P. JÄRVI
KODAI MATH. J.
13 (1990), 84–87

ON THE BOUNDARY BEHAVIOR OF HOLOMORPHIC MAPPINGS OF PLANE DOMAINS INTO RIEMANN SURFACES

BY PENTTI JÄRVI

1. Since the pioneering work of Ohtsuka ([7], [8]), several papers have dealt with Picard-type theorems for holomorphic mappings of plane domains into Riemann surfaces. See, *e.g.*, [2], [4], [5], [6], [9], [10], [12] and [13]. In this note we shall consider the behavior around null-sets of class N_B and N_D (in the familiar notation of Ahlfors-Beurling [1]) of holomorphic mappings into certain Riemann surfaces. Our result concerning the class N_B (Theorem 1) can be regarded as a generalization of a recent result of Shiga [12, Theorem 2].

2. We begin with some terminology. Let W be a Riemann surface and $E\subset W$ a compact totally disconnected set. We say that E is of class N_B (resp. N_D) in W if for each $p\in E$ there is a parametric disc (V, φ) in W such that $p\in V$, $E\cap\partial V=\emptyset$ and $\varphi(E\cap V)$ is of class N_B (resp. N_D). Let W^* stand for the Stoïlow compactification of W, and let $p\in\beta=W^*\setminus W$. We say that p is AB-removable (resp. AD-removable) if there is a planar end $V\subset W$ with $p\in\beta_V$, the relative ideal boundary of V, and a conformal map φ of $\bar{V}$ into the closed unit disc $\bar{U}\subset \boldsymbol{C}$ such that $\varphi(\partial V)=\partial\bar{U}$ and $\bar{U}\setminus\varphi(\bar{V})$ is of class N_B (resp. N_D). Obviously, $p\in\beta$ is AB-removable if and only if there is a Riemann surface $W'\supset W$ such that $p\in W'\setminus W$ and $W'\setminus W$ is of class N_B in W'. As usual, $\mathcal{O}_{AB}$ denotes the class of Riemann surfaces which do not carry nonconstant bounded holomorphic functions, while $\mathcal{O}_{MD^*}$ stands for the class of Riemann surfaces without non-constant meromorphic functions with a finite spherical Dirichlet integral.

THEOREM 1. *Let D be a plane domain and let $E\subset D$ be a compact set of class N_B. Let W be a Riemann surface which does not belong to $\mathcal{O}_{AB}$, and let $f: D\setminus E\to W$ be a holomorphic mapping. Then there exists a Riemann surface $W'\supset W$ such that*

(a) *$W'\setminus W$ is of class N_B in W' and*
(b) *f extends to a holomorphic mapping $f^*: D\to W'$.*

Proof. We may assume that f is nonconstant. Let g be a nonconstant bounded holomorphic function in W. Since $E\in N_B$, $g\circ f$ admits a holomorphic

Received June 26, 1989

extension to D. Fix $z \in E$. Then $Cl(g \circ f\,;\,z)$, the cluster set of $g \circ f$ attached to z, is a singleton. Since g is nonconstant, $Cl(f\,;\,z)$ must be totally disconnected. Hence, if taken with respect to the Stoïlow compactification W^*, $Cl(f\,;\,z)$ is a singleton. In other words, f extends to a continuous mapping $f^*: D \to W^*$.

Obviously, f^* is holomorphic in $D \setminus E'$, where E' stands for $(f^*)^{-1}(\beta)$. We claim that $W' = W \cup f^*(E')$ can be given a conformal structure, compatible with that of W. Fix $p_0 \in f^*(E')$ and pick out $z_0 \in E'$ such that $p_0 = f^*(z_0)$. Then choose a Jordan domain U with $z_0 \in U$ and $E' \cap \partial U = \emptyset$, and let V be an end of W such that $f^*(\partial U) \cap \bar{V} = \emptyset$ and $p_0 \in \beta_V$. Let $(\bar{V}_n)_{n \in N}$ be a relative exhaustion of $\bar{V}$. We may assume that ∂V_n consists of a finite number of Jordan curves and there is no branch point of f^* in $(f^*)^{-1}(\partial V_n) \cap U$, $n \in N$. Observe that the valence function of $f^* | U \setminus E'$ is finite and constant, say m, in V. Hence f^* defines a proper mapping of every component of $(f^*)^{-1}(V_n) \cap U$ onto V_n for each $n \in N$. Furthermore, it is easy to see that $(f^*)^{-1}(V_n) \cap U$ is connected for large n. Indeed, let $p \in V$ and let $K \subset U \setminus E'$ be a compact connected set which contains the preimages of p in U. Clearly, $(f^*)^{-1}(V_n) \cap U$ is then connected provided $f^*(K) \subset V_n$.

Let b_n(resp. b'_n) be the number of the boundary curves of V_n(resp. $(f^*)^{-1}(V_n) \cap U$), and let g_n stand for the genus of V_n. By the Riemann-Hurwitz formula, we have

$$b'_n - 2 \geqq m(2g_n + b_n - 2) \qquad \text{for large } n.$$

On the other hand, $b'_n \leqq m b_n$ for such n. Hence $m g_n \leqq m-1$, whence $g_n \leqq (m-1)/m < 1$. Thus, V_n is planar for each $n \in N$. In other words, we may realize V as a plane domain bounded by a finite number of Jordan curves (corresponding to ∂V) and a closed set F(corresponding to β_V). Clearly, all that remains is to show that F is of class N_B. But this follows from the corresponding property of E' and the fact that a nonconstant holomorphic function is a local homeomorphism off a discrete set; recall that a countable union of sets of class N_B is again of class N_B([11, p. 371]).

COROLLARY 1. *Let D be a plane domain and let $E \subset D$ be a compact set of class N_B. Let W be a Riemann surface which does not belong to O_{AB} and whose ideal boundary contains no AB-removable point. Let $f: D \setminus E \to W$ be a holomorphic mapping. Then f extends to a holomorphic mapping $f^*: D \to W$.*

We now indicate how a recent result of Shiga [12, Theorem 2] can be obtained from Theorem 1. Following [12] we say that a Riemann surface W is C-nondegenerate provided there exists $\varepsilon > 0$ such that the Carathéodory length of every nontrivial smooth closed curve on W exceeds ε.

COROLLARY 2 (Shiga [12]). *Let D be a plane domain and let $E \subset D$ be a compact set of class N_B. Let W be a C-nondegenerate Riemann surface and $\widetilde{W}$ a Riemann surface which is a (possibly branched) covering surface over W with the projection $\pi: \hat{W} \to W$. Suppose that for each p in W there exists a neighborhood*

V of p such that every component of $\pi^{-1}(V)$ is also C-nondegenerate. Let f: $D\setminus E\to\widetilde{W}$ be a holomorphic mapping. Then f extends to a holomorphic mapping f^*: $D\to\widetilde{W}$.

Proof. Since W is C-nondegenerate, $W\notin\mathcal{O}_{AB}$. Of course, $\widetilde{W}\notin\mathcal{O}_{AB}$ too. Hence by Corollary 1, it is enough to show that the ideal boundary of $\widetilde{W}$ does not possess AB-removable points. Assume it does have one, say p. Then there is a Riemann surface $\widetilde{W}'$ such that $p\in\widetilde{W}'\setminus\widetilde{W}$ and $\widetilde{W}'\setminus\widetilde{W}$ is of class N_B in $\widetilde{W}'$. By Theorem 1, there exists a Riemann surface $W'\supset W$ such that π extends to a holomorphic mapping π^*: $\widetilde{W}'\to W'$.

Suppose first that $f^*(p)\in W$. Then, given any neighborhood V of $f^*(p)$, we can find a parametric disc (U, φ) of $\widetilde{W}'$ such that $p\in U$, $U\cap\widetilde{W}$ is contained in a component of $\pi^{-1}(V)$ and $\partial U\cap(\widetilde{W}'\setminus\widetilde{W})=\emptyset$. Recalling that $U\cap(\widetilde{W}'\setminus\widetilde{W})$ is of class N_B in $\widetilde{W}'$ and making use of the relation $N_B\subset N_D$ as in [12] we see that $U\cap\widetilde{W}$ is not C-nondegenerate. Hence the same is true of the component of $\pi^{-1}(V)$ containing $U\cap\widetilde{W}$, contradicting the assumption.

There remains the case $f^*(p)\in W'\setminus W$. Because $W'\setminus W(\neq\emptyset)$ is of class N_B in W', the argument given above shows that W cannot be C-nondegenerate. This contradiction completes the proof.

THEOREM 2. *Let D be a plane domain and let $E\subset D$ be a compact set of class N_D. Let W be a Riemann surface which does not belong to $\mathcal{O}_{MD^*}$, and let f: $D\setminus E\to W$ be a holomorphic mapping of bounded valence. Then there exists a Riemann surface $W'\supset W$ such that*

(a) *$W'\setminus W$ is of class N_D in W' and*

(b) *f extends to a holomorphic mapping f^*: $D\to W'$.*

Proof. Let $g\in MD^*(W)$ be nonconstant. Since f has bounded valence, $g\circ f\in MD^*(D\setminus E)$. By [3, Theorem 2], $g\circ f$ extends to a meromorphic function in D. Hence $Cl(g\circ f;\, z)$ is a singleton for each $z\in E$. It follows, as in the proof of Theorem 1, that f extends to a continuous mapping f^*: $D\to W^*$. Let E' stand for $(f^*)^{-1}(\beta)$. Then $f^*(E\setminus E')\subset W$. Since $f^*|D\setminus E'$ is continuous and of bounded valence, f^* is actually holomorphic in $D\setminus E'$ [3, Theorem 2]. From now on the proof proceeds in complete analogy with the proof of Theorem 1. Hence we may omit the details.

COROLLARY. *Let D be a plane domain and let $E\subset D$ be a compact set of class N_D. Let W be a Riemann surface which does not belong to $\mathcal{O}_{MD^*}$ and whose ideal boundary contains no AD-removable point. Let f: $D\setminus E\to W$ be a holomorphic mapping of bounded valence. Then f extends to a holomorphic mapping f^*: $D\to W$.*

We conclude this note with two open problems.

(1) Let D be a plane domain and let $E\subset D$ be a compact set of class N_B. Let f be a holomorphic mapping of $D\setminus E$ into a Riemann surface W and suppose

that for some $z \in E$ $Cl(f; z)$ is neither empty nor a singleton. Must then $Cl(f; z)$ coincide with W? More generally, one may ask whether such mappings have the localizable Iversen property [11, p. 365]. Note that the latter question seems to be open even in the case that W is the Riemann sphere, while the former is of course trivial in this special case. The problem has relevance to Theorem 1, because there are Riemann surfaces in $\mathcal{O}_{AB}$ with big or even arbitrary "holes" (Myrberg, Kuramochi).

(2) Is Theorem 2 true without the assumption that $W \notin \mathcal{O}_{MD^*}$?

REFERENCES

[1] L. V. AHLFORS AND A. BEURLING, Conformal invariants and function-theoretic null-sets, Acta Math., **83** (1950), 101-129.

[2] M. HEINS, On Fuchsoid groups that contain parabolic transformations, Contributions to Function Theory, Tata Institute, Bombay (1960), 203-210.

[3] P. JÄRVI, On the continuation of meromorphic functions, Ann. Acad. Sci. Fenn. Ser. A I Math., **12** (1987), 177-184.

[4] P. JÄRVI, Generalizations of Picard's theorem for Riemann surfaces, to appear.

[5] A. MARDEN, I. RICHARDS AND B. RODIN, Analytic self-mappings of Riemann surfaces, J. Analyse Math., **18** (1967), 197-225.

[6] T. NISHINO, Prolongements analytiques au sens de Riemann, Bull. Sci. Math. France, **107** (1979), 97-112.

[7] M. OHTSUKA, On the behavior of an analytic function about an isolated boundary point, Nagoya Math, J., **4** (1952), 103-108.

[8] M. OHTSUKA, Boundary components of abstract Riemann surfaces, Lectures on Functions of a Complex Variable (W. Kaplan, ed.), Ann Arbor (1955), 303-307.

[9] H. RENGGLI, Remarks on the Picard theorem, Lecture Notes in Math., vol. 1351, Springer-Verlag, Berlin (1988), 279-284.

[10] H. L. ROYDEN, The Picard theorem for Riemann surfaces, Proc. Amer. Math. Soc., **90** (1984), 571-574.

[11] L. SARIO AND M. NAKAI, Classification theory of Riemann surfaces, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[12] H. SHIGA, On the boundary behavior of holomorphic mappings of plane domains to Riemann surfaces, J. Math. Kyoto Univ., to appear.

[13] M. SUZUKI, Comportement des applications holomorphes autour d'un ensemble polaire, C. R. Acad. Sci. Paris, **304** (1987), 191-194.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI,
SF-00100 HELSINKI 10, FINLAND