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On Umezawa's criteria for univalence.

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1. In an interesting paper of recent date, Umezawa [5] obtained some new criteria that a function analytic in a certain domain should be univalent there. Those criteria all involve the change in direction of the tangent vector to the image of the boundary.

In this note we extend slightly some of Umezawa's results, and we give what we believe are simpler proofs yielding slightly more precise results. We use a device introduced by Umezawa, and a result due to Kaplan and Umezawa, to show that the function $\int_0^z e^{-\zeta^2}\,d\zeta$ is univalent for $|z|<1.51$; this improves upon estimates due to Nehari [2] and Rogozin [4].

We shall make use of the results obtained by Kaplan [1] in a recent paper in which he introduced univalent close-to-convex functions.

2. The following result is Umezawa's fundamental lemma [5; p. 213]. Our proof avoids Umezawa's geometric argument and shows that Umezawa's result is equivalent to Kaplan's fundamental result [1; p. 173].

THEOREM I. *Let $f(z)$ be analytic inside and on the simple closed analytic curve Γ, and let $f'(z)$ have no zeros on Γ. If*

$$\int_\Gamma d\arg df(z)=2\pi\,, \tag{1}$$

and if for all arcs C on Γ we have

$$\int_C d\arg df(z)>-\pi\,, \tag{2}$$

then $f(z)$ is univalent inside Γ, and the image of Γ a simple close-to-convex curve.

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PROOF. Let $\mathfrak{D}$ denote the domain inside Γ, and let $z=g(t)$ be an analytic function that maps $\mathfrak{D}$ conformally onto $|t|<1$. A simple calculation shows that $F(t)\equiv f(g(t))$ has the following properties. If $t=e^{i\phi}$, then

$$\int_0^{2\pi} \mathfrak{R}\left(1+e^{i\phi}\frac{F''(e^{i\phi})}{F'(e^{i\phi})}\right)d\phi=\int_\Gamma d\arg df(z)=2\pi\,. \tag{3}$$

For all $0\leqq\phi_1<\phi_2\leqq 2\pi$, we have

$$\int_{\phi_1}^{\phi_2} \mathfrak{R}\left(1+e^{i\phi}\frac{F''(e^{i\phi})}{F'(e^{i\phi})}\right)d\phi=\int_C d\arg df(z)>-\pi\,. \tag{4}$$

Since $f'(z)$ has no zeros on Γ, it follows that $F'(t)$ has no zeros on the circle $|t|=1$. Also, from (3) we obtain

$$\int_{|t|=1}\frac{F''(t)}{F'(t)}dt=0\,,$$

from which conclude that $F'(t)$ has no zeros for $|t|\leqq 1$ (and hence $f'(z)$ has no zeros in $\overline{\mathfrak{D}}$). From (4) and the fact that $F'(t)\neq 0$ for $|t|\leqq 1$, we can conclude [1; pp. 175-176] that $F(t)$ is a close-to-convex univalent function for $|t|\leqq 1$, and therefore the image of Γ under $f(z)$ is a close-to-convex curve. This completes the proof.

As Umezawa points out, because (1) holds we can replace (2) by the condition that

$$\int_C d\arg df(z)<3\pi$$

hold for all arcs C on Γ.

COROLLARY. *Under the hypothesis of the preceding theorem, if $z_0\in\mathfrak{D}$, then the image curve of each level line of the Green's function $G(z,z_0)$ for $\mathfrak{D}$, with pole z_0, is again a close-to-convex curve.*

PROOF. Let $z=g(t)$ be the mapping function noted above. Then the image of a level line $G(z,z_0)=\lambda$ under $f(z)$ is the image of a certain circle in $|t|<1$ under $F(t)$, and these we have shown to be close-to-convex curves [3; p. 60].

In the light of the preceding results, it may be of some interest to state, without proof, the following equivalent form of Umezawa's fundamental lemma.

THEOREM II. *Let $\mathfrak{D}$ be a simply connected domain, which is not the whole plane, and let $f(z)$ be a function that is analytic with non-vanishing derivative in $\mathfrak{D}$. Let $z_0\in\mathfrak{D}$, and let $G(z,z_0)=\lambda$ be the level* *lines of the Green's function for $\mathcal{D}$. If $f(z)$ satisfies the following conditions for all level lines of $G(z, z_0)$,*

$$\int_{G(z,z_0)=\lambda} d \arg df(z) = 2\pi ,$$

and

$$\int_C d \arg df(z) > -\pi$$

for each arc C on $G(z, z_0) = \lambda$, then $f(z)$ is univalent in $\mathcal{D}$ and maps $\mathcal{D}$ onto a close-to-convex domain.

We remark, without going into details, that by the same methods as used in the proofs above, we can show that Umezawa's Theorem 3 [5; p. 215] is equivalent to his Theorem A [5; p. 212], and hence the image of Γ in his first theorem is a curve that is convex in one direction. We also call attention to early papers of Paatero, in which similar topics are discussed in great detail [3].

3. Now we shall employ a device introduced by Umezawa along with Theorem I [5; p. 214] in order to determine a radius of univalence for the function $\Phi(z) \equiv \int_0^z e^{-\zeta^2} d\zeta$.

Suppose it is known that $\Phi(z)$ maps the circle $|z| = R$ onto a simple close-to-convex curve; we must determine a value for R such that the relation (2) holds for all arcs on $|z| = R$. First, an easy calculation shows that the zeros of

$$\Psi(Re^{i\theta}) \equiv \mathcal{R} \left(1 + Re^{i\theta} \frac{\Phi''(Re^{i\theta})}{\Phi'(Re^{i\theta})} \right)$$

occur for $\theta \pm \alpha$, $\pi \pm \alpha$, where $0 < 2\alpha < \frac{\pi}{2}$ and $\cos 2\alpha = \frac{1}{2R^2}$. Hence we must assume that $2R^2 \geqq 1$. Another easy calculation shows that $\Psi(Re^{i\theta})$ is negative for $-\alpha < \theta < \alpha$, and for $\pi - \alpha < \theta < \pi + \alpha$, and positive on the other two arcs of the circle $|z| = R$. Now in order to satisfy (2) for the circle $|z| = R$, it is certainly sufficient that

$$\int_{-\alpha}^{\alpha} \Psi(Re^{i\theta}) \, d\theta = \int_{\pi-\alpha}^{\pi+\alpha} \Psi(Re^{i\theta}) \, d\theta > -\pi , \tag{5}$$

and

$$\int_{-\alpha}^{\pi+\alpha} \Psi(Re^{i\theta}) \, d\theta > -\pi , \tag{6}$$

hold. We find that the integrals in (5) are equal to $2\alpha - 2R^2 \sin 2\alpha$. Hence (5) becomes

$$\text{Arctan}\sqrt{4R^4-1} - \sqrt{4R^4-1} > -\pi\,. \tag{7}$$

Another calculation shows that if (7) holds, then so does (6). Hence the largest value of R for which (7), and hence (5), holds is that for which equality is achieved in (7). This last fact, plus the fact that $\Phi'(z)$ has no zeros, allows us to apply Theorem I, with Γ the circle $|z|=R$, to obtain the following result.

THEOREM III. *The function* $\int_0^z e^{-\zeta^2} d\zeta$ *is univalent and close-to-convex for* $|z|<R$, *where* R *is the largest positive root of the following equation:*

$$\text{Arctan}\sqrt{4R^4-1} - \sqrt{4R^4-1} = -\pi\,.$$

We find that $R=1.51$, approximately. This is slightly larger than a recent estimate of $R=\sqrt{\frac{\pi}{2}}$ due to Rogozin [4], which in turn is larger than Nehari's $R=\sqrt{\frac{\sqrt{\pi^2+1}-1}{2}}$ [2]. Since we have only determined a radius of univalence which is that for close-to-convexity, and since the image of $|z|=R$ is an analytic curve, it appears that the true value of the radius of convexity is somewhat larger than the value we have found. It would also be of some interest to determine a larger domain of univalence of $\Phi(z)$; this can certainly be done using Umezawa's fundamental lemma (Theorem I above).

4. We close with the observation that Umezawa's criteria for univalence of analytic functions defined in an annulus can be extended a little, much in the manner we have generalized some of his other results. We plan to return to that point on another occasion.

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