I. AMEMIYA AND K. MASUDA
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ON JORIS' THEOREM ON DIFFERENTIABILITY OF FUNCTIONS

BY ICHIRO AMEMIYA AND KAZUO MASUDA

1. Introduction.

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ be a function. If $f^2, f^3 \in C^\infty$, does it follow that $f \in C^\infty$? The Inverse Function Theorem does not immediately give the answer. In 1982 H. Joris answered this problem affirmatively by showing the following theorem.

THEOREM 1 (H. Joris [J]). *Let $n_1, n_2, \cdots, n_m$ be positive integers with g.c.d. $\{n_1, n_2, \cdots, n_m\}=1$. If $f: \boldsymbol{R}^n \to \boldsymbol{R}$ is a function such that $f^{n_i} \in C^\infty$ for $i=1, 2, \cdots, m$, then $f \in C^\infty$.*

In the same paper H. Joris proposed the next problem.

PROBLEM. *Find the other families of smooth functions $\{\phi_i: \boldsymbol{R} \to \boldsymbol{R} \mid i=1, 2, \cdots, m\}$ having the following property: For any function $f: \boldsymbol{R}^n \to \boldsymbol{R}$, f is smooth if and only if $\phi_i \circ f$ is smooth for $i=1, 2, \cdots, m$.*

If we assume the continuity of f, then we need only consider the germs at $x=0$ since the study of differentiability is a local problem. In 1985 J. Duncan, S.G. Krantz and H.R. Parks gave a certain family $\{\phi_i\}$ for continuous f.

THEOREM 2 (J. Duncan, S.G. Krantz and H.R. Parks, [D] Theorem 2). *Let $\phi_i: \boldsymbol{R} \to \boldsymbol{R}$ be smooth functions such that $\phi_i(x)=x^{n_i}+$ "higher order terms" near $x=0$ for $i=1, 2, \cdots, m$ with g.c.d. $\{n_1, n_2, \cdots, n_m\}=1$. Then $\{\phi_i\}$ has the following property: For any continuous function $f: \boldsymbol{R}^n \to \boldsymbol{R}$ with $f(0)=0$, f is smooth near $x=0$ if and only if $\phi_i \circ f$ is smooth near $x=0$ for $i=1, 2, \cdots, m$.*

In the present paper, we give a simple proof of Joris' Theorem (§2) and the necessary and sufficient condition for $\{\phi_i\}$ to have the property mentioned in Theorem 2 (§3 Theorem 3). In Appendix (§4), we discuss this condition further, especially for polynomials ϕ_i.

2. Simple proof of Joris' Theorem.

The essential part of our proof is the following algebraic lemma.

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LEMMA 1. *Let $\mathcal{R}$ be a ring and $\mathcal{S}$ a subring of $\mathcal{R}$ with the property:*

(P) if $a\in\mathcal{R}$ and $a^r\in\mathcal{S}$ for every sufficiently large positive integer r, then $a\in\mathcal{S}$. Then the ring $\mathcal{S}[\![x]\!]$ of formal power series of x with coefficients in $\mathcal{S}$ has the property (P) as a subring of $\mathcal{R}[\![x]\!]$.

Proof. Suppose $f=\sum_{i=0}^{\infty}a_i x^{k+i}$ for some $k\geqq 0$, $a_i\in\mathcal{R}$, and $f^r\in\mathcal{S}[\![x]\!]$ for every large r, then since a_0^r is the coefficient of the lowest term of f^r, $a_0^r\in\mathcal{S}$ and hence by (P) $a_0\in\mathcal{S}$. The set $\mathcal{S}'$ of all the elements a of $\mathcal{R}$ for which $a_0a^m\in\mathcal{S}$ for every positive integer m is a subring of $\mathcal{R}$ including $\mathcal{S}$, because a_0a^{rm}, $a_0b^{rm}\in\mathcal{S}$ implies $a_0^r a^{rm}b^{rm}\in\mathcal{S}$ for $r\geqq 2$ and hence $a_0(ab)^m\in\mathcal{S}$. Now we prove, by induction, $a_n\in\mathcal{S}'$ for every n. Suppose $a_0, a_1, \cdots, a_n\in\mathcal{S}'$, then for $f_n=\sum_{i=0}^{n}a_i x^{k+i}$, $a_0f_n^m\in\mathcal{S}[\![x]\!]$ for every m. So $a_0f^r(f-f_n)^{(r+1)m}=\sum_s \text{const.}\, f^{r+s}a_0f_n^{(r+1)m-s}\in\mathcal{S}[\![x]\!]$ and hence the coefficient $a_0^{r+1}a_{n+1}^{(r+1)m}$ of $x^{kr+(k+n+1)(r+1)m}$ in $a_0f^r(f-f_n)^{(r+1)m}$ is in $\mathcal{S}$. Therefore, by (P), $a_0a_{n+1}^m\in\mathcal{S}$, that is, $a_{n+1}\in\mathcal{S}'$. Since $a_n\in\mathcal{S}'$ for every n, $a_0f^m\in\mathcal{S}[\![x]\!]$ for every m. Then for $f-a_0x^k=a_1x^{k+1}+\cdots$, $(f-a_0x^k)^r=f^r+\sum_{s=1}^{r}\text{const.}\, a_0^s f^{r-s}x^{ks}\in\mathcal{S}[\![x]\!]$. This shows $a_1\in\mathcal{S}$ and repeating the same argument, we have $a_n\in\mathcal{S}$ for every n, that is, $f\in\mathcal{S}[\![x]\!]$, which completes the proof of Lemma 1.

Let $f:\boldsymbol{R}^n\to\boldsymbol{R}$ be a function such that $f^{n_i}\in C^\infty$ for $i=1, 2, \cdots, m$ with $g.c.d.\{n_1, n_2, \cdots, n_m\}=1$. We must show that $f\in C^\infty$. In [B] J. Boman showed that $f:\boldsymbol{R}^n\to\boldsymbol{R}$ is smooth if and only if $f\circ g$ is smooth for every smooth map $g:\boldsymbol{R}\to\boldsymbol{R}^n$. Hence we may assume that $n=1$, namely $f:\boldsymbol{R}\to\boldsymbol{R}$. Since $g.c.d.\{n_i\}=1$, there exists a positive integer p such that any integer $r\geqq p$ is written as $r=\sum\alpha_i n_i$ for suitable non-negative integers α_i and hence $f^r=\prod(f^{n_i})^{\alpha_i}$ is smooth. For any smooth function $g(x)$, the ∞-jet j_ag of g at $x=a$, i.e. $\sum_{n=0}^{\infty}(1/n!)g^{(n)}(a)x^n$, gives an element of the formal power series ring $\boldsymbol{R}[\![x]\!]$. We say g is *flat* at a if $j_ag=0$. It is easy to show that f is smooth near the non-flat point of f^p. In fact, assume that $j_0f^p\neq 0$. Choose an odd prime number $r>p$. Then we have $f^p=ax^n+\cdots$ and $f^r=bx^m+\cdots=x^mg$ where g is a smooth function with $g(0)\neq 0$. Since $(f^p)^r=(f^r)^p$ and $m=rl$ for some positive integer l, $f=x^lg^{1/r}$ is smooth near $x=0$. Let D be the set of all non-flat points of f^p, then D is an open subset of $\boldsymbol{R}$ and f^r is flat at every point in $\boldsymbol{R}-D$ for any $r\geqq p$. Now apply Lemma 1 to the ring $\mathcal{R}$ of all continuous functions on D and its subring $\mathcal{S}$ consisting of the restrictions of all continuous functions on $\boldsymbol{R}$ which vanish in $\boldsymbol{R}-D$. Obviously $\mathcal{S}$ has the property (P) in $\mathcal{R}$. For any smooth function g on D, $\{j_ag\,|\,a\in D\}$ can be considered as an element $J(g)$ of $\mathcal{R}[\![x]\!]$. Now for our f considered as a smooth function on D, $J(f)^r=J(f^r)\in\mathcal{S}[\![x]\!]$ for any $r\geqq p$. So, by Lemma 1, $J(f)\in\mathcal{S}[\![x]\!]$, in other words, for every n, the n-th derivative $f^{(n)}$ of f is the restriction of a continuous function on $\boldsymbol{R}$ vanishing in $\boldsymbol{R}-D$. The rest of the proof is covered by repeated applications of the following simple lemma.

LEMMA 2. *Let D be an open set of $\boldsymbol{R}$. If f and g are both continuous functions vanishing in $\boldsymbol{R}-D$, f is differentiable in D, and $f'(x)=g(x)$ for every $x\in D$, then f is differentiable in the whole $\boldsymbol{R}$ and $f'=g$.*

Proof. For $a \in \boldsymbol{R}-D$ and $\varepsilon>0$, there exists δ such that $|a-x|<\delta$ implies $|g(x)|<\varepsilon$. For any b with $|a-b|<\delta$, let c be the point closest to b among points of $\boldsymbol{R}-D$ between a and b. Then since any point x between b and c belongs to D and $|f'(x)|=|g(x)|<\varepsilon$, we have $|f(a)-f(b)|=|f(c)-f(b)| \leqq \varepsilon|c-b| \leqq \varepsilon|a-b|$. So f is differentiable at a and $f'(a)=0$. This completes the proof.

3. Condition for $\{\phi_i\}$.

Here we give an answer to Joris' Problem. In the sequel, $d(g)$ denotes, for $g=\sum_{n=0}^{\infty} a_n x^n \in \boldsymbol{R}[\![x]\!]$, the smallest n such that $a_n \neq 0$, $d(A)=g.c.d.\{d(g) \mid g \in A\}$ for any subset $A \subset \boldsymbol{R}[\![x]\!]$, and $[j_0\phi_1, j_0\phi_2, \cdots, j_0\phi_m]$ the subalgebra of $\boldsymbol{R}[\![x]\!]$ generated by $j_0\phi_i$, $i=1, 2, \cdots, m$.

THEOREM 3. *Let $\phi_i: \boldsymbol{R} \rightarrow \boldsymbol{R}$ be smooth functions with $\phi_i(0)=0$ for $i=1, 2, \cdots, m$. Then the following* (1) *and* (2) *are equivalent.*

(1) *For any continuous function $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ with $f(0)=0$, f is smooth near $x=0$ if and only if $\phi_i \circ f$ is smooth near $x=0$ for $i=1, 2, \cdots, m$.*

(2) $d([j_0\phi_1, j_0\phi_2, \cdots, j_0\phi_m])=1$.

Proof. As in section 2 we may assume that $n=1$. We consider two more conditions for $\{\phi_i\}$.

(3) *There exist smooth functions F_j, $j=1, 2, \cdots, l$ such that $F_j(\phi_1(x), \phi_2(x), \cdots, \phi_m(x))=x^{n_j}+$ "higher order terms" near $x=0$ with $g.c.d.\{n_1, n_2, \cdots, n_l\}=1$.*

(4) *There exist smooth functions F_1 and F_2 such that $F_j(\phi_1(x), \phi_2(x), \cdots, \phi_m(x)) =x^{n_j}$ near $x=0$ with $g.c.d.\{n_1, n_2\}=1$.*

Proof of (2)$\Leftrightarrow$(3). This follows from E. Borel's theorem which states that for any element $\tilde{g} \in \boldsymbol{R}[\![x]\!]$ there exists a smooth function $g(x)$ with $j_0 g=\tilde{g}$.

Proof of (3)$\Rightarrow$(4). This is given in Theorem 2 [D]. Here we give another proof. As in section 2 there exists a positive integer p such that any integer $r \geqq p$ is written as $r=\sum \alpha_j n_j$ for some non-negative integers α_j and hence $\prod F_j(\phi_1(x), \cdots)^{\alpha_j}=x^r+\cdots$. Then, for any odd integer $n \geqq p$, there exists $\tilde{F}(x_1, \cdots, x_m) \in \boldsymbol{R}[\![x_1, \cdots, x_m]\!]$ such that $\tilde{F}(j_0\phi_1, \cdots)=x^n$ in $\boldsymbol{R}[\![x]\!]$. By E. Borel's theorem there exist smooth functions F and g_1 such that $j_0 g_1=0$ and $F(\phi_1(x), \cdots) \doteq x^n+g_1(x)$ near $x=0$. We can easily find a smooth function G with $(G \circ F)(\phi_1(x), \cdots)=x^n$. In fact, since $y(x)=(F(\phi_1(x), \cdots))^{1/n}$ is smooth and $j_0 y=x$, we have $x=k(y)=y(1+g_2(y))$ for suitable smooth functions $k(y)$ and $g_2(y)$ with $j_0 g_2=0$. Put $G(Y)=k(Y^{1/n})^n$, then $G(Y)=(Y^{1/n}(1+g_2(Y^{1/n})))^n=Y(1+g_2(Y^{1/n}))^n$ is smooth since $j_0 g_2=0$ and $(G \circ F)(\phi_1(x), \cdots)=G(y(x)^n)=k(y(x))^n=x^n$ which completes the proof.

Proof of (4)$\Rightarrow$(1). This follows from Joris' Theorem.

To prove (1)$\Rightarrow$(2), we need the following algebraic lemma. $g=\sum a_n x^n \in \boldsymbol{R}[\![x]\!]$

is said to be *normal* if $a_{d(g)}=1$ and $a_{kd(g)}=0$ for every $k>1$, and for $g\in\boldsymbol{R}[\![x]\!]$ with $d(g)>0$, $[\![g]\!]$ denotes the closed subalgebra generated by g, namely $\{\sum_{i=1}^{\infty} c_i g^i \mid c_i\in\boldsymbol{R}\}$.

LEMMA 3. *For any subalgebra $A\subset\boldsymbol{R}[\![x]\!]$ with $d(A)>0$ and any positive integer d with $d\mid d(A)$, there exists a unique normal elemen $h\in\boldsymbol{R}[\![x]\!]$ such that $d(h)=d$ and $[\![h]\!]\supset A$.*

Proof. First we prove the following statement.

(i) If $d(h)=d$ and $[\![h]\!]\cap A\neq\{0\}$ then $[\![h]\!]\supset A$.

Since the coordinate transformation $y=h(x)^{1/d}$ induces the isomorphism $\boldsymbol{R}[\![x]\!]\cong\boldsymbol{R}[\![y]\!]$ and h corresponds to y^d, we may assume that $h=x^d$. If there exist elements $g=x^{rd}+\cdots+cx^{rd+a}+\cdots\in A$ with $c\neq 0$ and $a\nmid d$, we can choose g having the smallest a in the above representation. Take any $k=x^{sd}+\cdots\in[\![x^d]\!]\cap A\neq\{0\}$, then $g^s-k^r=bx^{td}+\cdots+scx^{srd+a}+\cdots\in A$ and $srd+a-td<a$ contradicting the choice of g. This completes the proof of (i). As a corollary we have

(ii) If $d(h)=d(h_1)>0$ and $[\![h]\!]\cap[\![h_1]\!]\neq\{0\}$ then $[\![h]\!]=[\![h_1]\!]$.

Now to show the existence of h in the lemma, choose any $g=x^{rd}+\cdots\in A$ and put $h_1=g^{1/r}=x^d+\cdots$, then $[\![h_1]\!]\cap A\ni g$ and hence $[\![h_1]\!]\supset A$ by (i). We can easily find real numbers c_i with $c_1=1$ such that $h=\sum_{i=1}^{\infty} c_i h_1^i$ is normal. We have also $[\![h]\!]=[\![h_1]\!]\supset A$ by (ii). To prove the uniqueness of h, let h_2 be any element satisfying the conditions of the lemma, then $[\![h]\!]=[\![h_2]\!]$ by (ii). If $h-h_2\neq 0$ then $d\nmid d(h-h_2)$ contradicting the fact $h-h_2\in[\![h]\!]$. This completes the proof of Lemma 3.

Proof of (1)⇒(2). Suppose $d=d([j_0\phi_1, \cdots])>1$. By Lemma 3 there exist $\tilde{h}=x^d+\cdots$ and $\tilde{F}_i\in\boldsymbol{R}[\![x]\!]$ such that $j_0\phi_i=\tilde{F}_i(\tilde{h})$ for $i=1, 2, \cdots, m$. By E. Borel's theorem we have $\phi_i(x)=F_i(h(x))+g_i(x)$ for suitable smooth functions F_i, h, g_i with $j_0 g_i=0$. So it is sufficient to show that there exists a non-smooth function f for which $h\circ f$ and $g_i\circ f$ are smooth. Since $h(x)=x^d(1+\cdots)=(x+\cdots)^d$, we can find a smooth function k with $h(k(x))=x^d$. Now we put $f(x)=k(x^{1/d})$ if d is odd and $f(x)=k(|x|)$ if d is even. Then f is not smooth and $g_i\circ f$ is smooth since $j_0 g_i=0$. Moreover $h\circ f$ is smooth, for $h(f(x))=x$ if d is odd and $h(f(x))=x^d$ if d is even. This contradicts (1). So the proof of Theorem 3 is completed.

4. Appendix.

In this section we discuss about the algorithm of computing $d([j_0\phi_1, \cdots])$ by use of jets $j_0\phi_i$. Let g be an element of $\boldsymbol{R}[\![x]\!]$ with $d(g)>0$ and d a positive integer with $d\mid d(g)$. Then, applying Lemma 3 to the subalgebra generated by g, we can find a unique normal element $h=h(g, d)\in\boldsymbol{R}[\![x]\!]$ such that $d(h)=d$ and $g=F(h)$ for some $F\in\boldsymbol{R}[\![x]\!]$. Note that each coefficient of $h(g, d)$ is given by some polynomial of $a_n/a_{d(g)}$ where $g=\sum a_n x^n$. Now, put $A=[j_0\phi_1, \cdots]$, $d_A=d(A)$ and $j_0\phi_i=\sum_{j=1}^{l_i} a_{ij}x^{n_{ij}}$ where $a_{ij}\neq 0$, $l_i\leqq\infty$ and $n_{i1}<n_{i2}<\cdots$ for $i=1, 2, \cdots, m$. It is clear that $g.c.d.\{n_{ij}|i, j\}\,|\,d_A$ and $d_A|g.c.d.\{n_{i1}|i\}$. In the finite number of integers d such that $d|g.c.d.\{n_{i1}|i\}$, d_A is characterized by the following proposition.

PROPOSITION 1. *For any positive integer d with $d|g.c.d.\{n_{i1}|i\}$, we have $d|d_A$ if and only if $h(j_0\phi_1, d)=\cdots=h(j_0\phi_m, d)$.*

Proof. Assume that $d_A=rd$ for some r. By Lemma 3, $[\![h_A]\!]\supset A$ for some h_A with $d(h_A)=rd$ and hence $j_0\phi_i=F_i(h_A)$ for some F_i. Let $h=h(h_A, d)$, then $h_A=F(h)$ for some F and hence $j_0\phi_i=(F_i\circ F)(h)$. So by the uniqueness of $h(j_0\phi_i, d)$, we have $h(j_0\phi_i, d)=h$ as desired. Conversely assume that $h(j_0\phi_1, d)=\cdots=h$. Then $[\![h]\!]\ni j_0\phi_i$ and hence $[\![h]\!]\supset A$ which implies $d|d_A$.

The following corollary corresponds to Theorem 3[D].

COROLLARY. *If $j_0\phi_1=x^{n_{11}}$ then $d_A=g.c.d.\{n_{ij}|i, j\}$.*

Proof. We have $h(j_0\phi_1, d_A)=\cdots=h$ and $j_0\phi_i=F_i(h)$ for suitable $F_i\in\boldsymbol{R}[\![x]\!]$. It follows from $j_0\phi_1=x^{n_{11}}=F_1(h)$ that $h=x^{d_A}$ and from $j_0\phi_i=F_i(x^{d_A})$, $d_A|n_{ij}$ which implies $d_A=g.c.d.\{n_{ij}|i, j\}$.

Proposition 1 gives an algorithm of computing d_A by use of jets $j_0\phi_i$, but it needs an infinite number of procedures to check whether $h(j_0\phi_1, d)=\cdots=h(j_0\phi_m, d)$ or not even though each ϕ_i is a polynomial. The following Proposition 2 gives a finite algorithm of computing d_A for polynomials ϕ_i.

LEMMA 4. (i) *Let $\mathcal{M}$ be the maximal ideal of $\boldsymbol{C}[\![x]\!]$ generated by x and $\Phi:\mathcal{M}\to\boldsymbol{C}[\![x]\!]^m$ a map given by $\Phi(g)=(j_0\phi_1\circ g, \cdots)$, then Φ is d_A to 1 map, namely for any $g\neq 0$ there exist exactly d_A elements k such that $\Phi(k)=\Phi(g)$.*

(ii) *Suppose that every $\phi_i(x)$ is analytic near $x=0$. Let $\Phi:U\to\boldsymbol{C}^m$ be a map given by $\Phi(x)=(\phi_1(x), \cdots)$ for some neighbourhood U of 0 in $\boldsymbol{C}$, then Φ is d_A to 1 map near $x=0$.*

Proof. (i) By Lemma 3, $[\![h]\!]\supset A$ for some h with $d(h)=d_A=d$. By the coordinate transformation we may assume that $h=x^d$ and hence $j_0\phi_i=\sum_j a_{ij}x^{jd}$ for suitable a_{ij}. By the definition of d_A there exists a positive integer p such that for any integer $r\geqq p$ we can find $f\in A$ with $d(f)=rd$ and hence $F\in\boldsymbol{R}[\![x_1, \cdots, x_m]\!]$ with $F(j_0\phi_1, \cdots, j_0\phi_m)=x^{rd}$. Therefore $\Phi(k)=\Phi(g)$ if and only if $k^d=g^d$, that is, $k=\varepsilon_d^n g$ for $\varepsilon_d=\exp(2\pi i/d)$ and $n=1, 2, \cdots, d$.

(ii) Suppose $\Phi(x)=\Phi(y)$. Then $\phi_i(x)=a_i x^{r_i d}(1+\cdots)=a_i y^{r_i d}(1+\cdots)$ and $y(1+\cdots)=\varepsilon_{r_i d}^n x(1+\cdots)$ and hence $y=\varepsilon_{r_i d}^n x+\cdots=y_{in}(x)$. Since $y_{in}(x)$ is holomorphic near $x=0$, it follows from (i) that the number of small solutions y of $\Phi(y)=\Phi(x)$ is d_A for small $x\neq 0$.

COROLLARY. *The following condition for $\{\phi_i\}$ is equivalent to* (1) *in Theorem* 1.

(5) *The map* $\Phi(g)=(j_0\phi_1\circ g, \cdots): \mathcal{M}\to\boldsymbol{C}[\![x]\!]^m$ *is injective.*

PROPOSITION 2. *Suppose that each* ϕ_i *is a polynomial. Let* $\phi(x, y)$ *be the greatest common divisor of* $\{\phi_i(x)-\phi_i(y)|i=1, 2, \cdots, m\}$ *in* $\boldsymbol{R}[x, y]$, *then* $d([j_0\phi_1, j_0\phi_2, \cdots, j_0\phi_m])=d(\phi(x, 0))$.

Proof. $\phi(x, y)$ is obtained by using Euclidean algorithm with respect to polynomials of x whose coefficients are rational functions of y. Then $\phi(x, a)$ is the greatest common divisor of $\{\phi_i(x)-\phi_i(a)|i\}$ in $\boldsymbol{R}[x]$ for any real number a with finite exceptions. Since the coefficient of the highest order term of $\phi(x, a)$ is independent of a, the solutions x of $\phi(x, a)=0$ depend continuously on a. So $d(\phi(x, 0))$ is equal to the number of solutions x of $\phi(x, a)=0$ such that $x\to0$ as $a\to0$. This number is equal to the number of common solutions x of $\phi_i(x)-\phi_i(a)=0$, $i=1, 2, \cdots, m$, such that $x\to0$ as $a\to0$. By Lemma 4 (ii) we have $d(\phi(x, 0))=d_A$.

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SAITAMA MEDICAL SCHOOL
KAWAKADO, MOROYAMA
SAITAMA 350-04, JAPAN

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU
TOKYO 152, JAPAN