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ON THE STABILITY OF A THREE-SPHERE

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1. Introduction.

Let (M^m, g) be an m-dimensional closed connected Riemannian manifold. The identity mapping id_M of M is a harmonic mapping, that is, a critical point of the first variation of the energy functional. (M^m, g) is said to be stable when the second variation of the energy functional at id_M is non-negative and otherwise, (M, g) is said to be unstable. The m-dimensional $(m \geqq 3)$ unit spheres are unstable. And unstable, simply connected compact irreducible symmetric spaces were determined (see Smith [6], Nagano [4], Ohnita [5] and Urakawa [11]).

Closed manifolds with negative Ricci curvature and closed Kaehler manifolds are examples of stable manifolds. Since Gao and Yau [1] proved the existence of a metric with negative Ricci curvature on every 3-dimensional closed manifold, there exists a stable metric on every 3-dimensional closed manifold.

Recently Urakawa [12] and Tanno [9] studied some deformation of the standard metric g_0 on S^{2n+1} $(n \geqq 1)$ with constant sectional curvature one. Let (CP^n, h) be the complex projective space with the Fubini-Study metric with constant holomorphic sectional curvature 4 and $\pi:(S^m, g_0) \to (CP^n, h)$ $(m=2n+1)$ be the Hopf fibration. Let ξ be the unit Killing vector field on S^m which is tangent to each fibre and η be the dual 1-form of ξ with respect to g_0. We define a one-parameter family $g(t)$, $0<t<\infty$, of Riemannian metrics on S^m by

$$g(t)=t^{-1}g_0+t^{-1}(t^m-1)\eta\otimes\eta .$$

THEOREM (Tanno [9]) *For $m=2n+1 \geqq 3$ and $t>t_0(m)$, $(S^m, g(t))$ is unstable, where $t_0(m)=[\{(m^2-4)^{1/2}-1\}/(m^2-5)]^{1/m}$.*

In this note, we show:

THEOREM A. *$(S^3, g(t))$ is stable if and only if $t \leqq t_0(3)=[\sqrt{5}-1/4]^{1/3}=0.676\cdots$.*

Remark 1. The sectional curvature $K_\sigma(t)$ of $g(t)$ is positive for $0<t<(4/3)^{1/3}$ (see Tanno [9]). In fact, for $t<1$, $t^4 \leqq K_\sigma(t) \leqq t(4-3t^3)$ and for $t \geqq 1$, $t(4-3t^3) \leqq K_\sigma(t) \leqq t^4$.

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Remark 2. The volume element of $(S^3, g(t))$ is invariant in $t\in(0, \infty)$.

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2. Preliminaries.

Let $g(t)$, $0<t<\infty$, be the family of metrics on S^3 defined in the introduction. Let ${}^{(t)}\nabla$ (resp. ${}^{(t)}R_{ij}$) be the Riemannian connection (resp. Ricci tensor field) of $g(t)$. $A^1(S^3)$ denotes the space of 1-forms on S^3. Let ${}^{(t)}\Delta$ be the Laplacian of $g(t)$ and let $\langle\,,\rangle_t$ (resp. $\|\ \|_t$) be the L^2-inner product (resp. L^2-norm) of forms on $(S^3, g(t))$. The Ricci transformation ${}^{(t)}Q: A^1(S^3)\to A^1(S^3)$ is defined by ${}^{(t)}Q\omega=({}^{(t)}R^k{}_j\omega_k)$ for $\omega\in A^1(S^3)$. For convenience, we set $\nabla={}^{(1)}\nabla$, $\Delta={}^{(1)}\Delta$, $\langle\,,\rangle=\langle\,,\rangle_1$, $\|\ \|=\|\ \|_1$ and $Q={}^{(1)}Q$. The Jacobi operator of the identity mapping acts on the space of vector fields. By the natural duality, the Jacobi operator $J(t)$ acting on the space of 1-forms is of the form (see Smith [6]):

$$J(t)=-{}^{(t)}\Delta-2\,{}^{(t)}Q\,. \tag{2.1}$$

Let λ_k be the k-th eigenvalue of the Laplacian Δ acting on the space of functions on S^3 with multiplicity $m(k)$. Then it is known that

$$\begin{aligned}\lambda_k&=k(k+2), & k&\geqq 0,\\ m(k)&=(k+1)^2, & k&\geqq 0.\end{aligned} \tag{2.2}$$

Let L_X be the Lie derivation with respect to a vector field X and V_k the space of eigenfunctions corresponding to the k-th eigenvalue. Then V_k has the following orthogonal decomposition with respect to g_0 (see Tanno [7]):

$$V_k=\textstyle\sum_\vartheta V_{k,\vartheta}\,,\qquad \vartheta=k,\ k-2,\ \cdots,\ k-2[k/2]\,. \tag{2.3}$$

Here for any $f\in V_{k,\vartheta}$, $L_\xi L_\xi f+\vartheta^2 f=0$.

Let $\{\xi_{(\alpha)}\}_{\alpha=1}^3$ be an orthonormal frame field of unit Killing vector fields of S^3 satisfying $\xi=\xi_{(1)}$ and $[\xi_{(\alpha)}, \xi_{(\beta)}]=2\xi_{(\gamma)}$ where (α, β, γ) is a cyclic permutation of (1, 2, 3) and $\{\eta_{(\alpha)}\}_{\alpha=1}^3$ the dual frame field of $\{\xi_{(\alpha)}\}_{\alpha=1}^3$ with respect to g_0. Set $\Phi_{(\alpha)}=-\nabla\xi_{(\alpha)}$ and $\Phi=\Phi_{(\alpha)}$. Then $\xi_{(\alpha)}$, $\eta_{(\alpha)}$ and $\Phi_{(\alpha)}$ satisfy the following equations: for any vector fields X, Y on $S^3(1)$,

$$\begin{aligned}
&\Phi_{(\alpha)}\xi_{(\alpha)}=0\,,\quad \eta_{(\alpha)}\circ\Phi_\alpha=0\,,\quad \eta_{(\alpha)}(\xi_{(\alpha)})=1\,,\\
&\Phi_{(\alpha)}\Phi_{(\alpha)}X=-X+\eta_{(\alpha)}(X)\xi_{(\alpha)}\,,\\
&g_0(X, Y)=g_0(\Phi_{(\alpha)}X, \Phi_{(\alpha)}Y)+\eta_{(\alpha)}(X)\eta_{(\alpha)}(Y)\,,\\
&(\nabla_X\Phi_{(\alpha)})(Y)=g_0(X, Y)\xi_{(\alpha)}-\eta_{(\alpha)}(Y)X\,,\\
&\Phi_{(\alpha)}\xi_{(\beta)}=-\Phi_{(\beta)}\xi_{(\alpha)}=\xi_{(\gamma)}\,,\\
&\Phi_{(\alpha)}\Phi_{(\beta)}-\xi_{(\alpha)}\otimes\eta_{(\beta)}=-\Phi_{(\beta)}\Phi_{(\alpha)}+\xi_{(\beta)}\otimes\eta_{(\alpha)}=\Phi_{(\gamma)}\,,
\end{aligned}$$

where (α, β, γ) is a cyclic permutation of (1, 2, 3). Then we get immediately:

LEMMA 1. $\Phi_{(\alpha)}{}^{rs}=-\Phi_{(\alpha)}{}^{sr}=\nabla^r\xi_{(\alpha)}{}^s$.

LEMMA 2. *For any* $t\in(0, \infty)$, *we have* $\langle\eta_{(\alpha)}, \eta_{(\beta)}\rangle_t=0$ $(\alpha\neq\beta)$, $\langle\eta_{(1)}, \eta_{(1)}\rangle_t=t^{-2}$ *and* $\langle\eta_{(2)}, \eta_{(2)}\rangle_t=\langle\eta_{(3)}, \eta_{(3)}\rangle_t=t$.

LEMMA 3. *We get* $L_{\xi_{(\alpha)}}(V_k)\subset V_k$, $L_{\xi_{(1)}}(V_{k,\vartheta})\subset V_{k,\vartheta}$ *and* $\langle L_{\xi_{(\alpha)}}f, h\rangle=-\langle f, L_{\xi_{(\alpha)}}f\rangle$ *for any smooth functions* f *and* h *on* S^3.

3. Proof of Theorem A.

By (2.1), Tanno [9] gave the Jacobi operator $J(t)$ of $g(t)$.

LEMMA 4 (Tanno [9]). *The Jacobi operator* $J(t)$ *of* $(S^m, g(t))$ $(m=2n+1\geqq 3)$ *is given by the following: For* $\omega\in A^1(S^m)$,

$$J(t)\omega=-t\Delta\omega+t(1-t^{-m})L_\xi L_\xi\omega+2t(t^m-1)(\Phi^{rs}\nabla_r\omega_s)\eta$$
$$-2t(m+1-2t^m)\omega-2(m+1)t(t^m-1)\omega(\xi)\eta\,.$$

We prepare some lemmas to prove theorem A.

LEMMA 5. *On* S^3, *the following equations hold.*

1) $\Delta(f\eta_{(\alpha)})=-(\lambda_k+4)f\eta_{(\alpha)}+2df\cdot\Phi_{(\alpha)}, \quad f\in V_k\,.$

2) $L_\xi L_\xi(f\eta_{(1)})=-\vartheta^2 f\eta_{(1)}, \quad f\in V_{k,\vartheta}\,.$

$L_\xi L_\xi(f\eta_{(2)})=-\vartheta^2 f\eta_{(2)}-4f\eta_{(2)}+4\xi_{(1)}f\eta_{(3)}, \quad f\in V_{k,\vartheta}\,.$

$L_\xi L_\xi(f\eta_{(3)})=-\vartheta^2 f\eta_{(3)}-4f\eta_{(3)}-4\xi_{(1)}f\eta_{(2)}, \quad f\in V_{k,\vartheta}\,.$

3) $\Phi^{rs}\nabla_r(\sum_\alpha f_\alpha\eta_{(\alpha)})_s=2f_1+\xi_{(3)}f_2-\xi_{(2)}f_3\,.$

4) *For any* $t\in(0, \infty)$, $\alpha=2, 3$, *we have* $J(t)\eta_{(1)}=0$ *and* $J(t)\eta_{(\alpha)}=4t(t^3-2+t^{-3})\eta_{(\alpha)}\,.$

Proof. 1) and 4) were proved by Tanno in [8] and [9]. 2) is easily verified by $[\xi_{(\alpha)}, \xi_{(\beta)}]=2\xi_{(\gamma)}$. Since we have $\Phi^{rs}\nabla_r\eta_{(\alpha)s}=\langle\nabla\xi_{(1)}, \nabla\xi_{(\alpha)}\rangle$ by Lemma 1, we have $\Phi^{rs}\nabla_r\eta_{(1)s}=2$ and $\Phi^{rs}\nabla_r\eta_{(\alpha)s}=0$ for $\alpha=2, 3$. By the definition of Φ, we obtain $\Phi^{rs}\nabla_r f_{(\alpha)}\eta_{(\alpha)s}=-g_0(\mathrm{grad}\, f_{(\alpha)}, \nabla_{\xi_{(\alpha)}}\xi_{(1)})$. Therefore 3) is proved.
q. e. d.

By the orthogonal decomposition (2.3) of V_k, any 1-form ω on S^3 can be represented by $\omega=\sum_{i,k,\vartheta}f_{i,k,\vartheta}\eta_{(i)}$, $f_{i,k,\vartheta}\in V_{k,\vartheta}$. By Lemmas 2, 3 and 4, we have the following lemma.

LEMMA 6. *For any* $\omega=\sum_{i,k,\vartheta}f_{i,k,\vartheta}\eta_{(i)}\in A^1(S^3)$ *and* $t\in(0, \infty)$, *we have*

$$\langle J(t)\omega, \omega\rangle_t=\sum_{k=0}^{\infty}S(\omega, k, t).$$

Here

$$\begin{aligned}S(\omega, k, t)=&\sum_{\vartheta}[\lambda_k+(t^{-3}-1)\vartheta^2]t^{-1}\|f_{1,k,\vartheta}\|^2\\&+\sum_{\vartheta}[\lambda_k+4(t^{-3}-2+t^3)+(t^{-3}-1)\vartheta^2]t^2\|f_{2,k,\vartheta}\|^2\\&+\sum_{\vartheta}[\lambda_k+4(t^{-3}-2+t^3)+(t^{-3}-1)\vartheta^2]t^2\|f_{3,k,\vartheta}\|^2\\&+(8t^{-1}-4t^2)\sum_{\vartheta}\langle\xi_{(1)}f_{3,k,\vartheta}, f_{2,k,\vartheta}\rangle\\&+4t^2\sum_{\vartheta,\alpha}\langle\xi_{(3)}f_{2,k,\vartheta}, f_{1,k,\alpha}\rangle\\&-4t^2\sum_{\vartheta,\alpha}\langle\xi_{(2)}f_{3,k,\vartheta}, f_{1,k,\alpha}\rangle.\end{aligned}$$

Proof. Using $df\circ\Phi_{(\alpha)}=\xi_{(\gamma)}f\eta_{(\beta)}-\xi_{(\beta)}f\eta_{(\gamma)}$ and Lemma 2, we have

$$\begin{aligned}J(t)\omega=&\sum_{k,\vartheta}[\lambda_k+(t^{-3}-1)\vartheta^2]f_{1,k,\vartheta}\eta_{(1)}\\&+\sum_{k,\vartheta}[\lambda_k+4(t^{-3}-2+t^3)+(t^{-3}-1)\vartheta^2]f_{2,k,\vartheta}\eta_{(2)}\\&+\sum_{k,\vartheta}[\lambda_k+4(t^{-3}-2+t^3)+(t^{-3}-1)\vartheta^2]f_{3,k,\vartheta}\eta_{(3)}\\&+4(t^{-3}-1)\sum_{k,\vartheta}\xi_{(1)}f_{3,k,\vartheta}\eta_{(2)}\\&+4(t^{-3}-1)\sum_{k,\vartheta}\xi_{(1)}f_{2,k,\vartheta}\eta_{(3)}\\&-2\sum_{k,\vartheta}[\xi_{(3)}f_{1,k,\vartheta}\eta_{(2)}-\xi_{(2)}f_{1,k,\vartheta}\eta_{(3)}]\\&-2\sum_{k,\vartheta}[\xi_{(1)}f_{1,k,\vartheta}\eta_{(3)}-\xi_{(3)}f_{1,k,\vartheta}\eta_{(1)}]\\&-2\sum_{k,\vartheta}[\xi_{(2)}f_{1,k,\vartheta}\eta_{(1)}-\xi_{(1)}f_{1,k,\vartheta}\eta_{(2)}]\end{aligned}$$

Therefore, by Lemmas 2 and 3, we have the above representation of $S(\omega, k, t)$.
q. e. d.

LEMMA 7. *For any* $\omega=\sum_{i,k,\vartheta}f_{i,k,\vartheta}\eta_{(i)}\in A^1(S^3)$, *set* $f_{i,k}=\sum_{\vartheta}f_{i,k,\vartheta}$. *Then for any* $\omega\in A^1(S^3)$, $t\in(0, 1]$ *and* $k\geqq 0$, *we have*

$$\begin{aligned}S(\omega, k, t)\geqq& t^{-1}(t^{-3}-1)\sum_{\vartheta}\vartheta^2\|f_{1,k,\vartheta}\|^2+t^{-1}k(k-2)\|f_{1,k}\|^2\\&+t^2(k-2)\{k+2(1-t^3)\}(\|f_{2,k}\|^2+\|f_{3,k}\|^2)\\&+2kt^{-1}\{(\|f_{1,k}\|-t^3\|f_{2,k}\|)^2+(\|f_{1,k}\|-t^3\|f_{3,k}\|)^2\}\\&+2t^2k\sum_{\vartheta}(\|f_{2,k,\vartheta}\|-\|f_{3,k,\vartheta}\|)^2\\&+t^2(t^{-3}-1)\sum_{\vartheta}[(\vartheta-2)^2(\|f_{2,k,\vartheta}\|^2+\|f_{3,k,\vartheta}\|^2)\\&\qquad+4t(\|f_{2,k,\vartheta}\|-\|f_{3,k,\vartheta}\|)^2].\end{aligned}$$

Proof. By the definition of $V_{k,\vartheta}$, we have that for any $\phi\in V_{k,\vartheta}$ and $A=2, 3$,

(3.1) $$\langle\xi_{(1)}f_{i,k,\vartheta}, f_{j,k,\vartheta}\rangle\leqq\vartheta\|f_{i,k,\vartheta}\|\|f_{j,k,\vartheta}\|,$$

(3.2) $$\sum_{\vartheta,\alpha}\langle\xi_{(A)}f_{i,k,\vartheta}, f_{j,k,\alpha}\rangle\leqq k\|f_{i,k}\|\|f_{j,k}\|.$$

By (3.1), (3.2) and Lemma 6, we have

$$
\begin{aligned}
S(\omega, k, t) = & t^{-1}(t^{-3}-1)\sum_{\vartheta}\vartheta^2\|f_{1,k,\vartheta}\|^2 \\
& + t^{-1}k(k-2)\|f_{1,k}\|^2 + t^{-1}4k\|f_{1,k}\|^2 \\
& + t^2[\lambda_k - 4(1-t^3)](\|f_{2,k}\|^2 + \|f_{3,k}\|^2) \\
& + t^2(t^{-3}-1)\sum_{\vartheta}(\vartheta^2+4)(\|f_{2,k,\vartheta}\|^2 + \|f_{3,k,\vartheta}\|^2) \\
& + 8t^2(t^{-3}-1)\sum_{\vartheta}\langle \xi_{(1)} f_{3,k,\vartheta}, f_{2,k,\vartheta}\rangle \\
& + 4t^2\sum_{\vartheta}\langle \xi_{(1)} f_{3,k,\vartheta}, f_{2,k,\vartheta}\rangle \\
& - 4t^2\sum_{\vartheta,\alpha}[\langle \xi_{(3)} f_{1,k,\vartheta}, f_{2,k,\alpha}\rangle + \langle \xi_{(2)} f_{3,k,\vartheta}, f_{1,k,\alpha}\rangle] \\
\geqq & t^{-1}(t^{-3}-1)\sum_{\vartheta}\vartheta^2\|f_{1,k,\vartheta}\|^2 \\
& + t^{-1}k(k-2)\|f_{1,k}\|^2 + t^{-1}4k\|f_{1,k}\|^2 \\
& + t^2[k^2+2k-4(1-t^3)](\|f_{2,k}\|^2 + \|f_{3,k}\|^2) \\
& + t^2(t^{-3}-1)\sum_{\vartheta}[(\vartheta-2)^2(\|f_{2,k,\vartheta}\|^2 + \|f_{3,k,\vartheta}\|^2) \\
& \qquad + 4\vartheta(\|f_{2,k,\vartheta}\| - \|f_{3,k,\vartheta}\|)^2] \\
& - 4t^2k\|f_{2,k}\|\|f_{3,k}\| \\
& - 4t^2k\|f_{1,k}\|(\|f_{2,k}\|^2 + \|f_{3,k}\|^2) \\
\geqq & t^{-1}(t^{-3}-1)\sum_{\vartheta}\vartheta^2\|f_{1,k,\vartheta}\|^2 + t^{-1}k(k-2)\|f_{1,k}\|^2 \\
& + 2t^{-1}k[(\|f_{1,k}\| - t^3\|f_{2,k}\|)^2 + (\|f_{1,k}\| - t^3\|f_{2,k}\|)^2] \\
& + t^2[k^2-2k-4(1-t^3)-2kt^3](\|f_{2,k}\|^2 + \|f_{3,k}\|^2) \\
& + 2t^2k(\|f_{2,k}\|^2 + \|f_{3,k}\|^2) \\
& + t^2(t^{-3}-1)\sum_{\vartheta}[(\vartheta-2)^2(\|f_{2,k,\vartheta}\|^2 + \|f_{3,k,\vartheta}\|^2) \\
& \qquad + 4\vartheta(\|f_{2,k,\vartheta}\| - \|f_{3,k,\vartheta}\|)^2].
\end{aligned}
$$

And this is the required inequality. q. e. d.

Set $W_1 = \{\sum_{i=1}^3 f_i\eta_{(i)} : f_i \in V_1\}$. To decompose W_1 into three linear subspaces, we define four forms in W_1. Let (x, y, z, w) be the canonical coordinate system in R^4 such that ξ is the restriction of $y\partial/\partial x - x\partial/\partial y + w\partial/\partial z - z\partial/\partial w$. Set

$$
\phi_1 = x\eta_{(2)} - y\eta_{(3)}, \qquad \phi_2 = y\eta_{(2)} + x\eta_{(3)},
$$
$$
\phi_3 = z\eta_{(2)} - w\eta_{(3)}, \qquad \phi_4 = w\eta_{(2)} + z\eta_{(3)}.
$$

LEMMA 8. *W_1 has the following orthogonal decomposition with respect to $g(t)$:*

$$
W_1 = W_{1,1} + W_{1,2} + W_{1,3}.
$$

Here

$$J(t)\omega_i=u_i(t)\omega_i, \qquad \omega_\iota\in W_{1,\iota}\ (\iota=1, 2, 3),$$
$$u_1(t)=t(2t^3+t^{-3}-1-\sqrt{(2t^3-1)^2+8}),$$
$$u_2(t)=t(2t^3+t^{-3}-1+\sqrt{(2t^3-1)^2+8}),$$
$$u_3(t)=t(9t^{-3}-8+4t^3),$$
$$W_{1,1}=\{f\eta+a_1(t)df\circ\Phi : f\in V_1\},$$
$$W_{1,2}=\{f\eta+a_2(t)df\circ\Phi : f\in V_1\},$$
$$W_{1,3}=\mathrm{Span}\{\phi_1, \phi_2, \phi_3, \phi_4\},$$
$$a_1(t)=4t^{-3}(3-2t^3-\sqrt{(2t^3-1)^2+8}),$$
$$a_2(t)=4t^{-3}(3-2t^3+\sqrt{(2t^3-1)^2+8}).$$

Proof. Tanno [9] proved that $W_{1,1}$ and $W_{1,2}$ are eigenspaces of $J(t)$ corresponding to $u_1(t)$ and $u_2(t)$. By Lemma 5, we obtain that $\Delta\omega=-9\omega$, $L_\xi^2\omega=-9\omega$ and $\Phi^{rs}\nabla_r\omega_s=0$. So, using Lemma 4, we see that $W_{1,3}$ is an eigenspace of $J(t)$ corresponding to $u_3(t)$. q.e.d.

Proof of Theorem A. When $t>t_0(3)$, by showing $u_1(t)<0$, Tanno proved that $(S^3, g(t))$ is unstable. From Lemma 7, we have $S(\omega, k, t)\geqq 0$ for any $\omega\in A^1(S^3)$, any $t\in(0, 1]$ and $k\neq 1$. And by Lemma 8, when $t\leqq t_0(3)$, $J(t)$ have no negative eigenvalue, that is, $(S^3, g(t))$ is stable. q.e.d.

We also have the nullities $\mathrm{Null}_t(id)$ and the indices $\mathrm{Index}_t(id)$ of the identity mapping on $(S^3, g(t))$ for $0<t\leqq 1$.

COROLLARY B. *We have*

$$\mathrm{Null}_t(id)=\begin{cases}4 & (0<t<t_0(3))\\ 8 & (t=t_0(3))\\ 4 & (t_0(3)<t<1)\\ 6 & (t=1)\end{cases} \qquad \mathrm{Index}_t(id)=\begin{cases}0 & (0<t\leqq t_0(3))\\ 4 & (t_0(3)<t\leqq 1).\end{cases}$$

Proof. Since indices are obtained by Lemmas 7 and 8, we give nullities of the identity mapping. From Lemma 7, we have that for any $t\in(0, 1]$, $k\neq 1$, $S(\omega, k, t)\geqq 0$ and moreover that if $f_{\iota,k}\neq 0$ for some i and $k\neq 0, 1, 2$, then $S(\omega, k, t)>0$. For $k=2$, set $S(\omega, k, t)=0$. Then we have that $f_{1,2,2}=f_{2,2,0}=f_{3,2,0}=0$ and $\|f_{2,2}\|=\|f_{3,2}\|=t^{-3}\|f_{1,2}\|$. Since we have $\xi_{(1)}f_{2,2}=2f_{3,2}$ by (3.1) and (3.2), we obtain that $\xi_{(3)}f_{1,2}=2t^3f_{2,2}$ and $\xi_{(2)}f_{1,2}=-2t^3f_{3,2}$. Therefore we have $f_{1,2}\eta_{(1)}+f_{2,2}\eta_{(2)}+f_{3,2}\eta_{(3)}=f_{1,2,0}\eta_{(1)}+2^{-1}t^{-3}d(f_{1,2,0})\circ\Phi$. By $\dim V_{2,0}=3$ and Lemma 8, we have nullities of the identity mapping of $(S^3, g(t))$ for $t\in(0, 1]$. q.e.d.

REFERENCES

[1] L. Z. GAO AND S. T. YAU, The existence of negatively Ricci curved metrics on three manifolds, Invent. math., **85** (1986), 637-652.

[2] H. MUTO, The first eigenvalue of the Laplacian on even dimensional spheres, Tôhoku Math. Journ., **32** (1980), 427-432.

[3] H. MUTO AND H. URAKAWA, On the least positive eigenvalue of the Laplacian for compact homogeneous spaces, Osaka Journ. Math., **17** (1980), 471-484.

[4] T. NAGANO, Stability of harmonic maps between symmetric spaces, Lect. Notes Math., **949**, Springer-Verlag, 1982, 130-137.

[5] Y. OHNITA, Stability of harmonic maps and standard minimal immersions, Tôhoku Math. Journ., **38** (1986), 259-267.

[6] R. T. SMITH, The second variation formula for harmonic mappings, Proc. Amer Math. Soc., **47** (1975), 229-236.

[7] S. TANNO, The first eigenvalue of the Laplacian on spheres, Tôhoku Math. Journ., **31** (1979), 179-185.

[8] S. TANNO, Geometric expressions of eigen 1-forms of the Laplacian on spheres, Spectra of Riemannian manifolds (Kaigai Pub.), Tokyo, 1983, 115-128.

[9] S. TANNO, Instability of spheres with deformed Riemannian metrics, Kodai Math Journ., **10** (1987), 250-257.

[10] H. URAKAWA, On the least positive eigenvalue of the Laplacian for compact group manifolds, Journ. Math. Soc. Japan, **31** (1979), 209-226.

[11] H. URAKAWA, The first eigenvalue of the Laplacian for positively curved homogeneous Riemannian manifold, Comp. Math., **59** (1986), 57-71.

[12] H. URAKAWA, Stability of harmonic maps and eigenvalues of the Laplacian, Trans. Amer. Math. Soc., **301** (1987), 557-589.

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