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EXTREMAL FUNCTIONS FOR BLOCH CONSTANTS

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1. Introduction.

Let $f(z)$ be an analytic function in the unit disc $\Delta=\{|z|<1\}$ with $f'(0)\neq 0$. We denote by $r_f(z)$ the radius of the largest schlicht disc with center at $f(z)$ in the Riemann image surface of $f:\Delta\to\mathbf{C}$ (the complex plane). We also denote by $\tilde{r}_f(z)$ the radius of the largest disc which is contained in $f(\Delta)\subset\mathbf{C}$. Define r_f and $\tilde{r}_f$ by

$$r_f=\sup_{z\in\Delta} r_f(z),$$

$$\tilde{r}_f=\sup_{z\in\Delta} \tilde{r}_f(z),$$

respectively. Let $\mathcal{A}$ be a family of all analytic functions f in Δ with $f'(0)\neq 0$ and $\mathcal{A}_0$ be a family of all analytic functions in Δ with $f'(z)\neq 0$, $z\in\Delta$. Let $\mathcal{S}$ be a family of all univalent analytic functions in Δ. Then the Bloch, Landau, locally univalent Bloch and univalent Bloch constants are defined respectively by

$$B=\inf_{f\in\mathcal{A}}\frac{r_f}{|f'(0)|},$$

$$\mathcal{L}=\inf_{f\in\mathcal{A}}\frac{\tilde{r}_f}{|f'(0)|},$$

$$B_0=\inf_{f\in\mathcal{A}_0}\frac{r_f}{|f'(0)|},$$

$$B_1=\inf_{f\in\mathcal{S}}\frac{r_f}{|f'(0)|}.$$

For terminologies our basic references are [3] and [4].

The purpose of the present article is to show the following;

THEOREM 1. *Let $f(z)$ be one of the extremal function for the Bloch, Landau, locally univalent Bloch and univalent Bloch constants. Then*

$$\left|\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|z|<1}\frac{\overline{f'(z)}}{z^2 f'(z)}dx\,dy\right|\leqq 1$$

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holds.

2. Proof of the Theorem.

Let $f(z)$ be one of the extremal function for the Bloch, Landau, locally univalent Bloch and univalent Bloch constants. Let t be a sfficiently small complex parameter and define an affine transformation $\phi_t(z)$ by

$$\phi_t(z)=z+bt\bar{z}. \tag{2.1}$$

We shall deform the image $f(\Delta)$ by ϕ_t as follows. Define $\mu_t \in L^\infty(\Delta)$ by

$$\mu_t(z)=\frac{\partial(\phi_t \circ f)}{\partial \bar{z}}(z) \Big/ \frac{\partial(\phi_t \circ f)}{\partial z}(z). \tag{2.2}$$

Then there exists a unique quasi-conformal automorphism g_t of $\bar{\Delta}$ with fixed points 0 and 1 satisfying $\partial g_t/\partial\bar{z}=\mu_t \partial g_t/\partial z$ almost everywhere. Set $f_t=\phi_t \circ f \circ g_t^{-1}$. An easy calculation shows that the Beltrami coefficient of f_t vanishes almost everywhere. Hence f_t is analytic in Δ. It is clear that $f_t(\Delta)=\phi_t(f(\Delta))$ and that $f_t(z)\to f(z)$ locally uniformly in Δ, as $t\to 0$.

LEMMA 1. *The asymptotic expansion of the $f_t'(0)^{-1}$ is given by;*

$$\frac{1}{|f_t'(0)|^2}=\frac{1}{|f'(0)|^2}\left[1-2\Re\left\{\frac{bt}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|z|<1}\frac{\overline{f'(z)}}{z^2 f'(z)}dx\,dy\right\}\right] +O(t^2), \tag{2.3}$$

as $t\to 0$.

For the proof see [7, Section 3 and 6]. Define r_t by

$$r_t=\begin{cases}\tilde{r}_{f_t}, & \text{if } f \text{ is extremal for the Landau constant,}\\ r_{f_t}, & \text{otherwise.}\end{cases}$$

Since the maximum modulus of the eigen values of ϕ_t is

$$\left|\frac{\partial\phi_t}{\partial z}(z)\right|+\left|\frac{\partial\phi_t}{\partial\bar{z}}(z)\right|=1+|bt|,$$

it is clear that

$$r_t\leqq(1+|bt|)r_0. \tag{2.4}$$

By the extremality of f, the inequality

$$\left\{\frac{r_0}{|f'(0)|}\right\}^2\leqq\left\{\frac{r_t}{|f_t'(0)|}\right\}^2 \leqq\left\{\frac{r_0}{|f'(0)|}\right\}^2\left[1+2\Re\left\{|bt|-\frac{bt}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|z|<1}\frac{\overline{f'(z)}}{z^2 f'(z)}dx\,dy\right\}\right] \tag{2.5}$$

$+O(t^2)$,

holds for sufficiently small $|t|$. Thus we have

$$\Re\left\{|bt|-\frac{bt}{\pi}\lim_{\varepsilon\to 0}\int_{\varepsilon<|z|<1}\frac{\overline{f'(z)}}{z^2 f'(z)}dxdy\right\}+O(t^2)\geqq 0, \tag{2.6}$$

as $t\to 0$. Since we can choose a complex constant b arbitrarily, we have the desired result.

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