UNIVALENT ANALYTIC FUNCTIONS AND THE POINCARÉ METRIC

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Abstract

Let Ω be a hyperbolic domain in the complex plane $\mathbb{C}$, let ρ_Ω be the density of the Poincaré metric in Ω , and let $\beta_\Omega=1/\rho_\Omega$. For g analytic in Ω we set $\|g\|_\Omega=\sup \beta_\Omega(w)|g(w)|$, $w\in\Omega$. Let $S(\Omega)$ be the family of functions f analytic and univalent in Ω . Criteria in terms of the partial derivatives of β_Ω for Ω to satisfy $\sup \|f''/f'\|_\Omega<+\infty$, where f ranges over $S(\Omega)$, are given. For example, $\sup \beta_\Omega(w)|(\beta_\Omega)_{ww}(w)|<+\infty$, $w\in\Omega$. If $f\in S(\Omega)$ is isolated in the sense that there is an $\varepsilon>0$ such that $0<\|f''/f'-g''/g'\|_\Omega<\varepsilon$ for no $g\in S(\Omega)$, then $\mathbb{C}\setminus f(\Omega)$ is of zero area. The domain Ω is simply connected if $\sup \beta_\Omega(w)|(\beta_\Omega)_{ww}(w)|\leq 1$, $w\in\Omega$, and Ω is convex (hence simply connected) if and only if $\sup |(\beta_\Omega)_w(w)|=1$, $w\in\Omega$.

1. Introduction.

By Ω we always mean a subdomain of the complex plane $\mathbb{C}=\{|z|<+\infty\}$ such that the complement contains at least two points. Let ρ_Ω be the density of the Poincaré metric $\rho_\Omega(w)|dw|$ in Ω so that $\rho_D(w)=(1-|w|^2)^{-1}$ if w is in the disk $D=\{|z|<1\}$. We shall call $\beta_\Omega=1/\rho_\Omega$ the weight function which appears in

$$\|f\|_\Omega=\sup_{w\in\Omega}\beta_\Omega(w)|f(w)|$$

for f analytic in Ω . For g analytic in Ω and locally univalent in Ω , namely, $g'(w)\neq 0$ at each $w\in\Omega$, we set $\lambda(g)=g''/g'$. Let $S(\Omega)$ be the family of functions f analytic and univalent in Ω . We shall call Ω of finite type if

$$a(\Omega)=\sup_{f\in S(\Omega)}\|\lambda(f)\|_\Omega$$

is finite. We have $a(\Omega)\leq 8$ for each simply connected Ω , $a(D)=6$ and $a(D\setminus\{0\})=+\infty$; see [6, Theorem 2] and [13, Theorem 1 and p. 452].

We begin with weight function criteria for Ω to be of finite type. For a complex function $g(w)$ of $w=u+iv\in\Omega$ we recall the definition of the partial derivatives:

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$$g_w = 2^{-1}(g_u - ig_v) \quad \text{and} \quad g_{\bar{w}} = 2^{-1}(g_u + ig_v).$$

If g is real-valued and of C^2 further, then we have

$$g_{w\bar{w}} = 4^{-1} \Delta g, \quad \overline{(g_w)} = g_{\bar{w}} \quad \text{and} \quad \overline{(g_{\bar{w}})} = g_{w\bar{w}}.$$

THEOREM 1. *The following are equivalent, where $\beta = \beta_\Omega$ for simplicity.*

- (1.1) Ω is of finite type.
- (1.2) β_w is bounded in Ω .
- (1.3) $\beta\beta_{w\bar{w}}$ is bounded in Ω .
- (1.4) $\beta\beta_{w\bar{w}}$ is bounded in Ω .

Note that B.G. Osgood proved essentially (1.2) $\Rightarrow$ (1.1), and it is not difficult to observe that (1.1) $\Rightarrow$ (1.2) [13, Theorem 5]. For the completeness we include the proof of (1.1) $\Leftrightarrow$ (1.2).

For f nonconstant and meromorphic in Ω we define the Schwarzian derivative of f by

$$\sigma(f) = (f''/f')' - 2^{-1}(f''/f')^2.$$

It is known that if f is meromorphic and univalent in Ω , then

$$\|\sigma(f)\|_\Omega^* = \sup_{w \in \Omega} \beta_\Omega(w)^2 |\sigma(f)(w)| \leq 12;$$

see [4, Theorem 1]; by a rotation of the Riemann sphere the meromorphic case is reduced to the analytic case. The notation $\|\sigma(f)\|_\Omega^*$ will be used also for f analytic and locally univalent in Ω . We shall prove

THEOREM 2. *If f is analytic and locally univalent in Ω of finite type, then*

$$\|\sigma(f)\|_\Omega^* \leq \|\lambda(f)\|_\Omega \{K + c(\Omega) + 2^{-1}\|\lambda(f)\|_\Omega\},$$

where $8/e \leq K < 10/3$ and $c(\Omega) \geq 0$ is a constant with

$$(1.5) \quad c(\Omega) \leq 2 \left\{ 1 + \sup_{w \in \Omega} |(\beta_\Omega)_w(w)| \right\}.$$

This is an Ω -analogue of the P.L. Duren, H.S. Shapiro and A.L. Shields estimate in D :

$$\|\sigma(f)\|_D^* \leq 4\|\lambda(f)\|_D + 2^{-1}(\|\lambda(f)\|_D)^2;$$

see [6, p. 251]. Thus, for example, if Ω is of finite type, then $\|\sigma(f)\|_\Omega^*$ becomes smaller as $\|\lambda(f)\|_\Omega$ becomes smaller.

In view of (1.2) and (1.5), the quantity $\omega(\Omega) = \sup_{w \in \Omega} |(\beta_\Omega)_w(w)|$, $w \in \Omega$, is important. We shall investigate this in Section 6.

Returning to general Ω we call $f \in S(\Omega)$ isolated if there exists $\varepsilon > 0$ such that $0 < \|\lambda(f) - \lambda(g)\|_\Omega < \varepsilon$ for no function $g \in S(\Omega)$. A set $E \subset C$ is called of full measure if $C \setminus E$ is of measure zero. The "measure" always means the two-

dimensional Lebesgue measure.

THEOREM 3. *If $f \in S(\Omega)$ is isolated, then $f(\Omega)$ is of full measure. The converse is false if Ω is simply connected.*

See [14] for the study of f meromorphic and univalent in Ω with $\sigma(f)$ and $\|\cdot\|_{\Omega}^*$ instead of $\lambda(f)$ and $\|\cdot\|_{\Omega}$. W.P. Thurston [17, p. 191] (see [3] also) found an Ω such that each Möbius transformation is isolated: There exists $\varepsilon > 0$ such that $0 < \|\sigma(f)\|_{\Omega}^* < \varepsilon$ for no f meromorphic and univalent in Ω . It is open to find Ω such that $S(\Omega)$ contains an isolated point in our sense.

Let $SD(\Omega)$ be the family of $f \in S(\Omega)$ with finite Dirichlet integral:

$$\iint_{\Omega} |f'(w)|^2 du dv < +\infty.$$

Theorem 3 shows in particular that each $f \in SD(\Omega)$ is not isolated. We can prove this fact in a somewhat stronger form in

THEOREM 4. *For each $f \in SD(\Omega)$ and each $\varepsilon > 0$ we can find distinct functions $f_k \in SD(\Omega)$ ($k=1, 2$) such that*

$$0 < \|\lambda(f) - \lambda(f_k)\|_{\Omega} < \varepsilon \quad (k=1, 2) \quad \text{and} \quad f = 2^{-1}(f_1 + f_2).$$

In the proofs of Theorems 3 and 4, X.U. Nguyen's existence theorem of Lipschitz functions and the method of D.H. Hamilton for constructing univalent functions are fundamental; see [12] and [7].

My esteemed colleagues, Hisao Sekigawa and Toshihiro Nakanishi gave me invaluable informations on the paper [13]. I wish to express my sincere gratitude to them.

2. A short survey on domains of finite type.

For each universal covering projection φ from D onto Ω we have

$$(2.1) \quad (1 - |z|^2) |\varphi'(z)| = \beta_{\Omega}(w)$$

at each $z \in D$ with $w = \varphi(z)$; see [1, Chapter 1] for example. In particular, φ' never vanishes in D and β_{Ω} is of C^{∞} . Set

$$\delta(w) = \inf_{z \in \partial D} |w - z|, \quad w \in \Omega, \quad \text{and} \quad b(\Omega) = \sup_{w \in \Omega} \beta_{\Omega}(w) \delta(w)^{-1}.$$

Osgood proved that $2b(\Omega) \leq a(\Omega) \leq 4b(\Omega)$ and

$$(2.2) \quad a(\Omega) < +\infty \iff \|\lambda(\varphi)\|_D < +\infty;$$

see [13, the proofs of Theorems 2 and 6]. Note that if $\|\lambda(\varphi)\|_D < +\infty$ for a φ , then $\|\lambda(\varphi)\|_D < +\infty$ for each φ ; for the proof, see the forthcoming expression (3.2).

In the special case $\partial\Omega$ is unbounded, C. Pommerenke proved that

$$\|\lambda(\varphi)\|_D < +\infty \iff b(\Omega) < +\infty;$$

see [16, Corollary 1, p. 195 and (4.2) in p. 196]. Actually, Pommerenke observed that the unbounded $\partial\Omega$ is uniformly perfect if and only if $\|\lambda(\varphi)\|_D < +\infty$. Here $\partial\Omega$ is called uniformly perfect if there exists a constant $0 < c \leq 1$ such that $\partial\Omega$ contains a point of the set $\{w; cr \leq |w-z| \leq r\}$ for each $z \in \partial\Omega$ and each $0 < r < +\infty$. If $\partial\Omega$ contains an isolated point, then $\partial\Omega$ is not uniformly perfect, so that Ω is not of finite type.

If Ω is of finite type, then the image $f(\Omega)$ by an $f \in S(\Omega)$ is again of finite type [13, Corollary 1, p. 457]. Actually, by the conformal invariance of the Poincaré metric one obtains

$$\beta_{f(\Omega)}(z) = |f'(w)|\beta_{\Omega}(w), \quad z = f(w).$$

Taking the logarithms of the both sides and partially differentiating them by w we have

$$(2.3) \quad (\beta_{f(\Omega)})_z(z) = \{|f'(w)|/f'(w)\} \{2^{-1}\beta_{\Omega}(w)\lambda(f)(w) + (\beta_{\Omega})_w(w)\}.$$

Therefore one obtains

$$\sup_{z \in f(\Omega)} |(\beta_{f(\Omega)})_z(z)| \leq 2^{-1}a(\Omega) + \sup_{w \in \Omega} |(\beta_{\Omega})_w(w)|,$$

which, combined with (1.1) $\Leftrightarrow$ (1.2), proves the property.

Let $w_0 \in \partial\Omega$. Then $f(w) = (w - w_0)^{-1}$ is in $S(\Omega)$. Thus, Ω is of finite type if and only if $\partial f(\Omega)$ is uniformly perfect.

3. Proof of Theorem 1.

Taking the logarithms of the both sides of (2.1), and then partially differentiating them by z , we obtain

$$(3.1) \quad \lambda(\varphi)(z) = 2\bar{z}/(1 - |z|^2) + 2\varphi'(z)(\beta_w/\beta)(w).$$

Although φ may not be in $S(D)$, the computation is essentially the same as that for (2.3). Setting $\chi_{\varphi}(z) = \varphi'(z)/|\varphi'(z)|$ we now have

$$(3.2) \quad (1 - |z|^2)\lambda(\varphi)(z) = 2\{\bar{z} + \chi_{\varphi}(z)\beta_w(w)\}.$$

Therefore,

$$\|\lambda(\varphi)\|_D < +\infty \iff \sup_{w \in \Omega} |\beta_w(w)| < +\infty,$$

which implies (1.1) $\Leftrightarrow$ (1.2) in view of (2.2).

Eliminating $\varphi''(z)$ from (3.1) and from the right hand side of

$$\lambda(\varphi)'(z) = 2\bar{z}^2/(1 - |z|^2)^2 + 2\varphi''(z)(\beta_w/\beta)(w) + 2\varphi'(z)^2(\beta_w/\beta)_w(w),$$

we have

$$\sigma(\varphi)(z) = 2\varphi'(z)^2 \{(\beta_w/\beta)_w(w) + (\beta_w/\beta)^2(w)\},$$

whence

$$(3.3) \quad (1 - |z|^2)^2 \sigma(\varphi)(z) = 2\chi_\varphi(z)^2 \beta(w) \beta_{w\bar{w}}(w).$$

It is known that $\|\lambda(\varphi)\|_D < +\infty$ if and only if $\|\sigma(\varphi)\|_B^* < +\infty$; see [19, Theorem 2], [18], [20] and [21]. We thus have (1.1) $\Leftrightarrow$ (1.3).

We recall the Gauss curvature identity in terms of β :

$$(3.4) \quad \beta^2 \Delta \log \beta \equiv -4 \quad \text{or} \quad \beta \beta_{w\bar{w}} - \beta_w \beta_{\bar{w}} \equiv -1.$$

This follows from the partial differentiation of (3.1) by $\bar{z}$. Then, $|\beta_w|^2 = \beta_w \beta_{\bar{w}}$ is bounded in Ω if and only if $\beta \beta_{w\bar{w}}$ is bounded in Ω .

Remark. If $\varphi \in S(D)$, then $\|\sigma(\varphi)\|_B^* \leq 6$. Suppose that $\|\sigma(\varphi)\|_B^* \leq 2q$. If $q=1$, then $\varphi \in S(D)$, while if $q < 1$, then φ is the restriction of a $(1+q)/(1-q)$ -quasiconformal mapping from $C \cup \{\infty\}$ onto $C \cup \{\infty\}$. See [8], [11] and [2]. Combining these with (3.3) we have the obvious criteria in terms of β_Ω for the property of Ω . For example, if

$$\sup_{w \in \Omega} \beta_\Omega(w) |(\beta_\Omega)_{ww}(w)| \leq 1,$$

then Ω is simply connected.

In case Ω is simply connected, we set $\delta(\Omega) = \|\sigma(f)\|_B^*$, where $f: \Omega \rightarrow D$ is an onto conformal homeomorphism; $\delta(\Omega)$ is independent of the choice of f and is called the distance of Ω from a disk [9, p. 61]. We can choose $f = \varphi^{-1}$ and we have $\|\sigma(\varphi^{-1})\|_B^* = \|\sigma(\varphi)\|_B^*$. A known result [9, Theorem 2.1, p. 63], together with (3.3), now shows: *If Ω is the image of a convex domain by a Möbius transformation, then*

$$\sup_{w \in \Omega} \beta_\Omega(w) |(\beta_\Omega)_{ww}(w)| \leq 1.$$

The equality holds for the domains specified in the cited theorem. See Proposition 2 in the forthcoming Section 6.

4. Proof of Theorem 2.

LEMMA 1. *For g analytic in D we have*

$$(4.1) \quad \sup_{z \in D} (1 - |z|^2)^2 |g'(z)| \leq K \|g\|_D,$$

where K is an absolute constant with $8/e \leq K < 10/3$.

This is due to M.-C. Liu [10, Theorem]; he actually proved that

$$K \leq 2^{-1}(\sqrt{5} + 1)(\sqrt{5} + 2)^{1/2} = 3.3301 \dots \quad [10, \text{p. 207}].$$

For the proof of Theorem 2 we let g be a branch of $\log(f' \circ \varphi)$ in D . Then,

$$\|g'\|_D = \|\lambda(f)\|_D,$$

which, combined with (4.1), yields

$$(4.2) \quad \sup_{z \in D} (1 - |z|^2) |g''(z)| \leq K \|\lambda(f)\|_D.$$

On the other hand,

$$(4.3) \quad \varphi'^2 \lambda(f)' \circ \varphi = g'' - \varphi' \cdot (\lambda(f) \circ \varphi) \lambda(\varphi).$$

It then follows from (4.2) and (4.3) that

$$\|\sigma(f)\|_D^2 \leq K \|\lambda(f)\|_D + \|\lambda(f)\|_D \|\lambda(\varphi)\|_D + 2^{-1} (\|\lambda(f)\|_D)^2.$$

Setting

$$c(\Omega) = \sup_{\varphi} \|\lambda(\varphi)\|_D,$$

we now have (1.5) from (3.2).

5. Proofs of Theorems 3 and 4.

LEMMA 2 (X. U. Nguyen [12]). *For each compact set E of positive measure, there exists a nonconstant analytic function F in the open set $E^c = (C \cup \{\infty\}) \setminus E$ such that F is bounded in E^c and F satisfies the Lipschitz condition in $C \setminus E$:*

$$A(F) = \sup_{\substack{z, w \in C \setminus E \\ z \neq w}} |F(z) - F(w)| / |z - w| < +\infty.$$

Note that, then $|F'(w)| \leq A(F)$ for each $w \in C \setminus E$.

LEMMA 3. *If f is analytic, bounded and bounded away from zero. $0 < A \leq |f| \leq B < +\infty$ in D , then*

$$\|f'/f\|_D \leq (2/\pi) \log(B/A).$$

Proof. Apply the Schwarz-Pick lemma

$$(1 - |z|^2) |h'(z)| / (1 - |h(z)|^2) \leq 1, \quad z \in D,$$

to $h = (iH + 1)/(iH - 1)$, where H is defined by

$$H = \exp[\{(\pi i)/\log(B/A)\} \log(f/A)].$$

Proof of Theorem 3. Suppose that $f(\Omega)$ is not of full measure. Let $E \subset C \setminus f(\Omega)$ be a compact set of positive measure and consider F of Lemma 2 with $A = A(F)$. Given $\varepsilon > 0$ we choose $\gamma > 0$ such that

$$(5.1) \quad \gamma A < (e^{\pi\varepsilon/2} - 1) / (e^{\pi\varepsilon/2} + 1).$$

For a complex number α , $0 < |\alpha| \leq \gamma$, we set

$$g_\alpha = f + \alpha F \circ f.$$

Then $g_\alpha \in S(\Omega)$ because

$$|g_\alpha(w) - g_\alpha(z)| \geq (1 - \gamma A) |f(w) - f(z)| > 0 \quad \text{for } w, z \in \Omega, w \neq z.$$

We shall show that

$$(5.2) \quad 0 < \|\lambda(f) - \lambda(g_\alpha)\|_\Omega < \varepsilon,$$

so that f is not isolated. Set

$$h_\alpha = 1 + \alpha F' \circ f \quad \text{and} \quad G_\alpha = h_\alpha \circ \varphi.$$

Then G_α is nonconstant and

$$0 < 1 - \gamma A < |G_\alpha| < 1 + \gamma A \quad \text{in } D,$$

so that, by Lemma 3, together with (5.1), we have

$$0 < \|G'_\alpha / G_\alpha\|_D < \varepsilon.$$

Therefore, (5.2) follows from $\lambda(g_\alpha) - \lambda(f) = h'_\alpha / h_\alpha$.

Suppose that Ω is simply connected and set $\phi = \varphi^{-1}$, the inverse map. Then, for $0 < p \leq 2$, each single-valued branch

$$f_p = \{(1 + \phi)/(1 - \phi)\}^p$$

is in $S(\Omega)$. Furthermore,

$$\|\lambda(f_p) - \lambda(f_2)\|_\Omega = 2(2 - p) \longrightarrow 0 \quad \text{as } p \rightarrow 2 - 0.$$

Therefore, $f_2(\Omega)$ is of full measure, yet f_2 is not isolated.

Corollary to Theorem 3. If Ω is not of full measure, then no linear function $L(z) = Az + B$ ($A \neq 0$ is isolated).

Proof. Suppose that $L(z) = Az + B$ is isolated. Then $L(\Omega)$ is of full measure, so that $\Omega = L^{-1} \circ L(\Omega)$ is of full measure.

Proof of Theorem 4. Since $f(\Omega)$ is not of full measure, we can construct g_α as in the proof of Theorem 3. Set

$$f_1 = g_\gamma \quad \text{and} \quad f_2 = g_{-\gamma}.$$

Then, $f_k \in S(\Omega)$ and $0 < \|\lambda(f) - \lambda(f_k)\|_\Omega < \varepsilon$, $k = 1, 2$. Since

$$|\lambda'_k| = |f'| \cdot |1 \pm \gamma F'(f)| \leq (1 + \gamma A) |f'|, \quad k = 1, 2,$$

it follows that $f_k \in SD(\Omega)$. Apparently, $f = 2^{-1}(f_1 + f_2)$.

6. The order of a locally nnivalent function.

Let g be a function analytic and locally univalent in D . The order $ord_D(g)$ of g is the supremum of $|a_2(z)|$, $z \in D$, where $a_2(z)$ is the Taylor coefficient in the expansion

$$\frac{g((\zeta+z)/(1+\bar{z}\zeta)) - g(z)}{(1-|z|^2)g'(z)} = \zeta + a_2(z)\zeta^2 + \dots, \quad \zeta \in D.$$

By simple computation, we have

$$ord_D(g) = \sup_{z \in D} |-\bar{z} + 2^{-1}(1-|z|^2)\lambda(g)(z)|.$$

Set $A = ord_D(g)$. Then $A \leq 2$ for $g \in S(D)$ by the coefficient theorem. Since for general g ,

$$|(\partial/\partial|z|) \log \{(1-|z|^2)g'(z)\}| \leq 2A/(1-|z|^2), \quad z \in D,$$

it follows from the familiar manipulation in the univalent function theory that

$$(1-|z|)^{4-1}/(1+|z|)^{4+1} \leq |g'(z)/g'(0)|, \quad z \in D.$$

The minimum modulus principle for g' , never vanishing in D , yields that $A \geq 1$. Furthermore,

$$\operatorname{Re}\{1+z\lambda(g)(z)\} \geq (1-2A|z|+|z|^2)/(1-|z|^2) > 0$$

if $|z| < A - (A^2 - 1)^{1/2}$, so that $A = 1$ implies $g \in S(D)$ and $g(D)$ is convex. Conversely, if $g \in S(D)$ and $g(D)$ is convex, then by the coefficient theorem, $|a_2(z)| \leq 1$, for $g, z \in D$, so that $A \leq 1$, or $A = 1$. See [5, pp. 33, 42 and 45] and [15, pp. 116, 117 and 133].

We note that $(\beta_D)_z(z) = -\bar{z}$ by $\beta_D(z) = 1 - |z|^2$. For f analytic and locally univalent in Ω , we set

$$ord_\Omega(f) = \sup_{w \in \Omega} |(\beta_\Omega)_w(w) + 2^{-1}\beta_\Omega(w)\lambda(f)(w)|$$

and call it the order of f in Ω . It then follows from (3.1), together with $\lambda(f \circ \varphi) = \varphi' \lambda(f) \circ \varphi + \lambda(\varphi)$, that

$$(6.1) \quad ord_\Omega(f) = ord_D(f \circ \varphi).$$

In particular, if $f_0(w) \equiv w$, then

$$(6.2) \quad \omega(\Omega) \equiv \sup_{w \in \Omega} |(\beta_\Omega)_w(w)| = ord_\Omega(f_0) = ord_D(\varphi) \geq 1.$$

Thus, Ω is of finite type if and only if $\omega(\Omega) < +\infty$. Furthermore, $2^{-1}c(\Omega) - 1 \leq \omega(\Omega) \leq 2^{-1}c(\Omega) + 1$; the left hand side is (1.5). In view of (6.1) it is now easy to prove the following

PROPOSITION 1. *Let f be analytic and locally univalent in Ω . Then $\text{ord}_\Omega(f)=1$ if and only if $f \in S(\Omega)$ and $f(\Omega)$ is convex.*

“Only if”. It follows from (6.1) that $f \circ \varphi \in S(D)$ and $f(\Omega) = f \circ \varphi(D)$ is convex. For the proof of the univalence of f in Ω , we let $w_1 \neq w_2$, both in Ω . Then there exist $z_1 \neq z_2$, both in D , such that $w_k = \varphi(z_k)$, $k=1, 2$. Therefore, $f(w_1) = (f \circ \varphi)(z_1) \neq (f \circ \varphi)(z_2) = f(w_2)$. Since $f(\Omega)$ is simply connected, Ω is simply connected. “If”. If $f(\Omega)$ by $f \in S(\Omega)$ is convex, then Ω must be simply connected. Thus, $f \circ \varphi \in S(D)$, and $\text{ord}_\Omega(f)=1$ follows from (6.1) with $\text{ord}_D(f \circ \varphi)=1$.

As a consequence of Proposition 1 we have: *If Ω is not simply connected, then $\text{ord}_\Omega(f) > 1$ for each f locally univalent in D .* It would be of interest to have a convex domain criterion in terms of $\omega(\Omega)$.

PROPOSITION 2. *A domain Ω is convex (and hence, simply connected) if and only if $\omega(\Omega)=1$.*

This is a consequence of (6.2) with $\varphi(D)=\Omega$. Proposition 2 has the following corollary: *A domain Ω is convex if and only if β_Ω is a superharmonic function in Ω .* Remember that $\beta_{w\bar{w}} = |\beta_w|^2 - 1$ ($\beta = \beta_\Omega$; see (3.4)). “Only if”. It follows from $\omega(\Omega)=1$ that $4^{-1}\Delta\beta = \beta_{w\bar{w}} \leq 0$. “If”. It follows from $\beta_{w\bar{w}} \leq 0$ that $|\beta_w| \leq 1$, whence $\omega(\Omega) \leq 1$, or $\omega(\Omega)=1$.

If Ω is simply connected or $\varphi \in S(D)$, then $\text{ord}_D(\varphi) \leq 2$, which, combined with (6.2), shows that $\omega(\Omega) \leq 2$. For the Koebe function κ we have $\text{ord}_D(\kappa)=2$. These are observed in [13, Theorem 3, (14), p. 454].

It would be interesting to have an upper bound of $a(\Omega)$ by $\omega(\Omega)$.

PROPOSITION 3. *For each Ω we have*

$$(6.3) \quad a(\Omega) \leq 8\omega(\Omega).$$

Actually, the same proof as that of [15, the left half of (1.11), p. 115], together with the property of the projection φ , that is, $\delta(w) = d(\varphi(z))$, $w = \varphi(z)$, teaches us that

$$\beta_\Omega(w) / \{2\text{ord}_D(\varphi)\} \leq \delta(w), \quad w \in \Omega,$$

which, combined with $a(\Omega) \leq 4b(\Omega)$ observed in Section 2, yields (6.3).

In the specified case where Ω is simply connected we have a better estimate $a(\Omega) \leq 8$ than that in Proposition 3. In the “convex” case we have

PROPOSITION 4. *If Ω is convex, then $a(\Omega) \leq 6$.*

As we observed the equality holds for $\Omega=D$. For each $w \in \Omega$ we choose a conformal homeomorphism φ from D onto Ω such that $w = \varphi(0)$. Then, for $f \in S(\Omega)$,

$$\beta_\Omega(w) |\lambda(f)(w)| = |\lambda(f \circ \varphi)(0) - \lambda(\varphi)(0)|.$$

Since $f \circ \varphi \in S(D)$, we have $|\lambda(f \circ \varphi)(0)| \leq 4$, while, since $\varphi \in S(D)$ and $\varphi(D)$ is convex, we have $|\lambda(\varphi)(0)| \leq 2$ [5, p. 45]. This completes the proof.

PROPOSITION 5.

$$(6.4) \quad \sup_{f \in S(\Omega)} |\omega(f(\Omega)) - \omega(\Omega)| \leq 2^{-1} a(\Omega).$$

Actually, in view of (2.3) one has

$$|(\beta_{f(\Omega)})_z(z)| \leq |(\beta_\Omega)_w(w)| + 2^{-1} \beta_\Omega(w) |\lambda(f)(w)|$$

and

$$|(\beta_\Omega)_w(w)| \leq |(\beta_{f(\Omega)})_z(z)| + 2^{-1} \beta_\Omega(w) |\lambda(f)(w)|.$$

It is now easy to have (6.4).

Propositions 3 and 5 yield

$$\sup_{f \in S(\Omega)} |\omega(f(\Omega)) - \omega(\Omega)| \leq 4 \omega(\Omega).$$

If Ω is simply connected, then

$$(6.5) \quad \sup_{f \in S(\Omega)} |\omega(f(\Omega)) - \omega(\Omega)| \leq 1;$$

actually since $f(\Omega)$ is simply connected, we have $1 \leq \omega(\Omega) \leq 2$ and $1 \leq \omega(f(\Omega)) \leq 2$. Note that $\omega(\kappa(D)) - \omega(D) = 1$. The sharp estimate (6.5) is unchanged if we further assume that Ω is convex. The Koebe function and D again show the sharpness.

We show that the set

$$\{\omega(\Omega); \Omega \text{ simply connected}\} = \{ord_D(f); f \in S(D)\}$$

is precisely the closed interval $[1, 2]$. Set

$$f_\alpha(z) = \{(1+z)/(1-z)\}^\alpha \quad (f_\alpha(0)=1, 1 \leq \alpha \leq 2).$$

Then, $f_\alpha \in S(D)$ and it suffices to prove that $ord_D(f_\alpha) = \alpha$. By a simple calculation we have

$$ord_D(f_\alpha) = \sup_{z \in D} F(z)^{1/2},$$

where

$$F(z) = \{\alpha^2(1-|z|^2)^2 + 4(\operatorname{Im} z)^2\} / \{(1-|z|^2)^2 + 4(\operatorname{Im} z)^2\}.$$

Then $F(z) \leq \alpha^2$ and the equality holds for real z .

If Ω is simply connected, then again

$$\{ord_\Omega(f); f \in S(\Omega)\} = [1, 2]$$

by (6.1). It would be an interesting problem to determine $\{ord_\Omega(f); f \in S(\Omega)\}$ for Ω of finite type yet not simply connected; as was remarked, this set does not contain 1, so that (6.1) shows that this set is contained in the interval $(1, +\infty)$.

Added in Proof. On the basis of the result of K.-J. Wirths in his paper: Über holomorphe Funktionen, die einer Wachstumsbeschränkung unterliegen; *Archiv der Mathematik* **30** (1978), 606–612, the constant K in Theorem 2 should be $K = (13\sqrt{3} + 55\sqrt{11})/64 = 3.20204\dots$. On the basis of the fact the author of [12] called himself N. X. Uy in his reference in the recent paper: A removable set for Lipschitz harmonic functions; *Michigan Mathematical Journal* **37** (1990), 45–51, the precise reference of Lemma 2 should be Uy's theorem.

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