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UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE THREE VALUES

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1. Introduction and Main Results.

Let f and g be two nonconstant meromorphic functions in the complex plane. If f and g have the same a-points with the same multiplicities, we say f and g share the value a CM. (see [1]). It is assumed that the reader is familiar with the notations of the Nevanlinna Theory (see, for example, [2]). Let E denote any set of finite linear measure of $0<r<\infty$. The notation $S(r, f)$ denotes any quantity satisfying

$$S(r, f)=o(T(r, f)) \qquad (r \to \infty, r \notin E).$$

H. Ueda proved the following theorem.

THEOREM A (*see* [3]). *Let f and g be two distinct nonconstant entire functions such that f and g share 0, 1 CM., and let a be a finite complex number, and $a \neq 0, 1$. If a is lacunary for f, then $1-a$ is lacunary for g, and*

$$(f-a)(g+a-1) \equiv a(1-a).$$

In [4] *Yi* Hong-Xun proved more generally the following theorem.

THEOREM B. *Let f and g be two distinct nonconstant entire functions such that f and g share 0, 1 CM., and let a be a finite complex number, and $a \neq 0, 1$. If $\delta(a, f)>1/3$, then a and $1-a$ are Picard exceptional values of f and g respectively, and*

$$(f-a)(g+a-1) \equiv a(1-a).$$

In this paper we extend the above theorems to meromorphic functions, and, in fact, prove the following theorem.

THEOREM 1. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1, ∞CM., and let a be a finite complex number, and $a \neq 0, 1$. If*

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$$\delta(a, f)+\delta(\infty, f)>\frac{4}{3},$$

then a and $1-a$ are Picard exceptional values of f and g respectively, and also ∞ is so, and

$$(f-a)(g+a-1)\equiv a(1-a).$$

In place of Theorem 1, we prove more generally the following theorem which is a generalization of Theorem A, Theorem B and Theorem 1.

THEOREM 2. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0, 1, \infty CM.$, and let $a_1, a_2, \cdots, a_p$ be p $(\geqq 1)$ distinct finite complex numbers, and $a_i \neq 0, 1$ $(i=1, 2, \cdots, p)$. If*

$$\sum_{i=1}^{p}\delta(a_i, f)+\delta(\infty, f)>\frac{2(p+1)}{p+2}, \tag{1}$$

then there exists one and only one a_j in $a_1, a_2, \cdots, a_p$ such that a_j and $1-a_j$ are Picard exceptional values of f and g respectively, and also ∞ is so, and

$$(f-a_j)(g+a_j-1)\equiv a_j(1-a_j).$$

2. Some Lemmas.

The following lemmas will be needed in the proof of our theorems.

LEMMA 1. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0, 1, \infty CM.$, then*

$$f=\frac{1-e^{\beta}}{1-e^{\beta-\alpha}}, \tag{2}$$

$$f-1=\frac{e^{\beta-\alpha}(1-e^{\alpha})}{1-e^{\beta-\alpha}}, \tag{3}$$

where α and β are entire functions and $e^{\alpha}\not\equiv 1$, $e^{\beta}\not\equiv 1$, $e^{\beta-\alpha}\not\equiv 1$ and

$$T(r, e^{\alpha})=0(T(r, f)) \qquad (r\notin E),$$

$$T(r, e^{\beta})=0(T(r, f)) \qquad (r\notin E).$$

Proof. By assumption, we have

$$f=ge^{\alpha} \tag{4}$$

and

$$f-1=(g-1)e^{\beta}, \tag{5}$$

where both e^{α} and e^{β} are entire functions, and

$$e^{\alpha}\not\equiv 1, \qquad e^{\beta}\not\equiv 1, \qquad e^{\beta-\alpha}\not\equiv 1.$$

It follows from (4) and (5) that (2) and (3) hold.

By Nevanlinna's second fundamental theorem, we obtain

$$T(r, g)<N\left(r, \frac{1}{g}\right)+N\left(r, \frac{1}{g-1}\right)+N(r, g)+S(r, g)<3T(r, f)+S(r, g).$$

Hence

$$T(r, g)<(3+o(1))T(r, f) \qquad (r\notin E).$$

It follows from Nevanlinna's first fundamental theorem that

$$T(r, e^{\alpha})\leqq T(r, f)+T\left(r, \frac{1}{g}\right)<(4+o(1))T(r, f) \qquad r\notin E)$$

and

$$T(r, e^{\beta})\leqq T(r, f-1)+T\left(r, \frac{1}{g-1}\right)<(4+o(1))T(r, f) \qquad (r\notin E).$$

This completes the proof of Lemma 1.

LEMMA 2 (*see* [5]). *Let $f_i(i=1, 2, \cdots, n)$ be n linearly independent meromorphic functions satisfying*

$$\sum_{i=1}^{n} f_i\equiv 1,$$

then for $j=1, 2, \cdots, n$ we have

$$T(r, f_j)<\sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right)+N(r, f_j)+N(r, D)-\sum_{i=1}^{n} N(r, f_i)-N\left(r, \frac{1}{D}\right)$$
$$+0(\log r+\log T_n(r)) \qquad (r\notin E),$$

where D denotes the Wronskian

$$D=\begin{vmatrix} f_1, & f_2, \cdots\cdots, & f_n \\ f_1', & f_2', \cdots\cdots, & f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)}, & f_2^{(n-1)}, \cdots, & f_n^{(n-1)} \end{vmatrix}$$

and $T_n(r)$ denotes the maximum of $T(r, f_i)$ $(i=1, 2, \cdots, n)$.

LEMMA 3. *Let b be a finite complex number, and $b\neq 0, 1$. Suppose that f and g are two distinct nonconstant meromorphic functions such that f and g share $0, 1, \infty$ CM.. Using the notations of Lemma 1, let $f_1=(1/1-b)(f-b)(1-e^{\beta-\alpha})$, $f_2=(1/1-b)e^{\beta}$, $f_3=(-b/1-b)e^{\beta-\alpha}$. If the $f_i(i=1, 2, 3)$ are linearly independent, then*

$$N\left(r, \frac{1}{f}\right)<N\left(r, \frac{1}{f-b}\right)+S(r, f),$$

$$N\left(r, \frac{1}{f-1}\right)<N\left(r, \frac{1}{f-b}\right)+S(r, f).$$

Proof. By assumption, from Lemma 1 we easily see that all $f_i(i=1, 2, 3)$ are entire functions, and

$$\sum_{i=1}^{3} f_i \equiv 1,$$

$$T_3(r)=0(T(r, f)),$$

where $T_3(r)$ denotes the maximum of $T(r, f_i)$ $(i=1, 2, 3)$.

Applying Lemma 2 to functions f_i $(i=1, 2, 3)$ we obtain

$$T(r, e^{\beta})<N\left(r, \frac{1}{f-b}\right)+N\left(r, \frac{1}{e^{\beta-\alpha}-1}\right)-N(r, f)+S(r, f).$$

Note that $e^{\beta}\not\equiv$const. and (2), then we have

$$N\left(r, \frac{1}{f}\right)=N\left(r, \frac{1}{e^{\beta}-1}\right)-N\left(r, \frac{1}{e^{\beta-\alpha}-1}\right)+N(r, f)$$

$$=T(r, e^{\beta})-N\left(r, \frac{1}{e^{\beta-\alpha}-1}\right)+N(r, f)+S(r, f)$$

$$<N\left(r, \frac{1}{f-b}\right)+S(r, f).$$

Let us put

$$g_1=\frac{1}{b}e^{\alpha-\beta}(f-b)(1-e^{\beta-\alpha}), \quad g_2=\frac{1}{b}e^{\alpha}, \quad g_3=-\frac{1-b}{b}e^{\alpha-\beta}, \quad \text{then } \sum_{i=1}^{3} g_i \equiv 1.$$

Assume that the $g_i(i=1, 2, 3)$ are linearly dependent, then there would be constants $d_i(i=1, 2, 3)$ which can't all equal zero, such that

$$d_1g_1+d_2g_2+d_3g_3=0.$$

Multiplying the above equation by $(b/1-b)e^{\beta-\alpha}$, and noting that $\sum_{i=1}^{3} f_i\equiv 1$, we obtain

$$(d_1-d_3)f_1+(d_2-d_3)f_2-d_3f_3=0.$$

Since d_1-d_3 and d_2-d_3 can't all equal zero, hence the $f_i(i=1, 2, 3)$ are also linearly dependent, contrary to the above assumption that the $f_i(i=1, 2, 3)$ are linearly independent. So the g_i $(i=1, 2, 3)$ must also be linearly independent. Noting that $e^{\alpha}\not\equiv$const. and (3), in a similar manner, we can prove that

$$N\left(r, \frac{1}{f-1}\right)<N\left(r, \frac{1}{f-b}\right)+S(r, f).$$

This completes the proof of Lemma 3.

By Nevanlinna's second fundamental theorem, we can easily prove the following lemma.

LEMMA 4. *Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If*

$$c_1 f + c_2 g \equiv c_3,$$

then

$$T(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}(r, f) + S(r, f).$$

3. Proof of Theorem 2.

In the following, we shall use the notations of Lemma 1 and Lemma 3.

Suppose that the $f_i (i=1, 2, 3)$ are linearly independent for any $b=a_i (i=1, 2, \cdots, p)$. By Lemma 3

$$N\left(r, \frac{1}{f}\right) < N\left(r, \frac{1}{f-a_i}\right) + S(r, f) \qquad (i=1, 2, \cdots, p)$$

and

$$N\left(r, \frac{1}{f-1}\right) < N\left(r, \frac{1}{f-a_i}\right) + S(r, f) \qquad (i=1, 2, \cdots, p).$$

Hence we have

$$N\left(r, \frac{1}{f}\right) < \frac{1}{p} \sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + S(r, f)$$

and

$$N\left(r, \frac{1}{f-1}\right) < \frac{1}{p} \sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + S(r, f).$$

By Nevanlinna's second fundamental theorem

$$\begin{aligned}(p+1)T(r, f) &< N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N(r, f) + \sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + S(r, f) \\ &< \left(1+\frac{2}{p}\right)\left(\sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + N(r, f)\right) + S(r, f) \\ &\leqq \frac{p+2}{p}\left\{p+1-\left(\sum_{i=1}^{p} \delta(a_i, f) + \delta(\infty, f)\right)\right\} T(r, f) + S(r, f). \qquad (6)\end{aligned}$$

Since

$$\frac{p+2}{p}\left\{p+1-\left(\sum_{i=1}^{p} \delta(a_i, f) + \delta(\infty, f)\right)\right\} < p+1,$$

so (6) is a contradiction. Hence the $f_i (i=1, 2, 3)$ are linearly dependent for at least one of the $a_i (i=1, 2, \cdots, p)$, say for $b=a_1$. Thus for the fixed value $b=a_1$ there would be three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{7}$$

If $c_1=0$, from (7) we have $c_2 \neq 0$, $c_3 \neq 0$ and $e^{\alpha} \equiv k_1$, where $k_1=(bc_3/c_2)$ clearly is a constant which depends on $b=a_1$, and $k_1 \neq 0, 1$. Then we have $f-1 \neq 0$ by (3). Hence by Nevanlinna's second fundamental theorem

$$pT(r, f) < N\left(r, \frac{1}{f-1}\right) + N(r, f) + \sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + S(r, f)$$

$$= N(r, f) + \sum_{i=1}^{p} N\left(r, \frac{1}{f-a_i}\right) + S(r, f).$$

So that

$$\sum_{i=1}^{p} \delta(a_i, f) + \delta(\infty, f) \leqq 1,$$

which contradicts the condition (1) of the theorem. Thus $c_1 \neq 0$.

In the following, suppose that $c_1 \neq 0$. From (7) we get

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3.$$

Hence

$$\left(1-\frac{c_2}{c_1}\right) f_2 + \left(1-\frac{c_3}{c_1}\right) f_3 \equiv 1. \tag{8}$$

We shall discuss the following four cases:

a) Assume $1-(c_2/c_1) \neq 0$ and $1-(c_3/c_1) \neq 0$.

If both e^{β} and $e^{\beta-\alpha}$ are nonconstants, then, by Lemma 4, we obtain

$$T(r, e^{\beta}) < \bar{N}\left(r, \frac{1}{e^{\beta}}\right) + \bar{N}\left(r, \frac{1}{e^{\beta-\alpha}}\right) + \bar{N}(r, e^{\beta}) + S(r, e^{\beta}) = o(T(r, e^{\beta})) \quad (r \notin E),$$

which is impossible. Thus at least one of $e^{\beta}=(1-b)f_2$ and $e^{\beta-\alpha}=-(1-b/b)f_3$ would equal a constant, so that both of them would be so by (8). Hence f and g are reduced to constants, which is a contradiction. Therefore this case is impossible.

b) Assume $1-(c_2/c_1)=0$ and $1-(c_3/c_1) \neq 0$.

Clearly $e^{\beta-\alpha} \equiv k_2$, where k_2 is a constant which depends on $b=a_1$, and $k_2 \neq 0, 1$.

Then we have $f=(1-e^{\beta}/1-k_2)$ by (2). For any complex number c we obtain

$$f-c = \frac{1}{1-k_2}\{1-c(1-k_2)-e^{\beta}\}.$$

If $1-c(1-k_2)\neq 0$, then

$$\delta(c,\ f)=0.$$

Since $\delta(\infty,\ f)=1$, it follows from (1)

$$\sum_{i=1}^{p}\delta(a_i,\ f)>\frac{p}{p+2}.$$

Hence there exists one and only one a_j in $a_1,\ a_2,\ \cdots,\ a_p$ such that

$$1-a_j(1-k_2)=0.$$

Thus

$$f=a_j(1-e^{\beta}),\quad g=(1-a_j)(1-e^{-\beta}).$$

Consequently

$$(f-a_j)(g+a_j-1)\equiv a_j(1-a_j),$$

and in this case a_j and $1-a_j$ are Picard exceptional values of f and g respectively, and also ∞ is so.

c) Assume $1-(c_2/c_1)\neq 0$ and $1-(c_3/c_1)=0$.

Clearly $e^{\beta}\equiv$const.. As the same as the case when $c_1=0$, this case is impossible too.

In fact, then we have $f\neq 0$ and

$$\sum_{i=1}^{p}\delta(a_i,\ f)+\delta(\infty,\ f)\leqq 1$$

by Nevanlinna's second fundamental theorem.

d) Assume $1-(c_2/c_1)=0$ and $1-(c_3/c_1)=0$.

Clearly we have $c_1=c_2=c_3$, which contradicts $\sum_{i=1}^{3}f_i\equiv 1$. Thus this case is also impossible.

Summarize the above, we conclude that under the hypotheses of the theorem, there exists one and only one a_j in $a_1,\ a_2,\ \cdots,\ a_p$ such that a_j and $1-a_j$ are Picard exceptional values of f and g respectively, and also ∞ is so, and

$$(f-a_j)(g+a_j-1)\equiv a_j(1-a_j).$$

This completes the proof of Theorem 2.

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