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ON THE STRATIFICATION OF GOOD HYPERSURFACES

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1. Statement of results.

Let $f(z)$ be a germ of an analytic function defined in a neighborhood of the origin and let $f(z)=\sum_{\nu} a_\nu z^\nu$ be the Taylor expansion. We consider the germ of the hypersurface $V=f^{-1}(0)$. We assume that f has a non-degenerate Newton boundary $\Gamma(f)$. The purpose of this paper is to construct a canonical Whitney b-regular stratification $\mathcal{S}$ of V which depends only on the Newton boundaries $\{\partial\Gamma(f)\}$. Under the non-degeneracy condition of the Newton boundary, the singular locus of V is the union of several coordinate subspaces $\boldsymbol{C}^{*I}$. However the b-regularity for $(V^*, \boldsymbol{C}^{*I})$ does not hold in general and we have to know the locus where the regularity fails. For this purpose, we introduce the concept of the *I-primary boundary components* which plays an important role for the stratification of V. Its rough description is as follows. Let $P={}^t(p_1, \cdots, p_n)$ be a positive rational dual vector and let $I(P)=\{1\leqq i\leqq n\,;\ p_i=0\}$. The face function $f_p(z)$ is defined by the partial sum $\sum' a_\nu z^\nu$ for ν such that $\nu\in\Delta(P)$. Here $\Delta(P)$ is the face of $\Gamma(f)$ where P takes its minimal value $d(P; f)$. We use the notations of [5]. Assume that $f_P(z)=z^L g(z_{I(P)})$ where $z_{I(P)}$ is the projection of z into the affine coordinate space $\boldsymbol{C}^{I(P)}$. In this case, we say that f_P is *essentially of* $z_{I(P)}$*-variables* and we denote $g(z_{I(P)})$ by $f_p^e(z_{I(P)})$. We consider the variety $V^*(P)$ and $\partial V^*(P)$ as follows. $V^*(P)=\{z\in\boldsymbol{C}^{*n}\,;\ f_P(z)=0\}$ and $\partial V^*(P)=\{z_{I(P)}\in\boldsymbol{C}^{*I(P)}\,;\ f_p^e(z_{I(P)})=0\}$. If f_P is not essensially of $z_{I(P)}$-variables, $\partial V^*(P)$ is $\boldsymbol{C}^{*I(P)}$ by definition. We call $\partial V^*(P)$ a *I-primary boundary component with respect to P if* $V^*(P)$ is not empty. Let V_{pr} be the closure of V^* in $\boldsymbol{C}^n$ and let $V^{*I}=V\cap\boldsymbol{C}^{*I}$ and let $V_{pr}^{*I}=V_{pr}\cap\boldsymbol{C}^{*I}$. Then V_{pr}^{*I} is a union of I-primary boundary components (Lemma (3.3)). We say that the hypersurface $V=f^{-1}(0)$ is *good* if for each subset I of $\{1, \cdots, n\}$ with $|I|>2$, there is at most one f_P among $\{f_P\,;\ I(P)=I\}$ such that f_P gives a proper I-primary boundary component. Here P may not unique. We assume that V is a good hypersurface hereafter. If V has a proper primary boundary component, we denote this component by ∂V_{pr}^{*I}. If V does not have proper primary boundary component, $\partial V_{pr}^{*I}=\phi$ by definition. Let P be a positive dual vector and let $I=I(P)$. We say that V satisfies *the primary non-degeneracy condition* or simply *the PND-condition* if the following conditions are satisfied for any P such that $V^*(P)\neq\phi$. Let $p_{\min}$

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$=$*minimum* $\{p_j;\ j\notin I\}$.

(**PND1**) Assume that f_P is essentially of z_I-variables and let $f=f_P+\hat{f}$. Write $f_P(z)=z^K f_P^e(z_I)$ where $K=(k_1, \cdots, k_n)$.

(a) (i) $d(P; f)=0$ or (ii) $d(P; f)>0$ and $d(P; \hat{f})\geqq d(P; f)+p_{\min}$ or (iii) the variety $\left\{z\in C^{*n};\ f_P(z)=0,\ z_j\dfrac{\partial \hat{f}_P}{\partial z_j}(z)-k_j\hat{f}_P(z)=0 \quad \text{for} \quad j\notin I\right\}$ is empty.

(b) $\partial V^*(P)$ is a non-degenerate hypersurface in C^{*I} in an ε-ball B_ε^I for some ε.

(**PND2**) Assume that f_P is not essentially of z_I-variables. For each $z_I\in C^{*I}\cap B_\varepsilon^I$, the fiber $q_I^{-1}(z_I)$ is a non-degenerate hypersurface in $C^{I^c}\times\{z_I\}$ where I^c is the complement of I in $\{1, \cdots, n\}$.

MAIN THEOREM. *We assume that V is a good hypersurface which satisfies the PND-condition. Let $\mathcal{S}(I)=\{V^{*I}-\partial V_{pr}^{*I}, \partial V_{pr}^{*I}\}$ and let $\mathcal{S}=\bigcup_I \mathcal{S}(I)$. Then $\mathcal{S}$ is a regular stratification of V.*

For the stratification of the hypersurfaces which is not good and the stratification of the complete intersection varieties, see [6].

2. Stratifications.

Let V be an analytic variety in an open set D of C^n. We recall the necessary notions of the stratification which is induced by Whitney and Thom. For further details, see [10, 7, 3]. Let $\mathcal{S}$ be a family of subsets of V such that V is covered disjointly by elements of $\mathcal{S}$. $\mathcal{S}$ is called *a Whitney stratification* if the following conditions are satisfied.

(i) (*D-strictness*) Each element M of $\mathcal{S}$ (which is called a *stratum*) is a connected smooth analytic variety such that $\bar{M}$ and $\bar{M}-M$ are closed analytic varieties in D. Here $\bar{M}$ is the closure of M in D.

(ii) (*Frontier property*) Let M and N be strata of $\mathcal{S}$ and assume that $M\neq N$ and $M\cap\bar{N}\neq\phi$. Then $M\subset\bar{N}-N$.

We recall the Whitney b-condition for a Whitney stratification $\mathcal{S}$. Let (N, M) be a pair of strata of $\mathcal{S}$ with $\bar{N}\supset M$ and let p be a point of M. Let p_i and q_i be sequences on N and M respectively. We assume that

$$p_i\to p,\quad q_i\to p,\quad T_{p_i}N\to\tau \quad\text{and}\quad [p_i-q_i]\to\lambda. \tag{2.1}$$

Here the arrows imply the convergence in the respective spaces and $[v]$ is the complex line generated by v. Thus $\tau\in G(r, n)$ ($r=\dim N$) and $\lambda\in G(1, n)=P^{n-1}$ where $G(r, n)$ is the Grassmannian manifold of r-planes in C^n. We say that (N, M) satisfies *Whitney b-condition* at p if $\lambda\in\tau$ for any such sequences. When each pair (N, M) with $M\subset\bar{N}$ satisfies the Whitney b-condition at any point p

of M, we call $\mathcal{S}$ *a b-regular Whitney stratification.* The following proposition is a direct consequence of the Curve Selection Lemma (§3 of [4] or [1]) and Theorem 17.5 of [10].

PROPOSITION (2.2). *Let p_i and q_i be as in* (2.1). *Then there are analytic curves $p(t)$ and $q(t)$ defined on the interval $(-\varepsilon, \varepsilon)$ $(\varepsilon>0)$ such that*

(i) $p(0)=q(0)=p$ *and* $p(t)\in N$ *for* $t\neq 0$ *and* $q(t)\in M$.

(ii) $T_{p(t)}N\to\tau$ *and* $[p(t)-q(t)]\to\lambda$.

It is known that the b-condition for analytic varieties follows from the ratio condition (R) by [2, 9]. There is also a weaker regularity condition which is called *Whitney a-condition* but this condition results from b-condition ([3]).

3. Non-degenerate hypersurface and primary boundary components.

Let $f(z)=\sum_{\nu} a_\nu z^\nu$ be an analytic function of n variables which is defined in a neighborhood of the origin. The Newton polyhedron $\Gamma_+(f)$ is the convex hull of the union of $\{\nu+\boldsymbol{R}_+^n\}$ for ν such that $a_\nu\neq 0$. The Newton boundary $\Gamma(f)$ is the union of the compact faces of the Newton polyhedron. We assume that the Newton boundary $\Gamma(f)$ is non-degenerate. As we are mainly interested in non-isolated singularities, we also use the notation $\partial\Gamma_+(f)$ which is the union of the boundaries of $\Gamma_+(f)$ which are not necessarily compact. The inclusion $\Gamma(f)\subset\partial\Gamma_+(f)$ is obvious by the definition.

Let Σ^* be a fixed unimodular simplicial subdivision which is compatible with the dual Newton diagrams $\{\Gamma^*(f)\}$ and let $\hat{\pi}: X\to\boldsymbol{C}^n$ be the associated modification map. See [8] and [5] for the definition. Let V_{pr} be the closure of V^* and let $\tilde{V}$ be the proper transform of V_{pr} by $\hat{\pi}$. Let $\pi: \tilde{V}\to V_{pr}$ be the restriction of $\hat{\pi}$ to $\tilde{V}$. For finite vertices $Q_1, \cdots, Q_s$ of Σ^*, we define a subvariety $E(Q_1, \cdots, Q_s)$ of $\tilde{V}$ by $E(Q_1)\cap\cdots\cap E(Q_s)$ and let $E(Q_1, \cdots, Q_s)^*=E(Q_1, \cdots, Q_s)-\bigcup_{P\neq Q_i} E(P)$ where $E(P)$ is the divisor of $\tilde{V}$ which corresponds to P. Note that $E(Q_1, \cdots, Q_s)^*$ is non-empty only if $Q_1, \cdots, Q_s$ are vertices of an $(n-1)$-simplex of Σ^*. The collection of $E(Q_1, \cdots, Q_s)^*$ gives a regular stratification $\tilde{\mathcal{S}}$ of $\tilde{V}$. Let $\sigma=(P_1, \cdots, P_n)$. Then we have

$$\tilde{V}\cap\boldsymbol{C}_\sigma^n=\{\boldsymbol{y}_\sigma\in\boldsymbol{C}_\sigma^n\,;\, f_\sigma(\boldsymbol{y}_\sigma)=0\} \tag{3.1}$$

where $f_\sigma(\boldsymbol{y}_\sigma)=f(\hat{\pi}(\boldsymbol{y}_\sigma))/\prod_{j=1}^n y_{\sigma j}^{d(P_j;f)}$.

THEOREM (3.2). *$\tilde{V}$ is a smooth complex manifold and $\pi: \tilde{V}\to V_{pr}$ is a proper modification of V_{pr} in the neighborhood of the origin.*

The assertion is well known if the origin is an isolated singular point of V_{pr}. The general case can be proved similarly. Let I be a subset of $\{1, \cdots, n\}$. We define the coordinate subspace $\boldsymbol{C}^I$ and $\boldsymbol{C}^{*I}$ by $\boldsymbol{C}^I=\{\boldsymbol{z}=(z_1, \cdots, z_n)\,;\, z_j=0$ if

$j\notin I\}$ and $\boldsymbol{C}^{*I}=\{z\in\boldsymbol{C}^n; z_j=0$ iff $j\notin I\}$ respectively. For simplicity we usually write $\boldsymbol{C}^{*n}$ instead of $\boldsymbol{C}^{*I}$ if $I=\{1, \cdots, n\}$. We define the *I-proper boundary* V^{*I}_{pr} of V in $\boldsymbol{C}^{*I}$ by $V_{pr}\cap\boldsymbol{C}^{*I}$. If I is empty, $V^{*I}_{pr}=\{0\}$ by definition. Then we claim:

LEMMA (3.3). *The I-proper boundary V^{*I}_{pr} of V is the union of the I-primary boundary components.*

Proof. Let $\pi: \tilde{V}\to V_{pr}$ be the resolution of V_{pr} constructed in §3. Let $\tilde{V}^{*I}$ be the union of the strata $E(P_1, \cdots, P_s)^*$ of the stratification $\tilde{S}$ of $\tilde{V}$ such that $\pi(E(P_1, \cdots, P_s)^*)\subset\boldsymbol{C}^{*I}$. As π is a proper surjective mapping, it is clear that $\pi(\tilde{V}^{*I})=V^{*I}$. Let $E(P_1, \cdots, P_s)^*$ be such a stratum and let $\sigma=(P_1, \cdots, P_n)$ be an $(n-1)$-simplex of Σ^*. Let $P=P_1+\cdots+P_s$. Then P is a positive dual vector with $I(P)=I$. We may assume that $I=\{m+1, \cdots, n\}$ $(m\geqq s)$ for simplicity and $\sigma=(p_{ij})$ has the following form.

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where A and B are unimodular matrixes of $m\times m$ and $(n-m)\times(n-m)$ respectively. Then Lemma (3.3) follows from the following.

SUBLEMMA (3.4). *The restriction of π to $E(P_1, \cdots, P_s)^*$ is a submersion onto $\partial V^*(P)$.*

Proof. Let $\boldsymbol{y}$ be an arbitrary point of $E(P_1, \cdots, P_s)^*$. Recall that $E(P_1, \cdots, P_s)^*$ is defined by

$$y_{\sigma 1}=\cdots=y_{\sigma s}=h(\boldsymbol{y}_\sigma)=0$$

where h is characterized by

$$h(\boldsymbol{y}_\sigma)\prod_{i=1}^{n} y_{\sigma i}^{d(f;P_i)}=f_P(\hat{\pi}(\boldsymbol{y}_\sigma)). \tag{3.5}$$

Note that $\Delta(P)=\bigcap_{i=1}^{s}\Delta(P_i)$. Thus $h(\boldsymbol{y}_\sigma)$ does not contain the variables $y_{\sigma 1}, \cdots, y_{\sigma s}$. Let $\boldsymbol{z}=\hat{\pi}(\boldsymbol{y}_\sigma)$. Then we have $\boldsymbol{z}_I=(\boldsymbol{y}_I)^B$ i.e.,

$$z_j=\prod_{i=m+1}^{n} y_{\sigma i}^{p_{ji}} \quad (j=m+1, \cdots, n). \tag{3.6}$$

In particular, $\{z_j\}$ $(m+1\leqq j\leqq n)$ depend only on $y_{\sigma(m+1)}, \cdots, y_{\sigma n}$. Let E^* be the subvariety of $\boldsymbol{C}^{*n}_\sigma$ defined by $h(\boldsymbol{y}_\sigma)=0$. E^* is nothing but the product of $\boldsymbol{C}^{*s}\times E(P_1, \cdots, P_s)^*$. Let $V^*(P)$ be the subvariety of the base space $\boldsymbol{C}^{*n}$ which is defined by

$$V^*(P)=\{\boldsymbol{z}\in\boldsymbol{C}^{*n}; f_P(\boldsymbol{z})=0\}.$$

It is clear that $\hat{\pi}: E^*\to V^*(P)$ is an isomorphism by (3.5). Let $q_I: V^*(P)\to\partial V^*(P)$

and $p: E^* \to E(P_1, \cdots, P_s)^*$ be the canonical projections. We have the commutative diagram:

$$\begin{array}{ccc} E^* & \xrightarrow{\hat{\pi}} & V^*(P) \\ \downarrow p & & \downarrow q_I \\ E(P_1, \cdots, P_s)^* & \xrightarrow{\pi} & \partial V^*(P) \end{array}$$

Let ϕ be the composition $q \circ \hat{\pi}: E^* \to \partial V^*(P)$. By the commutativity of the diagram, $\phi = \pi \circ p$. By the assumption PND1 and PND2, ϕ is a submersion. As $\phi = \pi \circ p$, this implies that $\pi: E(P_1, \cdots, P_s)^* \to \partial V^*(P)$ is a submersion. This completes the proofs of Sublemma (3.4) and Lemma (3.3).

Remark (3.7). Assume that $f(z_I)$ is not identically zero. Then V^{*I} is defined by $f(z_I)=0$. In this case, $f_P(z)=f(z_I)$ and for any P with $I(P)=I$. Thus V^{*I} itself is the unique I-primary boundary component. In this case, V is nonsingular on V^{*I}.

4. Key Lemma.

We first consider the following situation. Let $p(t)=(p_1(t), \cdots, p_n(t))$ be an analytic curve defined in the interval $(-1, 1)$ with the Taylor expansion $p_i(t)= a_i t^{b_i} + (\textit{higher terms})$. We assume that

(i) $f(p(t)) \equiv 0$,

(ii) $a_j \neq 0$ for each $j=1, \cdots, n$ and $b_i=0$ if and only if $i \in I$.

Let $B={}^t(b_1, \cdots, b_n)$, $\boldsymbol{a}=(a_1, \cdots, a_n)$. Let $b_{\min}=\textit{minimum}\ \{b_j\,;\, j \notin I\}$ and $J_{\min}=\{j\,;\, b_j=b_{\min}\}$. Let $q(t)$ be an analytic curve in $V^{*I}(B)$ with $q(0)=p(0)$. We assume that

(iii) $T_{p(t)}V^* \to \tau$ and $[p(t)-q(t)] \to \lambda$.

Then we assert

KEY LEMMA (4.1). *λ is contained in τ.*

Proof. It is well-known that the tangent space $T_z V^*$ is characterized by $df(z)^{\perp}=\{\boldsymbol{v} \in T_z \boldsymbol{C}^n\,;\, df(\boldsymbol{z})(\boldsymbol{v})=0\}$. Let us consider the limit of $df(p(t))$. For a real analytic function $k(t)$, we define an integer $ord(k(t))$ by the order of $k(t)$ at $t=0$. Similarly we define the order of a vector-valued analytic function by the minimum of the order of the coordinate functions. Thus $ord(df(p(t)))$ is the minimum of $ord(\partial f/\partial z_i(p(t)))$ for $i=1, \cdots, n$. Let $m=ord(df(p(t)))$ and let $\vec{\gamma}= df(p(t))/t^m|_{t=0}$. By the PND1-(b)-condition, $m \leqq d(B\,;\, f)$. Let $\vec{\gamma}=\sum_{i=1}^{n} \gamma_i dz_i$. Then we have an obvious equality $\tau=\vec{\gamma}^{\perp}$. Considering the leading term of (i), we obtain $f_B(\boldsymbol{a})=0$.

Case (a). Assume that $f_B(z)$ is not essentially of z_I-variables. Then $V^{*I}(B) = C^{*I}$ by the definition. Then by the PND2-condition, there exists an index j $(j \notin I)$ such that $\partial f_B/\partial z_j(a) \neq 0$ if $\sum_{i\in I} |a_i|^2$ is small enough. Thus we have $m \leqq d(B; f) - b_{\min}$. Assume that $m = d(B; f) - b_{\min}$. Then we must have

$$\frac{\partial f_B}{\partial z_j}(a) = 0 \quad \text{for } j \notin J_{\min} \cup I \quad \text{and} \quad \gamma_j = \frac{\partial f_B}{\partial z_j}(a) \quad \text{for } j \in J_{\min}. \tag{4.2}$$

If $m < d(B; f) - b_{\min}$, we have that

$$\gamma_j = 0 \qquad \text{for} \quad j \in J_{\min} \cup I\,. \tag{4.3}$$

Note that $\gamma_i = 0$ for $i \in I$ in both cases. This implies that $\vec{\gamma} | C^I = 0$.

Now we consider the line $[p(t) - q(t)]$. Let $k = ord(p(t) - q(t))$. As $q(t) \in C^{*I}$, it is easy to see that $1 \leqq k \leqq b_{\min}$. Let $\vec{\lambda} = (p(t) - q(t))/t^k |_{t=0}$. By the definition of λ, we have that $[\vec{\lambda}] = \lambda$. If $k < b_{\min}$, $\vec{\lambda}$ is a vector in C^I. In this case, it is clear that $\vec{\gamma}(\vec{\lambda}) = 0$. Assume that $k = b_{\min}$. Then $\lambda_j = a_j$ if $j \in J_{\min}$ and $\lambda_j = 0$ if $j \notin J_{\min} \cup I$. We consider the equality

$$0 \equiv \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p(t)) \frac{d p_j(t)}{dt}$$

$$\equiv \Big[\sum_{j \notin I} \frac{\partial f_B}{\partial z_j}(a) b_j a_j \Big] t^{d(B;f)-1} + (higher\ terms).$$

Thus we obtain the equality

$$\sum_{j \notin I} \frac{\partial f_B}{\partial z_j}(a) b_j a_j = 0\,. \tag{4.4}$$

If $m < d(B; f) - b_{\min}$, $\vec{\gamma}(\vec{\lambda}) = 0$ is immediate from (4.3). Assume that $m = d(B; f) - b_{\min}$. By (4.2) and (4.4), we can see easily that $\vec{\gamma}(\vec{\lambda}) = 0$. Here $\vec{\lambda}$ is identified with the tangent vector $\sum_{j=1}^{n} \lambda_j \frac{\partial}{\partial z_j}$ at $p(0)$.

Case (b). Assume that $f_B(z)$ is essentially of z_I-variables. Let $f_B(z) = z^L f_B^e(z)$ where z^L is a monomial in the variables $\{z_j;\ j \notin I\}$. Then $V^{*I}(B) = \{f_B^e(z_I) = 0\}$ and $ord(f_B(p(t))) = ord(p(t)^L) = d(B; f)$. We have two equalities:

$$\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}(p(t)) \frac{d p_j(t)}{dt} \equiv 0 \quad \text{and} \quad \sum_{i \in I} \frac{\partial f_B^e}{\partial z_i}(q(t)) \frac{d q_i(t)}{dt} \equiv 0\,. \tag{4.5}$$

Let $\beta = ord(f_B^e(p(t)))$ and $\delta = ord(\hat{f}(p(t)))$. First we assume that PND1-(a)-(ii) holds. As $f(p(t)) = f_B(p(t)) + \hat{f}(p(t)) \equiv 0$, we have

$$\beta + d(B; f) = \delta \geqq d(B; \hat{f}) \tag{4.6}$$

where $\hat{f}_B(z)$ is the secondary face function of f with respect to the weight B. The equality holds if and only if $\hat{f}_B(a) \neq 0$. We consider the equality which follows immediately from (4.5).

$$\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(p(t))\frac{d}{dt}\Big[p_j(t)-q_j(t)\Big]+ \tag{4.7}$$
$$\sum_{i\in I}\Big[\frac{\partial f}{\partial z_i}(p(t))-\frac{\partial f_B}{\partial z_i}(p(t))\Big]\frac{dq_i(t)}{dt}+$$
$$\sum_{i\in I}p(t)^L\Big[\frac{\partial f_B^e}{\partial z_i}(p(t))-\frac{\partial f_B^e}{\partial z_i}(q(t))\Big]\frac{dq_i(t)}{dt}\equiv 0.$$

By the assumption, $p_j(t)\equiv q_j(t)$ *modulo* (t^k) for any j. This implies that $ord\Big[\frac{\partial f_B^e}{\partial z_i}(p(t))-\frac{\partial f_B^e}{\partial z_i}(q(t))\Big]\geqq k$. Thus the order of the last sum is at least $d(B;f)+k$. On the other hand, we have

$$ord\Big(\frac{\partial f}{\partial z_i}(p(t))-\frac{\partial f_B}{\partial z_i}(p(t))\Big)\geqq d(B;\hat{f})\geqq d(B;f)+b_{\min}\quad (i\in I)$$

by PND1-(a)-(ii) where $\hat{f}=f-f_B$. As $k\leqq b_{\min}$, the order of the second sum in (4.7) is also at least $d(B;f)+k$. The order of the first sum in (4.7) is (at least) $m+k-1$. As $m\leqq d(B;f)$ by the PND1-(b)-condition and $k\leqq b_{\min}$, the coefficient of t^{m+k-1} of (4.7) is equal to $\vec{\gamma}(\vec{\lambda})$. Thus we conclude that $\vec{\gamma}(\vec{\lambda})=0$. Assume (a)-(i): $d(B;f)=0$. We consider the following equality instead of (4.7).

$$\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}(p(t))\frac{d}{dt}\Big[p_j(t)-q_j(t)\Big]+$$
$$\sum_{i\in I}\Big[\frac{\partial f}{\partial z_i}(p(t))-\frac{\partial f}{\partial z_i}(q(t))\Big]\frac{dq_i(t)}{dt}\equiv 0.$$

Here we have used the equality $\frac{\partial f}{\partial z_i}(q(t))=\frac{\partial f_B}{\partial z_i}(q(t))$. By the PND1-(b)-condition, $m=0$. Thus by a similar argument, we have $\vec{\gamma}(\vec{\lambda})=0$. Note that $m=d(B;f)$ if the PND1-(a)-condition is satisfied.

Assume that PND1-(a)-(iii) holds. We may assume that $d(B;\hat{f})<d(B;f)+b_{\min}$. We consider (4.7) again. The order of the last sum is at least $d(B;f)+k$. We can write $f_B^e(p(t))=\lambda t^{\theta}+(higher\ terms)$ by (4.6) where $\theta=d(B;\hat{f})-d(B;f)$. Note that $\theta\leqq\beta$. As $f(p(t))\equiv 0$, we have that $\hat{f}_B(\boldsymbol{a})+\lambda\boldsymbol{a}^K=0$. Thus we have

$$\frac{\partial f}{\partial z_j}(p(t))=\eta_j t^{d(B;\hat{f})-b_j}+(higher\ terms)\qquad \text{for } j\notin I$$

where $\eta_j=\frac{\partial\hat{f}_B}{\partial z_j}(\boldsymbol{a})+\lambda k_j\boldsymbol{a}^K/a_j=\Big(a_j\frac{\partial\hat{f}_B}{\partial z_j}(\boldsymbol{a})-k_j\hat{f}_B(\boldsymbol{a})\Big)/a_j$. As $f_B(\boldsymbol{a})=0$, there exists an index $j_0\notin I$ such that $\eta_{j_0}\neq 0$ by the PND1-(a)-(iii) condition. Thus the order of the first term of (4.7) is at most $d(B;\hat{f})-b_{j_0}+k-1$. The order of the second term is at least $d(B;\hat{f})$. As $k<b_{\min}$, we have the inequality: $d(B;\hat{f})-b_{j_0}+k-1<d(B;\hat{f})$. By the assumption that $d(B;\hat{f})<d(B;f)+b_{\min}$, we have

also the inequality: $d(B;\hat{f})-b_{j_0}+k-1<d(B;f)+k$. Therefore we conclude as before that $\vec{\gamma}(\vec{\lambda})=0$. This completes the proof of Lemma (4.1).

5. Proof of Main Theorem.

In this section, we will prove Main Theorem in §1. Let Y and Z be a pair of strata of $\mathcal{S}$ such that $\bar{Y}\cap Z\neq\phi$. We assume that $Y\in\mathcal{S}(J)$ and $Z\in\mathcal{S}(K)$. Then we must have $J\supset K$. If $J=K$, the b-regularity is obvious as V is good. Thus we may assume that $J\neq K$. If Y is an open dense stratum in $\boldsymbol{C}^{*J}$, the b-regularity for (Y, Z) is again obvious. Thus we assume that $\bar{Y}\neq\boldsymbol{C}^J$. Let $p(t)$ and $q(t)$ be real analytic curves defined on $(-1, 1)$ such that (i) $p(0)=q(0)\in Z$. (ii) $p(t)\in Y$ for $t>0$. (iii) $q(t)\in Z$ for $t\geqq 0$. Assume that the tangent space $T_{p(t)}Y$ converges to τ and the line $[p(t)-q(t)]$ converges to λ. Y is a non-degenerate hypersurface defined by $f_P^c(z_J)=0$ for some P with $I(P)=J$. Assume that $p_j(t)=a_jt^{b_j}+(\textit{higher terms})$ for $j\in J$. For brevity's sake, we assume that $J=\{1, \cdots, m\}$. Let $B={}^t(b_1, \cdots, b_m)$ and $\boldsymbol{a}=(a_1, \cdots, a_m)$. As $p(0)=q(0)=\boldsymbol{a}_I\in Z$, $K=I(P)$. By looking at the leading terms of the equality $h(p(t))\equiv 0$, we can see that $\boldsymbol{a}_K$ belongs to the K-primary component $Y^{*K}(B)$. Let $R=P+rQ$ for a sufficiently small $r>0$. Then it is an easy linear algebra to see the following.

(i) $(f_P)_B=f_R$. (ii) The secondary face function $\hat{f}_R$ of f with respect to R is equal to the secondary face function of f_P with respect to B.

Thus the PND-condition for f implies the PND-condition for f_P. Now we use Lemma (4.1) to obtain the regularity for the pair (Y, Z). This completes the proof of Main Theorem.

Example (5.1). Let $f(\boldsymbol{z})=(z_1z_2)^2(z_3^5+z_4^5)+(z_3z_4)^2(z_1^5+z_2^5)$. Then the singular locus of V is the union of the two dimensional coordinate planes $\boldsymbol{C}^I$ for $|I|=2$. Let $I=\{1, 2\}$. Then by an easy calculation, we have a proper primary boundary components defined by $C: z_1^5+z_2^5=0$. C consists of five lines, say $C_1, \cdots, C_5$. Thus $\mathcal{S}(I)=\{\boldsymbol{C}^{*I}-C, C_1, \cdots, C_5\}$. The same is true for $I=\{3, 4\}$. Thus the stratification of V consists of the following strata: V^*, $\boldsymbol{C}^{*I}$ $(I\neq\{1, 2\}, \{3, 4\})$, $\boldsymbol{C}^{*\{1,2\}}-C$, $\boldsymbol{C}^{*\{3,4\}}-D$, C_i, D_i $(i=1, \cdots, 5)$, $\boldsymbol{C}^{\{j\}}$ $(j=1, \cdots, 4)$, $\{0\}$ where $D=\bigcup_{i=1}^{5} D_i=\{z_3^5+z_4^5=0\}$.

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