic subset of $\boldsymbol{H}^1 \cong \boldsymbol{R}^{57}$.

PROPOSITION 2.3. *Let g be a Ricci flat metric on a $K3$ manifold X. Then the conformal group* $C_g^0(X)=\{\mathrm{id}_X\}$. *Moreover* $C_g{}'(X)=\{\mathrm{id}_X\}$ *where* $C_g{}'(X)= C_g \cap \mathrm{Diff}'(X)$.

Proof. As shown in the proof of Proposition 2.1, $C_g{}'$ consists of isometries of g. So, let ϕ be an isometry of g in Diff′. Since ϕ^* acts as the identity on $H^2(X, \boldsymbol{Z})$ and hence on $H^2(X, \boldsymbol{R})$, $\phi^*=\mathrm{id}$ on $H^2(g)$ and then ϕ must be an automorphism of X with respect to a complex structure $J=J_\alpha$ induced from a certain self-dual harmonic 2-form α. Therefore $\phi=\mathrm{id}_X$ by Proposition 11.3 (the weak Torelli theorem), § VIII in [4]. This completes the proof.

PROPOSITION 2.4 (Proposition 4.2, [11]). *The first cohomology group $\boldsymbol{H}^1$ at $[g]$, where g is Ricci flat, is isomorphic to the tensor product* $H^-(g)\otimes(H^+(g))^*$.

Proof. It suffices to show the following.

Let $\phi_a^+ \in H^+(g)$, $a=1, 2, 3$ and $\phi_b^- \in H^-(g)$, $b=1, \cdots, 19$ be harmonic self-dual (anti-self-dual) 2-forms which constitute orthonormal bases of $H^+(g)$ and $H^-(g)$, respectively. Then via the identification $H^+ \cong (H^+)^*$ the tensor products

$$\phi_b^- \otimes \phi_a^+ \in H^- \otimes H^+ , \qquad 1 \leqq a \leqq 3,\ 1 \leqq b \leqq 19$$

form an orthonormal basis of $\boldsymbol{H}^1$ with respect to the L^2-metric.

So, identify

$$\text{(2.2)} \qquad \Omega^- \otimes \Omega^+ (\cong \mathrm{Hom}(\Omega^+, \Omega^-)) \xrightarrow{\cong} \mathrm{S}_0^2(T^*)$$

$$(\eta^-, \eta^+) \longmapsto h=(h_{ij}),$$

by $h_{ij}=\sum_{kl} \bar{g}^{kl} \eta_{ik}^- \eta_{lj}^+$.

For $\phi^+ \in H^+$, $\phi^- \in H^-$ we let $h \in \Gamma(\mathrm{S}_0^2(T^*))$ be given by $\phi^- \otimes \phi^+$ via the above identification. To show $h \in \mathrm{Ker}\, L^* \cap \mathrm{Ker}\, D$ we first check that $L^*(h)=0$ and then $D(h)=0$.

We see easily $L^*(h)=0$ because $L^*(h)$ is given by $(L^*(h))_i=-\sum_{jk} g^{jk} \nabla_k h_{ji}$.

Since g is Ricci flat, $D(h)$ is for any h the $\mathrm{S}_0^2(\Omega^+)$-component of $U(h) \in$

$\Gamma(\Omega^2 \otimes \Omega^2)$, defined by

$$U(h)_{ijkl}=1/2(\nabla_k \nabla_j h_{il}-\nabla_l \nabla_j h_{ik}-\nabla_k \nabla_i h_{jl}+\nabla_l \nabla_i h_{jk}) \tag{2.3}$$

so that for our h

$$U(h)_{ijkl}=1/2\{(\nabla_k \nabla_s \phi_{\overline{ij}})(\phi^+)^s_l-(\nabla_l \nabla_s \phi_{\overline{ij}})(\phi^+)^s_k\}.$$

Now we may assume $\phi^+=\omega$ the Kähler form. So by using the complex indices we can show that $D(h)=0$.

It is not hard to show by Proposition 2.1 that $\phi_a^- \otimes \phi_b^+$, $1 \leqq a \leqq 19$, $1 \leqq b \leqq 3$ gives a basis, since the L^2-inner product of $\phi_a^- \otimes \phi_b^+$ and $\phi_c^- \otimes \phi_d^+$ is $\delta_{ac}\delta_{bd}$.

Now we divide the moduli into three parts according to the sign of the Yamabe invariant, in other words, the sign of constant scalar curvature;

$$\mathcal{M}(X)=\mathcal{M}^{(+)} \sqcup \mathcal{M}^{(0)} \sqcup \mathcal{M}^{(-)}, \tag{2.4}$$

where the Yamabe invariant is a conformal invariant of a conformal structure, which is essentially the value of the constant scalar curvature of a Yamabe metric (see [9]). A similar division is given for other moduli $\hat{\mathcal{M}}(X)$ and $\tilde{\mathcal{M}}(X)$.

Since a K3 manifold X has $b^+>0$, the Weitzenböck-Bochner formula for harmonic self-dual 2-forms assures that there is no anti-self-dual conformal structure of positive Yamabe invariant. On the other hand, from Proposition 0.1 in Introduction any anti-self-dual conformal structure of zero Yamabe invariant has the unique Ricci flat metric of volume one as a specific representative. So we can identify $\mathcal{M}^{(0)}(X)$, $\hat{\mathcal{M}}^{(0)}(X)$ and $\tilde{\mathcal{M}}^{(0)}(X)$ with $\mathcal{E}(X)$, $\hat{\mathcal{E}}(X)$ and $\tilde{\mathcal{E}}(X)$, respectively.

PROPOSITION 2.5. *The mapping class group Diff′/Diff^o acts freely on $\hat{\mathcal{M}}^{(0)}(X)$ and hence on $\hat{\mathcal{E}}(X)$.*

Proof. Assume $\phi \in$ Diff′ fixes a point of $\hat{\mathcal{M}}^{(0)}(X)$. It suffices to verify $\phi \in$ Diff^o. Let $[g]$ be a conformal structure representing this point. Then there is a $\phi_1 \in$ Diff^o such that $\phi^* g=\phi_1^* g$. So $\phi_2=(\phi \circ \phi_1^{-1}) \in$ Diff′ fixes g. Since g is Ricci flat, it follows from the argument in the proof of Proposition 2.3 that $\phi_2=id_X$, in other words, $\phi \in$ Diff^o. This proves Proposition 2.5.

We have the following diagram

$$\begin{array}{ccc} \hat{\mathcal{M}}^{(0)} & \xrightarrow{\ \cong\ } & \hat{\mathcal{E}} \\ {\scriptstyle /(\mathrm{Diff}'/\mathrm{Diff}^o)} \downarrow & & \downarrow {\scriptstyle /(\mathrm{Diff}'/\mathrm{Diff}^o)} \\ \tilde{\mathcal{M}}^{(0)} & \xrightarrow{\ \cong\ } & \tilde{\mathcal{E}} \end{array} \tag{2.5}$$

for which the Teichmüller moduli $\tilde{\mathcal{E}}$ is the quotient of $\hat{\mathcal{E}}$ by the fixed-point free action of the discrete group Diff′/Diff^o. Since, as explained in §1, $\hat{\mathcal{E}}$ has a smooth manifold structure, $\hat{\mathcal{E}}$ has also a smooth manifold structure. It is a covering space over a simply connected manifold $\tilde{\mathcal{E}}$ and is then considered as a union of copies of $\tilde{\mathcal{E}}$, where the number of copies is the order $\sharp\,\mathrm{Diff}'/\mathrm{Diff}^0$.

Return back to the moduli of conformal structures $\mathcal{M}^{(0)}(X)$ and $\mathcal{M}^{(0)}(X)$.

As was shown in Theorem 2.2 any point of $\mathcal{M}^{(0)}(X)$ has a neighborhood in $\mathcal{M}(X)$ of the form $\{h\in \boldsymbol{H}^1;\ |h|<\varepsilon,\ \Psi(h)=0\}$.

On the other hand, via the identification $\mathcal{M}^{(0)}(X)\cong\hat{\mathcal{E}}(X)$ and by the argument just above the point in $\hat{\mathcal{E}}(X)$ corresponding to this point has a neighborhood in the Grassmannian manifold $Gr^+_{3,22}$ from Theorem 1.2. This neighborhood is considered as an ε-neighborhood in $H^-(g)\otimes(H^+(g))^*$, which is from the Grassmannian space structure just the tangent space at the 3-plane $H^+(g)$.

Since $\boldsymbol{H}^1\cong H^-(g)\otimes(H^+(g))^*$ by Proposition 2.4 and an ε-neighborhood in $\boldsymbol{H}^1$ corresponds to an ε'-neighborhood of $H^-(g)\otimes(H^+(g))^*$, we have the

ASSERTION. *The map* $\Psi;\ \boldsymbol{H}^1\to\boldsymbol{H}^2$ *must be trivial for any Ricci flat metric* g.

Thus the following is verified.

THEOREM 2.6. *The moduli* $\mathcal{M}^{(0)}(X)$ *of anti-self-dual conformal structures of zero Yamabe invariant is an open subset of* $\mathcal{M}(X)$ *and has a smooth manifold structure (possibly not connected) of dimension* 57, *isomorphic to* $\hat{\mathcal{E}}(X)$.

Remark. It is not known whether $\mathcal{M}^{(-)}(X)$ is empty or not.

CONJECTURE. *The mapping class group of second kind* $\mathrm{Diff}'/\mathrm{Diff}^0$ *consists only of the identity, or equivalently* $\hat{\mathcal{E}}(X)$ *coincides with* $\tilde{\mathcal{E}}(X)$.

Remark that the mapping class group $\mathrm{Diff}^+/\mathrm{Diff}'$ of a K3 manifold X is identified with the index two subgroup $\mathrm{Aut}\,(H^2(X,\boldsymbol{Z}),\ q_X)\cap SO(3,19)$ of the group $\mathrm{Aut}\,(H^2(X,\boldsymbol{Z}),\ q_X)$ of automorphisms ([5]). There are certain criteria on the mapping class group relative to the lattice $(H^2(M,\boldsymbol{Z}),\ q_M)$ derived from the surgery theory.

Actually, Wall showed the following ([19]); let N be a simply connected, compact oriented 4-manifold with indefinite cup product q_N or of rank $H^2(N)\leqq 8$. If M is a connected sum $N\sharp(S^2\times S^2)$, then every positive q_M-automorphism of $H^2(M,\boldsymbol{Z})$ is induced by a $\phi\in\mathrm{Diff}^+(M)$.

On the other hand, M. Kreck obtained in [17] that if M is a simply connected compact, connected 4-manifold, then the group consisting of $\phi\in\mathrm{Diff}^+(M)$ which is pseudo-isotopic to id_M is isomorphic to $\mathrm{Diff}'(M)$. Here ϕ is pseudo-isotopic to ϕ' if there is a diffeomorphism F of the product space $M\times[0,1]$ such that $F(x,0)=(\phi(x),0)$ and $F(x,1)=(\phi'(x),1)$ (see [6]).

§ 3. L^2-metric on the moduli of conformal structures

In this section we will see how an L^2-metric is defined on the moduli of anti-self-dual conformal structures on a general 4-manifold M.

Since such a moduli is embedded in the ambient moduli, the moduli of all conformal structures, in order to get such an L^2-metric we will show that the space $\mathcal{C}(M)$ of all conformal structures on M admits an L^2-metric $\mathcal{G}$ which is diffeomorphism-invariant, namely $\phi^*(\mathcal{G}_{[g]})=\mathcal{G}_{\phi^*[g]}$, $[g]\in\mathcal{C}(M)$, $\phi\in\mathrm{Diff}^+$.

Let $[g]\in\mathcal{C}(M)$. The tangent space $T_{[g]}$ to $\mathcal{C}(M)$ at $[g]$ is then given by the space $\Gamma(M, S_0^2(T^*))$ of tracefree symmetric covariant tensors $h=(h_{ij})$. Moreover by the identification (2.3) $T_{[g]}$ is identified with $\Gamma(M, \mathrm{Hom}(\Omega_g^+, \Omega_g^-))$. So one can define an inner product $\mathcal{G}$ by

$$\mathcal{G}(A, B)=1/2\int_M -\mathrm{tr}\,(AB^*+BA^*)dV_g \tag{3.1}$$

for A, $B\in\Gamma(M, \mathrm{Hom}\,(\Omega^+, \Omega^-))$ (see Theorem 5 in [11]).

Here A^*, $B^*\in\Gamma(M, \mathrm{Hom}\,(\Omega^-, \Omega^+))$ mean the adjoint of A, B with respect to the wedge product, respectively, that is, $(A(\alpha))\wedge\beta=\alpha\wedge(A^*(\beta))$ for any $\alpha\in\Omega^+$ and $\beta\in\Omega^-$.

We need the minus sign in (3.1) since $\mathrm{tr}\,(AA^*)$ is negative definite. This negative definiteness stems from that the wedge product $\wedge$ is positive on Ω^+ and is negative on Ω^-.

The action of diffeomorphism on $\mathcal{C}(M)$ induces the differential map $T_{[g]}\to T_{[\phi^*g]}$, $A\mapsto\phi^*(A)=((\phi^*)^{-1}\circ A\circ\phi^*)$.

So one has $\mathrm{tr}\,((\phi^*A)(\phi^*B)^*)(x)=\mathrm{tr}\,(AB^*)(\phi(x))=(\phi^*\,\mathrm{tr}\,(AB^*))(x)$, $x\in M$.

In order that the $\mathcal{G}$ depends only on a conformal structure, not on a choice of representative metric g we require that the volume form dV, $g\mapsto dV_g$, appeared in the definition (3.1), must satisfy the conformally invariant property $dV_{fg}=dV_g$. Moreover we require that $dV_{\phi^*g}=\phi^*(dV_g)$, the diffeomorphism naturality, because of the diffeomorphism-invariance of $\mathcal{G}$.

We call such a volume form a *canonical volume form.*

PROPOSITION 3.1. *If M has $b^+(M)>0$ or $b^-(M)>0$, then M admits a canonical volume form dV.*

Proof. Assume for brevity $b^+>0$. Let $[g]\in\mathcal{C}(M)$ and g be a metric representing it. To prove the proposition we choose a basis $\{\psi_i\}$ of $H^+(M, g)$ which is q_M-orthonormal, that is, $q_M([\psi_i], [\psi_j])=\delta_{ij}$ as cohomology classes, or equivalently $\int_M\psi_i\wedge\psi_j=\delta_{ij}$.

We define then dV_g by

$$dV_g=\sum_{i=1}^{b+}\psi_i\wedge\psi_i\,. \tag{3.2}$$

Although this volume form dV_g may be degenerate, it is positive almost everywhere. This is because $\phi_i\wedge\phi_i=|\phi_i|^2dv_g$ and dV_g vanishes exactly at points where all ϕ_i's vanish, and any nonzero harmonic 2-form does not vanish almost everywhere.

To show the diffeomorphism naturality we first remark that any $\phi\in\mathrm{Diff}^+$ induces the q_M-isometry; $H^+(M, g)\to H^+(M, \phi^*g)$. So $\{\phi^*\phi_i\}$ is a q_M-orthonormal basis of $H^+(M, \phi^*g)$. Therefore,

$$dV_{\phi^*g}=\sum_i(\phi^*\phi_i\wedge\phi^*\phi_i)=\phi^*(\sum_i\phi_i\wedge\phi_i)=\phi^*(dV_g).$$

If $b^+(M)=0$ but $b^-(M)>0$, then we need a minor change in the definition (3.2), only the minus sign.

One can check the diffeomorphism-invariance of the L^2-metric $\mathcal{G}$, since the integration over M is preserved by the action of diffeomorphisms so that

$$\int_M\mathrm{tr}\{(\phi^*A)(\phi^*B)^*+(\phi^*B)(\phi^*A)^*\}\,dV_{\phi^*g}=\int_M\phi^*(\mathrm{tr}\,(AB^*+BA^*))\phi^*(dV_g)$$

reduces to $\int_M\mathrm{tr}\,(AB^*+BA^*)dV_g$.

The L^2-metric $\mathcal{G}$ on $\mathcal{C}(M)$ descends to the quotient spaces $\mathcal{C}(M)/\mathrm{Diff}^+$, $\mathcal{C}(M)/\mathrm{Diff}'$ and $\mathcal{C}(M)/\mathrm{Diff}^0$, respectively so that

THEOREM 3.2. *If $b^+(M)>0$ or $b^-(M)>0$, then the moduli admits an L^2-metric in such a way that* (i) *each of the following projections is isometric*;

$$\tilde{\tilde{\mathcal{M}}}(M)\longrightarrow\tilde{\mathcal{M}}(M)\longrightarrow\mathcal{M}(M)$$

and (ii) *the mapping class groups $\Gamma'(M)=\mathrm{Diff}'(M)/\mathrm{Diff}^0(M)$ and $\Gamma(M)=\mathrm{Diff}^+(M)/\mathrm{Diff}'(M)$ act as isometries on $\tilde{\tilde{\mathcal{M}}}(M)$ and $\tilde{\mathcal{M}}(M)$, respectively.*

The following is then obtained.

THEOREM 3.3. *The L^2-metric $\mathcal{G}$ on the moduli $\tilde{\mathcal{M}}^{(0)}(X)$ of anti-self-dual conformal structures of zero Yamabe invariant on a $K3$ manifold X is isometric up to constant to the invariant metric defined at* (2.4) *on the Teichmüller moduli $\tilde{\mathcal{E}}(X)$.*

Proof. For any Ricci flat metric g representing a conformal structure $[g]$ $H^+(g)$ consists only of parallel self-dual 2-forms so that each member of an orthonormal basis $\{\phi_i\}$ has the same constant norm. Thus the canonical volume form is just a constant multiple of the ordinary Riemannian volume form dv_g of g.

By Proposition 2.4 and Theorem 2.6 the basis $\{\phi_i\}$ together with an orthonormal basis $\{\phi_j^-\}$ of $H^-(g)$ give via the identification $\boldsymbol{H}^1\cong H^-(g)\otimes H^+(g)$ an orthonormal basis $\{\phi_j^-\otimes\phi_i^+\}$ of the tangent space $T_{[g]}$ of the moduli $\tilde{\mathcal{M}}^{(0)}$ with respect to the L^2-metric $\mathcal{G}$ and gives again by the period map theorem (Theorem 1.2) together with the formula (1.5) an orthonormal basis, up to constant, of the tangent space of the Grassmannian manifold $Gr^+_{3,22}$ at the corresponding point.

Remark 1. Since the invariant metric is complete on $Gr^+_{3,22}$, but not complete on T, the L^2-metric $\mathcal{G}$ is non-complete on $\mathcal{M}^{(0)}(X)$. The metric completion of $\mathcal{M}^{(0)}(X)$ or of $\tilde{\mathcal{E}}(X)$ relative to the L^2-metric is the space which is isometrically identified with the symmetric space $Gr^+_{3,22}$.

Remark 2. The complement of T, $Gr^+_{3,22}\setminus T$ consists of orbifold-singular Ricci flat metrics on X ([1], [13]). Here an orbifold-singular Ricci flat metric on X is a C^∞ symmetric covariant 2-tensor on X of the form π^*g, where $\pi: M\to V$ is a surjective real analytic map to a Ricci flat Einstein orbifold (V, g).

Moreover in the completion procedure we observe the bubbling off phenomena ([13], [3]). In fact from Theorem 21 in [13], if $\{g_i\}$ is a sequence in T having the limit $g\in Gr^+_{3,22}\setminus T$, then (i) the curvature of g_i concentrates near some configurations E of embedded 2-spheres of self-intersection number -2, (ii) if we rescale g_i by the local maximum value of the curvature, then the rescaled metrics $\tilde{g}_i$ converge to ALE gravitational instantons corresponding to the simple singularities obtained by contracting the configurations E and (iii) outside the singularities g_i converges to the orbifold-singular metric g.

§4. The moduli on an Enriques manifold

Let Y be an Enriques manifold, namely a compact 4-manifold underlying an Enriques surface. Here an Enriques surface is a compact complex surface obtained by a holomorphic $\boldsymbol{Z}_2$-quotient of a K3 surface. Then Y is a $\boldsymbol{Z}_2$-quotient of a K3 manifold X. Note that an Enriques surface has the trivial bundle $K^{\otimes 2}$ ([15, § VIII in 4]).

Let $\pi: X\to Y$ be a covering map yielding the Enriques manifold Y and let $\sigma: X\to X$ be the deck transformation of X, $\pi\circ\sigma=\pi$ such that $Y=X/\langle\sigma\rangle$.

The topological invariants of Y are $\chi(Y)=(1/2)\chi(X)=12$, $\tau(Y)=(1/2)\tau(X)=-8$ and hence $b_2(Y)=10$, $b^+(Y)=1$, $b^-(Y)=9$.

The deck transformation σ induces the cup product isometry of the cohomology group $L\equiv H^2(X, \boldsymbol{Z})\cong\oplus^2(-E_8)\oplus\oplus^3 H$ given by

$$x\oplus y\oplus z_1\oplus z_2\oplus z_3 \longmapsto y\oplus x\oplus -z_1\oplus z_3\oplus z_2 \tag{4.1}$$

so that the σ-invariant sublattice L^+ is isomorphic to $-2E_8\oplus 2H$ and the unimodular lattice $1/2\ L^+$, isomorphic to $-E_8\oplus H$, gives the cohomology group of an Enriques manifold Y with the cup product q_Y (see Lemma 19.1, § VIII in [4]) and thus $b^+=1$, $b^-=9$.

From Proposition 0.1 same as in the K3 manifold case we identify the moduli of unit volume Ricci flat metrics on Y with the moduli of anti-self-dual conformal structures of zero Yamabe invariant.

Let g be a Ricci flat metric on Y. Then, since g is anti-self-dual, as discussed in § 2, it provides an elliptic complex defining local deformation of anti-self-dual conformal structures:

$$(4.2) \qquad 0 \longrightarrow \Gamma(Y, T) \xrightarrow{L_g} \Gamma(Y, S_0^2(T^*)) \xrightarrow{D_g} \Gamma(Y, S_0^2(\Omega^+)) \longrightarrow 0 .$$

The index is $h^0 - h^1 + h^2 = -26$ from the topological invariants of Y.

PROPOSITION 4.1. *The elliptic complex* (4.2) *has the cohomology groups* $\boldsymbol{H}^0=0$, $\boldsymbol{H}^1 \cong \boldsymbol{R}^{29}$ *and* $\boldsymbol{H}^2 \cong \boldsymbol{R}^3$ *for an arbitrary Ricci flat metric.*

Proof. We apply the same proof of Proposition 2.1 to show $\boldsymbol{H}^0=0$. We postpone the calculation of $h^2 = \dim \boldsymbol{H}^2$ until just after Proposition 4.2. Actually we will see there $h^2=3$ and hence $h^1=29$.

The situation for an Enriques manifold is quite similar to the K3 manifold case. So, $\mathscr{\hat{M}}(Y)$ can be identified around the conformal structure $[g]$ with the C_g^0-quotient of a real analytic variety $\{h \in \boldsymbol{H}^1 ; |h| < \varepsilon, \Psi(h)=0\}$ where $\Psi : \boldsymbol{H}^1 \to \boldsymbol{H}^2$ is an analytic map, and $C_g^0 = C_g \cap \mathrm{Diff}^0(Y)$ denotes the group of conformal transformations of $[g]$.

ASSERTION. $C_g^0 = \{\mathrm{id}_Y\}$.

This is given as follows. Since g is Ricci flat, g is a Yamabe metric of zero Yamabe invariant so that C_g^0 consists only of isometries. Let ϕ be such an isometry. Then it lifts up as an isometry $\bar{\phi}$ of $\bar{g}$ commuting with σ where $\bar{g} = \pi^* g$. Since $\phi \in \mathrm{Diff}^0$, $\bar{\phi}$ is also in Diff^0 of X. It follows then from Proposition 2.3 that $\bar{\phi}$ is id_X and hence ϕ is id_Y.

On the other hand, a Ricci flat metric g on Y lifts up to a Ricci flat metric $\bar{g}$ on X which is deck transformation invariant and vice versa so that one has

OBSERVATION. *The moduli of Ricci flat metrics on Y is considered as the space of σ-invariant Ricci flat metrics $\bar{g}$ on X of $Vol(\bar{g})=2$.*

Suppose that $\bar{g}$ is a Ricci flat metric on a K3 manifold X such that $\bar{g} = \pi^* g$. Then we have the elliptic complex (2.1) over X associated to $\bar{g}$. Because $\bar{g}$ is σ-invariant, i.e., $\sigma^* \bar{g} = \bar{g}$, the deck transformation σ induces the involutive endomorphism of the elliptic complex (2.1).

If we restrict ourself to the σ-fixed parts, we derive the σ-invariant elliptic complex

$$(4.3) \qquad 0 \longrightarrow \Gamma_\sigma(X, T) \longrightarrow \Gamma_\sigma(X, S_0^2(T^*)) \longrightarrow \Gamma_\sigma(X, S_0^2(\Omega^+)) \longrightarrow 0$$

with cohomology groups $\boldsymbol{H}_\sigma^i$, $i=0, 1, 2$.

PROPOSITION 4.2. *Each cohomology group $\boldsymbol{H}_\sigma^i$ is canonically identified with the linear subspace of $\boldsymbol{H}^i(X)$ of X consisting of σ-fixed vectors.*

So, we identify $\boldsymbol{H}_\sigma^i$ with the elementwise σ-fixed linear subspace, $\boldsymbol{H}_\sigma^i \subset \boldsymbol{H}^i(X)$.

Proof. For brevity we show the case $i=1$. By definition each $\bar{h}$ of $\boldsymbol{H}_\sigma^1$ satisfies $D_{\bar{g}}\bar{h}=0$ and $(L_{\bar{g}}\bar{U}, \bar{h})=0$ for all $\bar{U}\in\Gamma_\sigma(X, T)$.

Denote by $(\boldsymbol{H}^1)^\sigma$ the linear subspace in $\boldsymbol{H}^1(X)$ of the above proposition. Then we observe $(\boldsymbol{H}^1)^\sigma\subset\boldsymbol{H}_\sigma^1$. Now we prove the converse implication. Let $\bar{h}\in\boldsymbol{H}_\sigma^1$. Consider this as a section of $S_0^2(T^*)$. So from the harmonic decomposition $\bar{h}=\bar{h}_1+\bar{h}_2+\bar{h}_3$, $\bar{h}_1\in\boldsymbol{H}^1(X)$, $\bar{h}_2\in\operatorname{Im} L_{\bar{g}}$, $\bar{h}_3\in\operatorname{Im} D_{\bar{g}}^*$. Since $D_{\bar{g}}\bar{h}=0$, it follows that $\bar{h}_3=0$. Then by the σ-invariance of $\bar{h}$ we can write $\bar{h}_2=L_{\bar{g}}\bar{U}$, $\bar{U}\in\Gamma_\sigma(T)$. Substitute this equality into $(L_{\bar{g}}\bar{V}, \bar{h})=0$ to get $L_{\bar{g}}\bar{U}=0$. So the proof is completed.

We are now ready to complete the proof of Proposition 4.1, namely to show $\dim \boldsymbol{H}^2=3$ for each Ricci flat metric on an Enriques manifold Y.

By Proposition 4.2 it is sufficient to assert $\dim \boldsymbol{H}_\sigma^2=3$ for any Ricci flat metric $\bar{g}$ such that $\sigma^*\bar{g}=\bar{g}$. As we showed in the proof of Proposition 2.1, $\boldsymbol{H}^2$ consists of parallel sections. Those sections of $S_0^2(\Omega^+)$ are of the form $\sum_{a,b}\phi_a^+\otimes\phi_b^+$, where ϕ_a^+, $a=1, 2, 3$ are parallel self-dual 2-forms giving a basis of $H^+(\bar{g})$.

Before counting the dimension we prepare the following

PROPOSITION 4.3. *For a σ-invariant Ricci flat metric $\bar{g}$ on X $H^+(\bar{g})$ and $H^-(\bar{g})$ split as*

$$H^+(\bar{g})=W_1^+\oplus W_2^+,$$

$$H^-(\bar{g})=W_1^-\oplus W_2^-, \tag{4.4}$$

into the subspaces of dimension $\dim W_1^+=1$, $\dim W_2^+=2$, $\dim W_1^-=9$ and $\dim W_2^-=10$ such that $\sigma^*=id$ on $W_1^\pm$ and $\sigma^*=-id$ on $W_2^\pm$.

Proof. This proposition is obvious, since the deck transformation σ acts on $H^2(X, \boldsymbol{Z})$ as (4.1), or equivalently $b^+=1$ and $b^-=9$ for an Enriques manifold.

Return back to the counting. From this proposition the action of σ on $H^+(\bar{g})$ is

$$\sigma^*\phi_1^+=\phi_1^+, \qquad \sigma^*\phi_a^+=-\phi_a^+, \qquad a=2, 3$$

for a certain basis $\{\phi_a^+\}$ so we see easily that $\phi_1^+\otimes\phi_1^+-\phi_2^+\otimes\phi_2^+$, $\phi_1^+\otimes\phi_1^+-\phi_3^+\otimes\phi_3^+$ and $\phi_2^+\otimes\phi_3^++\phi_3^+\otimes\phi_2^+$ give a basis of the σ-invariant linear subspace $\boldsymbol{H}_\sigma^2$.

The complex (4.3) is just the involution-invariant version of (4.2) so that $\mathcal{M}^{(0)}(Y)$ and $\mathcal{E}(Y)$ for an Enriques manifold Y are investigated by means of involution-invariant portions of the corresponding $\mathcal{M}^{(0)}(X)$ and $\mathcal{E}(X)$ for a K3 manifold X.

Let $[g]$ be an anti-self-dual conformal structure on an Enriques manifold Y. Assume that it has zero Yamabe invariant. Then it is represented by a Ricci flat metric g.

From the real analytic variety theorem a neighborhood of $[g]$ in the moduli $\mathscr{M}(Y)$ has the form of the zero locus of Ψ_σ in an ε ball $\subset \boldsymbol{H}_\sigma^1$, where $\Psi_\sigma: \boldsymbol{H}_\sigma^1 \to \boldsymbol{H}_\sigma^2$ is the analytic map associated to the σ-invariant Kuranishi map. By Proposition 4.2 we can consider Ψ_σ just as $\Psi: \boldsymbol{H}^1 \to \boldsymbol{H}^2$ over X restricted to the σ-fixed linear subspace. As was proved in § 2, we have $\Psi=0$ which assures that at $[g]$ $\mathscr{M}(Y)$ is isomorphic to $\{\bar{h} \in \boldsymbol{H}^1(X)\,;\, |\bar{h}| < \varepsilon,\ \sigma^*\bar{h} = \bar{h}\}$.

Because an ε-neighborhood of the first cohomology group over X gives a neighborhood of $\mathscr{M}^{(0)}(X)$, we can get a neighborhood of $[g]$ in $\mathscr{M}(Y)$ exactly inside the proper submoduli $\mathscr{M}^{(0)}(Y)$, the moduli of anti-self-dual conformal structures of zero Yamabe invariant. Thus we get

THEOREM 4.4. *The moduli $\mathscr{M}^{(0)}(Y)$, isomorphic to $\hat{\mathscr{E}}(Y)$, is a smooth manifold of dimension* 29, *whose tangent space is modelled by $\boldsymbol{H}_\sigma^1$, the elementwise σ-fixed linear subspace $\boldsymbol{H}^1(X)$.*

Since any Ricci flat metric g on Y induces a metric $\bar{g}$ on X which is Ricci flat, we have a natural map

$$\iota: \hat{\mathscr{E}}(Y) \longrightarrow \hat{\mathscr{E}}(X)$$

$$g \mod \mathrm{Diff}^0(Y) \longmapsto \bar{g} \mod \mathrm{Diff}^0(X)$$

In fact, arbitrary $\phi \in \mathrm{Diff}^0(Y)$ induces uniquely $\bar{\phi} \in \mathrm{Diff}^0(X)$, because ϕ is generated by finite number of vector fields on Y and these vector fields lift up on X.

THEOREM 4.5. *The map $\iota: \hat{\mathscr{E}}(Y) \to \hat{\mathscr{E}}(X)$ gives an embedding and moreover the image of this map is a totally geodesic submanifold of $\hat{\mathscr{E}}(X)$ equipped with the L^2-metric.*

Proof. Let g and g_1 be Ricci flat metrics on Y such that the lifted Ricci flat metrics $\bar{g}$ and $\bar{g}_1$ satisfy $\bar{g}_1 = \bar{\phi}^*\bar{g}$ for a $\bar{\phi} \in \mathrm{Diff}^0(X)$. Since $\bar{g}$ and $\bar{g}_1$ are σ-invariant, it holds $\sigma^*\bar{\phi}^*\bar{g} = \bar{\phi}^*\sigma^*\bar{g}$.

Hence $\bar{\phi} \circ \sigma \circ (\bar{\phi})^{-1} \circ \sigma \in C_{\bar{g}} \cap \mathrm{Diff}'(X)$. Because by Proposition 2.3 $C_{\bar{g}} \cap \mathrm{Diff}'(X)$ consists only of id_X, $\bar{\phi}$ commutes with the deck transformation σ so that $\bar{\phi}$ descends to a $\phi \in \mathrm{Diff}^0(Y)$ such that $g_1 = \phi^* g$. So the map ι is injective.

From the identification $\hat{\mathscr{E}} \cong \mathscr{M}^{(0)}$ for both Y and X and the first cohomology groups give their local coordinates, it is seen that ι is smooth and has at every point the maximal rank $\dim \boldsymbol{H}_\sigma^1$. Thus ι is an embedding.

To show the image is totally geodesic it suffices to verify that the image is exactly the fixed points of an isometry in $\hat{\mathscr{E}}(X)$. This isometry is just the action of the deck transformation σ; $\bar{g} \mod \mathrm{Diff}^0(X) \mapsto \sigma^*\bar{g} \mod \mathrm{Diff}^0(X)$. We must check that the action of σ is isometric. But we observed already at

Theorem 3.2 that any orientation-preserving diffeomorphism preserves the conformally defined L^2-metric $\mathcal{G}$ on $\hat{\mathcal{M}}^{(0)}$ and this metric agrees with the invariant metric on $\hat{\mathcal{E}}(X)$ via the period map (see Theorem 3.3 and also Theorem 6, [10]). Thus σ yields an isometric transformation of $\hat{\mathcal{M}}^{(0)}(X)\cong\hat{\mathcal{E}}(X)$ preserving this identification.

It is not hard to see that the image of ι is exactly the fixed points of this isometry.

The isometry $\sigma:\hat{\mathcal{E}}(X)\to\hat{\mathcal{E}}(X)$ gives rise to an involutive isometry of the Teichmüller moduli, denoted by $\sigma:\tilde{\mathcal{E}}(X)\to\tilde{\mathcal{E}}(X)$, because $\sigma\circ\phi\circ\sigma^{-1}\in\mathrm{Diff}'$ for any $\phi\in\mathrm{Diff}'$ in such a way that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{E}}(X) & \xrightarrow{\sigma} & \hat{\mathcal{E}}(X) \\ \downarrow & & \downarrow \\ \tilde{\mathcal{E}}(X) & \xrightarrow{\sigma} & \tilde{\mathcal{E}}(X)\,. \end{array} \tag{4.5}$$

To obtain geometrical feature of $\tilde{\mathcal{E}}(Y)$ for an Enriques manifold Y we investigate the space of σ-fixed points in $\tilde{\mathcal{E}}(X)$ which we denote by $\tilde{\mathcal{E}}_\sigma(X)$.

So, we consider the period map $pe:\tilde{\mathcal{E}}(X)\to SO(3,19)/SO(3)\times SO(19)$. Then σ acts naturally as an isometry on the symmetric space by sending any oriented positive 3-plane Π to $\sigma^*\Pi$ so that the actions of σ commute through the period map. Therefore, the image $pe(\tilde{\mathcal{E}}_\sigma(X))$ is an open dense subset of $(SO(3,19)/SO(3)\times SO(19))_\sigma$, where $(SO(3,19)/SO(3)\times SO(19))_\sigma$ is the fixedpoint set of the isometry σ.

Obviously this fixedpoint set is a symmetric space of noncompact type, totally geodesically embedded in the ambient symmetric space.

PROPOSITION 4.6 (Theorem 0.4 in Introduction). *The fixedpoint set has the structure of quotient space of the following form*

$$\begin{aligned}&(SO(3,19)/SO(3)\times SO(19))_\sigma\\&\quad=(SO(1,9)/SO(1)\times SO(9))\times(SO(2,10)/SO(2)\times SO(10))\,.\end{aligned}$$

The latter space is well embedded in $SO(3,19)/SO(3)\times SO(19)$.

Proof. Let us assume that Π is an arbitrary oriented positive 3-plane such that $\sigma^*\Pi=\Pi$. The q_X-orthogonal complement $\Pi^\perp$ is then σ-invariant. So we have splittings of these subspaces Π, $\Pi^\perp$ like Proposition 4.3. Therefore it is not difficult to get the proposition.

§5. A Z_2-quotient of Enriques manifold

Consider in this section the moduli of Ricci flat metrics on a 4-manifold Z last appeared in Theorem 0.2, Introduction, namely a $\boldsymbol{Z}_2$-quotient Z of an Enriques manifold Y.

The 4-manifold Z is written as $Z=Y/\langle\theta\rangle$ where $\theta\in\mathrm{Diff}^+(Y)$ is fixedpoint free and involutive. So, the Euler number $\chi(Z)$ is $\chi(Z)=(1/2)\chi(Y)=6$ and the index $\tau(Z)=(1/2)\tau(Y)=-4$. Since $\pi_1(Z)=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$ and hence $b_1(Z)=0$, we have $b^+(Z)=0$ and $b^-(Z)=4$ so that the cup product q_Z of Z is negative definite.

Let g be a Ricci flat metric on Z. Then it lifts up to a θ-invariant Ricci flat metric $\bar{g}$ on Y. Since $\theta^*\bar{g}=\bar{g}$, θ induces an involutive action of $H^+(Y, \bar{g})$.

From $b^+(Y)=1$ and $b^+(Z)=0$, θ acts as $-id$ on $H^+(Y, \bar{g})$, i.e., $\theta^*(\alpha)=-\alpha$ for $\alpha\in H^+(Y, \bar{g})$.

Since from Proposition 0.1 $\bar{g}$ is an anti-self-dual metric of zero scalar curvature, each element of $H^+(Y, \bar{g})$ is parallel in such a way that a certain $\alpha\in H^+(Y, \bar{g})$ gives the Kähler form to the metric $\bar{g}$ with respect to a certain complex structure J; $\alpha(u, v)=\bar{g}(J(u), v)$. In other words, $\bar{g}$ is a Kähler metric on a complex surface (Y, J). It follows then from $\theta^*\alpha=-\alpha$, $\theta^*\bar{g}=\bar{g}$ that $\theta^*J=-J$, i.e., θ is an anti-holomorphic involution.

On the other hand, from the topological invariants $b^-(Y)=9$, $b^-(Z)=4$, the space $H^-(Y, \bar{g})$ splits as

$$H^-(Y, \bar{g})=V_1^-\oplus V_2^- \qquad \dim V_1^-=4, \quad \dim V_2^-=5$$

where θ^* is id on V_1^- and $-id$ on V_2^-.

In the same way as in the argument for an Enriques surface we have for g an elliptic complex (2.1) whose index is $26/2=13$.

PROPOSITION 5.1. *The cohomology groups for a Ricci flat metric g on Z are* $\boldsymbol{H}^0=0$, $\boldsymbol{H}^1\cong\boldsymbol{R}^{15}$ *and* $\boldsymbol{H}^2\cong\boldsymbol{R}^2$.

Proof. We see $\boldsymbol{H}^0=0$ in a same way as in the proof of Proposition 2.1.

Now we compute the dimension $\dim\boldsymbol{H}^2$. For this we apply Proposition 4.2 to our situation. Actually $\boldsymbol{H}^2$ for Z is isomorphic to the θ-invariant linear subspace $(\boldsymbol{H}^2)_\theta$ of $\boldsymbol{H}^2$ for a Ricci flat Enriques manifold $(Y, \bar{g})$. We can then follow the argument given in [12]. We note that $\bar{g}$ is a Kähler metric on an Enriques surface (Y, J). The following decomposition of $S_0^2(\Omega^+)$ is valid for any complex Kähler surface M ([7], [10]). As a real vector bundle

$$S_0^2(\Omega^+)=\boldsymbol{R}\Phi\oplus(K_M)_{\boldsymbol{R}}\oplus(K_M^{\otimes 2})_{\boldsymbol{R}}$$

where Φ is a certain parallel section of $S_0^2(\Omega^+)$, and K_M and $K_M^{\otimes 2}$ are the canonical line bundle of M and its square, respectively. Moreover $(K_M)_{\boldsymbol{R}}$ means the rank two real vector bundle induced from K_M.

Since $\boldsymbol{H}^2=\mathrm{Ker}\,DD^*$ and $DD^*=(\nabla\nabla)^2$ for an arbitrary Ricci flat 4-manifold, the subspace $(\boldsymbol{H}^2)_\theta$ consists of θ-invariant parallel sections of the tracefree symmetric product bundle $S_0^2(\Omega_Y^+)$. We have seen in the above that θ is antiholomorphic and $\theta^*\alpha=-\alpha$ where α is the Kähler form of the Kähler metric $\bar{g}$ $(\alpha\in H^+(Y, \bar{g}))$.

Moreover $\theta^*\beta=c\bar{\beta}$, $c\in\boldsymbol{C}$ for a certain section β of unit norm of K_Y. Because θ is involutive, $|c|$ must be equal to 1.

Since Φ has the following form ([10])

$$\Phi=\frac{1}{4}\alpha^2-\frac{1}{2}\beta\cdot\bar{\beta},$$

Φ is θ-invariant.

On the other hand, for an Enriques surface Y K_Y is not trivial but $K_Y^{\otimes 2}$ is trivial as holomorphic bundles. So K_Y does not admit but $K_Y^{\otimes 2}$ admits a global holomorphic and hence parallel section. This section may be identified with β^2. Thus we have two parallel sections of $(K_Y^{\otimes 2})_{\boldsymbol{R}}$ the real part $\Phi_1=1/2(\beta^2+\bar{\beta}^2)$ and the imaginary part $\Phi_2=1/2\sqrt{-1}(\beta^2-\bar{\beta}^2)$.

Of course $\theta^*\Phi_1$, $\theta^*\Phi_2$ are parallel in $\Gamma(Y, (K_Y^{\otimes 2})_{\boldsymbol{R}})$. Since $\theta^*\beta=c\bar{\beta}$ with c of $|c|=1$, the 2×2 coefficient matrix of $\theta^*\Phi_i$ relative to Φ_i has trace zero and determinant -1 so that this matrix has eigenvalues $+1$, -1. Therefore, $\Gamma(Y, (K_Y^{\otimes 2})_{\boldsymbol{R}})$ has a 1-dim linear subspace generated by a $+1$-eigensection. Thus we see $\dim\boldsymbol{H}^2=\dim(\boldsymbol{H}^2(Y))_\theta=2$.

$\boldsymbol{H}^1$ has the dimension $\dim\boldsymbol{H}^1=-(\text{index})+\dim\boldsymbol{H}^0+\dim\boldsymbol{H}^2=15$.

Therefore, in a quite similar way to the argument given for Theorem 4.2 we get

THEOREM 5.2. *The isotopy-Teichmüller moduli* $\mathcal{M}^{(0)}(Z)$ *of anti-self-dual conformal structures of zero Yamabe invariant, which is isomorphic to the isotopy-Teichmüller moduli* $\hat{\mathcal{E}}(Z)$ *of Ricci flat metrics on* Z, *admits a* 15 dim *smooth manifold structure whose tangent space at each point is modelled by* $\boldsymbol{H}_\theta^1(Y)$, *the elementwise* θ*-fixed linear subspace of the first cohomology group* $\boldsymbol{H}^1(Y)$ *for an Enriques manifold* Y.

Since $\pi_1(Z)=\boldsymbol{Z}_2\times\boldsymbol{Z}_2$, we can further regard Z as a $\boldsymbol{Z}_2\times\boldsymbol{Z}_2$-quotient of a K3 manifold X, $Z=X/\langle\sigma, \theta\rangle$ where σ, θ are involutive diffeomorphisms of X satisfying $\sigma\circ\theta=\theta\circ\sigma$.

So, via the covering map; $X\to Z$ an arbitrary Ricci flat metric g on Z is considered as a Ricci flat metric $\bar{g}$ on X which is $\langle\sigma, \theta\rangle$-invariant, i.e., $\sigma^*\bar{g}=\theta^*\bar{g}=\bar{g}$ and we have a natural map like the Ricci flat Enriques manifold case

$$j:\hat{\mathcal{E}}(Z)\longrightarrow\hat{\mathcal{E}}(X)$$

$$g \bmod \mathrm{Diff}^0(Z)\longmapsto\bar{g} \bmod \mathrm{Diff}^0(X)$$

THEOREM 5.3. *The map* $j:\hat{\mathcal{E}}(Z)\to\hat{\mathcal{E}}(X)$ *enjoys a totally geodesic embedding* *and the image* $\jmath(\hat{\mathcal{E}}(Z))$ *is a fixedpoint set of the isometries induced from the deck transformations* $\langle\sigma, \theta\rangle$ *in* $\hat{\mathcal{E}}(X)$.

The isometries $\sigma, \theta : \hat{\mathcal{E}}(X)\to\hat{\mathcal{E}}(X)$ yield isometries of the Teichmüller moduli $\tilde{\mathcal{E}}(X)$ and also of the Grassmannian manifold $SO(3, 19)/SO(3)\times SO(19)$ such that the actions of these isometries commute via the period map $pe : \tilde{\mathcal{E}}(X)\to SO(3, 19)/SO(3)\times SO(19)$. Thus we obtain Theorem 0.5 in the introduction.

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