S. SASAO AND J.-S. CHA
KODAI MATH. J.
13 (1990), 231—240

ON SELF-HOMOTOPY EQUIVALENCES OF COVERING SPACES

BY SEIYA SASAO AND CHA JOON-SIM

§ 0. Introduction.

Let X be a G-space and we denote by KX the space of continuous maps from K into X endowed with Compact-Open topology. Since a G-action on the space KX is naturally induced we may regard the path-connected components of KX, $\pi_0(KX)$, as a G-set. Then we are interested in the isotropy subgroup $G\langle f\rangle$ at the homotopy class of a map $f: K\to X$. This relates to othor problems as follows:

(1) When K is considered as a trivial G-space a map f is equivariant up to homotopy if and only if $G\langle f\rangle=G$, namely f is a fixed element.

(2) Since the G -action on X is given by a continuous map $\Phi: G\to XX$ we have an induced homomorphism $\Phi_*: G\to\varepsilon(X)$, the group of homotopy classes of self-homotopy equivalences. Then $G\langle 1_x\rangle$ is just the kernel of Φ.

(3) Of course the determination of $G\langle f\rangle$ for all f gives us some informations on the structure of the set $\pi_0(KX)$.

As the first step of our program, in this paper we are mainly concerned with the case of covering spaces and their deck transformation groups. Then there are a few points of view about categories:

(1) The category of 0-connected CW-complexes and maps of base-point free.
(2) The sub-category of fibre-preserving maps.
(3) The sub-category of equivariant maps.

We work in these categories to investigate the kernel of $\Phi_*: G\to\varepsilon(X)$. As results, we obtain some exact sequences for a regular covering $p: X\to Y$ with its deck transformation group G as follows:

(1) $\{1\}\to\Gamma(X, Y; p)/\Gamma(X)\to G\to\varepsilon(X)$

(2) $\{1\}\to\Gamma(Y)/\Gamma_F(X)\to G\to\varepsilon_F(X)\to\varepsilon_L(Y)\to\{1\}$

(3) $\{3\}\to\Gamma(Y)/\Gamma_G(X)\to Z[G]\to\varepsilon_G(X)$

(see the context about notations)

For example, let $p: R^n\to Y$ be a universal covering and G be the grour $\pi_1(Y, *)$, then we have an exact sequence

Received October 13, 1989; Revised December 7, 1989,

$$\{1\} \longrightarrow Z[G] \longrightarrow G \longrightarrow \varepsilon_F(R^n) \longrightarrow \text{Aut.}\, G \longrightarrow \{1\}.$$

§ 1. $\pi_1(X, *)$-action on the set $[K, X]_0$

First we recall a notion from the homotopy theory [5], [8]. Let us denote by $[K, X]_0$ the set of homotopy classes of base-point preserving maps. Then, for every loop ω of X at $*$ and a base-point preserving map $f: K \to X$, there exists a map $\phi: I \times K \to X$ which is an extension of the map

$$\omega \cup f : I \times * \cup 0 \times K \longrightarrow X.$$

Since the homotopy class of the restriction of ϕ on $1 \times K$ depends on only homotopy classes of ω and f this defines an action $\omega^\#$ of $\pi_1(X, *)$ on the set $[K, X]_0$. On the other hand this action can be reformulated as follows:

Let $p: KX \to X$ be the fibring defined by $p(f) = f(*)$. Clearly the fibre over $*$ is the space of base-point preserving maps, $\{K, X\}_0$, and we have the part of the homotopy exact sequence

$$S_1: \pi_1(KX, f) \underset{p_*}{\longrightarrow} \pi_1(X, *) \underset{\partial_f}{\longrightarrow} \pi_0(\{K, X\}_0) \longrightarrow \pi_0(KX, f) \longrightarrow \{f\}.$$

Then it holds $\partial_f(\omega) = \omega^\#(f)$.

Since it is clear that a loop ω is contained in the image of p_* if and only if there exists a map: $S^1 \times K \to X$ of type (ω, f) we have

LEMMA 1.1. *$\omega^\#(f) = f$ holds if and only if there exists a* map: *$S^1 \times K \to X$ of type (ω, f),*

Here we note a property of the π_1-action above which easily follows from the definition.

LEMMA 1.2. *For two maps $f: (X, *) \to (Y, *)$ and $g: (Y, *) \to (Z, *)$ we have*

$$\tau^\#(gf) = \tau^\#(g)f \quad \textit{ond} \quad g_*(\omega)^\#(gf) = g(\omega^\#(f))$$

where τ and ω are elements of $\pi_1(Y, *)$ and $\pi_1(Z, *)$ respectively.

For example we prove

PROPOSITION 1.3. *If $f: (Y, *) \to (X, *)$ is a homotopy equivalence then $\omega^\#(f)$ is also a homotopy equivalence for any ω of $\pi_1(X, *)$.*

Proof. First it is shown that $\omega^\#(1_X)$ is a homotopy equivalence because we have

$$(\omega^{-1})^\#(1_X)\omega^\#(1_X) = (\omega^{-1})^\#(1_X \omega^\#(1_X)) = (\omega^{-1})^\# \omega^\#(1_X) = 1_X$$

and similarly $\omega^{\#}(1_X)(\omega^{-1})^{\#}(1_X)=1_X$. Secondly, let g be a homotopy inverse of f. Then we have

$$g\{\omega^{\#}(f)\}=g_*(\omega)^{\#}(gf)=\{g_*(\omega)\}^{\#}(1_X) \quad \text{and} \quad \omega^{\#}(f)g=\omega^{\#}(fg)=\omega^{\#}(1_X).$$

and Thus it follows from the first case that $\omega^{\#}(f)$ has a right and left inverse respectively and hence $\omega^{\#}(f)$ is a homotopy equivalence.

As an example we consider the case of $K=X$ and $f=1_X$ in the based category. Then the exact sequence S_1 is turned into the sequence

$$S_2: \pi_1(XX, 1_X) \longrightarrow \pi_1(X, *) \longrightarrow ([X, X]_0, 1_X) \longrightarrow ([X, X], 1_X) \longrightarrow \{1_X\}.$$

Now we define a multiplication in the set $[X, X]$ by the composite of maps, which makes the set a semi-group with 1_X as unit. Since we have

$$\partial(\omega\cdot\tau)=(\omega\tau)^{\#}(1_X)=\omega^{\#}(\tau^{\#}(1_X))=\omega^{\#}(1_X\tau^{\#}(1_X))=\omega^{\#}(1_X)\tau^{\#}(1_X)=(\partial\omega)(\partial\tau)$$

the following lemma holds.

LEMMA 1.4. *∂ is homomorphic in the sequence S_2.*

Since, for a class h of a homotopy equivalence: $(X, *)\to(X, *)$, $\omega^{\#}(h)$ is also a homotopy equivalence the sequence S_2 is transformed by Proposition 1.3 into an exact sequence in the category of groups and homomorphisms

$$S_3: \pi_1(XX, 1_X) \longrightarrow \pi_1(X, *) \longrightarrow \varepsilon_0(X) \longrightarrow \varepsilon(X) \longrightarrow \{1_X\}$$

where $\varepsilon_0(X)$ denotes the group consisting of invertible elements of $[X, X]_0$.

Now we define a (normal) subgroup of $\pi_1(X, *)$ by

$$\Gamma(X)=\{\omega \mid \text{there exists a map}: S^1\times X\to X \text{ of type } (\omega, 1_X)\}.$$

LEMMA 1.5. *$\Gamma(X)$ is contained in the centre of $\pi_1(X, *)$ (see page 843 of [2]).*

Proof. For τ of $\pi_1(X, *)$ and ω of $\Gamma(X)$ a map: $S^1\times S^1\to X$ of type (ω, τ) is given by the composite $S^1\times S^1\to S^1\times X\to X$. Hence Whitehead product $[\tau, \omega]$ is trivial, i.e. $\tau\omega=\omega\tau$.

Since we know $\partial^{-1}(1_X)=\Gamma(X)$ from lemma 1.2 we have

THEOREM 1.6. *There exists an exact sequence*

$$\{1\} \longrightarrow \pi_1(X, *)/\Gamma(X) \longrightarrow \varepsilon_0(X) \longrightarrow \varepsilon(X) \longrightarrow \{1\}.$$

Thus Theorem 1.6 and lemma 1.5 give

COROLLARY 1.7. *If the centre of $\pi_1(X, *)$ is trivial we have an exact sequence*

$$\{1\} \longrightarrow \pi_1(X, *) \longrightarrow \varepsilon_0(X) \longrightarrow \varepsilon(X) \longrightarrow \{1\}\,.$$

As another example, let P_m be the pseud-projective plane $S^1 \cup_m e^2$. Since it follows from a cohomological considration that $\Gamma(P_m)$ is trivial we have a short exact sequence ([1], [4])

$$\{1\} \longrightarrow Z_m \longrightarrow \varepsilon_0(X) \longrightarrow \varepsilon(X) \longrightarrow \{1\}\,.$$

§2. Regular Covering spaces.

In this section our argumrnt is related to the paper [7]. Let $p: X \to Y$ be a regular covering, i.e. $p_*(\pi_1(X, *))$ is a normal subgroup of $\pi_1(Y, *)$ and Let G be the deck transformation group of p. Then for any locally compact and locally path-connected Hausdorff space K we have

LEMMA 2.1. *The naturally induces map $p^K: KX \to KY$ is a fibre space whose fibre over pf is Gf for any map $f: K \to X$ where the action of G on KX is given by $G \times KX \to KX: (g, h) \to gh$.*

Consider a part of the homotopy exact sequence of p^K

$$S_4: \pi_1(KX, f) \longrightarrow \pi_1(KY, pf) \longrightarrow (G, *) \longrightarrow \pi_0(KX, f) \longrightarrow \pi_0(KY, pf)$$

for a map $f: K \to X$ where we identify Gf with G. Then a standard argument gives

LEMMA 2.2. *The boundary $\pi_1(KY, pf) \to (G, *)$ is homomorphic, and the correspondence $(G, *) \to \pi_0(KX, f)$ is naturally induced by the action of G on KX.*

Now consider the following commutative diagram obtained from fibrings

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \pi_1(KX, f) & \longrightarrow & \pi_1(KY, pf) & \longrightarrow & (G, *) & \longrightarrow & \pi_0(KX, f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \pi_1(X, *) & \longrightarrow & \pi_1(Y, pf) & \longrightarrow & (G, *) & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\{f\} & \longrightarrow & \{[K, X]_0, f\} & \longrightarrow & \{[K, Y]_0, pf\} & & & &
\end{array}$$

and use the following notation for a map $h: (A, *) \to (B, *)$ $\Gamma(A, B: h) = \{\omega | \omega \in \pi_1(B, *)$ and there exists a map $S^1 \times A \to B$ of type $(\omega, h)\}$.

Then, using lemma 2.2, we can easily obtain

PROPOSITION 2.3. *Let $p: X \to Y$ be a regular covering whose deck transformation group is G. Then $G\langle f \rangle$ is isomorphic to $\Gamma(K, Y: pf)/\Gamma(K, X: f)$*

For a regular covering $p: X \to Y$ we have as applications of PROP. 2.3

COLLORARY 2.4. *There exists an exact sequence*

$$\{1\} \longrightarrow \Gamma(K, Y: pf)/\Gamma(K, X: f) \longrightarrow \pi^1(Y, *) \longrightarrow \pi_1(X, *) \longrightarrow \{[K, X], f\}$$

Since $\Gamma(X, X: 1_X) = \Gamma(X)$ (see § 1), as a special case, we have

COLLORARY 2.5. *There exists an exact sequence*

$$\{1\} \longrightarrow \Gamma(X, Y: p)/\Gamma(X) \longrightarrow G \longrightarrow \varepsilon(X) \longrightarrow \{[X, Y], p\}.$$

As another application we have

COLLORARY 2.6. *A map $f: S^n \to X (n \geqq 2)$ is G-equivariant up to homotopy ıf and only if all Whitehead products $[\pi_1(Y, *), pf]$ vanish.*

Let $p: X \to Y$ be a covering space which is not necessarly regular. Then, noting $p(k)^{-1}(pf) = Gf$, the sequence S_4 turns out the sequence,

$$\pi_1(KX, f) \longrightarrow \pi_1(KY, pf) \longrightarrow (Gf, e_0 f) \longrightarrow \pi_0(KX, f) \longrightarrow \pi_0(KY, pf)$$

which relates to other sequences as follows:

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \pi_1(KX, f) & \longrightarrow & \pi_1(KY, pf) & \longrightarrow & (Gf, e_0 f) & \longrightarrow & \pi_0(KX, f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \pi_1(X, *) & \longrightarrow & \pi_1(Y, *) & \longrightarrow & (p^{-1}(*), *) & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\{1\} & \longrightarrow & \{[K, X]_0\ 'f\} & \longrightarrow & \{K, Y]_0, pf]\}. & & & &
\end{array}$$

Let denote by $N(\pi_1(X, *))$ the normalizer of $\pi_1(X, *)$ in $\pi_1(Y, *)$ and $\Gamma(K, Y: f)$ be the intersection $\Gamma(K, Y: pf) \cap N(\pi_1(X, *))$. Since G is isomorphic to $N(\pi_1(X, *))/\pi_1(X, *)$, using the above diagram and argument similar to the case of regular coverings we can obtain the following

PROPOSITION 2.7. *Let $p: X \to Y$be a covering space with its deck transformation group G and f be a map $(X, *) \to (Y, *)$. Then $G\langle f \rangle$ is isomorphic to $\Gamma(K, Y: pf)/\Gamma(K, X: f)$.*

§ 3. Orbits (fibre)-preserving maps.

For a regular covering $p: X \to Y$ we denote by $F(X)$ the space of orbits preserving maps, i.e. $f: X \to X$ satisfying $pf(gx) = pf(x)$ for all x, g. Then we have the pull-buck diagram of fibrings derived from the covering

$$\begin{array}{ccc} F(X) & \longrightarrow & XX \\ q\downarrow & & \downarrow p^X \\ YY & \longrightarrow & XY \end{array}$$

where $YY \to XY$ is given by composite $X \to Y \to Y$. Since we may consider G as the fibre $q^{-1}(1_Y)$ we have the commutative diagram of a part of homotopy exact sequences;

$$\begin{array}{ccccccc} \pi_1(XX, 1_X) & \longrightarrow & \pi_1(XY, p) & \longrightarrow & (G, *) & \longrightarrow & \pi_0(XX, 1_X) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \pi_1(F(X), 1_X) & \longrightarrow & \pi_1(YY, 1_Y) & \longrightarrow & (G, *) & \longrightarrow & \pi_0(F(X), 1_X). \end{array}$$

Then, as the same as the case of the upper sequqnce, we can know that the lower sequence is an exact sequence of semi-groups and homomorphisms, We denote by $\varepsilon_F(X)$ the group consisting of invertible elements of $\pi_0(F(X), 1_X)$, and obtain the following diagram from the above one

$$\begin{array}{ccccc} \pi_1(XY, p) & \longrightarrow & (G, *) & \longrightarrow & \varepsilon(X) \\ \uparrow & & \uparrow & & \uparrow \\ \pi_1(YY, 1_Y) & \longrightarrow & (G, *) & \longrightarrow & \varepsilon_F(X). \end{array}$$

We define subgroup of $\pi_1(X, *)$ by

$$\Gamma_F(X) = \{\tau \mid \text{there exist an orbits-preserving map } S^1 \times X \to X \text{ of type } (\tau, 1_X)\}$$

PROPOSITION 3.1. *The image of the boundary* $\pi_1(YY, 1_Y) \to (G, *)$ *in the lower sequence is isomorphic to* $\Gamma(Y)/\Gamma_F(X)$.

Proof. The proof follows from the argument analogus to PROP. 2.3 and the diagram,

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & \pi_1(X, *) & \longrightarrow & \pi_1(Y, *) & \longrightarrow & (G, *) & \longrightarrow & \{1\} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \pi_1(F(X), 1_X) & \longrightarrow & \pi_1(YY, 1_Y) & \longrightarrow & (G, *). & & \end{array}$$

COROLLARY 3.2. *The image*: $\pi_1(YY, 1_Y) \to (G, *)$ *is contained in the center of* G.

Proof. For ω of $\pi_1(Y, *)$, assume that there exists a map: $S^1 \times Y \to Y$ of type $(\omega, 1_Y)$. Then for any σ of $\pi_1(Y, *)$, a map: $S^1 \times S^1 \to Y$ of type (ω, σ) is given by composite $S^1 \times S^1 \to S^1 \times Y \to Y$. Hence Whitehead procuct $[\omega, \sigma]$ is trivial, i.e. $\omega\sigma = \sigma\omega$. Since $\pi_1(Y, *) \to G$ is onto the proof is completed.

Example 3.3. Let $X \to Y$ be the universal covering. If the centre of $\pi_1(Y, *) = G$ is trivial (e.g. G: simple) then $\varepsilon_F(X)$ contains $\pi_1(Y, *)$ as a subgroup.

Next we consider the homomorphism $q_\cdot : \pi_0(F(X), 1_X) \to \pi_0(YY, 1_Y)$ induced by the projection q. Let us denote by $L(Y)$ the image of $q_\cdot$ and by $\varepsilon_L(Y)$ the group consisting of invertible elements of $L(Y)$.

LEMMA 3.4. *The homomorphism* $\varepsilon_F(X) \to \varepsilon_L(Y)$ *is surjective.*

Proof. Consider the commutative diagarm

$$\begin{array}{ccc} f: X & \longrightarrow & X \\ p\downarrow & & \downarrow p \\ \tilde{f}: Y & \longrightarrow & Y \end{array}$$

and let $\tilde{g}$ be a homotopy inverse of $\tilde{f}$. Since we may assume teat $\tilde{f}$ and $\tilde{g}$ are base-point preserving maps the following equalities hold for a loop ω at $*$

$$\omega^{-1} p_* \pi_1(X, *)\omega = \omega^{\#} p_* \pi_1(X, *) = \tilde{g}_* f_* p_* \pi_1(X, *) = \tilde{g}_* p_* \pi_1(X, *).$$

Thus we have $\tilde{g}_* p_* \pi_1(X, *) = p_* \pi_1(X, *)$ from the normality of $p_* \pi_1(X, *)$ in $\pi_1(Y, *)$, and this means that $\tilde{g}$ is liftable, i.e. there exists a map $g: X \to X$ such that $\tilde{g}p = pg$. Then, from $q_*(fg) = 1_Y$ and exactness of the sequence, we can know that f is invertible in $\pi_0(F(X), 1_X)$

Now combining PROP. 3.1 with lemma 3.4 we have

THEOREM 3.5. *For a regular covering* $p: X \to Y$ *there exists an exact sequence*

$$\{1\} \longrightarrow \Gamma(Y)/\Gamma_F(X) \longrightarrow \varepsilon_F(X) \longrightarrow \varepsilon_L(Y) \longrightarrow \{1\}.$$

Example 3.6. For the universal covering $p: X \to Y$ we have an exact sequence: $\{1\} \to \Gamma(Y) \to \pi_1(Y, *) \to \varepsilon_F(X) \to \varepsilon_L(Y) \to \{1\}$.

Example 3.7. Let $p: \boldsymbol{R}^n \to Y$ be the universal covering and let G be $\pi_1(Y, *)$. Since it can be easily shown that the center of G, $Z[G]$, is isomorphic to $\Gamma(Y)$ and that $\varepsilon_L(Y)$ is also isomorphic to AUT. G we have an exact sequence

$$\{1\} \longrightarrow Z[G] \longrightarrow G \longrightarrow \varepsilon_F(\boldsymbol{R}^n) \longrightarrow \text{AUT}.\,G \longrightarrow \{1\}.$$

Especially if G is abelian we have an isomorphism $\varepsilon_F(\boldsymbol{R}^n) \simeq \text{AUT}.\,G$.

§ 4. Equivariant maps.

Since equivariant maps are a kind of typical orbits-preserving maps we shall study the space of those maps, which is denoted by $Eq(X)$.

First by James's result (Theorem (2.1) of [2]) we have a commutative diagram of fibrings

$$\begin{array}{ccccc} Z[G] & \longrightarrow & Eq(X) & \longrightarrow & YY \\ \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & F(X) & \longrightarrow & YY. \end{array}$$

Remark. This diagram contains an easy proof of Corollary 3.2.

Define a subgroup of $\pi_1(X, *)$ by

$\Gamma_G(X)=\{\tau \mid$ there exists an equivariant map: $S^1\times X\to X$ of type $(\tau, 1_X)\}$

Then using the diagram above and argument analogus to the preceeding section we have

PROPOSITION 4.1. *There exists an exact sequence*

$$\{1\} \longrightarrow \Gamma(Y)/\Gamma_G(X) \longrightarrow Z[G] \longrightarrow \varepsilon(X).$$

where $\varepsilon_G(X)$ denotes the subgroup of $\pi_0(Eq(X))$ consisting of invertible elements, i.e. homotopy classes of self-homotopy equivalences in the equivariant category.

Now we prove

PROPOSITION 4.2. *The homomorphism: $\varepsilon_G(X)\to\varepsilon_F(X)$ is injective.*

For the proof we note

LEMMA 4.4. *Let X_k be a properly discontinuous free G-space $(k=1,2)$ and f be a map$\cdot$ $X_1\to X_2$ such that $p_2f(gx)=p_2f(x)$ where $p_k: X_k\to Y_k$ is the projection onto the space of orbits. Clearly f defines a correspondence $\rho_f: G\times X_1\to G$ by $f(gx)=\rho_f(g, x)f(x)$. Then ρ_f is continuous.*

Proof of Proposition 4.2. Let f be an equivariant map: $X\to X$ and H be a homotopy between f and 1_X in the space $F(X)$, i.e. $H: I\times X\to X$ satisfies

$$H(0, x)=f(x), \quad H(1, x)=x \quad \text{and} \quad pH(l, gx)=pH(t, x).$$

Applying lemma 4.3 to the case of $X_1=I\times X$, $X_2=X$, H defines a continuous map $\rho: I\times X\times G\to G$ satisfying

$$H(g(t, x))=H(t, gx)=\rho(t, x, g)H(t, x).$$

Since G is discreate we know $\rho(t, x, g)=\rho(0, *, g)$ for all t and x. On the other hand, from equalities:

$$\rho(0, *, g)f(*)=\rho(0, *, g), \qquad H(0, *)=H(0, g*)=f(g*)=gf(*)$$

it follows $\rho(0, *, g)=g$. Thus we have $H(t, gx)=gH(t, x)$, which means H is an equivariant homotopy between f and 1_X. This completes the proof.

Now let $f: X \to X$ be a map satisfying $pf(x)=p(f(gx))$, i.e. $f \in F(X)$. By lemma 4.3 there exists a continuous map $\rho_f : G \times X \to G$ with $f(gx)=\rho_f(g, x)f(x)$. Again, since G is discrete this turns out a correspoudence

$$\rho_f^* : G \longrightarrow G \qquad \text{with} \quad f(gx)=\rho_f^*(g)f(x).$$

LEMMA 4.4. *ρ_f^* is an endmorphism of G, and $\rho_{f_1}^*=\rho_{f_2}^*$ if f_1 is homotopic to f_2 in the space $F(X)$.*

Proof. The first follows from

$$\rho_f^*(g_1, g_2)f(*)=f(g_1g_2*)=\rho_f^*(g_1)f(g_2*)=\rho_f^*(g_1)\rho_f^*(g_2)f(*)$$

and the second is easily shown by an argument anaolgus to the proof of Proposition 4.2

Thus ρ_f^* gives another correspondence

$$[\rho] : \pi_0(F(X)) \longrightarrow \text{End.}\, G$$

defined by $[\rho](f)=\rho_f^*$.

LEMMA 4.5. *$[\rho]$ is homomorphic, and hence this induces a homomorphism: $\varepsilon_F(X) \to \text{Aut.}\, G$, whose kernel is isomorphic to $\varepsilon_G(X)$.*

Proof. Let $f_k : X \to X$ $(k=1, 2)$ be maps in the space $F(X)$. Then equalities

$$\rho_{f_1f_2}^*(g)(f_1f_2)(*)=(f_1f_2)(g*)=f_1(f_2(g*))=f_1(\rho_{f_2}^*(g)f_2(*)=\rho_{f_1}^*(\rho_{f_2}^*(g))f_1f_2(*))$$

gives the proof.

Thus, from lemma 4.5 and Proposition 4.2, we obtain

PROPOSITION 4.6. *There exists an exact sequence:*

$$\{1\} \longrightarrow \varepsilon_G(X) \longrightarrow \varepsilon_F(X) \longrightarrow \text{Act.}\, G.$$

In general it seems to be difficult to obtain some characterization of the image: $\varepsilon_F(X) \to \text{Aut.}\, G$.

Remark. There is another interpretation of the homomorphism $[\rho]$ above, namely consider two covering spaces with base point as follows:

$$\begin{array}{ccccc} G* & \longrightarrow & (X, *) & \longrightarrow & (Y, *) \\ \downarrow & & \downarrow & & \downarrow \\ Gf(*) & \longrightarrow & (X, f(*)) & \longrightarrow & (Y, \tilde{f}(*)). \end{array}$$

Then we have a commutative diagram:

$$\begin{array}{ccccc} \pi_1(Y,*) & \longrightarrow & G & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \rho_f{}^* & & \\ \pi_1(Y,f(*)) & \longrightarrow & G & \longrightarrow & \{1\}\,. \end{array}$$

Example 4.7. If $\boldsymbol{R}^n \to Y$ is a regular covering and $\pi_1(Y,*)$ is abelian then $\varepsilon_G(\boldsymbol{R}^n)$ is trivial ($G=\pi_1(Y,*)$) (Example 3.1 of [6]).

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DEPT. OF MATH.
TOKYO INSTITUTE OF TECHNOLOGY
OHOKAYAMA MEGURO-KU TOKYO
JAPAN