THE HARNACK AND OTHER CONFORMALLY INVARIANT METRICS

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1. Introduction. This paper investigates the *Harnack metric* h_D in proper subdomains D of euclidean space $\mathbf{R}^n$. By definition

$$h_D(x_1, x_2) = \sup_u \left| \log \frac{u(x_1)}{u(x_2)} \right|,$$

where the supremum is taken over all positive functions u which are harmonic in D . Also useful is the *quasihyperbolic metric* [GP] given by

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{d(z, \partial D)},$$

where the infimum is taken over all locally rectifiable arcs γ joining x_1, x_2 in D .

First, we prove that $h_D \leq n k_D$, and n is the smallest constant for which this is true in all subdomains D of $\mathbf{R}^n$. Next we show that if G is a domain which has Green's functions $g(x, y)$ and if $D = G - E$, where E is a relatively closed polar subset of G , then for all $x_1, x_2 \in D$

$$h_D(x_1, x_2) = \max \left\{ h_G(x_1, x_2), \sup_{y \in E} \left| \log \frac{g(x_1, y)}{g(x_2, y)} \right| \right\}.$$

Then we give a formula for the distance between two antipodal points on the circle of symmetry in an annulus. See [K], [L] and [V] for similar results.

Finally, we consider plane domains D and we compare the Bergman metric b_D , the Carathéodory metric c_D , the Harnack metric h_D , and the Poincaré metric p_D . These metrics are all conformally invariant, and it is known that

$$b_D \leq p_D \quad \text{and} \quad c_D \leq h_D \leq p_D;$$

see [AB], [BG], [K], [T]. We give examples to show that these inequalities cannot be combined, and that for general plane domains the second set of inequalities is best possible. We also show that the strict inequality $h_D < p_D$ holds when D has two or more isolated boundary points.

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Throughout this paper D denotes a proper subdomain of euclidean space $\mathbf{R}^n$. The euclidean ball centered at x of radius r is denoted $B^n(x, r)$ and we use the abbreviations $B^n = B^n(0, 1)$, $B = B^2$. For $0 < q < 1$ fixed we let A denote the annulus $A = \{z \in \mathbf{R}^2: \sqrt{q} < |z| < 1/\sqrt{q}\}$, and we work with the complete elliptic integrals and the Jacobian elliptic functions with modulus $k = k(q)$; see [BF]. Explicitly, $k = 4\sqrt{q} \prod_1^\infty \left(\frac{1+q^{2n}}{1-q^{2n}} \right)^4$ [N, p. 289, (30)].

By a *metric* we mean a pseudo-distance function. With each non-negative measurable function λ defined in D we associate a metric ρ_λ by

$$\rho_\lambda(x_1, x_2) = \inf_\gamma \int_\gamma \lambda(x) |dx|,$$

where the infimum is taken over all locally rectifiable arcs γ joining x_1, x_2 in D . With each metric ρ defined in D we associate a non-negative function $\partial\rho$, the *metric-density* of ρ , by

$$\partial\rho(x) = \limsup_{y \rightarrow x} \frac{\rho(x, y)}{|x - y|}.$$

It is not difficult to show that when λ is continuous

$$(1) \quad \partial\rho_\lambda = \lambda,$$

and that when $\partial\rho$ is continuous

$$(2) \quad \rho \leq \rho_{\partial\rho}.$$

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2. The Harnack Metric. The *Harnack metric*, abbreviated *H-metric*, is defined in D by

$$(3) \quad h_D(x_1, x_2) = \sup_u \left| \log \frac{u(x_1)}{u(x_2)} \right|,$$

where the supremum is taken over all functions u in the class $\mathcal{H}^+(D)$ of positive functions harmonic in D ; see [K], [L], [T].

Employing normal family arguments and Taylor's theorem, one can prove

THEOREM 1. For $x \in D \subset \mathbf{R}^n$,

$$\partial h_D(x) = \sup_u |\nabla \log u(x)|,$$

where the supremum is taken over all functions $u \in \mathcal{H}^+(D)$.

We now derive a sharp upper bound for the H -metric. Notice that we have

$$h_D(x_1, x_2) \leq \log M$$

if and only if we have the Harnack type inequality

$$\frac{1}{M} \leq \frac{u(x_1)}{u(x_2)} \leq M \quad \text{for all } u \in \mathcal{H}^+(D).$$

The *quasihyperbolic metric* is defined in D by

$$k_D(x_1, x_2) = \rho_\lambda(x_1, x_2), \quad \lambda(x) = d(x, \partial D)^{-1},$$

where $d(x, \partial D)$ is the euclidean distance from x to ∂D . When D is a half-space this is the classical hyperbolic, or Poincaré, metric.

THEOREM 2. For $x_1, x_2 \in D \subset \mathbf{R}^n$,

$$h_D(x_1, x_2) \leq n k_D(x_1, x_2),$$

and n is the smallest constant for which this is true in all subdomains D of $\mathbf{R}^n$.

Proof. We first establish the infinitesimal version of our result,

$$(4) \quad \frac{|\nabla u(x_0)|}{u(x_0)} \leq \frac{n}{r} \quad \text{for all } u \in \mathcal{H}^+(B^n(x_0, r)).$$

Fix $u \in \mathcal{H}^+(B^n(x_0, r))$. Then by Harnack's inequality [CH2]

$$\left(\frac{r}{r+\rho}\right)^{n-2} \frac{r-\rho}{r+\rho} \leq \frac{u(x)}{u(x_0)} \leq \left(\frac{r}{r-\rho}\right)^{n-2} \frac{r+\rho}{r-\rho}$$

for all x, ρ with $|x - x_0| \leq \rho < r$. Letting $t = |x - x_0|/r$ for $x \in B^n(x_0, r)$ we obtain

$$1 - nt + O(t^2) \leq \frac{u(x)}{u(x_0)} \leq 1 + nt + O(t^2)$$

as $t \rightarrow 0$, and hence

$$-\frac{n}{r} + O(|x - x_0|) \leq \frac{u(x) - u(x_0)}{u(x_0)|x - x_0|} \leq \frac{n}{r} + O(|x - x_0|)$$

as $|x - x_0| \rightarrow 0$, which establishes (4).

Now Theorem 1 and (4) yield

$$\partial h_D \leq n \partial k_D,$$

whence by (1) and (2)

$$h_D \leq \rho_{\partial h_D} \leq \rho_{n \partial k_D} = n k_D.$$

To see that n is the smallest such constant we compute the distances between the origin and a nearby point in B^n . We have

$$k_{B^n}(0, x) = \int_{[0, x]} \frac{|dz|}{1-|z|} = \log \frac{1}{1-|x|},$$

and (from [K] or [L])

$$h_{B^n}(0, x) = \log \frac{1+|x|}{(1-|x|)^{n-1}}.$$

Thus

$$h_{B^n}(0, x) - nk_{B^n}(0, x) = \log(1-|x|^2),$$

and given $c < n$ one can find an $x \in B^n$ such that

$$h_{B^n}(0, x) > ck_{B^n}(0, x).$$

Leutwiler has shown [L, Lemma 5.4] that a lower bound for the H -metric is given by

$$h_D(x_1, x_2) \geq (n-2) \left| \log \frac{d(x_1, \partial D)}{d(x_2, \partial D)} \right|.$$

If D has isolated boundary points, then Theorem 3 below also gives lower bounds. Moreover, since the H -metric is known in balls and half-spaces, Theorem 3 gives a large class of domains for which we can compute the H -metric.

A set $E \subset \mathbf{R}^n$ is *polar* if there exist an open set $U \supset E$ and a subharmonic function u defined on U with $u|_E \equiv -\infty$.

THEOREM 3. *Let $G \subset \mathbf{R}^n$ be a domain which has a Green's function $g(x, y)$. Let $E \subset G$ be a relatively closed polar set, and let $D = G - E$. Then*

$$(5) \quad h_D(x_1, x_2) = \max \left\{ h_G(x_1, x_2), \sup_{y \in E} \left| \log \frac{g(x_1, y)}{g(x_2, y)} \right| \right\}$$

for all $x_1, x_2 \in D$.

Proof. Fix $x_1, x_2 \in D$. Since $D \subset G$,

$$h_D(x_1, x_2) \geq h_G(x_1, x_2).$$

If $y \in E$, then $g(\cdot, y) \in \mathcal{H}^+(D)$, so

$$h_D(x_1, x_2) \geq \left| \log \frac{g(x_1, y)}{g(x_2, y)} \right|.$$

Thus

$$h_D(x_1, x_2) \geq \log M,$$

where $\log M$ denotes the right-hand side of (5).

For the reverse inequality, fix $u \in \mathcal{H}^+(D)$. Then u has a unique superharmonic extension to G , and there exists a unique measure μ on G such that

$$u(x) = v(x) + \int_G g(x, y) d\mu(y),$$

where v is the greatest harmonic minorant of u in G ; see [He]. Now since u is harmonic in D , μ is supported in E . Thus, using the definition of M we obtain

$$u(x_1) = v(x_1) + \int_E g(x_1, y) d\mu(y) \leq Mv(x_2) + M \int_E g(x_2, y) d\mu(y) = Mu(x_2).$$

Hence by symmetry

$$\frac{1}{M} \leq \frac{u(x_1)}{u(x_2)} \leq M$$

and thus

$$h_D(x_1, x_2) \leq \log M.$$

Next, suppose D has a Poisson kernel, that is, suppose there exists a positive function $P(\xi, x)$, defined for $\xi \in \partial D$, $x \in D$ such that

$$u(x) = \int_{\partial D} P(\xi, x) u(\xi) d\sigma(\xi)$$

for all u harmonic in D and continuous in $\bar{D}$, where σ is surface area measure on ∂D . We remark that for each fixed $\xi \in \partial D$, $P(\xi, x)$ is a harmonic function of $x \in D$.

Fix $x, y \in D$ and consider

$$f(\xi) = \frac{P(\xi, x)}{P(\xi, y)}, \quad \xi \in \partial D.$$

Choose $\xi_1, \xi_2 \in \partial D$ so that

$$m = f(\xi_1) = \min_{\partial D} f, \quad M = f(\xi_2) = \max_{\partial D} f.$$

Suppose u is positive, harmonic in D , and continuous in $\bar{D}$. Then

$$mu(y) = m \int_{\partial D} P(\xi, y) u(\xi) d\sigma(\xi) \leq \int_{\partial D} P(\xi, x) u(\xi) d\sigma(\xi) = u(x) \leq Mu(y).$$

As there is equality in the first, second inequality for

$$u_1(z) = \frac{P(\xi_1, z)}{P(\xi_1, y)}, \quad u_2(z) = \frac{P(\xi_2, z)}{P(\xi_2, y)},$$

respectively, we conclude

THEOREM 4.

$$h_D(x, y) = \log \max \left\{ M, \frac{1}{m} \right\}.$$

This is the method we use to study the H -metric in an annulus.

THEOREM 5. *The Harnack distance between the points $1, -1 \in A$ is given by*

$$h_A(1, -1) = \log \frac{1+k}{1-k}.$$

Proof. By Theorem 4 and symmetry

$$h_D(1, e^{i\theta}) = \log M,$$

where M is the maximum value of

$$f(\zeta) = \frac{P(\zeta, 1)}{P(\zeta, e^{i\theta})} \quad (\zeta \in \partial A)$$

and $P(\zeta, z)$ is the Poisson kernel for A .

By [CH1] the Green's function for A with a pole at the point 1 is

$$g(\zeta, 1) = -\operatorname{Re} \log \left(\sqrt{k} i \operatorname{sn} \left(\frac{K}{\pi i} \log \zeta \right) \right),$$

and hence

$$P(\zeta, 1) = \frac{\mp K}{2\pi^2 |\zeta|} \operatorname{Im} \left(\frac{dn u}{tn u} \right)$$

where $\zeta = q^{\pm 1/2} e^{i\phi}$, $u = (K/\pi)\phi \pm (K'/2)i$. Thus

$$f(\zeta) = \frac{\operatorname{Im} \left(\frac{dn u}{tn u} \right)}{\operatorname{Im} \left(\frac{dn v}{tn v} \right)},$$

where $\zeta = q^{\pm 1/2} e^{i\phi}$, $u = (K/\pi)\phi \pm (K'/2)i$, $v = (K/\pi)(\phi - \theta) \pm (K'/2)i$. From [BF]

$$dn(u-K) = k'/dn u, \quad tn(u-K) = -1/k' tn u,$$

whence

$$f(\zeta) = \frac{\operatorname{Im} \left(\frac{dn u}{tn u} \right)}{-k'^2 \operatorname{Im} \left(\frac{tn u}{dn u} \right)} = \frac{1}{1-k^2} \left| \frac{dn u}{tn u} \right|^2$$

and the result follows as soon as we establish

$$\max_{x \in \mathbf{R}} \left| \frac{dn \left(x + \frac{K'}{2} i \right)}{tn \left(x + \frac{K'}{2} i \right)} \right| = 1+k.$$

Using symmetry and standard mapping properties of $w = sn z$ it is not difficult to show that for $x \in \mathbf{R}$

$$\left| sn\left(x + \frac{K'}{2}i\right) \right| = 1/\sqrt{k}.$$

Then using the definitions of $cn z$, $dn z$ and mapping properties of $w = (1 - z^2)^{1/2}$ we find that for $x \in \mathbf{R}$

$$\left| dn\left(x + \frac{K'}{2}i\right) \right| = \sqrt{k} \left| cn\left(x + \frac{K'}{2}i\right) \right| \leq \sqrt{k} \left| cn\left(\frac{K'}{2}i\right) \right| = (1+k)^{1/2}.$$

Thus

$$\left| \frac{dn}{tn}\left(x + \frac{K'}{2}i\right) \right| = k \left| cn\left(x + \frac{K'}{2}i\right) \right|^2 \leq 1+k$$

as desired.

3. Other Metrics. We now restrict ourselves to proper subdomains D of the complex plane $\mathbf{C} = \mathbf{R}^2$, where the Harnack metric is conformally invariant. That is, if $\phi: D \rightarrow D'$ is a conformal mapping of D onto D' , then

$$h_{D'}(\phi(z_1), \phi(z_2)) = h_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. We examine three other conformally invariant metrics: the Bergman, Carathéodory, and Poincaré metrics; and we compare all four metrics. Each metric arises as a solution to an extremal problem and relations among the metrics give information relating these extremal problems.

In the unit disk B there is essentially *one* conformally invariant metric:

$$\rho(z_1, z_2) = \log \frac{1+r}{1-r}, \quad r = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|.$$

The *Carathéodory metric* $c_D(z_1, z_2)$ and the *Poincaré, or hyperbolic, metric* $p_D(z_1, z_2)$ are defined in D by

$$\sup_f \rho(f(z_1), f(z_2)),$$

where the suprema are taken over all functions f in the classes $\mathcal{B}$ and $\mathcal{M}$ respectively. Here $\mathcal{M}$ is the class of all multiple-valued analytic functions $f: D \rightarrow B$ and $\mathcal{B}$ is the subclass of $\mathcal{M}$ of single-valued functions. Thus $\mathcal{M}$ consists of analytic continuations f , along curves in D , of single-valued analytic functions $f_0: B^2(z_0, r) \subset D \rightarrow B$, with each value of $f(z)$ in B for all z in D .

We have given a global definition for the Poincaré metric. In fact

$$p_D = \rho_{\lambda_D}, \quad \lambda_D = \partial p_D,$$

where

$$ds^2 = \lambda_D(z)^2 |dz|^2$$

defines a Riemannian metric in D with constant Gaussian curvature -1 . The *Bergman metric* is defined in D by

$$b_D(z_1, z_2) = \rho_{\kappa_D}(z_1, z_2), \quad \kappa_D(z) = 2(\pi \tilde{K}_D(z, \bar{z}))^{1/2},$$

where $\tilde{K}_D$ is the reduced Bergman reproducing kernel for the Hilbert space

$$\left\{ f' : f \text{ analytic in } D, \iint_D |f'(z)|^2 dx dy < +\infty \right\};$$

see [BG], [B].

3.1. Relations between metric-densities. Using (1), an extremal property of $\tilde{K}_D$, Theorem 1 and its proof, and Schwarz' lemma we find that the metric-densities ∂b_D , ∂c_D , ∂h_D , ∂p_D are each given by

$$(6) \quad 2 \sup_f |f'(z)|,$$

where the suprema are taken over all functions f in $\mathcal{D}$, $\mathcal{B}$, $\mathcal{H}$, $\mathcal{M}$ respectively. Here

$$\mathcal{D} = \left\{ f \text{ analytic in } D : \iint_D |f'(z)|^2 dx dy \leq \pi \right\}$$

and

$$\mathcal{H} = \left\{ f \in \mathcal{M} : \operatorname{Re} \left(\frac{1+f}{1-f} \right) \text{ is single-valued} \right\}.$$

From (6), [AB], [BG], and the inclusions $\mathcal{B} \subset \mathcal{H} \subset \mathcal{M}$ we obtain

$$(7) \quad \partial b_D \leq \partial c_D \leq \partial h_D \leq \partial p_D,$$

and this is valid throughout each domain $D \subset C$.

At each point $e^{i\theta}$ in A we have

$$(8) \quad \begin{cases} \partial b_A = \frac{2}{\pi} (K^2 - KE)^{1/2}, \\ \partial c_A = \frac{2}{\pi} Kk, \\ \partial p_A = \frac{K}{K'}; \end{cases}$$

see [BG], [Ha], [Si]. From [BF] it follows that

$$\partial b_A < \partial c_A < \partial p_A$$

at each $e^{i\theta}$ in A . In fact for a non-simply, finitely connected domain D with no degenerate boundary components we have

$$\partial b_D < \partial c_D < \partial p_D;$$

see [Sa1], [Sa2], [A].

3.2. Relations between metrics. In each simply connected domain $D \subset C$

$$b_D = c_D = h_D = p_D.$$

For a general subdomain D of C we see from (7), [Sa3] that

$$(9) \quad b_D \leq p_D,$$

with equality if and only if D is simply connected or $p_D = 0$; and from [K], [T] we have

$$(10) \quad c_D \leq h_D \leq p_D.$$

From (7) we expect that, locally,

$$b_D \leq c_D \leq h_D \leq p_D.$$

However, we shall see that such inequalities are not, in general, valid globally. That is, (9) and (10) cannot be combined.

First we consider the punctured disk $D = B - \{0\}$. It is easy to see that

$$b_D = c_D = p_B.$$

By Theorem 3

$$h_D(z_1, z_2) = \max\{p_D(|z_1|, |z_2|), p_B(z_1, z_2)\}.$$

Thus on concentric circles

$$c_D = h_D < p_D,$$

while on radial segments

$$c_D < h_D = p_D.$$

Therefore (10) cannot, in general, be improved.

THEOREM 6. *Suppose D is a domain with (at least) two isolated boundary points. Then*

$$(11) \quad h_D < p_D.$$

Proof. First, suppose $G \subset C$ is a domain which has a Green's function $g(z, \zeta)$. Fix $\zeta_0 \in G$ and set $G_0 = G - \{\zeta_0\}$. Then it is not hard to show that

$$\left| \log \frac{g(z_1, \zeta_0)}{g(z_2, \zeta_0)} \right| \leq p_{G_0}(z_1, z_2)$$

for all $z_1, z_2 \in G_0$.

To establish (11) we may assume that D has a Green's function $g(z, \zeta)$. Choose isolated points $\zeta_1, \zeta_2 \in \partial D$ and let

$$G = D \cup \{\zeta_1, \zeta_2\}, \quad G_j = G - \{\zeta_j\} \quad (j=1, 2).$$

Then by Theorem 3 and the above we obtain

$$\begin{aligned} h_D(z_1, z_2) &= \max \left\{ \left| \log \frac{g(z_1, \zeta_1)}{g(z_2, \zeta_1)} \right|, \left| \log \frac{g(z_1, \zeta_2)}{g(z_2, \zeta_2)} \right|, h_G(z_1, z_2) \right\} \\ &\leq \max \{ p_{G_1}(z_1, z_2), p_{G_2}(z_1, z_2), p_G(z_1, z_2) \} \\ &< p_D(z_1, z_2) \end{aligned}$$

for all $z_1, z_2 \in D$.

Next we consider the annulus A . We remark that for $0 \leq \theta \leq \pi$ we have

$$(12) \quad \begin{cases} b_A(1, e^{i\theta}) = \frac{2\theta}{\pi} (K^2 - KE)^{1/2}, \\ c_A(1, e^{i\theta}) = \log \frac{1 + k \operatorname{sn}\left(\frac{\theta}{\pi}\right) K}{1 - k \operatorname{sn}\left(\frac{\theta}{\pi}\right) K} \\ p_A(1, e^{i\theta}) = \theta \frac{K}{K'}. \end{cases}$$

The first and last of these follow from (8) and the fact that the circle $\{z: |z|=1\}$ is a geodesic for any conformally invariant metric in A . The middle identity is contained in [Si], and it and Theorem 5 imply

$$c_A(1, -1) = h_A(1, -1).$$

Finally, for the special case $q = e^{-\pi}$ we have

$$k = 1/\sqrt{2}, \quad K = 1.85 \dots, \quad E = 1.35 \dots,$$

whence by (12) and [BF]

$$b_A(1, i) = 0.96 \dots < c_A(1, i) = 1.21 \dots$$

while

$$b_A(1, -1) = 1.93 \dots > c_A(1, -1) = 1.76 \dots$$

This shows that (9) and (10) cannot be combined, and that there is no simple relationship between either the Bergman and Carathéodory metrics or between the Bergman and Harnack metrics.

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