INFINITESIMAL VARIATIONS OF SUBMANIFOLDS OF AN EVEN-DIMENSIONAL SPHERE

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§ 0. Introduction

The present authors [3] have recently studied submanifolds of an even-dimensional sphere and one of the present authors [2] studied infinitesimal variations of submanifolds of Riemannian manifolds.

The purpose of the present paper is to study infinitesimal variations of submanifolds of an even-dimensional sphere.

In § 1, we state some of known results on the structure ([4], [5]) which an even-dimensional sphere admits and in § 2 those on various kinds of submanifolds of an even-dimensional sphere (see [3]).

§ 3 is devoted to the study of infinitesimal variations of these submanifolds, § 4 to the study of infinitesimal variations carrying an invariant submanifold of an even-dimensional sphere into an invariant one and § 5 to the study of those carrying an anti-invariant submanifold of an even-dimensional sphere into an anti-invariant one.

§ 1. Even-dimensional spheres

Let E^{2m+2} be a $(2m+2)$ -dimensional Euclidean space with standard complex structure F . We denote a point P in E^{2m+2} by its position vector X .

Let M^{2m} be a $2m$ -dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and denote by g_{ji} and ∇_j the metric tensor of M^{2m} and the operator of covariant differentiation with respect to g_{ji} respectively. Here and in the sequel the indices $h, i, j, k, \dots$ run over the range $\{1, 2, \dots, 2m\}$.

We assume that M^{2m} is isometrically immersed in E^{2m+2} by the immersion $\iota: M^{2m} \rightarrow E^{2m+2}$, identify $i(M^{2m})$ with M^{2m} and represent the immersion by

$$(1.1) \quad X = X(x).$$

Then the immersion being isometric, we have

$$(1.2) \quad g_{ji} = X_j \cdot X_i,$$

where $X_j = \partial_j X$ ($\partial_j = \partial/\partial x^j$) and the dot indicates the inner product in E^{2m+2} . The

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inner product in E^{2m+2} satisfies $(FY) \cdot (FZ) = Y \cdot Z$ for any vector fields Y and Z .

We denote by C and D two mutually orthogonal unit vectors normal to M^{2m} . Then for the transforms FX_i of X_i by F , we have equations of the form

$$(1.3) \quad FX_i = f_i^h X_h + u_i C + v_i D,$$

where f_i^h is a tensor field of type (1,1) and u_i, v_i are 1-forms of M^{2m} .

For FC and FD , we have equations of the form

$$(1.4) \quad FC = -u^h X_h + \lambda D$$

and

$$(1.5) \quad FD = -v^h X_h - \lambda C$$

respectively, where $u^h = u_i g^{ih}$, $v^h = v_i g^{ih}$ and λ is a function, g^{ih} being contravariant components of the metric tensor of M^{2m} .

Applying F to (1.3), (1.4) and (1.5) and using $F^2 = -1$ and these equations, we find

$$(1.6) \quad f_i^t f_t^h = -\delta_i^h + u_i u^h + v_i v^h,$$

$$(1.7) \quad u_i f_i^t = \lambda v_t, \quad v_i f_i^t = -\lambda u_t,$$

$$(1.8) \quad f_i^h u^t = -\lambda v^h, \quad f_i^h v^t = \lambda u^h,$$

$$(1.9) \quad u_i u^t = 1 - \lambda^2, \quad u_i v^t = 0, \quad v_i v^t = 1 - \lambda^2.$$

Also, from (1.3) and $(FX_j) \cdot (FX_i) = X_j \cdot X_i$, we find

$$(1.10) \quad f_j^t f_i^s g_{ts} = g_{ji} - u_j u_i - v_j v_i.$$

A structure defined by the set $(f_i^h, g_{ji}, u_i, v_i, \lambda)$ satisfying (1.6)~(1.10) is called an (f, g, u, v, λ) -structure [4], [5]. Thus a submanifold M^{2m} of codimension 2 of a $(2m+2)$ -dimensional Euclidean space admits an (f, g, u, v, λ) -structure depending on the choice of normals C and D .

From (1.6)~(1.10), we see that $f_{ji} = f_j^t g_{ti}$ satisfies $f_{ji} = -f_{ij}$.

Equations of Gauss and those of Weingarten for M^{2m} are respectively

$$(1.11) \quad \nabla_j X_i = h_{ji} C + k_{ji} D$$

and

$$(1.12) \quad \nabla_i C = -h_i^h X_h + \mu D, \quad \nabla_i D = -k_i^h X_h - \mu C,$$

where h_{ji} and k_{ji} are the second fundamental tensors of M^{2m} with respect to C and D respectively, $h_i^h = h_{it} g^{th}$, $k_i^h = k_{it} g^{th}$ and μ is a function.

We now suppose that M^{2m} is a sphere $S^{2m}(1)$ with radius 1 and with its center at the origin of E^{2m+2} . Thus we can put $C = -X$ and choose D as a unit vector field perpendicular to the hyperplane E^{2m+1} containing $S^{2m}(1)$. Then

C is a concurrent vector field and D is a parallel vector field. Thus from (1.12) we have $h_i{}^h = \delta_i^h$, $k_i{}^h = 0$, $\mu = 0$ and consequently $h_{ji} = g_{ji}$, $k_{ji} = 0$. Thus equations of Gauss and Weingarten reduce to

$$(1.13) \quad \nabla_j X_i = g_{ji} C, \quad \nabla_j C = -X_j, \quad \nabla_j D = 0.$$

Thus differentiating (1.3) covariantly along $S^{2m}(1)$ and using (1.4) and (1.13), we find

$$(1.14) \quad \nabla_j f_i{}^h = -g_{ji} u^h + \delta_i^h u_j,$$

$$(1.15) \quad \nabla_j u_i = f_{ji},$$

$$(1.16) \quad \nabla_j v_i = \lambda g_{ji}.$$

Equation (1.15) shows that u^h is a Killing vector field and (1.16) shows that v^h is a conformal Killing vector field.

Differentiating (1.4) and (1.5) covariantly along $S^{2m}(1)$ and using (1.3) and (1.13), we find

$$(1.17) \quad \nabla_i u^h = f_i{}^h,$$

$$(1.18) \quad \nabla_i v^h = \lambda \delta_i^h,$$

$$(1.19) \quad v_i = -\nabla_i \lambda,$$

As we see from (1.9), the vanishing of u^h or v^h in a domain of $S^{2m}(1)$ is equivalent to the fact that $\lambda = \pm 1$. But as (1.16) shows, the case in which $v^h = 0$ and $\lambda = \pm 1$ never happens in a domain of $S^{2m}(1)$. Thus, $u^h \neq 0$, $v^h \neq 0$ and $\lambda \neq \pm 1$ almost everywhere in $S^{2m}(1)$.

§ 2. Submanifolds of $S^{2m}(1)$

Let M^n be an n -dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and denote by g_{cb} and ∇_c the fundamental metric tensor and the operator of covariant differentiation with respect to g_{cb} respectively. Here and in the sequel, the indices $a, b, c, \dots$ run over the range $\{1', 2', \dots, n'\}$.

We assume that M^n is isometrically immersed in $S^{2m}(1)$ by the immersion $i: M^n \rightarrow S^{2m}(1)$ and identify $i(M^n)$ with M^n . We represent the immersion by

$$(2.1) \quad x^h = x^h(y^a).$$

If we put

$$(2.2) \quad B_b{}^h = \partial_b x^h \quad (\partial_b = \partial / \partial y^b),$$

then the immersion being isometric, we have

$$(2.3) \quad g_{cb} = B_c^j B_b^i g_{ji}.$$

We choose $2m-n$ mutually orthogonal unit vectors normal to M^n and denote them by C_y^h , where, here and in the sequel, the indices x, y, z run over the range $\{(n+1)', \dots, (2m)'\}$.

The transforms $f_i^h B_b^i$ of B_b^i by f_i^h are written in the form

$$(2.4) \quad f_i^h B_b^i = f_b^a B_a^h - f_b^z C_z^h$$

and the transforms $f_i^h C_y^i$ of C_y^i by f_i^h in the form

$$(2.5) \quad f_i^h C_y^i = f_y^a B_a^h + f_y^x C_x^h,$$

f_b^x and f_y^a being related by

$$(2.6) \quad f_{by} = f_{yb},$$

where $f_{by} = f_b^z g_{zy}$ and $f_{yb} = f_y^c g_{cb}$, g_{zy} being the metric tensor of the normal bundle of M^n .

When $f_i^h B_b^i$ are always tangent to M^n , that is, when $f_b^x = 0$, M^n is said to be *invariant*. In order for M^n to be invariant, it is necessary and sufficient that

$$(2.7) \quad f_{bx} = f_b^y g_{yx} = -f_{ji} B_b^j C_x^i = 0.$$

When $f_i^h B_b^i$ are always normal to M^n , that is, $f_b^a = 0$, M^n is said to be *anti-invariant*. In order for M^n to be anti-invariant, it is necessary and sufficient that

$$(2.8) \quad f_{ba} = f_b^c g_{ca} = f_{ji} B_b^j B_a^i = 0.$$

Equations of Gauss and those of Weingarten for M^n are respectively written as

$$(2.9) \quad \nabla_c B_b^h = h_{cb}^x C_x^h$$

and

$$(2.10) \quad \nabla_c C_y^h = -h_c^a{}_y B_a^h,$$

∇_c denoting the van der Waerden-Bortolotti covariant differentiation along M^n , where h_{cb}^x are the second fundamental tensors of M^n with respect to C_x^h , $h_c^a{}_y = h_{cb} y g^{ba}$ and $h_{cb} y = h_{cb}^z g_{zy}$.

(1) *Submanifolds of $S^{2m}(1)$ tangent to u^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is tangent to u^h . Then we have equation of the form $B_a^h u^a = u^h$, from which, differentiating covariantly along M^n and using (1.17) and (2.4), we find

$$h_{cb}^x u^b C_x^h + B_a^h \nabla_c u^a = f_c^a B_a^h - f_c^x C_x^h,$$

and consequently

$$(2.11) \quad \nabla_c u_b = f_{cb}, \quad h_{cb} u^b = -f_{cy},$$

f_{cb} being skew-symmetric, we have

PROPOSITION 2.1 ([3]). *Suppose that a submanifold M^n of $S^{2m}(1)$ is tangent to u^h , then u^h is a Killing vector field of M^n . Moreover, in order for M^n to be invariant in $S^{2m}(1)$, it is necessary and sufficient that u^h is in the principal direction with principal curvature zero.*

PROPOSITION 2.2 ([3]). *In order for a submanifold M^n of $S^{2m}(1)$ tangent to u^h to be anti-invariant in $S^{2m}(1)$, it is necessary and sufficient that u^h is a parallel vector field of M^n .*

(2) *Submanifolds of $S^{2m}(1)$ normal to u^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is normal to u^h . Then we have $B_b{}^i u_i = 0$, from which, differentiating covariantly along M^n and using (1.15) and (2.4), we find

$$h_{cb}{}^x C_x{}^i u_i + B_b{}^i (f_c{}^a B_{ai} - f_c{}^x C_{xi}) = 0,$$

where $B_{ai} = B_a{}^j g_{ji}$ and $C_{xi} = C_x{}^j g_{ji}$ and consequently

$$h_{cb}{}^x C_x{}^i u_i + f_{cb} = 0.$$

Thus, $h_{cb}{}^x C_x{}^i u_i$ being symmetric and f_{cb} being skew-symmetric in c and b , we have

$$(2.12) \quad h_{cb}{}^x C_x{}^i u_i = 0, \quad f_{cb} = 0.$$

Thus we have

PROPOSITION 2.3 ([3]). *If a submanifold M^n of $S^{2m}(1)$ is normal to u^h , then M^n is geodesic with respect to u^h and is anti-invariant.*

(3) *Submanifolds of $S^{2m}(1)$ tangent to v^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is tangent to v^h . Then we have equation of the form $B_a{}^h v^a = v^h$, from which, differentiating covariantly along M^n and using (1.18), we have

$$h_{cb}{}^x v^b C_x{}^h + B_a{}^h \nabla_c v^a = \lambda B_c{}^h,$$

and consequently

$$(2.13) \quad \nabla_c v_b = \lambda g_{cb}, \quad h_{cb}{}^x v^b = 0.$$

On the other hand, from (1.19), we have $v_b = -\nabla_b \lambda$ and consequently, from the first equation of (2.13), we have $\nabla_c \nabla_b \lambda = -\lambda g_{cb}$. Thus by a theorem of Obata [1], we have

PROPOSITION 2.4 ([3]). *If a complete connected submanifold M^n of $S^{2m}(1)$ is*

tangent to v^h , then M^n is a sphere of radius 1 and v^h is in the principal direction of M^n with principal curvature zero.

(4) Submanifolds of $S^{2m}(1)$ normal to v^h

Suppose that a submanifold M^n of $S^{2m}(1)$ is normal to v^h . Then we have $B_b^i v_i = 0$ or $B_b^i \nabla_i \lambda = 0$, which shows that λ is constant along M^n . Differentiating $B_b^i v_i = 0$ covariantly along M^n and using (1.16), we find

$$(2.14) \quad h_{cb}^x C_x^i v_i + \lambda g_{cb} = 0.$$

We also have $\nabla_b v^h = \lambda B_b^h$ which shows that the normal v^h to M^n is parallel or concurrent according as $\lambda = 0$ or $\lambda \neq 0$ along M^n . Thus we have

PROPOSITION 2.5 ([3]). Suppose that a submanifold M^n of $S^{2m}(1)$ is normal to v^h . Then $\lambda = \text{constant}$ along M^n .

If $\lambda = 0$ along M^n , then M^n is on a great sphere of $S^{2m}(1)$ and is geodesic with respect to v^h . Moreover v^h is parallel along M^n .

If $\lambda \neq 0$ along M^n , then M^n is on a small sphere of $S^{2m}(1)$ and is umbilical with respect to v^h . Moreover v^h is concurrent along M^n .

§ 3. Infinitesimal variations of submanifolds of $S^{2m}(1)$ [2]

We now consider an infinitesimal variation

$$(3.1) \quad \bar{x}^h = x^h + \xi^h(y) \varepsilon$$

of a submanifold M^n of $S^{2m}(1)$, where ξ^h is a vector field of $S^{2m}(1)$ defined along M^n and ε is an infinitesimal. We then have $\bar{B}_b^h = B_b^h + \partial_b \xi^h \varepsilon$, where $\bar{B}_b^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold at varied point $(\bar{x}^h)$. We displace $\bar{B}_b^h$ back parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h) and obtain $\tilde{B}_b^h = B_b^h + (\nabla_b \xi^h) \varepsilon$, neglecting terms of order higher than one with respect to ε . In the sequel, we neglect always terms of order higher than one with respect to ε . Thus putting $\delta B_b^h = \tilde{B}_b^h - B_b^h$, we obtain

$$(3.2) \quad \delta B_b^h = (\nabla_b \xi^h) \varepsilon.$$

On the other hand, putting

$$(3.3) \quad \xi^h = \xi^a B_a^h + \xi^x C_x^h,$$

we have

$$(3.4) \quad \nabla_b \xi^h = (\nabla_b \xi^a - h_b^a \xi^x) B_a^h + (\nabla_b \xi^x + h_{ba}^x \xi^a) C_x^h.$$

When the tangent space at a point (x^h) of the submanifold and that at the corresponding point $(\bar{x}^h)$ of the varied submanifold are parallel, the variation is said to be *parallel*. From (3.2) and (3.4) we see that in order for an infinitesimal variation to be parallel, it is necessary and sufficient that

$$(3.5) \quad \nabla_b \xi^x + h_{ba}^x \xi^a = 0.$$

We denote by $\bar{C}_y^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y^h$ vectors obtained from $\bar{C}_y^h$ by parallel displacement of $\bar{C}_y^h$ from the varied point $(\bar{x}^h)$ back to the original point (x^h) . Putting $\delta C_y^h = \tilde{C}_y^h - C_y^h$, we find [2]

$$(3.6) \quad \bar{C}_y^h = C_y^h - \Gamma_{ji}^h \xi^j C_y^i + \delta C_y^h,$$

where Γ_{ji}^h are Christoffel symbols formed with g_{ji} . Assuming that δC_y^h are infinitesimals of order one with respect to ε and putting

$$(3.7) \quad \delta C_y^h = (\eta_y^a B_a^h + \eta_y^x C_x^h) \varepsilon,$$

we have

$$(3.8) \quad \eta_y^a = -(\nabla_y^a \xi_y + h_b^a \eta_y^b),$$

where $\xi_y = g_{yx} \xi^x$. Then η_y^x appearing in (3.7) is a tensor field of the normal bundle of M^n satisfying $\eta_{yx} + \eta_{xy} = 0$, η_{yx} being defined by $\eta_{yx} = \eta_y^z g_{zx}$.

(1) *Infinitesimal variations of submanifolds of $S^{2m}(1)$ tangent to u^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is tangent to u^h . Then we have equation of the form $B_a^h u^a = u^h$. In order that the varied submanifold is also tangent to $u^h(\bar{x})$, it is necessary and sufficient that we have equation of the form $\bar{B}_a^h \bar{u}^a = u^h(\bar{x})$, from which, putting $\bar{u}^a = u^a + \delta u^a$,

$$(B_b^h + \partial_b \xi^h \varepsilon)(u^b + \delta u^b) = u^h + \xi^j \partial_j u^h \varepsilon.$$

Thus using (1.17), we find

$$(3.9) \quad B_b^h \delta u^b + [(\nabla_b \xi^h) u^b - f_i^h \xi^i] \varepsilon = 0.$$

On the other hand, using (2.4), (2.5) and (3.3), we find

$$(3.10) \quad f_i^h \xi^i = (f_b^a \xi^b + f_y^a \xi^y) B_a^h + (-f_b^x \xi^b + f_y^x \xi^y) C_x^h.$$

Thus, substituting (3.4) and (3.10) into (3.9), we find

$$\begin{aligned} B_b^h \delta u^b + [(\nabla_b \xi^a - h_{ba}^x \xi^x) u^b - (f_b^a \xi^b + f_y^a \xi^y)] B_a^h \varepsilon \\ + [(\nabla_b \xi^x + h_{be}^x \xi^e) u^b + (f_b^x \xi^b - f_y^x \xi^y)] C_x^h \varepsilon = 0, \end{aligned}$$

or using (2.11)

$$\begin{aligned} B_b^h \delta u^b + (u^b \nabla_b \xi^a - \xi^b \nabla_b u^a) B_a^h \varepsilon \\ + (u^b \nabla_b \xi^x - f_y^x \xi^y) C_x^h \varepsilon = 0. \end{aligned}$$

from which

$$(3.11) \quad \delta u^a = (\mathcal{L} u^a) \varepsilon,$$

$\mathcal{L}$ denoting the Lie derivative with respect to ξ^a and

$$(3.12) \quad u^b \nabla_b \xi^x - f_y^x \xi^y = 0.$$

Thus we have

PROPOSITION 3.1. *In order for an infinitesimal variation (3.1) to carry a submanifold M^n of $S^{2m}(1)$ tangent to u^h into a submanifold $\bar{M}^n$ also tangent to $u^h(\bar{x})$, it is necessary and sufficient that (3.12) holds, the variation of u^a being given by (3.11).*

Since $f_{yx} = -f_{xy}$, we have, from (3.12),

$$u^b \nabla_b (\xi^x \xi_x) = 0.$$

Thus we have

COROLLARY. *Suppose that an infinitesimal variation carries a submanifold M^n of $S^{2m}(1)$ tangent to u^h into a submanifold $\bar{M}^n$ also tangent to $u^h(\bar{x})$. Then the length of the normal part of the variation vector is constant in the direction of u^h .*

(2) *Infinitesimal variations of submanifolds of $S^{2m}(1)$ normal to u^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is normal to u^h . Then we have the equation $B_b^i u_i = 0$. In order that the varied submanifold is also normal to $u^h(\bar{x})$, it is necessary and sufficient that we have $\bar{B}_b^i u_i(\bar{x}) = 0$, from which

$$[B_b^i + (\partial_b \xi^i) \varepsilon] [u_i + \xi^j \partial_j u_i \varepsilon] = 0.$$

Thus, using (1.15), we find

$$(\nabla_b \xi^i) u_i + f_{ji} \xi^j B_b^i = 0.$$

Substituting (3.4) and (3.10) into this equation, we have

$$\begin{aligned} & [(\nabla_b \xi^a - h_b^a \xi^x) B_a^i + (\nabla_b \xi^x + h_{be}^x \xi^e) C_x^i] u_i \\ & + [(f_c^a \xi^c + f_y^a \xi^y) B_{ai} + (-f_c^x \xi^c + f_y^x \xi^y) C_{xi}] B_b^i = 0, \end{aligned}$$

from which

$$(\nabla_b \xi^x + h_{be}^x \xi^e) C_x^i u_i + f_{cb} \xi^c + f_{yb} \xi^y = 0.$$

But we know that in this case $h_{be}^x u_x = 0$ and $f_{cb} = 0$. Thus the above equation becomes

$$(3.13) \quad (\nabla_b \xi^x) u_x + f_{xb} \xi^x = 0.$$

Thus we have

PROPOSITION 3.2. *In order for an infinitesimal variation (3.1) to carry a submanifold M^n of $S^{2m}(1)$ normal to u^h into a submanifold normal to $u^h(\bar{x})$, it is*

necessary and sufficient that (3.13) holds.

(3) *Infinitesimal variations of submanifolds of $S^{2m}(1)$ tangent to v^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is tangent to v^h . Then we have equation of the form $B_a{}^h v^a = v^h$. In order that the varied submanifold is also tangent to $v^h(\bar{x})$, it is necessary and sufficient that we have equation of the form $\bar{B}_a{}^h \bar{v}^a = v^h(\bar{x})$, from which, putting $\bar{v}^a = v^a + \delta v^a$,

$$[B_b{}^h + \partial_b \xi^h \varepsilon][v^b + \delta v^b] = v^h + (\xi^j \partial_j v^h) \varepsilon.$$

Thus, using (1.18), we have

$$B_a{}^h \delta v^a + [(\nabla_b \xi^h) v^b - \lambda \xi^h] \varepsilon = 0.$$

Thus, substituting (3.3) and (3.4) into this equation, we find

$$\begin{aligned} B_a{}^h \delta v^a + [(\nabla_b \xi^a - h_b{}^a x \xi^x) v^b - \lambda \xi^a] B_a{}^h \varepsilon \\ + [(\nabla_b \xi^x + h_{be}{}^x \xi^e) v^b - \lambda \xi^x] C_x{}^h \varepsilon = 0, \end{aligned}$$

from which, using (2.13),

$$B_a{}^h \delta v^a + (v^b \nabla_b \xi^a - \lambda \xi^a) B_a{}^h \varepsilon + (v^b \nabla_b \xi^x - \lambda \xi^x) C_x{}^h \varepsilon = 0,$$

and consequently, using $\lambda \xi^a = (\nabla_b v^a) \xi^b$,

$$(3.14) \quad \delta v^a = \mathcal{L} v^a \varepsilon,$$

$\mathcal{L}$ denoting the Lie derivative with respect to ξ^a and

$$(3.15) \quad v^b \nabla_b \xi^x - \lambda \xi^x = 0.$$

Thus we have

PROPOSITION 3.3. *In order for an infinitesimal variation (3.1) to carry a submanifold M^n of $S^{2m}(1)$ tangent to v^h into a submanifold tangent to $v^h(\bar{x})$, it is necessary and sufficient that (3.15) holds, the variation of v^a being given by (3.14).*

(4) *Infinitesimal variations of submanifolds of $S^{2m}(1)$ normal to v^h*

Suppose that a submanifold M^n of $S^{2m}(1)$ is normal to v^h . Then we have equation of the form $B_b{}^i v_i = 0$. In order that the varied submanifold is also normal to $v^h(\bar{x})$, it is necessary and sufficient that we have $\bar{B}_b{}^i \bar{v}_i = 0$, from which

$$[B_b{}^i + (\partial_b \xi^i) \varepsilon][v_i + (\xi^j \partial_j v_i) \varepsilon] = 0.$$

Thus using (1.16), we have

$$(\nabla_b \xi^i) v_i + \lambda B_b{}^i \xi_i = 0.$$

Therefore substituting (3.4) into this equation, we have

$$(\nabla_b \xi^x + h_{be}^x \xi^e) v_x + \lambda \xi_b = 0,$$

or, using (2.14),

$$(3.16) \quad (\nabla_b \xi^x) v_x = 0.$$

PROPOSITION 3.4. *In order for an infinitesimal variation (3.1) to carry a submanifold M^n of $S^{2m}(1)$ normal to v^h into a submanifold normal to $v^h(\bar{x})$, it is necessary and sufficient that (3.16) holds.*

From (3.5) and (3.16), we have

COROLLARY. *An infinitesimal normal parallel variation carries a submanifold M^n of $S^{2m}(1)$ normal to v^h into a submanifold normal to $v^h(\bar{x})$.*

§ 4. Infinitesimal variations carrying an invariant submanifold of $S^{2m}(1)$ into an invariant one

We assume that M^n is an invariant submanifold of $S^{2m}(1)$. We then have

$$(4.1) \quad f_i^h B_b^i = f_b^a B_a^h, \quad f_i^h C_y^i = f_y^x C_x^h.$$

Differentiating the first equation of (4.1) covariantly along M^n and using (1.14), we find

$$\begin{aligned} & (-g_{ji} u^h + \delta_j^h u_i) B_c^j B_b^i + f_i^h h_{cb}^x C_x^i \\ & = (\nabla_c f_b^a) B_a^h + f_b^e h_{ce}^x C_x^h, \end{aligned}$$

from which, putting $u^h = u^a B_a^h + u^x C_x^h$ and using (4.1),

$$\begin{aligned} & -g_{cb} (u^a B_a^h + u^x C_x^h) + B_c^h u_b + h_{cb}^y f_y^x C_x^h \\ & = (\nabla_c f_b^a) B_a^h + h_{ce}^x f_b^e C_x^h, \end{aligned}$$

and consequently, comparing the tangential and normal parts,

$$(4.2) \quad \nabla_c f_b^a = -g_{cb} u^a + \delta_c^a u_b$$

and

$$(4.3) \quad h_{cb}^y f_y^x - h_{ce}^x f_b^e = g_{cb} u^x,$$

from which, taking the skew-symmetric part,

$$(4.4) \quad h_{ce}^x f_b^e = h_{be}^x f_c^e.$$

Differentiating the second equation of (4.1) covariantly along M^n and using (1.14), we find

$$\begin{aligned} & (-g_{ji} u^h + \delta_j^h u_i) B_c^j C_y^i - f_i^h h_c^a{}_y B_a^i \\ & = (\nabla_c f_y^x) C_x^h - f_y^x h_c^a{}_x B_a^h, \end{aligned}$$

from which, using (4.1),

$$B_c^h u_y - h_c^e f_e^a B_a^h = (\nabla_c f_y^x) C_x^h - h_c^a f_y^x B_a^h,$$

and consequently

$$(4.5) \quad h_c^e f_e^a - h_c^a f_y^x = \delta_c^a u_y,$$

which is equivalent to (4.3) and

$$(4.6) \quad \nabla_c f_y^x = 0.$$

We now consider an infinitesimal variation (3.1) and assume that it carries the invariant submanifold M^n of $S^{2m}(1)$ into an invariant submanifold. Then we have

$$f_i^h(x + \xi \varepsilon) \bar{B}_b^i = (f_b^a + \delta f_b^a) \bar{B}_a^h,$$

that is,

$$\begin{aligned} (f_i^h + \xi^j \partial_j f_i^h \varepsilon) (B_b^i + \partial_b \xi^i \varepsilon) \\ = (f_b^a + \delta f_b^a) (B_a^h + \partial_a \xi^h \varepsilon), \end{aligned}$$

from which, using (1.14), we obtain

$$\begin{aligned} [f_i^h + \xi^j (-\Gamma_{ji}^h f_i^t + \Gamma_{ji}^t f_t^h - g_{ji} u^h + \delta_j^h u_i) \varepsilon] (B_b^i + \partial_b \xi^i \varepsilon) \\ = (f_b^a + \delta f_b^a) (B_a^h + \partial_a \xi^h \varepsilon), \end{aligned}$$

or

$$(f_i^h \nabla_b \xi^i - f_b^a \nabla_a \xi^h - \xi_b u^h + u_b \xi^h) = (\delta f_b^a) B_a^h.$$

Thus substituting (3.4) into this equation, we have

$$\begin{aligned} [(\nabla_b \xi^e - h_b^e f_e^x) f_e^a B_a^h + (\nabla_b \xi^y + h_{be}^y \xi^e) f_y^x C_x^h \\ - f_b^e (\nabla_e \xi^a - h_e^a f_e^x) B_a^h - f_b^e (\nabla_e \xi^x + h_{ed}^x \xi^d) C_x^h \\ - \xi_b (u^a B_a^h + u^x C_x^h) + u_b (\xi^a B_a^h + \xi^x C_x^h)] \varepsilon = (\delta f_b^a) B_a^h, \end{aligned}$$

where we have put $u^h = u^a B_a^h + u^x C_x^h$, from which

$$\begin{aligned} \delta f_b^a = [(\nabla_b \xi^e - h_b^e f_e^x) f_e^a - f_b^e (\nabla_e \xi^a - h_e^a f_e^x) \\ - \xi_b u^a + u_b \xi^a] \varepsilon \end{aligned}$$

or, using (4.2) and (4.4),

$$(4.7) \quad \delta f_b^a = [(\mathcal{L} f_b^a) - 2h_b^e f_e^x \xi^a] \varepsilon$$

and

$$(\nabla_b \xi^y + h_{be}^y \xi^e) f_y^x - f_b^e (\nabla_e \xi^x + h_{ed}^x \xi^d) - \xi_b u^x + u_b \xi^x = 0,$$

or, using (4.3),

$$(4.8) \quad (\nabla_b \xi^y) f_y^x - f_b^e (\nabla_e \xi^x) + u_b \xi^x = 0.$$

Thus we have

PROPOSITION 4.1. *In order for an infinitesimal variation (3.1) to carry an invariant submanifold M^n of $S^{2m}(1)$ into an invariant one, it is necessary and sufficient that (4.8) holds, the variation of f_b^a being given by (4.7).*

§ 5. Infinitesimal variations carrying an anti-invariant submanifold into an anti-invariant one

We assume that M^n is an anti-invariant submanifold of $S^{2m}(1)$. We then have

$$(5.1) \quad f_i^h B_b^i = -f_b^x C_x^h, \quad f_i^h C_y^i = f_y^a B_a^h + f_y^x C_x^h.$$

Differentiating the first equation of (5.1) covariantly along M^n and using (1.14), we find

$$\begin{aligned} & (-g_{ji} u^h + \delta_j^h u_i) B_c^j B_b^i + f_i^h h_{cb}^x C_x^i \\ & = -(\nabla_c f_b^x) C_x^h + f_b^x h_c^a{}^x B_a^h, \end{aligned}$$

from which, using (5.1),

$$\begin{aligned} & -g_{cb} (u^a B_a^h + u^x C_x^h) + B_c^h u_b + h_{cb}^y (f_y^a B_a^h + f_y^x C_x^h) \\ & = -(\nabla_c f_b^x) C_x^h + h_c^a{}^x f_b^x B_a^h, \end{aligned}$$

and consequently, comparing the tangential and normal parts

$$(5.2) \quad h_{cb}^x f_x^a - h_c^a{}^x f_b^x - g_{cb} u^a + \delta_c^a u_b = 0$$

and

$$(5.3) \quad \nabla_c f_b^x = g_{cb} u^x - h_{cb}^y f_y^x.$$

From (5.2), taking the skew-symmetric part, we have

$$(5.4) \quad h_c^a{}^x f_b^x - h_b^a{}^x f_c^x = \delta_c^a u_b - \delta_b^a u_c.$$

Differentiating the second equation of (5.1) covariantly along M^n and using (1.14), we find

$$\begin{aligned} & (-g_{ji} u^h + \delta_j^h u_i) B_c^j C_y^i - f_i^h h_c^a{}^y B_a^i \\ & = (\nabla_c f_y^a) B_a^h + f_y^e h_{ce}^x C_x^h + (\nabla_c f_y^x) C_x^h - f_y^x h_c^a{}^x B_a^h, \end{aligned}$$

from which, using (5.1),

$$\begin{aligned} & B_c^h u_y + h_c^a{}_y f_a^x C_x^h \\ &= (\nabla_c f_y^a) B_a^h + h_{ce}^x f_y^e C_x^h + (\nabla_c f_y^x) C_x^h - h_c^a{}_x f_y^x B_a^h. \end{aligned}$$

Thus, comparing the tangential and normal parts, we have

$$(5.5) \quad \nabla_c f_y^a = \delta_c^a u_y + h_c^a{}_x f_y^x,$$

which is equivalent to (5.3) and

$$(5.6) \quad \nabla_c f_y^x = h_c^a{}_y f_a^x - h_{ce}^x f_y^e.$$

We now consider an infinitesimal variation (3.1) and assume that it carries the anti-invariant submanifold M^n into an anti-invariant submanifold. Then we have

$$f_i^h(x + \xi \varepsilon) \bar{B}_b^i = -(f_b^x + \delta f_b^x) \bar{C}_x^h,$$

that is, using (3.6) and (3.7),

$$\begin{aligned} & (f_i^h + \xi^j \partial_j f_i^h)(B_b^i + \partial_b \xi^i \varepsilon) \\ &= -(f_b^x + \delta f_b^x)[C_x^h - \Gamma_{ji}^h \xi^j C_x^i \varepsilon + (\eta_x^a B_a^h + \eta_x^y C_y^h) \varepsilon], \end{aligned}$$

from which, using (1.14), we obtain

$$\begin{aligned} & [f_i^h + \xi^j (-\Gamma_{ji}^h f_i^t + \Gamma_{ji}^t f_i^h - g_{ji} u^h + \delta_j^h u_i) \varepsilon][B_b^i + \partial_b \xi^i \varepsilon] \\ &= -(f_b^y + \delta f_b^y)[C_y^h - \Gamma_{ji}^h \xi^j C_y^i \varepsilon + (\eta_y^a B_a^h + \eta_y^x C_x^h) \varepsilon], \end{aligned}$$

that is,

$$\begin{aligned} & [f_i^h (\nabla_b \xi^i) - \xi_b u^h + u_b \xi^h] \\ &= -f_b^y (\eta_y^a B_a^h + \eta_y^x C_x^h) \varepsilon - (\delta f_b^y) C_y^h. \end{aligned}$$

Thus, substituting (3.4) into the above equation, we find

$$\begin{aligned} & [-(\nabla_b \xi^a - h_b^a{}_y \xi^y) f_a^x C_x^h + (\nabla_b \xi^y + h_{be}^y \xi^e) (f_y^a B_a^h + f_y^x C_x^h) \\ & \quad - \xi_b (u^a B_a^h + u^x C_x^h) + u_b (\xi^a B_a^h + \xi^x C_x^h)] \varepsilon \\ &= -f_b^y (\eta_y^a B_a^h + \eta_y^x C_x^h) \varepsilon - (\delta f_b^y) C_y^h, \end{aligned}$$

from which, comparing tangential and normal parts and using (3.8),

$$\begin{aligned} (5.7) \quad & (\nabla_b \xi^y + h_{be}^y \xi^e) f_y^a - \xi_b u^a + u_b \xi^a \\ &= f_b^y (\nabla^a \xi_y + h_e^a{}_y \xi^e), \end{aligned}$$

which is, according to (5.2), equivalent to

$$(5.8) \quad (\nabla_b \xi^y) f_y^a = f_b^y \nabla^a \xi_y,$$

and

$$(5.9) \quad \delta f_b{}^x = [(\nabla_b \xi^a - h_b{}^a{}_y \xi^y) f_a{}^x - (\nabla_b \xi^y + h_{be}{}^y \xi^e) f_y{}^x + \xi_b u^x + u_b \xi^x - f_b{}^y \eta_y{}^x] \varepsilon.$$

Thus we have

PROPOSITION 5.1. *In order for an infinitesimal variation (3.1) to carry an anti-invariant submanifold M^n of $S^{2m}(1)$ into an anti-invariant one, it is necessary and sufficient that (5.8) holds, the variation of $f_b{}^x$ being given by (5.9).*

From (3.5) and (5.7), we have

PROPOSITION 5.2. *An infinitesimal parallel variation carries an anti-invariant submanifold M^n of $S^{2m}(1)$ into an anti-invariant one if and only if ξ^a is in the direction of u^a .*

From (3.5) and (5.8), we have

PROPOSITION 5.3. *An infinitesimal normal parallel variation carries always an anti-invariant submanifold M^n of $S^{2m}(1)$ into an anti-invariant one.*

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