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UNICITY THEOREMS FOR ENTIRE FUNCTIONS

BY HONG-XUN YI

1. Introduction

For any set S and any meromorphic function f let

$$E_f(S)=\bigcup_{a\in S}\{z\mid f(z)-a=0\},$$

where each zero of $f-a$ with multiplicity m is repeated m times in $E_f(S)$ (cf. [1]). It is assumed that the reader is familiar with the standard notations of Nevanlinna's theory that can be found, for instance, in [2]. It will be convenient to let E denote any set of finite linear measure on $0<r<\infty$, not necessarily the same at each occurrence. We denote by $S(r, f)$ any qtantity satisfying $S(r, f)=o(T(r, f))$ $(r\to\infty,\ r\notin E)$.

R. Nevanlinna proved the following well-known theorem.

THEOREM A (see [3], [4]). *Let $S_j=\{a_j\}$ $(j=1, 2, 3, 4)$, where a_1, a_2, a_3 and a_4 are four distinct complex numbers ($a_j=\infty$ is allowed). Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j)=E_g(S_j)$ for $j=1, 2, 3, 4$. Then either $f\equiv g$, or f is a linear fractional transformation of g, two of the values, say a_1 and a_2, must Picard values, and the cross ratio $(a_1, a_2, a_3, a_4)=-1$.*

It is easy to see from Theorem A that there exist three finite sets S_j $(j=1, 2, 3)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1, 2, 3$ must be identical. In [5] F. Gross asked the following open question (Question 6): Can one find two finite sets S_j $(j=1, 2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1, 2$ must be identical? In [5] F. Gross wrote: "The author and S. Koont have studied pairs of sets, each containing no more than two elements. In these cases one can probably prove that Question 6 can be answered negatively. If the answer to Question 6 is affirmative, it would be interesting to know how large both sets would have to be."

Throughout this paper we shall use w and u to denote the constants $\exp(2\pi i/n)$ and $\exp(2\pi i/m)$ respectively, where n and m are positive integers.

In this paper we answer the question posed by F. Gross. In fact, we prove

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more generally the following theorems.

THEOREM 1. *Let* $S_1=\{a+b,\ a+bw,\ \cdots,\ a+bw^{n-1}\}$, $S_2=\{c\}$, *where* $n>4$, a, b *and* c *are constants such that* $b\neq 0$, $c\neq a$ *and* $(c-a)^{2n}\neq b^{2n}$. *Suppose that* f *and* g *are nonconstant entire functions satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1, 2$. *Then* $f=g$.

THEOREM 2. *Let* $S_1=\{a_1+b_1,\ a_1+b_1w,\ \cdots,\ a_1+b_1w^{n-1}\}$, $S_2=\{a_2+b_2,\ a_2+b_2u,\ \cdots,\ a_2+b_2u^{m-1}\}$, *where* $n>4$, $m>4$, a_1, b_1, a_2 *and* b_2 *are constants such that* $b_1b_2\neq 0$ *and* $a_1\neq a_2$. *Suppose that* f *and* g *are nonconstant entire functions satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1, 2$. *Then* $f=g$.

Using Theorem A, we can prove that there exist four finite sets S_j ($j=1, 2, 3, 4$) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1, 2, 3, 4$ must be identical. Now it is natural to ask the following question: Can one find three finite sets S_j ($j=1, 2, 3$) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j)=E_g(S_j)$ for $j=1, 2, 3$ must be identical? In this paper we answer the above question. In fact, we prove the following theorem which is an extension of Theorem 2.

THEOREM 3. *Let* $S_1=\{a_1+b_1,\ a_1+b_1w,\ \cdots,\ a_1+b_1w^{n-1}\}$, $S_2=\{a_2+b_2,\ a_2+b_2u,\ \cdots,\ a_2+b_2u^{m-1}\}$ *and* $S_3=\{\infty\}$, *where* $n>6$, $m>6$, a_1, b_1, a_2 *and* b_2 *are constants such that* $b_1b_2\neq 0$ *and* $a_1\neq a_2$. *Suppose that* f *and* g *are nonconstant meromorphic functions satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1, 2, 3$. *Then* $f=g$.

2. Some Lemmas

LEMMA 1 (see [3]). *Let* $f_1, f_2, \cdots, f_n$ *be linearly independent meromorphic functions satisfying* $\sum_{j=1}^{n} f_j=1$. *Then for* $k=1, 2, \cdots, n$ *we have*

$$T(r, f_k)<\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right)+N(r, f_k)+N(r, D)-\sum_{j=1}^{n} N(r, f_j)$$
$$-N\left(r, \frac{1}{D}\right)+o(T(r)) \qquad (r\notin E),$$

where D *denotes the Wronskian*

$$D=\begin{vmatrix} f_1 & f_2 & \cdots f_n \\ f_1' & f_2' & \cdots f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots f_n^{(n-1)} \end{vmatrix}$$

and $T(r)$ *denotes the maximum of* $T(r, f_j)$, $j=1, 2, \cdots, n$.

Using the second fundamental theorem, it is easy to deduce the following result which is a special case ($n=2$) of Lemma 1.

LEMMA 2. *Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If $c_1f+c_2g=c_3$, then*

$$T(r, f)<\overline{N}\left(r, \frac{1}{f}\right)+\overline{N}\left(r, \frac{1}{g}\right)+\overline{N}(r, f)+S(r, f).$$

LEMMA 3 (see [6]). *Let f_1, f_2 and f_3 be three meromorphic functions satisfying $\sum_{j=1}^{3} f_j=1$, and let $g_1=-f_3/f_2$, $g_2=1/f_2$ and $g_3=-f_1/f_2$. If f_1, f_2 and f_3 are linearly independent, then g_1, g_2 and g_3 are linearly independent.*

3. Preliminary Theorems

In [7] F. Gross and C. F. Osgood proved the following theorem.

THEOREM B. *Let $S_1=\{-1, 1\}$, $S_2=\{0\}$. If f and g are entire functions of finite order such that $E_f(S_j)=E_g(S_j)$ $(j=1, 2)$, then $f=\pm g$ or $fg=\pm 1$.*

The present author [8] and independently G. Brosch [9] proved the following result which is an improvement of Theorem B.

THEOREM C. *Let $S_1=\{-1, 1\}$, $S_2=\{0\}$, $S_3=\{\infty\}$. If f and g are nonconstant meromorphic functions such that $E_f(S_j)=E_g(S_j)$ $(j=1, 2, 3)$, then $f=\pm g$ or $fg=\pm 1$.*

The present author [10] and independently K. Tohge [11] proved the following result which is an extension of the above results.

THEOREM D. *Let $S_1=\{a+b, a+bw, \cdots, a+bw^{n-1}\}$, $S_2=\{a\}$ and $S_3=\{\infty\}$, where $n>1$, a and b $(\neq 0)$ are constants. If f and g are meromorphic functions such that $E_f(S_j)=E_g(S_j)$ $(j=1, 2, 3)$, then $f-a=t(g-a)$, where $t^n=1$, or $(f-a)(g-a)=s$, where $s^n=b^{2n}$.*

In this paper we prove the following interesting results which are some improvements of the above theorems. These results will be needed in the proof of our theorems.

THEOREM 4. *Let $S_1=\{a+b, a+bw, \cdots, a+bw^{n-1}\}$, $S_2=\{\infty\}$, where $n>6$, a and b $(\neq 0)$ are constants. If f and g are meromorphic functions such that $E_f(S_j)=E_g(S_j)$ $(j=1, 2)$, then $f-a=t(g-a)$, where $t^n=1$, or $(f-a)(g-a)=s$, where a and ∞ are Picard values of f and g, and $s^n=b^{2n}$.*

Proof. Let $S_3=\{1, w, \cdots, w^{n-1}\}$, and let $F=(f-a)/b$ and $G=(g-a)/b$. By $E_f(S_j)=E_g(S_j)$ $(j=1, 2)$, we obtain $E_F(S_j)=E_G(S_j)$ $(j=2, 3)$. Then, from Nevanlinna's second fundamental theorem, we have

$$(n-1)T(r, G)<\sum_{k=0}^{n-1}N\Big(r, \frac{1}{G-w^k}\Big)+N(r, G)+S(r, G)$$

$$=\sum_{k=0}^{n-1}N\Big(r, \frac{1}{F-w^k}\Big)+N(r, F)+S(r, G)$$

$$<(n+1)T(r, F)+S(r, G). \qquad (1)$$

Thus

$$T(r, G)=O(T(r, F)) \qquad (r\notin E). \qquad (2)$$

Again by $E_F(S_j)=E_G(S_j)$ $(j=2, 3)$, we obtain

$$F^n-1=e^h(G^n-1), \qquad (3)$$

where h is an entire function. From (1) and (3), we have

$$T(r, e^h)=T\Big(r, \frac{F^n-1}{G^n-1}\Big)$$

$$<T(r, F^n)+T(r, G^n)+O(1)$$

$$<nT(r, F)+\frac{n(n+1)}{n-1}T(r, F)+S(r, F).$$

Thus

$$T(r, e^h)=O(T(r, F)) \qquad (r\notin E). \qquad (4)$$

Let us put $f_1=F^n$, $f_2=e^h$, $f_3=-e^hG^n$, and $T(r)$ denote the maximum of $T(r, f_j)$, $j=1, 2, 3$. From (2), (3) and (4), we obtain

$$\sum_{j=1}^{3}f_j=1 \qquad (5)$$

and

$$T(r)=O(T(r, F)) \qquad (r\notin E). \qquad (6)$$

We discuss the following three cases.

a) Suppose neither f_2 nor f_3 is a constant.

If f_1, f_2 and f_3 are linearly independent, applying Lemma 1 to functions f_j $(j=1, 2, 3)$, from (5) and (6) we have

$$T(r, f_1)<\sum_{j=1}^{3}N\Big(r, \frac{1}{f_j}\Big)-N\Big(r, \frac{1}{D}\Big)+N(r, D)-N(r, f_2)-N(r, f_3)+S(r, F), \qquad (7)$$

where

$$D=\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}. \qquad (8)$$

We note that

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right)=nN\left(r, \frac{1}{F}\right)+nN\left(r, \frac{1}{G}\right) \tag{9}$$

and

$$N\left(r, \frac{1}{D}\right)\geqq nN\left(r, \frac{1}{F}\right)-2\overline{N}\left(r, \frac{1}{F}\right)+nN\left(r, \frac{1}{G}\right)-2\overline{N}\left(r, \frac{1}{G}\right). \tag{10}$$

From (5) and (8) we get

$$D=\begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}$$

and hence

$$\begin{aligned} N(r, D)-N(r, f_2)-N(r, f_3) &\leqq N(r, (G^n)'')-N(r, G^n) \\ &=2\overline{N}(r, G). \end{aligned} \tag{11}$$

From (7), (9), (10) and (11) we deduce

$$\begin{aligned} nT(r, F) &< 2\overline{N}\left(r, \frac{1}{F}\right)+2\overline{N}\left(r, \frac{1}{G}\right)+2\overline{N}(r, G)+S(r, F) \\ &< 2T(r, F)+4T(r, G)+S(r, F). \end{aligned} \tag{12}$$

Let $g_1=-f_3/f_2=G^n$, $g_2=1/f_2=e^{-h}$ and $g_3=-f_1/f_2=-e^{-h}F^n$. From (5) we obtain

$$\sum_{j=1}^{3} g_j=1.$$

By Lemma 3 we know that g_1, g_2 and g_3 are linearly independent. In the same manner as above, we have

$$nT(r, G)<4T(r, F)+2T(r, G)+S(r, F). \tag{13}$$

Combining (12) and (13) we get

$$(n-6)T(r, F)+(n-6)T(r, G)<S(r, F). \tag{14}$$

Since $n>6$, (14) is absurd. Hence f_1, f_2 and f_3 are linearly dependent. Then, there exist three constants $(c_1, c_2, c_3)\neq(0, 0, 0)$ such that

$$c_1f_1+c_2f_2+c_3f_3=0. \tag{15}$$

If $c_1=0$, from (15) $c_2\neq0$, $c_3\neq0$ and

$$f_3=-\frac{c_2}{c_3}f_2$$

and hence

$$G^n=\frac{c_2}{c_3},$$

which is impossible. Thus $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{16}$$

Now combining (5) and (16) we get

$$\left(1-\frac{c_2}{c_1}\right) f_2 + \left(1-\frac{c_3}{c_1}\right) f_3 = 1. \tag{17}$$

Since neither f_2 nor f_3 is a constant, from (17) we have $c_1 \neq c_2$ and $c_1 \neq c_3$. Again from (17) we obtain

$$\left(1-\frac{c_3}{c_1}\right) G^n + e^{-h} = 1 - \frac{c_2}{c_1}. \tag{18}$$

By Lemma 2 and (18) we get

$$nT(r, G) < \overline{N}\left(r, \frac{1}{G}\right) + S(r, G)$$

$$< T(r, G) + S(r, G),$$

which is again a contradiction.

b) Suppose that $f_2 = c$ $(\neq 0)$.

If $c \neq 1$, from (5) we have

$$f_1 + f_3 = 1 - c$$

that is

$$F^n - cG^n = 1 - c. \tag{19}$$

By Lemma 2 we have

$$nT(r, F) < \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + S(r, F)$$

$$< 2T(r, F) + T(r, G) + S(r, F),$$

and

$$nT(r, G) < T(r, F) + 2T(r, G) + S(r, G).$$

Hence,

$$(n-3)T(r, F) + (n-3)T(r, G) < S(r, F) + S(r, G),$$

which is impossible. Thus $c=1$. From (19) we deduce $F^n = G^n$ and $F = tG$, where $t^n = 1$. Thus $f - a = t(g - a)$, where $t^n = 1$.

c) Suppose that $f_3 = c$ $(c \neq 0)$.

If $c \neq 1$, from (5) we have

$$f_1 + f_2 = 1 - c$$

that is

$$F^n + e^h = 1 - c. \tag{20}$$

By Lemma 2 we have

$$nT(r, F)<\overline{N}\left(r, \frac{1}{F}\right)+S(r, F)$$
$$<T(r, F)+S(r, F),$$

which is impossible. Thus $c=1$. From (20) we have $F^n=-e^h$, $G^n=-e^{-h}$ and $F^nG^n=1$. Thus $(f-a)(g-a)=s$, where a and ∞ are Picard values of f and g, and $s^n=b^{2n}$.

This completes the proof of Theorem 4.

When f and g are nonconstant entire functions, $N(r, f)=N(r, g)=0$. Using the above result, and proceeding as in the proof of Theorem 4, we can prove the following theorem.

THEOREM 5. *Let* $S=\{a+b, a+bw, \cdots, a+bw^{n-1}\}$, *where* $n>4$, a *and* b $(\neq 0)$ *are constants. If* f *and* g *are nonconstant entire functions such that* $E_f(S)=E_g(S)$, *then* $f-a=t(g-a)$, *where* $t^n=1$, *or* $(f-a)(g-a)=s$, *where* a *is a Picard value of* f *and* g, *and* $s^n=b^{2n}$.

4. Proof of Theorem 1

By the assumption $E_f(S_1)=E_g(S_1)$, we have from Theorem 5

$$f-a=t(g-a), \tag{21}$$

where $t^n=1$, or

$$(f-a)(g-a)=s, \tag{22}$$

where a is a Picard value of f and g, and $s^n=b^{2n}$. We discuss the following two cases.

a) Suppose that f and g satisfy (21).

If c is a Picard value of f, by the assumption $E_f(S_2)=E_g(S_2)$, we know that c is a Picard value of g. Again from (21), we know that $a+t(c-a)$ is a Picard value of f. Since f is an entire function, we have $c=a+t(c-a)$. Thus $t=1$, and hence $f=g$.

If c is not a Picard value of f, then exist z_0 such that $f(z_0)=g(z_0)=c$. By (21), we obtain $c-a=t(c-a)$. Thus $t=1$, and hence $f=g$.

b) Suppose that f and g satisfy (22).

It is easy to see that c is not a Picard value of f. Then exist z_0 such that $f(z_0)=g(z_0)=c$. By (22), we obtain $(c-a)^2=s$. Thus $(c-a)^{2n}=s^n=b^{2n}$, this contradicts the assumption.

This completes the proof of Theorem 1.

5. Proof of Theorems 2 and 3

5.1. Proof of Theorem 3

By the assumption $E_f(S_j)=E_g(S_j)$ $(j=1, 3)$, we have from Theorem 4

$$f-a_1=t_1(g-a_1), \tag{23}$$

where $t_1^n=1$, or

$$(f-a_1)(g-a_1)=s_1, \tag{24}$$

where a_1 and ∞ are Picard values of f and g, and $s_1^n=b_1^{2n}$. In the same manner as above, by the assumption $E_f(S_j)=E_g(S_j)$ $(j=2, 3)$, we have

$$f-a_2=t_2(g-a_2), \tag{25}$$

where $t_2^m=1$, or

$$(f-a_2)(g-a_2)=s_2, \tag{26}$$

where a_2 and ∞ are Picard values of f and g, and $s_2^m=b_2^{2m}$.

We discuss the following four cases.

a) Suppose that f and g satisfy (23) and (25). Then

$$a_2-a_1=(t_1-t_2)g+(t_2a_2-t_1a_1). \tag{27}$$

Since g is not a constant, and $a_1\neq a_2$, we have from (27), $t_1=t_2=1$. Thus $f=g$.

b) Suppose that f and g satisfy (23) and (26). Then a_2 and ∞ are Picard values of f and g. From (26), we know that $f\neq g$. Again from (23), we know that $t_1\neq 1$ and $a_1+t_1(a_2-a_1)$ is a Picard value of f. Thus a_2, $a_1+t_1(a_2-a_1)$ and ∞ are Picard values of f, which is impossible.

c) Suppose that f and g satisfy (24) and (25). Similar to the case b), we have again a contradiction.

d) Suppose that f and g satisfy (24) and (26). Then, a_1, a_2 and ∞ are Picard values of f, which is impossible.

This completes the proof of Theorem 3.

5.2. Proof of Theorem 2

Using Theorem 5, and proceeding as in the proof of Theorem 3, we can prove Theorem 2.

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