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SELF-DUAL AND ANTI-SELF-DUAL HERMITIAN SURFACES

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1. Introduction.

Let (M, g) be a 4-dimensional oriented Riemannian manifold. The star operator $*$ defined on the space of 2-forms $\Lambda^2 M$ satisfies $*\circ *=id$. So $\Lambda^2 M$ splits into two eigenspaces as $\Lambda^2 M=\Lambda^2_+ M \oplus \Lambda^2_- M$, where $\Lambda^2_+ M$ and $\Lambda^2_- M$ are the eigenspaces corresponding to eigenvalues $+1$ and -1, respectively. Let W be Weyl's conformal curvature tensor of g. For each point $p \in M$, we may regard W_p as a symmetric linear endomorphism of $\Lambda^2_p M$. And let W_+ (resp. W_-) be the restriction of W to $\Lambda^2_+ M$ (resp. $\Lambda^2_- M$). A 4-dimensional oriented Riemannian manifold (M, g) is called *self-dual* (resp. *anti-self-dual*) if $W_-=0$ (resp. $W_+=0$). B. Y. Chen proposed the following problem:

Problem. *Classify all self-dual and anti-self-dual Hermitian surfaces.*

B. Y. Chen ([2]) classified compact self-dual Kähler surfaces, thereafter J. P. Bourguignon ([1]) and A. Derdzinski ([3]) reproved it independently by different methods. On one hand, M. Itoh ([5]) gave a classification of compact anti-self-dual Kähler surfaces. Hence in the case of Kähler surfaces, the above problem is completely solved. So it will be in turn a problem to classify self-dual, anti-self-dual Hermitian surfaces. In the present paper, we shall prove the followings

THEOREM A. *Let (M, J, g) be a 4-dimensional almost Hermitian manifold. If it is self-dual and Einstein, then it is of pointwise constant holomorphic sectional curvature.*

THEOREM B. *A compact Hermitian surface M is anti-self-dual if and only if M is a locally conformal Kähler manifold with $\tau=3\tau^*$, where τ and τ^* denote the scalar curvature and the $*$-scalar curvature of M respectively.*

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2. Preliminaries.

Let $M=(M, J, g)$ be a 4-dimensional almost Hermitian manifold. We assume that M is oriented by the volume form $\frac{1}{2}\Omega^2$, where Ω is the Kähler form defined by $\Omega(X, Y)=g(X, JY)$ for $X, Y\in\mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M). We denote by ∇, R, ρ, τ and W the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature and the Weyl's conformal curvature tensor of M respectively. The Riemannian curvature tensor R and the Weyl's conformal curvature tensor W are defined respectively by

$$R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]},$$

$$W(X, Y)=R(X, Y)-\frac{1}{2}\{AX\wedge Y+X\wedge AY\}+\frac{\tau}{6}X\wedge Y,$$

where A denotes a field of symmetric endomorphism which corresponds to the Ricci tensor ρ, and $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Y, Z)X-g(X, Z)Y$, for $X, Y, Z\in\mathfrak{X}(M)$. Furthermore, we denote by ρ^* and τ^* the Ricci $*$-tensor and the $*$-scalar curvature of M respectively (cf. [11] p. 367).

Let $\{e_i\}=\{e_1, e_2=Je_1, e_3, e_4=Je_3\}$ be a positively oriented orthonormal basis of the tangent space T_pM at a point $p\in M$, and $\{e^i\}$ the dual basis. We denote by T_p^CM the complexification of the tangent space T_pM ($p\in M$). We put

$$f_1=(e_1-\sqrt{-1}e_2)/\sqrt{2} \qquad f_{\bar{1}}=(e_1+\sqrt{-1}e_2)/\sqrt{2}\,, \tag{2.1}$$
$$f_2=(e_3-\sqrt{-1}e_4)/\sqrt{2} \qquad f_{\bar{2}}=(e_3+\sqrt{-1}e_4)/\sqrt{2}\,.$$

Then $\{f_1, f_2\}$ becomes a unitary basis of T_p^CM, and its dual basis $\{f^A\}$ is given by

$$f^1=(e^1+\sqrt{-1}e^2)/\sqrt{2} \qquad f^{\bar{1}}=(e^1-\sqrt{-1}e^2)/\sqrt{2}\,, \tag{2.2}$$
$$f^2=(e^3+\sqrt{-1}e^4)/\sqrt{2} \qquad f^{\bar{2}}=(e^3-\sqrt{-1}e^4)/\sqrt{2}\,.$$

In the sequel, we shall adopt the following notational conventions

$$R_{ijkl}=g(R(e_i, e_j)e_k, e_l)\,, \tag{2.3}$$

$$K_{ABCD}=g(R(f_A, f_B)f_C, f_D)\,,$$

$$R_{ij}=\rho(e_i, e_j)\,,$$

$$K_{AB}=\rho(f_A, f_B)\,,$$

$$W_{ijkl}=g(W(e_i, e_j)e_k, e_l)$$
$$=R_{ijkl}-\frac{1}{2}\{g_{il}R_{jk}-R_{ik}g_{jl}+R_{il}g_{jk}-g_{ik}R_{jl}\}$$
$$+\frac{\tau}{6}\{g_{il}g_{jk}-g_{ik}g_{jl}\},$$
$$W_{ABCD}=g(W(f_A, f_B)f_C, f_D),$$

where $i, j, k, l\in\{1, 2, 3, 4\}$ and $A, B, C, D\in\{1, 2, \bar{1}, \bar{2}\}$.

The Weyl's conformal curvature operator (also denoted by W) is the symmetric endomorphism of the vector bundle Λ^2M defined by

$$g(W(\iota(x)\wedge\iota(y)), \iota(z)\wedge\iota(w))=-g(W(x, y)z, w)$$

for $x, y, z, w\in T_pM$, $p\in M$, where ι denotes the duality $TM\to T^*M$ (the cotangent bundle of M) defined by means of the metric g.

3. Self-dual and anti-self-dual Kähler surfaces.

Since the Weyl's conformal curvature tensor W is invariant under any conformal change of the Riemannian metric, the notion of self-duality (resp. anti-self-duality) is conformal invariant. On one hand, if (M, J, g) is a Hermitian surface, then (M, J, fg) is also a Hermitian surface, for any positive-valued smooth function f on M. However, this is not valid for Kähler surfaces. So, the self-duality (resp. anti-self-duality) gives a strong restriction for Kähler surfaces.

We shall recall some results about self-dual, anti-self-dual Kähler surfaces ([1], [2], [3], [5]).

THEOREM 3.1 ([5]). *Let (M, J, g) be a Kähler surface. If it is self-dual with respect to the canonical orientation and it is Einstein, then it is of constant holomorphic sectional curvature.*

THEOREM 3.2 ([5]). *Let (M, J, g) be a Kähler surface. Then it is anti-self-dual if and only if its scalar curvature vanishes everywhere.*

4. Curvature conditions.

First, we shall write the curvature conditions for a 4-dimensional almost Hermitian manifold to be self-dual. Let M be a 4-dimensional almost Hermitian manifold, p any point of M, $\{e_1, e_2=Je_1, e_3, e_4=Je_3\}$ any positively oriented orthonormal basis of T_pM and $\{e^i\}$ the dual basis. We take $\{f_A\}$ and $\{f^A\}$ as in (2.1) and (2.2) respectively. Then we see easily that $\{e^1\wedge e^2-e^3\wedge e^4=\sqrt{-1}(f^1\wedge f^{\bar{1}}-f^2\wedge f^{\bar{2}})$, $e^1\wedge e^3-e^4\wedge e^2=f^1\wedge f^2+f^{\bar{1}}\wedge f^{\bar{2}}$, $e^1\wedge e^4-e^2\wedge e^3=\sqrt{-1}(f^1\wedge f^2-f^{\bar{1}}\wedge f^{\bar{2}})\}$ forms a basis of Λ^2_-M at $p\in M$. Thus, by the definition, M is self-dual if and only if

$$W_{1\bar{1}AB}-W_{2\bar{2}AB}=0, \quad W_{1\bar{2}AB}=W_{\bar{1}2AB}=0 \tag{4.1}$$

for any A and B in $\{1, 2, \bar{1}, \bar{2}\}$. Hence we have

PROPOSITION 4.1. *A 4-dimensional almost Hermitian manifold is self-dual with respect to the canonical orientation if and only if*

$$12K_{1\bar{2}2\bar{1}}=\tau, \quad K_{1\bar{1}1\bar{2}}-K_{2\bar{2}1\bar{2}}=0, \quad K_{1\bar{2}1\bar{2}}=0 \tag{4.2}$$

for any basis $\{f_A\}$ of $T_p^C M$ of the form (2.1) *at each point $p\in M$.*

Proof. We may see that $W_{1\bar{1}AB}-W_{2\bar{2}AB}=0$ for any A and B in $\{1, 2, \bar{1}, \bar{2}\}$ if and only if $12K_{1\bar{2}2\bar{1}}=\tau$, $K_{1\bar{1}1\bar{2}}-K_{2\bar{2}1\bar{2}}=0$, and also that $W_{1\bar{2}AB}=0$ for any A and B in $\{1, 2, \bar{1}, \bar{2}\}$ if and only if $K_{1\bar{2}1\bar{2}}=0$, $K_{1\bar{1}1\bar{2}}-K_{2\bar{2}1\bar{2}}=0$, $12K_{1\bar{2}2\bar{1}}=\tau$. Q. E. D.

Next, we shall consider a 4-dimensional anti-self-dual almost Hermitian manifold (M, J, g). We see easily that $\{e^1\wedge e^2+e^3\wedge e^4=\sqrt{-1}(f^1\wedge f^{\bar{1}}+f^2\wedge f^{\bar{2}}),\ e^1\wedge e^3+e^4\wedge e^2 =f^1\wedge f^2+f^{\bar{1}}\wedge f^{\bar{2}},\ e^1\wedge e^4+e^2\wedge e^3=-\sqrt{-1}(f^1\wedge f^2-f^{\bar{1}}\wedge f^{\bar{2}})\}$ forms a basis of $\Lambda_+^2 M$ at $p\in M$. Thus, by the definition, we see that M is anti-self-dual if and only if

$$W_{1\bar{1}AB}+W_{2\bar{2}AB}=0, \quad W_{12AB}=W_{\bar{1}\bar{2}AB}=0 \tag{4.3}$$

for any A and B in $\{1, 2, \bar{1}, \bar{2}\}$.

In contrast with Proposition 4.1, we have easily

PROPOSITION 4.2. *A 4-dimensional almost Hermitian manifold (M, J, g) is anti-self-dual with respect to the canonical orientation if and only if*

$$\tau=3\tau^*, \quad K_{1\bar{1}12}+K_{2\bar{2}12}=0, \quad K_{1212}=0 \tag{4.4}$$

for any basis $\{f_A\}$ of $T_p^C M$ of the form (2.1) *at each point $p\in M$.*

Proof. The proof is similar to the one of Proposition 4.1. But we will use the followings

$$\tau=2(K_{1\bar{1}}+K_{2\bar{2}})=2(K_{1\bar{1}1\bar{1}}+K_{2\bar{2}2\bar{2}}+2K_{21\bar{1}\bar{2}}+2K_{1\bar{2}2\bar{1}}),$$

$$\tau^*=2(K_{1\bar{1}1\bar{1}}+K_{2\bar{2}2\bar{2}}+2K_{12\bar{1}\bar{2}}+2K_{1\bar{2}2\bar{1}})$$

and $\tau^*-\tau=8K_{12\bar{1}\bar{2}}$. Q. E. D.

5. Self-dual almost Hermitian manifolds.

In this section, we shall prove Theorem A. First, we prepare the following result by S. Tanno ([9]).

PROPOSITION. *An almost Hermitian manifold (M^m, J, g) is of constant holomorphic sectional curvature at $p\in M$, if and only if*

$$
\begin{aligned}
&R(e_i, Je_j, Je_k, e_l)+R(e_i, Je_k, Je_j, e_l)+R(e_i, Je_j, Je_l, e_k) \\
&\quad +R(e_i, Je_l, Je_j, e_k)+R(e_i, Je_k, Je_l, e_j)+R(e_i, Je_l, Je_k, e_j) \\
&\quad +R(e_j, Je_i, Je_k, e_l)+R(e_j, Je_k, Je_i, e_l)+R(e_j, Je_i, Je_l, e_k) \\
&\quad +R(e_j, Je_l, Je_i, e_k)+R(e_k, Je_j, Je_i, e_l)+R(e_k, Je_i, Je_j, e_l) \\
&=4H(g_{jk}g_{il}+g_{kl}g_{ij}+g_{jl}g_{ik})
\end{aligned}
$$

for any basis $\{e_i\}_{i=1}^m$ *of* T_pM, *where* $R(x, y, z, w)=g(R(x, y)z, w)$ *for* $x, y, z, w\in T_pM$.

By (2.1), (2.3) and the above proposition, we have immediately the following

LEMMA 4.3. *A 4-dimensional almost Hermitian manifold* (M, J, g) *is of constant holomorphic sectional curvature* H *at a point* p *in* M, *if and only if*

$$
\begin{aligned}
&K_{1\bar{1}1\bar{1}}=K_{2\bar{2}2\bar{2}}=H, \quad K_{1\bar{1}1\bar{2}}=K_{2\bar{2}1\bar{2}}=0, \\
&K_{1\bar{1}2\bar{2}}+K_{1\bar{2}2\bar{1}}=H, \quad K_{1\bar{2}1\bar{2}}=0
\end{aligned} \tag{5.1}
$$

for any basis $\{f_A\}$ *of* T_p^CM *of the form* (2.1).

We are now in a position to prove Theorem A.

We suppose that $M=(M, J, g)$ is a 4-dimensional self-dual, Einstein almost Hermitian manifold. From $\rho=\dfrac{\tau}{4}g$, by the straightforward calculation, we get

$$
K_{1\bar{1}1\bar{1}}=K_{2\bar{2}2\bar{2}}, \quad K_{211\bar{2}}=K_{122\bar{1}}=0, \quad K_{1\bar{1}1\bar{2}}+K_{2\bar{2}1\bar{2}}=0 \tag{5.2}
$$

for any basis $\{f_A\}$ of T_p^CM of the form (2.1). By the second equation of (4.2) and the third equation of (5.2), we get the second equation of (5.1). Next, we take any basis $\{f'_A\}$ of T_p^CM of the form (2.1). Then we may express $f'_1=af_1+bf_2$, $f'_2=cf_1+df_2$ for some (a, b, c, d) such that $|a|^2+|b|^2=|c|^2+|d|^2=1$, $a\bar{c}+b\bar{d}=0$. By taking account of (4.2) and (5.2), we see easily $g(R(f'_1, f'_{\bar{1}})f'_1, f'_{\bar{1}})=g(R(f_1, f_{\bar{1}})f_1, f_{\bar{1}})$. Hence we get the first equation of (5.1). Since $\tau=2(K_{1\bar{1}1\bar{1}}+K_{2\bar{2}2\bar{2}}+2K_{21\bar{1}\bar{2}}+2K_{1\bar{2}2\bar{1}})$, by the first equation of (4.2), we get the third equation of (5.1). The last equation of (5.1) is nothing but the last equation of (4.2). This completes the proof.

Theorem A is a generalization of Theorem 3.1.

6. Anti-self-dual Hermitian surfaces.

In this section, we shall prove Theorem B. Let $M=(M, J, g)$ be a Hermitian surface. We shall make use of the same notational conventions as in § 5. First, we prepare the following result by A. Gray ([4]).

PROPOSITION. *Let M be a Hermitian manifold. Then we have*

$$R(W, X, Y, Z)+R(JW, JX, JY, JZ)-R(JW, JX, Y, Z)$$
$$-R(JW, X, JY, Z)-R(JW, X, Y, JZ)-R(W, JX, JY, Z)$$
$$-R(W, JX, Y, JZ)-R(W, X, JY, JZ)$$
$$=0 \tag{6.1}$$

for any $W, X, Y, Z\in\mathfrak{X}(M)$, where $R(W, X, Y, Z)=g(R(W, X)Y, Z)$.

LEMMA 6.1. *Let (M, J, g) be a Hermitian surface. Then we have*

$$K_{1212}=0.$$

Proof. Putting $W=Y=e_1$ and $X=Z=e_3$, we get $\mathrm{Re}(K_{1212})=0$ ($\mathrm{Re}(K_{1212})$ denotes the real part of K_{1212}). Similarly, putting $W=Y=e_1$, $X=e_3$ and $Z=e_4$, we get $\mathrm{Im}(K_{1212})=0$ ($\mathrm{Im}(K_{1212})$ denotes the imaginary part of K_{1212}). Thus finally $K_{1212}=0$. Q. E. D.

By Proposition 4.2 and Lemma 6.1, we have immediately

PROPOSITION 6.2. *A Hermitian surface $M=(M, J, g)$ is anti-self-dual if and only if*

$$\tau=3\tau^*, \quad K_{1\bar{1}12}+K_{2\bar{2}12}=0 \tag{6.2}$$

for any basis $\{f_A\}$ of T_p^CM of the form (2.1) *at each point $p\in M$.*

It is well known that the Kähler form Ω is integrable in the following sense ([10]),

$$d\Omega=\omega\wedge\Omega \quad \text{with } \omega=\delta\Omega\circ J. \tag{6.3}$$

The 1-form ω appeared in (6.3) is called the *Lee form* of (J, g). The Lee form ω satisfies

$$\sum_{i=1}^{4}(\nabla_{e_i}\omega)(Je_i)=0 \tag{6.4}$$

where $\{e_i\}$ is any orthonormal basis of T_pM at each $p\in M$.

Next, we shall consider the second condition of (6.2). Taking account of the formula by K. Sekigawa ([8]), we have the following

PROPOSITION. *Let (M, J, g) be a Hermitian surface. Then we have*

$$2\{g(R(W, X)JY, Z)+g(R(W, X)Y, JZ)\}$$
$$=g(X, Z)\Big\{(\nabla_W\omega)(JY)+\frac{1}{2}\omega(JY)\omega(W)-\frac{1}{2}\Omega(W, Y)\|\omega\|^2\Big\} \tag{6.5}$$

$$-g(W, Z)\Big\{(\nabla_X\omega)(JY)+\frac{1}{2}\omega(JY)\omega(X)-\frac{1}{2}\Omega(X, Y)\|\omega\|^2\Big\}$$
$$-g(X, Y)\Big\{(\nabla_W\omega)(JZ)+\frac{1}{2}\omega(JZ)\omega(W)-\frac{1}{2}\Omega(W, Z)\|\omega\|^2\Big\}$$
$$+g(W, Y)\Big\{(\nabla_X\omega)(JZ)+\frac{1}{2}\omega(JZ)\omega(X)-\frac{1}{2}\Omega(X, Z)\|\omega\|^2\Big\}$$
$$+\Omega(X, Z)\Big\{(\nabla_W\omega)(Y)+\frac{1}{2}\omega(W)\omega(Y)\Big\}$$
$$-\Omega(W, Z)\Big\{(\nabla_X\omega)(Y)+\frac{1}{2}\omega(X)\omega(Y)\Big\}$$
$$-\Omega(X, Y)\Big\{(\nabla_W\omega)(Z)+\frac{1}{2}\omega(W)\omega(Z)\Big\}$$
$$+\Omega(W, Y)\Big\{(\nabla_X\omega)(Z)+\frac{1}{2}\omega(X)\omega(Z)\Big\}$$

for $W, X, Y, Z\in\mathfrak{X}(M)$.

LEMMA 6.3. $K_{1\bar{1}12}+K_{2\bar{2}12}=0$ *if and only if* $d\omega$ *is an anti-self-dual* 2-*form.*

Proof. Taking account of (2.2) and (2.3), we see easily that $K_{1\bar{1}12}+K_{2\bar{2}12}=0$ if and only if

$$(6.6)\qquad \begin{aligned} &R_{1214}+R_{1223}+R_{3414}+R_{3423}=0,\\ &R_{1213}-R_{1224}+R_{3413}-R_{3424}=0. \end{aligned}$$

But, from the above proposition, we may see that (6.6) holds if and only if

$$(6.7)\qquad \begin{aligned} &(\nabla_{e_1}\omega)(e_4)-(\nabla_{e_4}\omega)(e_1)+(\nabla_{e_2}\omega)(e_3)-(\nabla_{e_3}\omega)(e_2)=0,\\ &(\nabla_{e_1}\omega)(e_3)-(\nabla_{e_3}\omega)(e_1)+(\nabla_{e_4}\omega)(e_2)-(\nabla_{e_2}\omega)(e_4)=0. \end{aligned}$$

By (6.4) and (6.7), we may easily show that the 2-form $d\omega$ is anti-self-dual.

Q. E. D.

From Proposition 6.2 and Lemma 6.3, we have

PROPOSITION 6.4. *A Hermitian surface* $M=(M, J, g)$ *is anti-self-dual if and only if*

$$(6.8)\qquad \tau=3\tau^*,\quad d\omega \text{ is an anti-self-dual 2-form.}$$

If the Lee form ω of (J, g) is closed (i.e. $d\omega=0$), then a 4-dimensional Hermitian manifold (M, J, g) is a *locally conformal Kähler* manifold. We are now in a crucial position to prove Theorem B.

We assume that M is an anti-self-dual compact Hermitian surface. Then, from (6.7), we have

$$(d\omega, d\omega)=\int_M d\omega\wedge *(d\omega)=\int_M d\omega\wedge(-d\omega)=-\int_M d(\omega\wedge d\omega)=0.$$

So, $(d\omega, d\omega)=0$. Hence $d\omega=0$.

Conversely, if M is a locally conformal Kähler manifold with $\tau=3\tau^*$, then $d\omega$ is anti-self-dual 2-form. So, from Proposition 6.4, M is anti-self-dual. This completes the proof of Theorem B.

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