T. KUCZUMOW AND A. STACHURA
KODAI MATH. J.
12 (1989), 423—428

COMMON FIXED POINTS OF COMMUTING HOLOMORPHIC MAPPINGS

BY T. KUCZUMOW AND A. STACHURA

Abstract

Let B_k be a unit open ball in a k-dimensional complex Hilbert space. If T_μ is a commuting family of continuous functions mapping $\bar{B}_k^n$ into itself and holomorphic in B_k^n, then there exists a common fixed point for all functions of this family.

Introduction. Let us consider a family $\mathcal{A}$ of mappings of some set into itself. If $Tx=x$ for all T in $\mathcal{A}$ and some x we say that x is a common fixed point for $\mathcal{A}$ (or for the mapping T in $\mathcal{A}$). In this paper we are concerned with the existence of common fixed points for families of mappings.

For many years it was unknown whether two commuting continuous mappings of a compact convex set into itself necessarily have a common fixed point. In 1969 Boyce ([2]) and Huneke ([13]) independently gave counterexamples: there exist two commuting continuous mappings of [0, 1] into itself without a common fixed point. In view of this it is not surprising that the positive results must involve some additional restrictions on the family $\mathcal{A}$. Throughout this paper $\mathcal{A}$ denotes a subfamily of a family of all mappings from $\bar{B}^n$ into $\bar{B}^n$ (B^n is a cartesian product of n open unit balls B of a Hilbert space H). The mappings in $\mathcal{A}$ are holomorphic on B^n and continuous on $\bar{B}^n$.

In [21] (see also [3] and [4]) Shields proved that if $\mathcal{A}$ is a family of commuting functions which are continuous on the closed disc $\bar{\Delta}$ of the complex plane and are holomorphic on the open disc Δ and map the closed disc into itself, then there exists a common fixed point for all the functions of the family $\mathcal{A}$. This result was extended to polydiscs in C^2 by Eustice ([6]) (see also [24]) and to the unit ball of a finite dimensional inner product space by Suffridge ([23]). In [11] Heath and Suffridge gave the following theorem. If T_1 and T_2 are continuous mappings of a polydisc $\bar{\Delta}^n$ into itself that are holomorphic on Δ^n and $T_1 \circ T_2 = T_2 \circ T_1$, then they have a common fixed point in $\bar{\Delta}^n$. However we think that the proof in [11] is not complete.

In each mentioned above paper the proof is based on the fact that if a closure of the iterates of T (denoted by $\Gamma(T)$) is a compact topological semigroup then it contains a unique idempotent. Because of this our problem reduces

Received September 19, 1988; Revised March 2, 1989.

to a study of idempotents in $\mathcal{A}$. The absolutely different methods allow us to prove the following two facts. If B is an open unit ball of a Hilbert space and $\mathcal{A}$ is a commuting family of continuous functions mapping $\bar{B}$ into itself and holomorphic in B, then there exists a common fixed point for $\mathcal{A}$ ([14], [17]) and if $T_1, \cdots, T_m : B^n - B^n$ are commuting and holomorphic and every mapping has a fixed point, then they have a common fixed point ([16], [17]).

Basic notations and facts. We shall use the following notations:

(i) H (H_k) is a complex Hilbert space (a complex Hilbert space with $\dim H_k = k$) with an inner product $(\cdot, \cdot)$ and a norm $\|\cdot\|$.

(ii) B (B_k) is a unit open ball in H (in H_k) and $\bar{B}$ $(\bar{B}_k)$ is a closure of B (B_k) in H (H_k).

(iii) In B^n we introduce the following CRF metric ρ_n ([7], [8], [9], [10], [26]):

$$\rho_1(x, y) = \tanh^{-1}\left(1 - \frac{(1-\|x\|^2)(1-\|y\|^2)}{|1-(x, y)|^2}\right)^{1/2}$$

for $x, y \in B^1 = B$ and

$$\rho_n((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = \max_{1 \le i \le n} \rho_1(x_i, y_i)$$

for $(x_1, \cdots, x_n), (y_1, \cdots, y_n) \in B^n$.

(iv) $\mathcal{H}(B^n)$ $(\mathcal{H}(B_k^n))$ is the set of holomorphic mappings of B^n into B^n (of B_k^n into B_k^n). $\mathcal{H}(B_k^n, H_k^n)$ is the set of holomorphic mappings of B_k^n into H_k^n.

(v) For $T \in \mathcal{H}(B_k^n)$ $\Gamma(T)$ denotes the closure in $\mathcal{H}(B_k^n, H_k^n)$ of the iterates of T in the topology of the uniform convergence on compact subsets of B_k^n. $\Gamma(T)$ is a compact set in this topology.

(vi) For $T \in \mathcal{H}(B_k^n)$ $\Gamma'(T)$ denotes the set $(\subset \mathcal{H}(B_k^n, H_k^n))$ of all subsequential limits of $\{T^p\}$ in the topology of the uniform convergence on compact subsets of B_k^n.

If $\Gamma(T) \subset \mathcal{H}(B_k^n)$, then it forms a compact topological semigroup. Let us notice that this topological semigroup $\Gamma(T)$ contains exactly one idempotent R_T ([12], [28]). This holomorphic idempotent in $\Gamma(T)$ is a holomorphic retraction of B_k^n. In [1], [6], [11], [16], [23] and [27] it is shown what such a retraction looks like.

Every holomorphic mapping $T \in \mathcal{H}(B^n)$ is nonexpansive in (B^n, ρ_n) and has a fixed point if and only if there exists $x \in B^n$ such that a sequence of its iterates $\{T^p x\}$ is bounded in (B^n, ρ_n) ([8], [26]).

Common fixed points. First we require the following results concerning $\Gamma(T)$ and $\Gamma'(T)$.

THEOREM 1. *Let $T : B_k^n \to B_k^n$ be a holomorphic mapping. The following statements are equivalent:*

(i) *T has a fixed point,*
(ii) *$\Gamma(T)\subset\mathcal{H}(B_k^n)$,*
(iii) *$\Gamma(T)$ contains a holomorphic retraction $R_T\in\mathcal{H}(B_k^n)$,*
(iv) *$\Gamma'(T)$ contains $F\in\mathcal{H}(B_k^n)$,*
(v) *there exist x_0 and $\{p_i\}$ such that $\sup_i \|T^{p_i}(x_0)\|<1$.*

Proof.
(i)⇒(ii). If T has a fixed point, then for every $x\in B_k^n$ the sequence $\{T^p x\}$ is ρ_n-bounded and it gives $\Gamma(T)\subset\mathcal{H}(B_k^n)$.

(ii)⇒(iii). If $\Gamma(T)\subset\mathcal{H}(B_k^n)$ then $\Gamma(T)$ is a compact abelian semigroup and therefore $\Gamma(T)$ contains a holomorphic idempotent which is a holomorphic retraction.

(iii)⇒(iv). Obvious.

(iv)⇒(v). Obvious.

(v)⇒(i). If for $x_0\in B_k^n$ there exists $\{p_i\}$ such that $\sup_i \|T^{p_i}(x_0)\|<1$ then the sequence $\{T^{p_i}(x_0)\}$ is ρ_n-bounded. By the theorem of Całka ([5]) this subsequence $\{T^{p_i}(x_0)\}$ of $\{T^p(x_0)\}$ guarantees the ρ_n-boundedness of $\{T^p(x_0)\}$ and hence T has a fixed point.

Remark 1. It is worth noticing that if $B\subset H$ and $\dim H=+\infty$ the implication (v)⇒(i) is not true ([22]).

Directly from the above proof we get

THEOREM 2. *Let $T: B_k^n\to B_k^n$ be a holomorphic mapping. T is fixed point free if and only if for every $F\in\Gamma'(T)$ the image $F(B_k^n)$ is contained in the boundary of B_k^n.*

Theorem 1 allows us to prove the main theorem about the existence of common fixed points.

THEOREM 3. *Let $\{T_\mu\}$ be a commuting family of continuous functions mapping $\bar{B}_k^n$ into itself and holomorphic in B_k^n. Then there is a common fixed point for all functions of the family.*

Proof. It is sufficient to prove this theorem for a finite family $\{T_1, \cdots, T_m\}$.

Case 1. Every $T_{i|B_k^n}$ has a fixed point in B_k^n. Here the existence of a common fixed points is proved in [14] and [15]. For the reader's convenience we give a sketch of the proof of this fact.

Every $\mathrm{Fix}(T_{i|B_k^n})$ is a holomorphic retract of B_k^n ([16], [27]). Since $\{T_1, \cdots, T_m\}$ is a commuting family of functions we have

$$T_j(\mathrm{Fix}(T_{i|B_k^n}))\subset\mathrm{Fix}(T_{i|B_k^n})$$

for $1\leqq i,\ j\leqq m$.

Next if $R: B_k^n - A \subset B_k^n$ is a holomorphic retraction and $T: B_k^n - B_k^n$ is a holomorphic mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $T(A) \subset A$, then $A \cap \mathrm{Fix}(T)$ is a nonempty holomorphic retract of B_k^n. Indeed $T \circ R$ has a fixed point (see (v) in Theorem 1) and every such a point lies in A. It is easy to observe that $\mathrm{Fix}(T \circ R) = A \cap \mathrm{Fix}(T)$.

Now it is sufficient to apply a mathematical induction with respect to m to obtain a common fixed point of $\{T_1, \cdots, T_m\}$.

In the next two cases we proceed by induction with respect to n. For $n=1$ see [14] and [23].

Case 2. $T_{1|B_k^n} \in \mathcal{H}(B_k^n)$ and $\Gamma(T_{1|B_k^n}) \not\subset \mathcal{H}(B_k^n)$. Then there exists $T \in \Gamma(T_{1|B_k^n})$ such that (after applying of an appropriate linear mapping L that permutes coordinates)

$$T(B_k^n) \subset \{e_1\} \times \cdots \times \{e_q\} \times B_k^{n-q}$$

$(1 \leqq q \leqq n)$ and $T, T_1, \cdots, T_m$ commute. It is clear that

$$T_i(\{e_1\} \times \cdots \times \{e_q\} \times B_k^{n-q}) \subset \{e_1\} \times \cdots \times \{e_q\} \times \bar{B}_k^{n-q}$$

$(1 \leqq i \leqq m)$ and therefore $T_1, \cdots, T_m$ have a common fixed point in $\{e_1\} \times \cdots \times \{e_q\} \times \bar{B}_k^{n-q}$ by the induction hypothesis.

Case 3. $T_{1|B_k^n} \notin \mathcal{H}(B_k^n)$. By the induction hypothesis $T_1, \cdots, T_m$ have a common fixed point (see Case 2).

Remark 2. A continuous mapping $T: \bar{B}^n \to \bar{B}^n$ which satisfies the following condition

$$\rho_n(tTx, tTy) \leqq \rho_n(x, y)$$

for all $x, y \in B^n$ and every $0 \leqq t < 1$ is called a nonexpansive mapping in $\bar{B}^n$. Theorem 1 and 2 are still true if we replace the assumption of holomorphy of mappings in B_k^n by nonexpansiveness in $\bar{B}_k^n$.

Open problem. If we have a mapping $T: \bar{B}^n \to \bar{B}^n$ which is nonexpansive in $\bar{B}^n$, then T has a fixed point ([8], [9], [10], [15], [19], [20]). Is Theorem 2 true if we replace $\bar{B}_k^n$ by $\bar{B}^n$? If either $n=1$ or every T_μ is a ρ_n-isometry it is known that the answer is positive ([14], [17], [18]).

References

[1] M. Abd-Alla, Sur l'ensemble des points fixes d'une application holomorphe, C.R. Acad. Sci. Paris Sér. I Math. **302** (1986), 451-454.

[2] W.M. Boyce, Commuting functions with no common fixed point, Trans. Amer. Math. Soc. **137** (1969), 77-92.

[3] R.B. Burckel, Iterating analytic self-maps of disc, Amer. Math. Monthly **88** (1981), 396-407.

[4] R.B. Burckel, An introduction to classical complex analysis, Vol. I., Birkhäuser

Verlag, Basel and Stuttgart, 1979.

[5] A. Całka, On conditions under which isometries have bounded orbits, Colloq. Math. XLVIII (1984), 219-227.

[6] D. J. Eustice, Holomorphic idempotents and common fixed points on the 2-disk, Michigan Math. J. **19** (1972), 347-352.

[7] T. Franzoni and E. Vesentini, Holomorphic maps and invariant distances, North Holland, Amsterdam 1980.

[8] K. Goebel, Uniform convexity of Carathéodory's metric on the Hilbert ball and its consequences, Institute Nazionale di Alta Matematica Francesco Severi, Symposia Mathematica XXVI (1982), 163-179.

[9] K. Goebel and S. Reich, Uniform convexity, hyperbolic geometry and nonexpansive mappings, Marcel Dekker, New York and Basel, 1984.

[10] K. Goebel, T. Sekowski and A. Stachura, Uniform convexity of hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball, Nonlinear Analysis **4** (1980), 1011-1021.

[11] L. F. Heath and T. J. Suffridge, Holomorphic retracts in complex n-space, Illinois J. Math. **25** (1981), 125-135.

[12] E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. I, Academic Press, New York, 1963.

[13] J. P. Huneke, On common fixed points of commuting continuous functions on an interval, Trans. Amer. Math. Soc. **139** (1969), 371-381.

[14] T. Kuczumow, Common fixed points of commuting holomorphic mappings in Hilbert ball and polydisc, Nonlinear Analysis **8** (1984), 417-419.

[15] T. Kuczumow, Nonexpansive retracts and fixed points of nonepansive mappings in the cartesian product of n Hilbert balls, Nonlinear Analysis **9** (1985), 601-604.

[16] T. Kuczumow, Holomorphic retracts of polyballs, Proc. Amer. Math. Soc. **98** (1986), 374-375.

[17] T. Kuczumow, Nonexpansive mappings and isometries of Hilbert n-balls with hyperbolic metrics, Habilitation Dissertation, UMCS Lublin, 1987.

[18] T. Kuczumow and W. O. Ray, Isometries in the cartesian product of n unit open Hilbert balls with a hyperbolis metric, Ann. Mat. Pura Appl. (to appear.).

[19] T. Kuczumow and A. Stachura, Convexity and fixed points of holomorphic mappings in Hilbert ball and polydisc, Bull. Polish Acad. Sci. Math. **34** (1986), 189-193.

[20] T. Kuczumow and A. Stachura, Fixed points of holomorphic mappings in the cartesian product of n unit Hilbert balls, Canad. Math. Bull. **29** (1986), 281-286.

[21] A. L. Shields, On fixed points of commuting analytic functions, Proc. Amer. Math. Soc. **15** (1964), 703-706.

[22] A. Stachura, Iterates of holomorphic self-maps of the unit ball in Hilbert space, Proc. Amer. Math. Soc. **93** (1985), 88-90.

[23] T. J. Suffridge, Common fixed points of commuting holomorphic maps of the hyperball, Michigan Math. J. **21** (1974), 309-314.

[24] M. Suzuki, The fixed point set and the iterational limits of a holomorphic self-maps, Kodai Math. J. **10** (1987), 298-306.

[25] E. Vesentini, Complex geodesics and holomorphic maps, Instituto Nazionale di Alta Matematica Francesco Severi, Symposia Mathematica XXVI (1982), 211-230.

[26] J.-P. Vigué, Points fixes d'applications holomorphes dans un produit fini de boules-unités d'espaces de Hilbert, Ann. Mat. Pura Appl. CXXXVII (1984), 245-256.

[27] J.-P. Vigué, Points fixes d'applications holomorphes dans un domaine borné convexe de C^n, Trans. Amer. Math. Soc. **289** (1985), 345–353.

[28] A. D. Wallace, The structure of topological semigroups, Bull. Amer. Math. Soc. **61** (1955), 95–112.

Instytut Matematyki UMCS
Plac Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland