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ON THE ADELE RINGS AND ZETA-FUNCTIONS OF ALGEBRAIC NUMBER FIELDS

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Throughout this paper Q and Z denote the rational number field and the rational integer ring, respectively. An algebraic number field always means an algebraic number field of finite degree, an integer means a rational integer and a prime number means a rational prime number. For an algebraic number field k, we denote by k_A the adele ring of k, by $\zeta_k(s)$ the Dedekind zeta-function of k and O_k the integer ring of k. For any prime ideal $\mathfrak{p}$ of k, $k_{\mathfrak{p}}$ denotes the completion of k by $\mathfrak{p}$-adic valuation, for real place $\mathfrak{p}$ of k, $k_{\mathfrak{p}}$ denotes the real number field and for imaginary place $\mathfrak{p}$ of k, $k_{\mathfrak{p}}$ denotes the complex number field. We write $N_{k/Q}$ for the norm of an ideal in k. For a Galois extension L of a field F, we denote by $Gal(L/F)$ the Galois group of L/F. For a set S, we denote by card (S) the cardinarity of S. We write $[G ; H]$ for the index of a subgroup H in a finite group G. The word "isomorphism" for topological groups, toplogical rings and topological fields, means a topological isomorphism. The main purpose of this paper is to prove the following theorem, which is a refinement of our previous paper [4]:

THEOREM. *Let m be a square free integer such that $m \neq \pm 1, \pm 2$, and n an integer such that $n \geqq 3$. Put $k=Q(\sqrt[2n]{m})$ and $k'=Q(\sqrt{2} \times \sqrt[2n]{m})$. If*

$$m \equiv 1, 3, 5, 6, 9, 10, 11, 13 \pmod{16},$$

then k_A is not isomorphic to k'_A and $\zeta_k(s)=\zeta_{k'}(s)$. If

$$m \equiv 2, 7, 14, 15 \pmod{16},$$

then k_A is isomorphic to k'_A and k is not isomorphic to k'.

For two algebraic number fields K and K', we should notide that $K \cong K'$ implies $K_A \cong K'_A$ and that $K_A \cong K'_A$ implies $\zeta_K(s)=\zeta_{K'}(s)$. Now we describe the following lemma, which plays an important role in this paper:

LEMMA 1. (cf. lemma 7 of [3] and lemma 3 of [4]) *Let k be an algebraic number field, V_k the set of places of k and W_k the set of non-zero prime ideals of k. We adopt similar notations for an algebraic number field k'. Then the following conditions are equivalent*:

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(1) k_A and k'_A are isomorphic.

(2) There exists a bijection Φ of V_k onto $V_{k'}$ such that $k_{\mathfrak{p}}$ and $k'_{\Phi(\mathfrak{p})}$ are isomorphic for every $\mathfrak{p}\in V_k$.

(3) There exists a bijection Ψ of W_k onto $W_{k'}$ such that $k_{\mathfrak{p}}$ and $k'_{\Psi(\mathfrak{p})}$ are isomorphic for every $\mathfrak{p}\in W_k$.

An immediate consequence of the above lemma is the following proposition:

PROPOSITION. (cf. Corollary of lemma 3 in [4]) *Let k and k' be algebraic number fields.* Then $k_A\cong k'_A$ *implies* $\zeta_k(s)=\zeta_{k'}(s)$.

LEMMA 2. *Let L be a finite Galois extension of Q, let $G=Gal(L/Q)$, and let H and H' be subgroups of G. Let k and k' be subfields of L corresponding to the subgroups H and H' of G, respectively. For every element σ of G, let $C(\sigma)=\{\tau^{-1}\sigma\tau\,|\,\tau\in G\}$. Then the following conditions are equivalent:*

(1) For every element σ of G, card $(C(\sigma)\cap H)=\text{card}(C(\sigma)\cap H')$.

(2) For every prime number p, the collection of degrees of factors of p in k is identical with the collection of degrees of the factors of p in k'.

(3) The zeta-functions $\zeta_k(s)$ and $\zeta_{k'}(s)$ are the same.

Let G be a group. An automorphism f of G is called to be an element-wise inner automorphism, if for every element σ of G, σ and $f(\sigma)$ are conjugate in G.

LEMMA 3. *Let G be a finite group, H a subgroup of G and f an element-wise inner automorphism of G. Then for every element σ of G, we have card $(C(\sigma)\cap H)$ $=$card $(C(\sigma)\cap f(H))$.*

Proof. For any element σ of G, we have $f(C(\sigma))=C(\sigma)$. This shows $f(C(\sigma)\cap H)=C(\sigma)\cap f(H)$. So we have our assertion.

The following lemma owes to Gerst [2].

LEMMA 4. *Let m $(\neq\pm1,\ \pm2)$ be a square free integer, $n(\geqq 3)$ an integer and η a primitive 2^n-th root of 1. If k, k' and L are $Q(\sqrt[2^n]{m})$, $Q(\sqrt{2}\times\sqrt[2^n]{m})$ and $k(\eta)$, respectively, then the conditions* (1), (2) *and* (3) *of lemma 2 hold and $k\ncong k'$.*

Proof. Put $K=Q(\eta)$. Suppose that $k\cong k'$. Then there exists an integer b such that $k'=Q(\sqrt[2^n]{m}\,\eta^b)$. This shows that $\sqrt{2}\,\eta^{-b}$ is contained in $Q(\sqrt[2^n]{m}\,\eta^b)$. On the other hand, for any integer a, we have $K\cap Q(\sqrt[2^n]{m}\,\eta^a)=Q$, which shows $\sqrt{2}\,\eta^{-b}\in Q$. This is a contradiction. Therefore we have $k\ncong k'$. Let $N=Gal(L/K)$. Then we have

$$N=\{\tau_b\in G\,|\,b\in \boldsymbol{Z},\ \eta^{\tau_b}=\eta \text{ and } \sqrt[2^n]{m}^{\tau_b}=\sqrt[2^n]{m}\,\eta^b\}$$

$$H=\{\sigma_a\in G\,|\,a\in \boldsymbol{Z},\ a \text{ is prime to } 2,\ \eta^{\sigma_a}=\eta^a \text{ and } \sqrt[2^n]{m}^{\sigma_a}=\sqrt[2^n]{m}\}$$

and

$$H'=\{\sigma'_a\in G\,|\,a\in \boldsymbol{Z},\ a \text{ is prime to } 2,\ \eta^{\sigma'_a}=\eta^a \text{ and } (\sqrt{2}\times\sqrt[2^n]{m})^{\sigma'_a}=\sqrt{2}\times\sqrt[2^n]{m}\}$$

The subgroup N of G is normal in G, $H\cap N=H'\cap N$ is trivial and $G=HN=H'N$.

Further for any elements $\sigma_a \in H$, $\sigma'_a \in H'$, $\tau_b \in N$, we have

$$\sigma_a^{-1}\tau_b\sigma_a = \tau_{ab} \quad \text{and} \quad \sigma_a'^{-1}\tau_b\sigma'_a = \tau_{ab}\,.$$

Therefore we can define an automorphism f of G such that

$$f(\sigma_a\tau_b) = \sigma'_a\tau_b \quad \text{for} \quad \sigma_a \in H,\ \tau_b \in N\,.$$

Since $\sqrt{2} = \eta^{2^{n-3}} + \eta^{-2^{n-3}}$, we have

$$(\sqrt{2} \times \sqrt[2^n]{m})^{\sigma_a} = \sqrt[2^n]{m}(\eta^{2^{n-3}a} + \eta^{-2^{n-3}a})\,.$$

Hence, if $a \equiv 1, 7 \pmod 8$, then $(\sqrt{2} \times \sqrt[2^n]{m})^{\sigma_a} = \sqrt{2} \times \sqrt[2^n]{m}$, and if $a \equiv 3, 5 \pmod 8$, then $(\sqrt{2} \times \sqrt[2^n]{m})^{\sigma_a} = -\sqrt{2} \times \sqrt[2^n]{m}$. On the other hand, for any element $\tau_c \in N$, we have

$$(\sqrt{2} \times \sqrt[2^n]{m})^{\tau_c\sigma'_a\tau_c^{-1}} = \sqrt{2} \times \sqrt[2^n]{m}\,\eta^{ac-c}\,.$$

The above consideration shows the following:

$$f(\sigma_a) = \begin{cases} \sigma_a & \text{for} \quad a \equiv 1, 7 \pmod 8 \\ \tau_{2^{n-2}}^{-1}\sigma_a\tau_{2^{n-2}} & \text{for} \quad a \equiv 3 \pmod 8 \\ \tau_{2^{n-3}}^{-1}\sigma_a\tau_{2^{n-3}} & \text{for} \quad a \equiv 5 \pmod 8\,. \end{cases}$$

Hence the automorphism f is an element-wise inner automorphism of G. Therefore our assertion follows from lemma 2 and lemma 3.

LEMMA 5. *Notations and assumptions being as in lemma 4, if*

$$m \equiv 2, 7, 14, 15 \pmod{16}\,,$$

then $k_A \cong k'_A$ *and* $k \not\cong k'$.

Proof. Let p be a prime number. Suppose that the decomposition of the ideal pO_k in k is as follows:

$$pO_k = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}\,, \qquad N_{k/Q}(\mathfrak{p}_i) = p^{f_i} \quad \text{for } i = 1, \cdots, g\,,$$

where $\mathfrak{p}_1, \cdots, \mathfrak{p}_g$ are distinct prime ideals of k. From lemma 4, there exist distinct prime ideals $\mathfrak{p}'_1, \cdots, \mathfrak{p}'_g$ in k' such that

$$pO_{k'} = \mathfrak{p}_1'^{e_1'} \cdots \mathfrak{p}_g'^{e_g'} \quad \text{and that} \quad N_{k'/Q}(\mathfrak{p}'_i) = p^{f_i} \quad \text{for } i = 1, \cdots, g\,.$$

If p is unramified in k/Q, then p is unramified in k'/Q. Therefore, for the prime number p which is unramified in k/Q, we have $k_{\mathfrak{p}_i} \cong k'_{\mathfrak{p}'_i}$ for $i = 1, \cdots, g$. Now we assume that p is ramified in k/Q and that $p \neq 2$. Since p divides m and since m is square free, p is totally ramified in k/Q and in k'/Q. If $p \equiv 1, 7 \pmod 8$, then $Q_p(\sqrt[2^n]{m}) = Q_p(\sqrt{2} \times \sqrt[2^n]{m})$ follows from that Q_p contains $\sqrt{2}$. If $p \equiv 3, 5 \pmod 8$, then $Q_p(\sqrt[2^n]{m}) \cong Q_p(\sqrt{-1} \times \sqrt[2^n]{m}) = Q_p(\sqrt{2} \times \sqrt[2^n]{m})$ follows from that Q_p contains

$\sqrt{-2}$. Suppose that $p=2$. One can easily see that p is totally ramified in k/Q. Hence from lemma 1, it is sufficient to prove $Q_p(\sqrt[2n]{m})\cong Q_p(\sqrt{2}\times\sqrt[2n]{m})$. There are three cases:

(1) $m\equiv 2 \pmod{16}$. Writing $m=2u$, we see that $\sqrt{u}\in Q_2$. Hence we have $Q_2(\sqrt[2n]{m})=Q_2(\sqrt{2}\times\sqrt[2n]{m})$.

(2) $m\equiv 14 \pmod{16}$. Writing $m=2u$, we see that $\sqrt{-u}\in Q_2$. So we have $\sqrt{-2}\in Q_2(\sqrt[2n]{m})\cap Q_2(\sqrt{-1}\times\sqrt[2n]{m})$. Hence we have $Q_2(\sqrt[2n]{m})\cong Q_2(\sqrt{-1}\times\sqrt[2n]{m})=Q_2(\sqrt{2}\times\sqrt[2n]{m})$.

(3) $m\equiv 7 \pmod 8$. Since $-m\equiv 1 \pmod 8$ shows $\sqrt{-m}\in Q_2$, we have $\sqrt{-1}\in Q_2(\sqrt{m})$. Hence we have $Q_2(\sqrt[2n]{m})\cong Q_2\left(\frac{\sqrt{2}}{2}\sqrt[2n]{m}(1+\sqrt{-1})\right)=Q_2(\sqrt{2}\times\sqrt[2n]{m})$.

LEMMA 6. *Notations and assumptions being as in lemma* 4, *if* $m\equiv 3, 5, 6, 10, 11, 13 \pmod{16}$, *then* $k_A\not\cong k'_A$ *and* $\zeta_k(s)=\zeta_{k'}(s)$.

Proof. The lemma 4 shows $\zeta_k(s)=\zeta_{k'}(s)$. Considering the structure of G, we see that the quandratic number fields in L are $Q(\sqrt{-1})$, $Q(\sqrt{2})$. $Q(\sqrt{-2})$, $(Q\sqrt{m})$, $Q(\sqrt{-m})$, $Q(\sqrt{2m})$ and $Q(\sqrt{-2m})$. In none of them, the ideal $2Z$ splits completely. Let $\mathfrak{P}$ be a prime divisor of the ideal $2O_L$, D the decomposition group of $\mathfrak{P}$ with respect to L/Q and F the decomposition field of $\mathfrak{P}$ with respect to L/Q. Suppose that $G\neq D$. As G is a 2-group, there exists a maximal proper subgroup N of G such that $N\supset D$ and that $[G\,;N]=2$. Let k_1 be the subfield of L corresponding to N. The ramification index and the degree of the ideal $\mathfrak{P}\cap k_1$ in k_1/Q are equal to 1. Since k_1/Q is a Galoi extension, the ideal $2Z$ splits completely in k_1/Q. This is a contradiction. Hence we have $G=D$. Let $L_{\mathfrak{p}}$ be the completion of L by $\mathfrak{P}$-adic valuation. We put $\mathfrak{p}=\mathfrak{P}\cap k$ and $\mathfrak{p}'=\mathfrak{P}\cap k'$. Let K(resp. K') be the topological closure of k(resp. k') in $L_{\mathfrak{P}}$. We should notice $K\cong k_{\mathfrak{p}}$ and $K'\cong k'_{\mathfrak{p}}$. Since $G=D$, there exists a natural isomorphism φ of $Gal(L/Q_2)$ onto G, where Q_2 is the topological closure of Q in $L_{\mathfrak{P}}$. We have $\varphi(Gal(L_{\mathfrak{P}}/K))=H$ and $\varphi(Gal(L_{\mathfrak{P}}/K'))=H'$. Since $k\not\cong k'$, H and H' are not conjugate in G. This shows $K\not\cong K'$, which means $k_{\mathfrak{p}}\not\cong k'_{\mathfrak{p}}$. Hence we have $k_A\not\cong k'_A$ from lemma 1.

LEMMA 7. *Let* u *be an element of* Q_2 *and* s $(\geqq 1)$ *an integer. If* $\sqrt{\pm u}\notin Q_2$, *then a polynomial* $x^{2^s}-u$ *is irreducible over* Q_2.

Proof. Suppose that there exist two polynomials

$$f(x)=x^r+\cdots+a \quad\text{and}\quad g(x)=x^t+\cdots+b \quad\text{in } Q_2[x]$$

such that $x^{2^s}-u=f(x)g(x)$. Let η be a primitive 2^s-th root of 1, and let v be an integer such that $a=\sqrt[2^s]{u}\,\eta^v$. We assume that $1\leqq r<2^s$. We can put $r=2^e c$. where c is a positive odd integer and where e is an integer such that $O\leqq e<s$. As $a^{2^s}=u^r$, we have $\pm a^{2^{s-e}}=u^c$. Since 2 is prime to c, there exist two integers α, β such that $2^{s-e}\alpha+c\beta=1$. So we have

$$u=u^{2^{s-e}\alpha+\beta c}=(u^\alpha)^{2^{s-e}}(\pm a^{2^{s-e}})^\beta\,,$$

which is a contradiction.

The following lemma is an elementary property of an algebraic number theory:

LEMMA 8. *Let n be a positive integer, p a prime number and α a p-adic integer. Suppose that p^a exactly divides n. For any integer s with $s>\frac{1}{p-1}+a$, if $\alpha\equiv 1 \pmod{p^s}$, then we have $\sqrt[n]{\alpha}\in Q_p$.*

LEMMA 9. *Let u be a 2-adic integer such that $u\equiv\pm 3 \pmod 8$. Then for an integer $s(\geqq 1)$, $Q_2(\sqrt[2^s]{u})\not\cong Q_2(\sqrt{2}\sqrt[2^s]{u})$.*

Proof. We should notice that $\sqrt{\pm u}\notin Q_2$. There are three cases:

(1) $s=1$. Suppose $Q_2(\sqrt{u})\cong Q_2(\sqrt{2u})$, which means $Q_2(\sqrt{u})=Q_2(\sqrt{2u})$. Then there exist elements a, b in Q_2 such that $\sqrt{2u}=a+b\sqrt{u}$. As

$$2u=a^2+ub^2+2ab\sqrt{u},$$

we conclude $ab=0$. If $a=0$, then $\sqrt{2}\in Q_2$, which is a contradiction. If $b=0$, then $\sqrt{2u}\in Q_2$, which is also a contradiction.

(2) $s=2$. Polynomials x^4-u and x^4-4u are irreducible over Q_2 from lemma 7. Suppose that $Q_2(\sqrt[4]{u})\cong Q_2(\sqrt{2}\sqrt[4]{u})$. Then $Q_2(\sqrt{2}\sqrt[4]{u})=Q_2(\sqrt[4]{u})$ or $Q_2(\sqrt{2}\sqrt[4]{u})=Q_2(\sqrt{-1}\sqrt[4]{u})$. Suppose $Q_2(\sqrt{2}\sqrt[4]{u})=Q_2(\sqrt[4]{u})$. Then there exist elements a, $b\in Q_2(\sqrt{u})$ such that $\sqrt{2}\sqrt[4]{u}=a+b\sqrt[4]{u}$. Since

$$2\sqrt{u}=a^2+b^2\sqrt{u}+2ab\sqrt[4]{u},$$

we conclude $ab=0$. If $a=0$, then $\sqrt{2}\in Q_2(\sqrt{u})$, which is a contradiction. If $b=0$, then $\sqrt{2}\sqrt[4]{u}\in Q_2(\sqrt{u})$, which is also a contradiction. Assuming $Q_2(\sqrt{2}\sqrt[4]{u})=Q_2(\sqrt{-1}\sqrt[4]{u})$, we have also a contradiction in an analogious way.

(3) $s\geqq 3$. Let η be a primitive 2^s-th root of 1. There exists a prime number p such that $p\equiv u \pmod{2^{s+2}}$. From lemma 8, we have $Q_2(\sqrt[2^s]{p})=Q_2(\sqrt[2^s]{u})$ and $Q_2(\sqrt{2}\times\sqrt[2^s]{p})=Q_2(\sqrt{2}\times\sqrt[2^s]{u})$. Since $p\equiv\pm 3 \pmod 8$, it follows from the proof of lemma 6 that $Q_2(\sqrt[2^s]{p})\not\cong Q_2(\sqrt{2}\times\sqrt[2^s]{p})$.

LEMMA 10. *Let m $(\neq\pm 1, \pm 2)$ be a square free integer, n $(\geqq 3)$ an integer and s $(\leqq n-1)$ a non-negative integer. Suppose that there exists a 2-adic integer u such that $m=u^{2^s}$ and such that $u\equiv\pm 3 \pmod 8$. Then*

$$x^{2^n}-m=(x^{2^{n-s}}-u)\prod_{\nu=0}^{s-1}(x^{2^{n-s+\nu}}+u^{2^\nu}) \quad \text{and}$$

$$x^{2^n}-2^{2^n-1}m=(x^{2^{n-s}}-2^{2^{n-1-s}}u)\prod_{\nu=0}^{s-1}(x^{2^{n-s+\nu}}+2^{2^{n-1-s+\nu}}u^{2^\nu})$$

are factorizations in irreducible polynomials of $Q_2[x]$.

Proof. We denote by Z_2 the 2-adic integer ring. Since $\sqrt{\pm u}\notin Q_2$, both polynomials $x^{2^{n-s}}-u$ and $x^{2^{n-s}}+u$ are irreducible over Q_2 from lemma 7. Now, $u\equiv\pm3 \pmod{2^3}$ implies that $u^{2^\nu}\equiv1 \pmod{2^{\nu+1}}$ for $\nu=1, 2, \cdots, s-1$. Hence a polynomial $(x+1)^{2^{n-\nu}}+u^{2^{s-\nu}}$ is an Eisenstein polynomial in $Z_2[x]$, for $\nu=1, \cdots, s-1$. This shows that a polynomial $x^{2^{n-\nu}}+u^{2^{s-\nu}}$ is irreducible over Q_2. Using lemma 4, we have that the number of the prime factors of 2 in k is $s+1$. This shows that polynomials $x^{2^{n-s}}-2^{n-1-s}u$ and $x^{2^{n-s+\nu}}+2^{2^{n-1-s+\nu}}u^{2^\nu}$ are irreducible over Q_2, for $\nu=0, 1, \cdots, s-1$.

LEMMA 11. *Let $m(\neq\pm1, \pm2)$ be a square free integer and $n(\geqq3)$ an integer. Further we put $k=Q(\sqrt[2^n]{m})$ and $k'=Q(\sqrt{2}\times\sqrt[2^n]{m})$. If*

$$m\equiv1 \pmod{2^{n+2}}, \text{ then } k_A\not\cong k'_A \text{ and } \zeta_k(s)=\zeta_{k'}(s).$$

Proof. It follows from lemma 4 that $\zeta_k(s)=\zeta_{k'}(s)$. By lemma 8, we can see that there exists a 2-adic integer u such that $m=u^{2^n}$. So we have

$$x^{2^n}-m=(x-u)(x+u)\prod_{i=2}^{n-1}(x^{2^i}+u^{2^i}) \quad \text{and}$$

$$x^{2^n}-2^{2^{n-1}}m=(x^2-2u)(x^2+2u)(x^2-2ux+2u)(x^2+2ux+2u)\prod_{i=3}^{n-1}(x^{2^i}+2^{2^{i-1}}u^{2^i})$$

are factorizations in irreducible polynomials of $Q_2[x]$. This shows that there exists a prime factor $\mathfrak{p}$ of 2 in k such that the ramification index and the degree of $\mathfrak{p}$ are 1. Considering the above factorization of the polynomial $x^{2^n}-2^{2^{n-1}}m$, we have a contradiction.

We should notice that for the above fields k and k', the collection of ramification indexes of 2 in k is not identical with the collection of ramification indexes of factors of 2 in k'.

LEMMA 12. *Notations and assumptions being as in lemma 11, if $m\equiv1 \pmod 8$, we have $k_A\not\cong k'_A$ and $\zeta_k(s)=\zeta_{k'}(s)$.*

Proof. It follows from lemma 4 that $\zeta_k(s)=\zeta_{k'}(s)$. We may assume that $m\equiv1 \pmod{2^{n+2-t}}$, where t is a non-negative integer such that $t\leqq n-1$. We use induction on t. If $t=0$, the lemma is an immediate consequence of lemma 11. Suppose $t\geqq1$. From lemma 8, there exists a 2-adic integer u such that $m=u^{2^{n-t}}$. Suppose $u\equiv\pm1 \pmod 8$. We can easily see that $m\equiv1 \pmod{2^{n-(t-1)+2}}$, which proves the lemma by applying the induction assumption. Suppose $u\equiv\pm3 \pmod 8$. We put $s=n-t$. Then we have $m=u^{2^s}$. From lemma 10, polynomials $x^{2^n}-m$ and $x^{2^n}-2^{n-1}m$ have the following factorizations in irreducible polynomials of $Q_2[x]$:

$$x^{2^n}-m=(x^{2^{n-s}}-u)\prod_{i=0}^{s-1}(x^{2^{n-s+i}}+u^{2^i})$$

$$x^{2^n}-2^{2^{n-1}}m=(x^{2^{n-s}}-2^{2^{n-1-s}}u)\prod_{i=0}^{s-1}(x^{2^{n-s+i}}+2^{n-1-s+i}u^{2^i}).$$

Therefore in order to prove this lemma, it is sufficient to show $Q_2(\sqrt[2^{n-s}]{u}) \not\cong Q_2(\sqrt{2} \times \sqrt[2^{n-s}]{u})$ and $Q_2(\sqrt[2^{n-s}]{u}) \not\cong Q_2(\sqrt{2} \times \sqrt[2^{n-s}]{-u})$ for $u \equiv 3 \pmod 8$. We have $Q_2(\sqrt[2^{n-s}]{u}) \not\cong Q_2(\sqrt{2} \times \sqrt[2^{n-s}]{u})$ from lemma 9. If $u \equiv 3 \pmod 8$, the ideal $2Z_2$ is fully ramified in $Q_2(\sqrt[2^{n-s}]{u})/Q_2$. If $u \equiv 3 \pmod 8$ and $n-s>1$, then the degree of the ideal $2Z_2$ in $Q_2(\sqrt{2} \times \sqrt[2^{n-s}]{-u})/Q_2$ is greater than 1. From the above consideration one can easily see $Q_2(\sqrt[2^{n-s}]{u}) \not\cong Q_2(\sqrt{2} \times \sqrt[2^{n-s}]{-u})$. This completes our proof.

The above lemmas show our theorem.

REFERENCES

[1] CASSELS-FROHLICH, Algebraic Number Theory, Academic Press, London NEW-York, (1967).

[2] I. GERST, On the theory of n-th power residue and a conjecture of Kronecker, Acta Arithmetica XII, 121-138, (1970).

[3] K. IWASAWA, On the rings of valuation vectors, Ann. of Math., **57**, 331-356, (1953).

[4] K. KOMATSU, On the adele rings of algebraic number fields, Kōdai Math. Sem. Rep., **28**, 78-84, (1976).

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