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THE INVARIANT PSEUDO-METRIC RELATED TO NEGATIVE PLURISUBHARMONIC FUNCTIONS

Dedicated to Professor Tadashi Kuroda on his 60th birthday

BY KAZUO AZUKAWA

Introduction.

In [1], using a family of negative plurisubharmonic functions on a complex manifold M, the author defined an invariant pseudo-metric P^M on M whose indicatrices are always pseudoconvex domains in the holomorphic tangent spaces. On the other hand, Klimek [5] defined an extremal plurisubharmonic function g_p^M with pole at a given point p of M.

The aim of the present note is to clarify the relationship between P^M and g_p^M (Proposition 2.4), and to simplify the original construction of P^M in [1] (Lemma 2.1, Corollary 2.5). We also show that P^M is a higher-dimensional generalization of the pseudo-metric $c_\beta^z|dz|$ induced from the capacity $c_\beta^z=\exp(-k_\beta^z)$ on an open Riemann surface M (cf. [11]), where $k_\beta^z(p)$ is the Robin constant at a point p of M with respect to a local coordinate z around p (Proposition 3.1). Finally, we derive some results related to the pseudo-metric P^M for Riemann surfaces M.

§ 1. Klimek's extremal plulisubharmonic functions.

Let p be a point of a complex manifold M. We denote by $PS^M(p)$ the family of all $[-\infty, 0)$-valued plurisubharmonic functions f on M such that the function $f-\log\|z\|$ is bounded from above in a deleted neighborhood of p for some holomorphic local coordinate z with $z(p)=0$. We note that every $f\in PS^M(p)$ takes the value $-\infty$ at p, and that $PS^M(p)$ always contains the constant function $-\infty$. The definition of the family $PS^M(p)$ does not depend on the choice of the coordinate z with $z(p)=0$. According to Klimek [5], we define the extremal function g_p^M on M by

$$g_p^M(q)=\sup\{f(q)\,;\ f\in PS^M(p)\}$$

for $q\in M$.

We quote from [5] some results on g_p^M. In [5], Klimek dealt with the case when M is a domain in C^m. However, one can see that these assertions hold also for prescribed manifolds M.

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LEMMA 1.1 ([5; Theorem 1.1]). (Decreasing property) *If $\Phi: M \to M'$ is a holomorphic mapping between complex manifolds M and M', then $g^{M'}_{\Phi(p)} \circ \Phi \leqq g^M_p$ on M for any $p \in M$.*

LEMMA 1.2 ([5; Corollary 1.3]). *For every point p of a complex manifold M, the function g^M_p belongs to $PS^M(p)$.*

LEMMA 1.3 ([5; Theorem 1.5]). *If $g = g^M_p \in L^\infty_{loc}(M - \{p\})$, then g satisfies the homogeneous Monge-Ampère equation $(dd^c g)^m = 0$ in $M - \{p\}$, where $d^c = i(\bar\partial - \partial)$ and m is the dimension of a complex manifold M* (cf. [2]).

In one complex variable, the Monge-Ampère equation is reduced to the Laplace equation, so that when M is one-dimensional the conclusion of Lemma 1.3 means that g^M_p is harmonic in $M - \{p\}$.

LEMMA 1.4 ([5; Proposition 1.6]). *If M is a pseudoconvex, relatively compact domain of a Stein manifold with C^1-boundary, then $g^M_p \in L^\infty_{loc}(M - \{p\})$, and $g^M_p(q) \to 0$ if q approaches any boundary point of M* (cf. [4]).

We note that every open Riemann surface is a Stein manifold, and that every domain of an open Riemann surface is pseudoconvex.

§ 2. Invariant pseudo-metrics.

For a holomorphic tangent vector $X \in T_p(M)$ at a point p of a complex manifold M, we denote by LHC(X) the totality of local holomorphic curves contacting with X at p, that is, $\varphi \in$ LHC(X) if and only if φ is a holomorphic mapping from $\varepsilon U = \{\lambda \in \boldsymbol{C}; |\lambda| < \varepsilon\}$ with some $\varepsilon > 0$ into M satisfying $\varphi(0) = p$ and $\varphi_*(d/d\lambda)_0 = X$. For $f \in PS^M(p)$ and $\varphi \in$ LHC(X), we set

$$L_f[\varphi] = \limsup_{\lambda \to 0,\, \lambda \neq 0} (\exp f \circ \varphi)(\lambda)/|\lambda|.$$

We shall show the following key lemma for the argument in this note, which was proved in [1; Remark 3.1] in the case when M is one-dimensional.

LEMMA 2.1. *If $f \in PS^M(p)$ and $\varphi_i \in$ LHC(X) $(i = 0, 1)$ with $X \in T_p(M)$, then $L_f[\varphi_0] = L_f[\varphi_1]$.*

Proof. Take a holomorphic chart (z, U_z) with $z(p) = 0$. We may assume that φ_i are defined in εU with $\varphi_i(\varepsilon U) \subset U_z$. Since $\varphi_i(0) = p$, the open subset

$$D = \{(\lambda, \xi) \in \varepsilon U \times \boldsymbol{C};\ (1-\xi) z \circ \varphi_0(\lambda) + \xi z \circ \varphi_1(\lambda) \in z(U_z)\}$$

of $\varepsilon U \times \boldsymbol{C}$ includes the line $\{0\} \times \boldsymbol{C}$. For every $\xi \in \boldsymbol{C}$, the mapping

$$\tilde\varphi_\xi(\lambda) = z^{-1}((1-\xi) z \circ \varphi_0(\lambda) + \xi z \circ \varphi_1(\lambda))$$

defined for $\lambda\in\varepsilon U$ with $(\lambda, \xi)\in D$ belongs to LHC(X) and satisfies $\tilde{\varphi}_0=\varphi_0$, $\tilde{\varphi}_1=\varphi_1$. We consider the function

$$g(\lambda, \xi)=f\circ\tilde{\varphi}_\xi(\lambda)-\log|\lambda|, \qquad (\lambda, \xi)\in D-(\{0\}\times \boldsymbol{C}),$$

which is plurisubharmonic on $D-(\{0\}\times\boldsymbol{C})$. Since $f\in PS^M(p)$, there exists a positive number η such that $(\exp f\circ z^{-1})(v)\leqq\eta\|v\|$ for all sufficiently small $v\in\boldsymbol{C}^m$, where m is the dimension of M. For every $\xi\in\boldsymbol{C}$, using $f\circ\tilde{\varphi}_\xi=(f\circ z^{-1})\circ(z\circ\tilde{\varphi}_\xi)$, we see that

$$(\exp f\circ\tilde{\varphi}_\xi)(\lambda)/|\lambda|\leqq\eta\|z\circ\tilde{\varphi}_\xi(\lambda)/\lambda\|$$

for all sufficiently small $\lambda\in\boldsymbol{C}-\{0\}$. If we take a $u=(u^1, \cdots, u^m)\in\boldsymbol{C}^m$ with

$$X=(\partial_u^z)_p:=\textstyle\sum_{\nu=1}^m u^\nu(\partial/\partial z^\nu)_p, \tag{2.1}$$

where $z=(z^1, \cdots, z^m)$, it then follows that

$$\limsup_{(\lambda', \xi')\to(0,\xi),\, \lambda'\neq 0} g(\lambda', \xi')\leqq\log(\eta\|u\|) \tag{2.2}$$

for any $\xi\in\boldsymbol{C}$. Therefore, g is uniquely extended to a plurisubharmonic function $\tilde{g}$ on D. Furthermore, the value $\tilde{g}(0, \xi)$ coincides with the left hand side of (2.2). Using the fact that the restriction of $\tilde{g}$ over the intersection of a complex line with D is a subharmonic function there, we get the desired assertion as follows: First, for every $\xi\in\boldsymbol{C}$, the function $\tilde{g}(\cdot, \xi)$ is subharmonic in a neighborhood of 0 in $\boldsymbol{C}$. From this we have

$$\begin{aligned}\tilde{g}(0, \xi)&=\limsup_{\lambda\to 0,\, \lambda\neq 0}\tilde{g}(\lambda, \xi)\\ &=\limsup_{\lambda\to 0,\, \lambda\neq 0} g(\lambda, \xi)\\ &=\log L_f[\tilde{\varphi}_\xi].\end{aligned} \tag{2.3}$$

Secondly, the function $\tilde{g}(0, \cdot)$ is subharmonic on $\boldsymbol{C}$. Furthermore, it follows from (2.2) that $\tilde{g}(0, \cdot)$ is bounded from above on $\boldsymbol{C}$, so that it must be constant. Combining this with (2.3), we have

$$\log L_f[\varphi_0]=\tilde{g}(0, 0)=\tilde{g}(0, 1)=\log L_f[\varphi_1],$$

as desired.

Lemma 2.1 implies that the family $PS^M(p)$ defined in the present paper coincides with the one originally defined in [1].

By virtue of Lemma 2.1, for every $f\in PS^M(p)$, we may define a function L_f on $T_p(M)$ by

$$L_f(X)=L_f[\varphi], \qquad X\in T_p(M),$$

where $\varphi\in$ LHC(X).

LEMMA 2.2 ([1; Lemma 3.3]). *If $f\in PS^M(p)$, $\varphi\in$ LHC(X), and φ is defined on εU, then the function $a(r)$, $0<r<\varepsilon$, given by*

$$a(r)=(2\pi)^{-1}\int_0^{2\pi} f\circ\varphi(re^{i\theta})d\theta-\log r$$

is monotone-increasing in the interval $(0, \varepsilon)$ *and converges to* $\log L_f(X)$ *as* $r\to 0$.

LEMMA 2.3. *For every* $f\in PS^M(p)$, *the function* $\log L_f$ *is plurisubharmonic on* $T_p(M)$.

Proof. Take a holomorphic chart (z, U_z) around p so that $z(p)=0$ and $z(U_z)$ is a ball, and set $l(u)=\log L_f((\partial_u^z)_p)$ for $u\in \boldsymbol{C}^m$ (see (2.1)), where $m=\dim M$. We must show that l is plurisubharmonic on $\boldsymbol{C}^m$.

To prove the upper semi-continuity of l, consider the function h on $D=\{(\lambda, u)\in \boldsymbol{C}\times \boldsymbol{C}^m\,;\ \lambda u\in z(U_z)\}$ defined by $h(\lambda, u)=f\circ z^{-1}(\lambda u)$, $(\lambda, u)\in D$. Fix a vector $u_0\in \boldsymbol{C}^m$, and take a real number $\eta>l(u_0)$. Since

$$l(u_0)=\limsup_{\lambda\to 0,\,\lambda\neq 0}(h(\lambda, u_0)-\log|\lambda|),$$

one can find a positive number δ such that $h(\lambda, u_0)-\log|\lambda|<\eta$ for any $\lambda\in \boldsymbol{C}$ with $0<|\lambda|\leqq\delta$. Since h is upper semi-continuous, using the compactness of the set $\delta T=\{\lambda\in \boldsymbol{C};\ |\lambda|=\delta\}$, we can find a neighborhood W of u_0 such that $h(\lambda, u)-\log\delta<\eta$ for any $\lambda\in\delta T$ and $u\in W$. It follows from Lemma 2.2 that

$$l(u)\leqq(2\pi)^{-1}\int_0^{2\pi} h(\delta e^{i\theta}, u)d\theta-\log\delta<\eta$$

for any $u\in W$. This means that l is upper semi-continuous at u_0.

We next show that

$$l(u)\leqq(2\pi)^{-1}\int_0^{2\pi} l(u+e^{i\xi}v)d\xi$$

for any $u, v\in \boldsymbol{C}^m$. By Lemma 2.2 we have

$$l(u)=\lim_{r\to 0+}\Big((2\pi)^{-1}\int_0^{2\pi} f\circ z^{-1}(re^{i\theta}u)d\theta-\log r\Big),$$

$$l(u+e^{i\xi}v)=\lim_{r\to 0+}\Big((2\pi)^{-1}\int_0^{2\pi} f\circ z^{-1}(re^{i\theta}u+re^{i(\theta+\xi)}v)d\theta-\log r\Big).$$

Thus, the desired inequality follows from the monotone convergence theorem, Fubini's theorem, and the plurisubharmonicity of $f\circ z^{-1}$ (cf. the proof of [1; Lemma 3.8]). This completes the proof.

For every $X\in T_p(M)$, we define

$$P^M(X)=\sup\{L_f(X)\,;\ f\in PS^M(p)\}.$$

PROPOSITION 2.4. *If* $g=g_p^M$ *is the extremal plurisubharmonic function on a complex manifold* M *with pole at* $p\in M$, *defined in the preceding section, then* $P^M=L_g$ *on* $T_p(M)$.

Proof. Let $X\in T_p(M)$. Since $g\in PS^M(p)$ (Lemma 1.2), we have $P^M(X)\geqq L_g(X)$. On the other hand, if $f\in PS^M(p)$ and $\varphi\in \mathrm{LHC}(X)$, then

$$(\exp f\circ\varphi)(\lambda)/|\lambda|\leqq(\exp g\circ\varphi)(\lambda)/|\lambda|$$

for all sufficiently small $\lambda\in \boldsymbol{C}-\{0\}$. It follows that $L_f(X)\leqq L_g(X)$, so that $P^M(X)\leqq L_g(X)$. This completes the proof.

Combining Proposition 2.4 with Lemma 2.3, we get the following.

COROLLARY 2.5. *For every point p of a complex manifold M, the function $\log P^M|_{T_p(M)}$ is plurisubharmonic on $T_p(M)$.*

In particular, this corollary asserts that $\log P^M|_{T_p(M)}$ is upper semi-continuous on $T_p(M)$. Therefore, the function P^M defined in the present paper coincides with the one originally defined in [1]. According to [1; Proposition 3.8, Theorem 4.3], we review some fundamental properties on P^M in the following:

For every complex manifold M, P^M is a pseudo-metric on M, that is, P^M is a $[0,+\infty)$-valued function on the holomorphic tangent bundle $T(M)$ of M satisfying $P^M(\lambda X)=|\lambda|P^M(X)$ for any $X\in T(M)$ and $\lambda\in \boldsymbol{C}$.

For a holomorphic mapping Φ from M to M', it holds that

$$\Phi^*P^{M'}\leqq P^M \tag{2.4}$$

(Decreasing property).

Let C^M and K^M be the Carathéodory and Kobayashi pseudo-metrics on M, respectively (for the definitions, cf., *e.g.*, [7], [3], [1]). Then, it follows that

$$C^M\leqq P^M\leqq K^M. \tag{2.5}$$

For every $p\in M$, the indicatrix $\{X\in T_p(M)\,;\,P^M(X)<1\}$ of P^M at p is a pseudoconvex domain in $T_p(M)$.

Let M be a starlike circular domain in $\boldsymbol{C}^m$, *i.e.*, a domain satisfying $\lambda M\subset M$ for any $\lambda\in\boldsymbol{C}$ with $|\lambda|\leqq 1$, and let $N^M(u)=\inf\{\lambda>0\,;\,u\in\lambda M\}$, $P_0^M(u)=P^M((\partial_u^z)_0)$ for $u\in\boldsymbol{C}^m$ (see (2.1)), where $z(u)=u$, $u\in M$, is the natural coordinate on M. Then, $P_0^M\leqq N^M$, and the equality holds if and only if M is pseudoconvex. Furthermore, the indicatrix $\{u\in\boldsymbol{C}^m\,;\,P_0^M(u)<1\}$ of P_0^M coincides with the holomorphic hull of M.

Recently, Nishihara, Shon, and Sugawara [9] introduced, in the same manner as in [1], the pseudo-metric P^M for a class of infinite-dimensional complex manifolds M, and showed that the above-mentioned properties hold also for such manifolds.

We close this section by a useful lemma, which will be employed later.

LEMMA 2.6. *Let $(M_n)_{n=1}^\infty$ be a sequence of domains in a complex manifold M such that $M_{n+1}\supset M_n$, $M=\bigcup_{n=1}^\infty M_n$. Then, the following hold:*

(i) *For every $p\in M$, the sequence of functions $g_p^{M_n}$ is decreasing and con-*

verges to g_p^M *(see Lemma* 1.1).

(ii) *For every* $X\in T(M)$, *the sequence of numbers* $P^{M_n}(X)$ *is decreasing and converges to* $P^M(X)$ *(see* (2.4)).

Proof. Fix $p\in M$, take n_0 with $p\in M_{n_0}$, and set $g_n=g_p^{M_n}$, $g=g_p^M$, $P_n=P^{M_n}|_{T_p(M)}$, $P=P^M|_{T_p(M)}$ for $n\geqq n_0$.

(i) By the decreasing property (Lemma 1.1) we see that $g_n(q)\geqq g_{n+1}(q)\geqq g(q)$ for $q\in M_n$, $n\geqq n_0$. It follows that the function $f=\lim_{n\to\infty}g_n$ is well-defined on M and satisfies $f\geqq g$. Since f is the limit of a decreasing sequence of plurisubharmonic functions, it follows that $f\in PS^M(p)$, so that $f\leqq g$. Therefore, $f=g$.

(ii) Assume $X\in T_p(M)$. Let $\varphi\in\mathrm{LHC}(X)$, $\varphi:\varepsilon U\to M$, and set

$$a_n(r)=(2\pi)^{-1}\int_0^{2\pi}g_n\circ\varphi(re^{i\theta})d\theta-\log r\,,$$

$$a(r)=(2\pi)^{-1}\int_0^{2\pi}g\circ\varphi(re^{i\theta})d\theta-\log r$$

for $r\in(0,\varepsilon)$, $n\geqq n_0$. By Proposition 2.4 as well as Lemma 2.2, we have $\log P(X)=\lim_{r\to0+}a(r)$, $\log P_n(X)=\lim_{r\to0+}a_n(r)$. On the other hand, using the monotone convergence theorem, by part (i) we have $a(r)=\lim_{n\to\infty}a_n(r)$. However, using the monotonicity of $a_n(r)$ in each variable of n and r, we see that

$$\lim_{r\to0+}\lim_{n\to\infty}a_n(r)=\lim_{n\to\infty}\lim_{r\to0+}a_n(r)\,;$$

this means that $\log P(X)=\lim_{n\to\infty}\log P_n(X)$. The proof is completed.

It is well-known that the same assertion (ii) of Lemma 2.6 for C^M or K^M in place of P^M holds true.

§ 3. One-dimensional cases.

Throughout this section, we assume that the manifold M under consideration is one-dimensional, *i.e.*, M is a Riemann surface. Let C^M and K^M be the Carathéodory and Kobayashi pseudo-metrics on M, respectively. If we express C^M as $c_B^z|dz|$ using a local coordinate z, the quantity $c_B^z(p)$ is called the analytic capacity at $p\in M$ with respect to z. On the other hand, if the universal covering of M is holomorphically equivalent to the unit disc U in $\boldsymbol{C}$, then K^M is the metric induced from the Poincaré metric of U; otherwise $K^M=0$.

We next investigate the pseudo-metric P^M on a Riemann surface. When M is compact, it is immediately seen by definition that $P^M=0$. To clarify P^M on an open Riemann surface M, we review the definition of the capacity, according to Sario and Oikawa [11; pp. 54-55]. Let $(M_n)_{n=1}^\infty$ be an exhaustion of M by regular subdomains with respect to the Dirichlet problem for the Laplace equation. Let $p\in M$, and z a local coordinate around p. For n with $p\in M_n$, let g_n and $k_n^z(p)$ be the Green function on M_n and the Robin constant at p with respect to z, respectively, *i.e.*, $g_n(\ ,p)$ and $k_n^z(p)$ be a unique function and a unique real constant, respectively, such that $g_n(\ ,p)$ is harmonic on $M_n-\{p\}$, $g_n(q,p)+\log|z(q)-z(p)|\to k_n^z(p)$ as $q\to p$, and $g_n(q,p)\to 0$ as q approaches any boundary point of M_n. Set

$$g(\ ,p)=\lim_{n\to\infty}g_n(\ ,p), \qquad k_\beta^z(p)=\lim_{n\to\infty}k_n^z(p).$$

The quantities $k_\beta^z(p)$ and $c_\beta^z(p)=\exp(-k_\beta^z(p))$ are called the Robin constant and the capacity (of the ideal boundary β) at p with respect to the coordinate z, respectively. By Lemmas 1.3 and 1.4 and the remarks after them, we have $g_p^{M_n}=-g_n(\ ,p)$. Furthermore, by Proposition 2.4 we see $\log P^{M_n}((d/dz)_p)=-k_n^z(p)$. Therefore, Proposition 2.6 implies that $g_p^M=-g(\ ,p)$, $P^M((d/dz)_p)=c_\beta^z(p)$. We thus get the following.

PROPOSITION 3.1. *If M is an open Riemann surface, then the pseudo-metric P^M coincides with $c_\beta^z|dz|$, where $c_\beta^z=\exp(-k_\beta^z)$ is the capacity and k_β^z is the Robin constant with respect to a local coordinate z.*

Now, we have noted in (2.5) that $C^M\leqq P^M\leqq K^M$. Since M is one-dimensional, the quantities C^M/P^M and P^M/K^M are well-defined $[0,1]$-valued functions on M, provided that $P^M>0$ and $K^M>0$, respectively. Of course, these functions are biholomorphically invariant. We also note that both the functions converge to 1 as the point approaches any boundary point of M when M is a strongly pseudoconvex domain in $\boldsymbol{C}$ (cf. Graham [3], also cf. [13]).

To establish a formula for P^M/K^M, we review the argument in Suita [13] based on Myrberg's theorem [8]. Let M be an open Riemann surface with $M\notin O_G$, *i.e.*, with $P^M>0$. Then, the universal covering of M is holomorphically equivalent to the unit disc $U=\{\lambda\in\boldsymbol{C};\ |\lambda|<1\}$. Assume that M is not simply connected. Let π be a covering projection from U onto M. Let $p\in M$. Take a connected neighborhood W of p such that for every component W_n of $\pi^{-1}(W)$ $(n=0,1,\cdots)$, the restriction $\pi|_{W_n}:W_n\to W$ is homeomorphic. Let $z=(\pi|_{W_0})^{-1}$, and $z_n=(\pi|_{W_n})^{-1}$ for $n\geqq 1$. By Myrberg's theorem [8] the Green function g of M can be expressed as

$$g(q,p)=\log\left|\frac{1-\overline{z(p)}z(q)}{z(q)-z(p)}\right|+\sum_{n=1}^{\infty}\log\left|\frac{1-\overline{z_n(p)}z(q)}{z(q)-z_n(p)}\right|$$

for $q\in W$. It follows that

$$c_\beta^z=\frac{1}{1-|z|^2}\prod_{n=1}^{\infty}\left|\frac{z-z_n}{1-\bar{z}_n z}\right|$$

on W. Since $|dz|/(1-|z|^2)$ is the restriction to W of the Kobayashi metric on M, we get the following.

LEMMA 3.2. *Let $\pi:U\to M$ be a universal covering of an open Riemann surface M with $M\notin O_G$. For every $p\in M$, let $\{\zeta_n\}_{n=0}^{\infty}$ be a numbering of the fibre $\pi^{-1}(p)$. Then,*

$$P^M/K^M(p)=\prod_{n=1}^{\infty}|\zeta_0-\zeta_n|/|1-\bar{\zeta}_n\zeta_0|.$$

COROLLARY 3.3. *An open Riemann surface M with $M\notin O_G$ is simply connected, i.e., holomorphically equivalent to U, if and only if $P^M=K^M$, or equivalently, $P^M=K^M$ on some tangent space $T_p(M)$.*

As an example, we consider the functions $C^A/P^A, P^A/K^A$ for the annulus $A=A_q=\{\lambda\in \boldsymbol{C};\ q<|\lambda|<1\}$ with $0<q<1$. For $F=C$, P, or K, the same symbol F stands for $F^A((d/dz)_\lambda)$, where $z(\lambda)=\lambda$, $\lambda\in A$ is the natural coordinate on A. Then, these values are explicitly given by

$$C=\frac{\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1+2t})(1+q^{2n-1-2t})}{\prod_{n=1}^{\infty}(1+q^{2n-1})^2(1-q^{2(n-1)+2t})(1-q^{2n-2t})}, \tag{3.1}$$

$$P=\frac{\prod_{n=1}^{\infty}(1-q^{2n})^2}{q^{t^2}\prod_{n=1}^{\infty}(1-q^{2(n-1)+2t})(1-q^{2n-2t})}, \tag{3.2}$$

$$K=\frac{\pi}{2q^t(-\log q)\sin \pi t}, \tag{3.3}$$

where $|\lambda|=q^t$ $(0<t<1)$ for $\lambda\in A$. The formula (3.1) was given by Robinson [10] and Simha [12]. The formula (3.2) was given by Suita [13], or is obtained from the formulas of the Green function given in [5], [10]. The formula (3.3) is obtained from the explicit form (as in the proof of Proposition 3.4 below) of a covering projection from the unit disc onto A (cf. Kobayashi [6; pp. 14-15]).

To formulate our assertion, set $\alpha(t)=C/P$, $\beta(t)=P/K$ $(t\in(0, 1))$ with $|\lambda|=q^t$, $\lambda\in A$. It is noted that $\alpha(1-t)=\alpha(t)$, $\beta(1-t)=\beta(t)$ for $t\in(0, 1)$. Let ϑ_q be Jacobi's theta function given by

$$\begin{aligned}\vartheta_q(z)&=\sum_{n\in Z} q^{n^2}z^n\\ &=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}).\end{aligned}$$

We shall show the following.

PROPOSITION 3.4. *The functions α, β are strictly decreasing in the interval $(0, 1/2]$. In particular, the minimums of C/P and P/K are both taken in the middle circle $|\lambda|=\sqrt{q}$ of the annulus A_q. The minimums of C/P and P/K are given by*

$$\alpha(1/2)=q^{1/4}\vartheta_q(q)\vartheta_q(1)^{-1} \tag{3.4}$$

and

$$\beta(1/2)=q^{1/4}(-\log q)\pi^{-1}\vartheta_q(1)\vartheta_q(q)=\vartheta_r(1)\vartheta_r(-1), \tag{3.5}$$

respectively, where $r\in(0, 1)$ is the number determined by

$$\pi/\log q=(\log r)/\pi. \tag{3.6}$$

We remark that the value (3.4) is the square root of the modulus in the theory of Jacobi elliptic functions with respect to the period basis $(2\pi i, 2\log q)$. Furthermore, Robinson proved in [10] that this value is realized as the exponential of the minimum value $-g(\sqrt{q}, -\sqrt{q})$ of $-g(s, -t)$ when both s and t run over the subset $(q, 1)\subset A_q$, where g is the Green function of the annulus A_q.

Proof of Proposition 3.4. By (3.1) and (3.2) we find that

$$\alpha(t)=Q(t)/Q(0), \qquad Q(t):=\sum_{n\in Z} q^{(n+t)^2}.$$

It is known ([10; p. 348]) that the function $Q(t)$ is strictly decreasing in the interval $[0, 1/2]$. Therefore, all the assertions for α and C/P follow.

To prove the assertions for β and P/K, we consider the domains $B=\{\xi\in \boldsymbol{C};\ 0<\mathrm{Im}\,\xi<-\log q\}$, $H=\{\eta\in \boldsymbol{C};\ \mathrm{Im}\,\eta>0\}$, $U=\{\zeta\in \boldsymbol{C};\ |\zeta|<1\}$, and the mappings $\Phi: U\to H$, $\Psi: H\to B$, $\pi_1: B\to A$, given by $\eta=i(\zeta+1)/(\zeta-1)$, $\xi=(-\log q)(\log\eta)/\pi$, and $\lambda=e^{i\xi}$, respectively. Then, $\pi=\pi_1\circ\Psi\circ\Phi: U\to A$ is a covering projection onto A. Let $\lambda\in(q, 1)\subset A$ be fixed. For $n\in \boldsymbol{Z}$, set $\xi_n=2n\pi-i\log\lambda$, $\eta_n=\Psi^{-1}(\xi_n)$, and $\zeta_n=\Phi^{-1}(\eta_n)$. Since $\pi^{-1}(\lambda)=\{\zeta_n;\ n\in \boldsymbol{Z}\}$, $\zeta_n=(\eta_n-i)/(\eta_n+i)$, it follows from Myrberg's formula (Lemma 3.2) that

$$P/K=\prod_{n\neq 0}|\zeta_n-\zeta_0|/|1-\bar{\zeta}_n\zeta_0|=\prod_{n\neq 0}|\eta_n-\eta_0|/|\bar{\eta}_n-\eta_0|.$$

Using the number r given by (3.6) we see $\eta_n=r^{-2n}e^{t\pi i}$. Therefore, for every $t\in(0, 1)$ we see

$$\beta(t)=\prod_{n=1}^{\infty}(1-r^{2n})^2|e^{2t\pi i}-r^{2n}|^{-2}.$$

For every n, it is easily seen that the function $R(t)=|e^{2t\pi i}-r^{2n}|$ is strictly increasing in the interval $[0, 1/2]$. Therefore, the function β is strictly decreasing in $(0, 1/2]$, and P/K takes the minimum

$$\beta(1/2)=\prod_{n=1}^{\infty}(1-r^{2n})^2(1+r^{2n})^{-2}=\vartheta_r(1)\vartheta_r(-1)$$

at $\lambda\in A$ with $|\lambda|=\sqrt{q}$. Furthermore, it follows from formulas (3.2) and (3.3) that

$$\begin{aligned}\beta(1/2)&=2q^{1/4}(-\log q)\pi^{-1}\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{2n-1})^{-2}\\&=q^{1/4}(-\log q)\pi^{-1}\vartheta_q(1)\vartheta_q(q).\end{aligned}$$

This gives the first expression in (3.5) for the value $\beta(1/2)$. Thus, the proof of Proposition 3.4 is completed.

Remark 3.5. We make some comments on the relation (3.5). First, we assume $q=r$, *i.e.*, $q=e^{-\pi}$. Then, the relation (3.5) implies that the number $q=e^{-\pi}$ satisfies the equation $\vartheta_q(q)q^{1/4}=\vartheta_q(-1)$, that is,

$$2\sum_{n=1}^{\infty}\exp(-\pi(n-1/2)^2)=1+2\sum_{n=1}^{\infty}(-1)^n\exp(-\pi n^2)$$

or

$$e^{\pi/8}=\sqrt{2}\prod_{n=1}^{\infty}(1+e^{-n\pi}+e^{-2n\pi}+e^{-3n\pi}).$$

On the other hand, since

$$\beta(1/2)=\frac{2q^{1/4}(-\log q)}{\pi(1-q)}\prod_{n=1}^{\infty}\frac{(1+q+\cdots+q^{2n-1})^2}{(1+q+\cdots+q^{2n-2})(1+q+\cdots+q^{2n})},$$

taking the limits as $q\to 1-0$ in both sides of (3.5), noting $r\to 0+$, we obtain Wallis' formula

$$(2/\pi)\prod_{n=1}^{\infty}(2n)^2/(2n-1)(2n+1)=1.$$

Thus, the formula (3.5) can be seen as an extension of Wallis' formula.

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