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Generalized virtualization on welded links

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Abstract. The aim of this paper is to study two local moves $V(n)$ and V^n on welded links for a positive integer n, which are generalizations of the crossing virtualization. We show that the $V(n)$-move is an unknotting operation on welded knots for any n, and give a classification of welded links up to $V(n)$-moves. On the other hand, we give a necessary condition for two welded links to be equivalent up to V^n-moves. This leads us to show that the V^n-move is not an unknotting operation on welded knots except for $n = 1$. We also discuss relations among V^n-moves, associated core groups and the multiplexing of crossings.

1. Introduction.

A μ-component *virtual link diagram* is the image of an immersion of ordered and oriented μ circles in the plane, whose transverse double points admit not only *classical crossings* but also *virtual crossings* illustrated in Figure 1.1.[1] We emphasize that a virtual link diagram is always ordered and oriented unless otherwise specified.

Figure 1.1.

A *virtual link* is an equivalence class of virtual link diagrams under *generalized Reidemeister moves*, which consist of Reidemeister moves R1–R3 and virtual moves VR1–VR4 illustrated in Figure 1.2 [**11**]. In the virtual context, there are two forbidden local moves OC and UC (meaning *over-crossings* and *under-crossings commute*, respectively) illustrated in Figure 1.3. The extension of the generalized Reidemeister moves which also allows the OC-move is called *welded Reidemeister moves*, and a sequence of welded Reidemeister moves is called a *welded isotopy*. A *welded link* is an equivalence class of virtual

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[1] For simplicity, we do not use here the usual drawing convention for virtual crossings, which is a small circle around the corresponding double point.

link diagrams under welded isotopy [**5**]. A 1-component (virtual or welded) link is also called a (virtual or welded) *knot*.

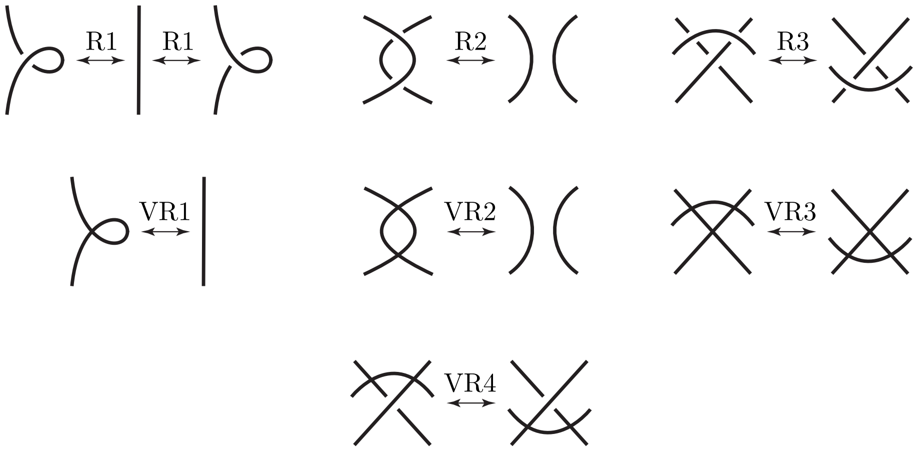

Figure 1.2. Generalized Reidemeister moves.

Figure 1.3. Forbidden moves OC and UC.

A virtual link diagram is *classical* if it has no virtual crossings, and a welded link is *classical* if it has a classical link diagram. Goussarov, Polyak and Viro [**7**] essentially proved that welded isotopic classical link diagrams can be related by Reidemeister moves R1–R3. Therefore, welded links can be viewed as a natural extension of classical links. We remark that any virtual knot diagram can be unknotted by UC-moves and welded Reidemeister moves [**7**], [**10**], [**21**]. That is, the UC-move is an unknotting operation on welded knots. This result is one reason why the UC-move is still forbidden in the welded context.

In classical knot theory, local moves have played important roles and hence have been widely studied; see for example [**1**], [**8**], [**15**], [**17**], [**18**], [**23**]. Recently, some "classical" local moves, which exchange classical tangle diagrams, have been studied for welded knots and links [**2**], [**3**], [**19**], [**25**]. In this paper, we will study "non-classical" local moves for welded links. A typical non-classical local move is the crossing virtualization. The *crossing virtualization* V is a local move on virtual link diagrams replacing a classical crossing with a virtual one; see the left-hand side of Figure 1.4. We remark that any virtual link diagram can be deformed into a diagram of the trivial link by applying the crossing virtualization repeatedly. The crossing virtualization is equivalent to the local move illustrated in the right-hand side of Figure 1.4. Here, two local moves are *equivalent* if each move is realized by a sequence of the other moves and welded Reidemeister moves.

Figure 1.4. Crossing virtualization.

Let n be a positive integer. The aim of this paper is to study the two oriented local moves $V(n)$ and V^n illustrated in the upper and lower sides of Figure 1.5, respectively. Both are considered as generalizations of the crossing virtualization. In fact, both $V(1)$- and V^1-moves are equivalent to the crossing virtualization. Note that if n is even, then a $V(n)$-move may change the number of components. Two welded links are *$V(n)$-equivalent* (resp. *V^n-equivalent*) if their diagrams are related by $V(n)$-moves (resp. V^n-moves) and welded Reidemeister moves.

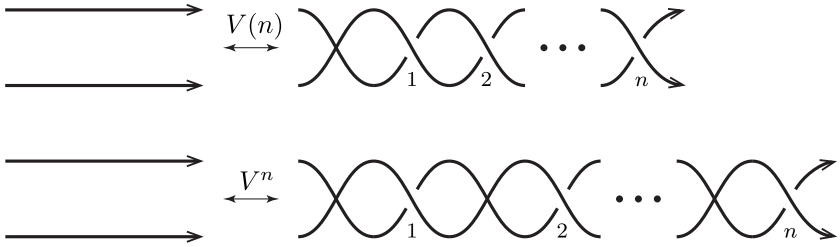

Figure 1.5. $V(n)$- and V^n-moves.

We obtain that the $V(n)$-move is an unknotting operation on welded knots for any n because a UC-move is realized by a sequence of $V(n)$-moves and welded Reidemeister moves (Proposition 3.4). Moreover, we give a classification of welded links up to $V(n)$-equivalence in the sense of Theorems 1.1 and 1.2.

Theorem 1.1. *Let n be an even integer. Any welded link is $V(n)$-equivalent to the trivial knot.*

Let D be a virtual link diagram. For any i, j $(i \neq j)$, let $\lambda_{ij}(D)$ denote the sum of the signs of all classical crossings of D where the ith component passes over the jth component. The integer $\lambda_{ij}(D)$ is a welded link invariant and is also preserved by UC-moves. For a welded link L, the *ordered linking number* $\lambda_{ij}(L)$ is defined to be $\lambda_{ij}(D)$ for a diagram D of L.

It is not hard to see that if n is odd, then the modulo-n reduction of $\lambda_{ij}(L)+\lambda_{ji}(L)$ is preserved by $V(n)$-moves. Using these invariants we have the following.

Theorem 1.2. *Let n be an odd integer. Two μ-component welded links L and L' are $V(n)$-equivalent if and only if $\lambda_{ij}(L)+\lambda_{ji}(L) \equiv \lambda_{ij}(L')+\lambda_{ji}(L') \pmod n$ for any i, j $(1 \leq i < j \leq \mu)$.*

On the other hand, V^n-moves preserve the modulo-n reduction of $\lambda_{ij}(L)$ for any positive integer n. However, these invariants are not strong enough to classify welded links up to V^n-equivalence because the V^n-move is not an unknotting operation on welded knots except for $n = 1$ (Proposition 5.2). Considering the UC-move, which is an unknotting operation for welded knots, we have the following.

Theorem 1.3. *Let n be a positive integer. Two μ-component welded links L and L' are $(V^n + \mathrm{UC})$-equivalent if and only if $\lambda_{ij}(L) \equiv \lambda_{ij}(L') \pmod{n}$ for any i, j $(1 \leq i \neq j \leq \mu)$.*

Here, two welded links are $(V^n + \mathrm{UC})$-*equivalent* if their diagrams are related by V^n-moves, UC-moves and welded Reidemeister moves.

Remark 1.4. Theorem 1.3 easily follows from the classification of welded links up to UC-moves given in [**22**, Theorem 8], see also [**20**, Theorem 4.7] and [**3**, Proposition 3.6]. In this paper, we will prove Theorem 1.3 without using the classification result. Our proof of the theorem provides an alternative proof for this classification, which is similar to that of [**3**, Proposition 3.6].

We also discuss the relations among *unoriented* V^n-moves, associated core groups and the multiplexing of crossings. The *associated core group* is known as an unoriented classical link invariant [**6**], [**9**], [**12**], [**26**]. This group extends naturally to an unoriented welded link invariant, and furthermore, it is preserved by unoriented V^n-moves for any even integer n (Proposition 6.2). In [**16**] the authors introduced the notion of *multiplexing* of crossings for an unoriented μ-component welded link L, which yields a new unoriented welded link $L(m_1, \ldots, m_\mu)$ associated with a μ-tuple $(m_1, \ldots, m_\mu)$ of integers. For any μ-tuple $(m_1, \ldots, m_\mu)$ of even integers, $L(m_1, \ldots, m_\mu)$ is deformed into the μ-component trivial link by unoriented V^2-moves (Proposition 6.3). As a consequence, we obtain that there are infinitely many nontrivial welded knots whose associated core groups are isomorphic to that of the trivial knot (Theorem 6.4).

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2. Arrow calculus.

To show Theorems 1.1, 1.2 and 1.3, we will use the *arrow calculus* introduced by Meilhan and the third author in [**14**]. In this section, we will briefly recall the basic definitions of arrow calculus from [**14**]. We only need the notion of w-arrow, and refer the reader to [**14**] for more details of arrow calculus.

Definition 2.1. Let D be a virtual link diagram. A w-*arrow* γ for D is an oriented arc immersed in the plane of the diagram such that

(1) the endpoints of γ are contained in $D \setminus \{\text{crossings of } D\}$,

(2) all singularities of γ are virtual crossings,

(3) all singularities between D and γ are virtual crossings, and

(4) γ has a number (possibly zero) of decorations $\bullet$ on the interior of γ, called *twists*, which are disjoint from all crossings.

The initial and terminal points of γ are called the *tail* and the *head*, respectively. For a union of w-arrows for D, all crossings among w-arrows are assumed to be virtual.

Hereafter, diagrams are drawn with bold lines while w-arrows are drawn with thin lines.

Let A be a union of w-arrows for D. We define *surgery along* A on D, which yields a new virtual link diagram denoted by D_A, as follows. Suppose that there is a disk in the plane which intersects $D \cup A$ as illustrated in Figure 2.1. Then the figure indicates the result of surgery along a w-arrow of A on D. We emphasize that the surgery move depends on the orientation of the strand of D containing the tail of the w-arrow.

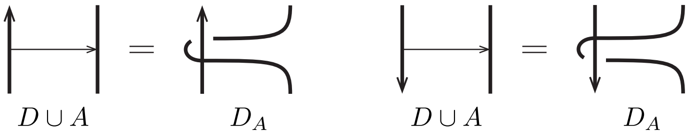

Figure 2.1. Surgery along a w-arrow of A on D.

If a w-arrow of A intersects a (possibly the same) w-arrow (resp. D), then the result of surgery is essentially the same as above but each intersection introduces virtual crossings as illustrated on the left-hand side (resp. center) of Figure 2.2. Furthermore, if a w-arrow of A has some twists, then each twist is converted to a half-twist whose crossing is virtual; see the right-hand side of Figure 2.2.

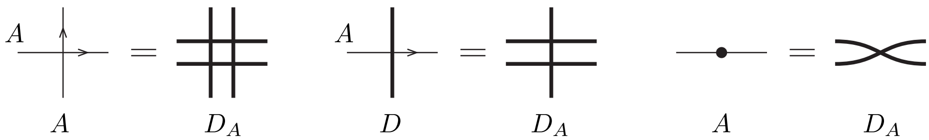

Figure 2.2.

An *arrow presentation* for a virtual link diagram D is a pair (T, A) of a virtual link diagram T without classical crossings and a union A of w-arrows for T such that T_A is welded isotopic to D. Every virtual link diagram has an arrow presentation because any classical crossing can be replaced by a virtual one with a w-arrow; see Figure 2.3.

Figure 2.3.

Two arrow presentations (T, A) and (T', A') are *equivalent* if T_A and $T'_{A'}$ are welded isotopic. *Arrow moves* consist of virtual moves VR1–VR3 involving w-arrows and/or strands of D and the local moves AR1–AR10 on arrow presentations illustrated in Figure 2.4. Here, each vertical strand in AR1–AR3 is either a strand of D or a w-arrow, and the symbol $\circ$ on a w-arrow in AR8 and AR10 means that the w-arrow may or may not contain a twist. Two arrow presentations are equivalent if and only if they are related by arrow moves [**14**, Theorem 4.5].

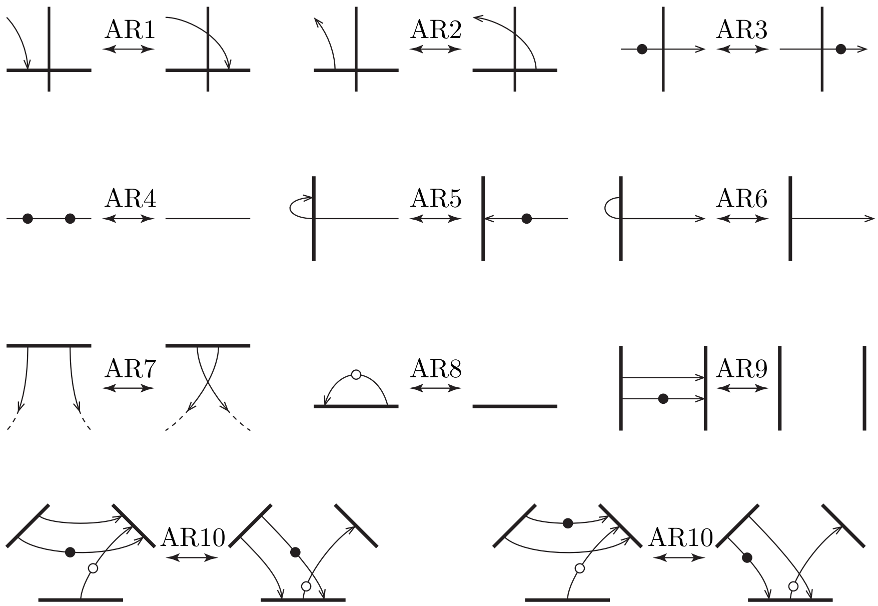

Figure 2.4. Arrow moves AR1–AR10.

Remark 2.2. One may wonder if we need the additional moves obtained from AR9 and AR10, respectively, by reflecting each arrow presentation along a vertical line. The reflected AR9 and AR10 are realized by sequences of arrow moves. See for example Figures 2.5 and 2.6, where $\overset{\mathrm{AR}}{\sim}$ in the figures denotes a sequence of arrow moves. In the following, we will call the reflected AR9 and AR10 also AR9 and AR10, respectively.

In the rest of this section, we will introduce several local moves on arrow presentations. We first consider two *allowable moves* AR11 and AR12 illustrated in Figures 2.7 and 2.8, respectively.[2] Each of the moves is realized by a sequence of arrow moves. Figure 2.9 shows that the left-hand side moves in Figures 2.7 and 2.8 are realized by arrow moves. The other cases are shown similarly.

The *heads exchange move*[3] is a local move on arrow presentations exchanging the positions of two consecutive heads of w-arrows; see Figure 2.10. While there are several kinds of heads exchange moves depending on the orientation of the strand containing the tail and existence or nonexistence of a twist for a w-arrow, we have the following.

[2]The moves AR11 and AR12 are close to Gauss diagram versions of R3, see [**7**, Fig. 6].
[3]Note that our definition slightly differs from the one given in [**14**, Lemma 5.14].

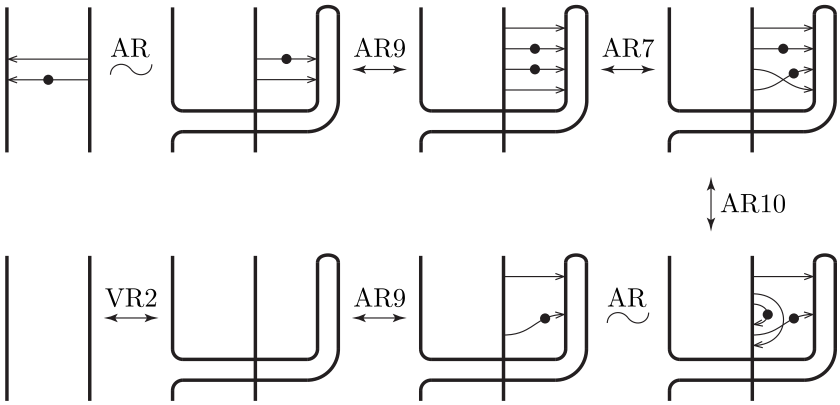

Figure 2.5.

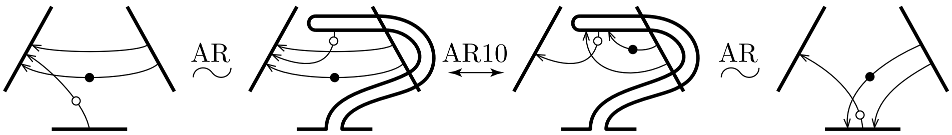

Figure 2.6.

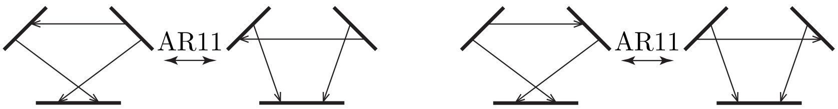

Figure 2.7. Allowable move AR11.

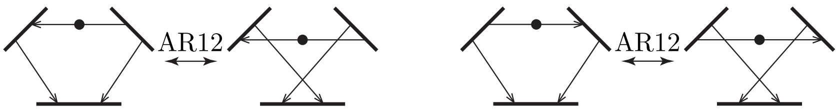

Figure 2.8. Allowable move AR12.

Sublemma 2.3. *A heads exchange move is realized by a sequence of* H*-moves and arrow moves, where the* H*-move is a local move on arrow presentations illustrated in Figure* 2.11*.*

Proof. We prove the result for two types of heads exchange moves. The upper side of Figure 2.12 indicates the proof when the orientation of the strand containing the tail of a single w-arrow is opposite to that of the H-move. The lower side of Figure 2.12 indicates the proof for the case where one of the w-arrows has a twist. It is not hard to show the other cases. □

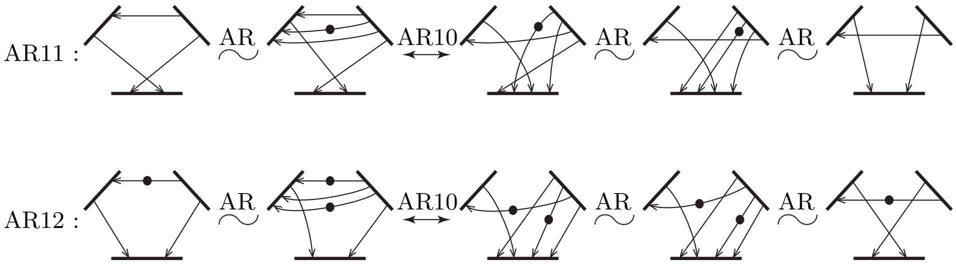

Figure 2.9.

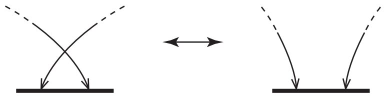

Figure 2.10. Heads exchange move.

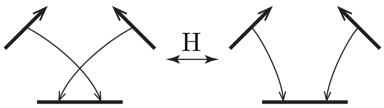

Figure 2.11. H-move.

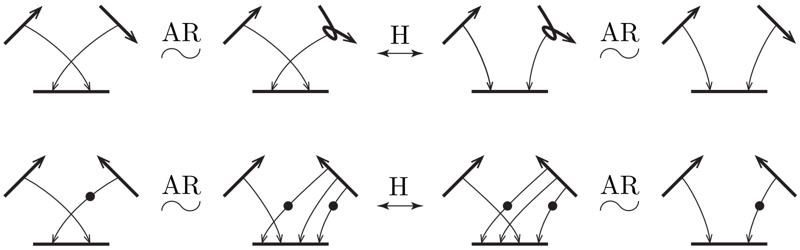

Figure 2.12.

The *head-tail exchange move*[4] is a local move on arrow presentations exchanging the positions of a pair of consecutive head and tail of w-arrows; see Figure 2.13.

Sublemma 2.4. *A head-tail exchange move is realized by a sequence of heads exchange moves and arrow moves.*

Proof. See Figure 2.14. □

The *ends exchange moves* are of the following three kinds of moves: AR7, heads exchange and head-tail exchange moves. From Sublemmas 2.3 and 2.4 we have the following.

[4]Note that our definition slightly differs from the one given in [**14**, Lemma 5.16].

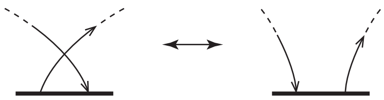

Figure 2.13. Head-tail exchange move.

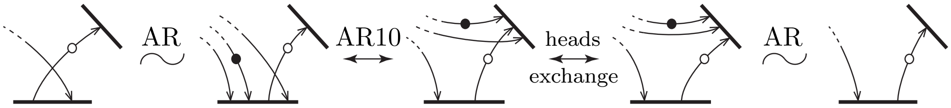

Figure 2.14.

Lemma 2.5. *An ends exchange move is realized by a sequence of* H*-moves and arrow moves.*

3. $V(n)$-moves and UC-moves.

In this section, we will show that the $V(n)$-move is an unknotting operation on welded knots. We start with the following lemma concerning the UC-move.

Lemma 3.1. *An arrow presentation for a* UC*-move is realized by a sequence of heads exchange moves and arrow moves. Conversely, surgery along a heads exchange move is realized by a sequence of* UC*-moves and welded Reidemeister moves.*

Proof. Figure 3.1 shows that an arrow presentation for a UC-move is realized by a sequence of heads exchange moves and arrow moves. In the figure, we choose certain orientations of the two strands at the virtual crossing. The other cases are shown similarly.

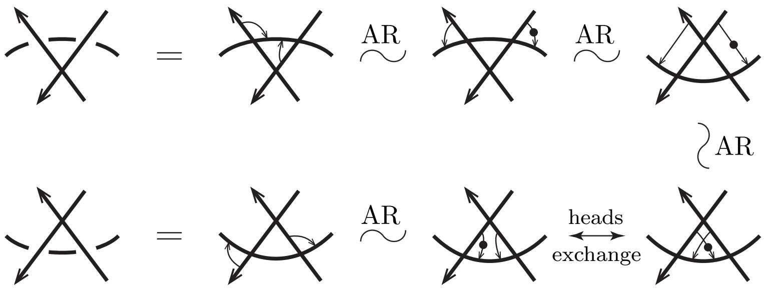

Figure 3.1.

Conversely, Figure 3.2 shows that surgery along an H-move is realized by a sequence of UC-moves and welded Reidemeister moves, where $\overset{\mathrm{w}}{\sim}$ in the figure denotes a welded isotopy. This and Sublemma 2.3 complete the proof. □

We define the $A(n)$*-move* as a local move on arrow presentations depending on the parity of n. The $A(n)$-move is illustrated in Figure 3.3 (resp. Figure 3.4) when n is odd (resp. even).

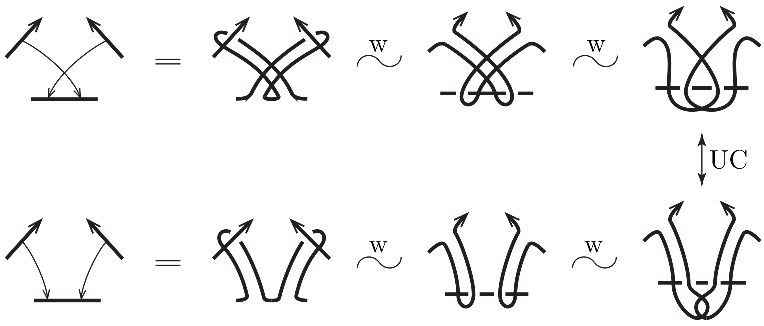

Figure 3.2.

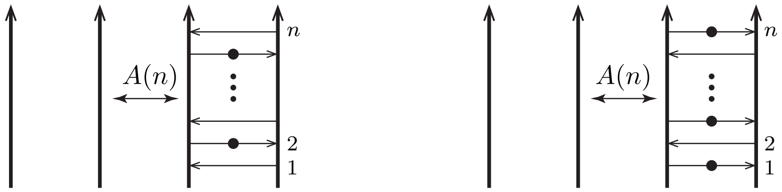

Figure 3.3. $A(n)$-move when n is odd.

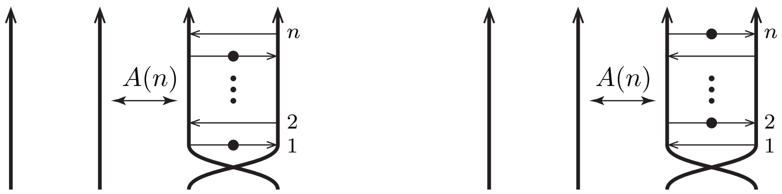

Figure 3.4. $A(n)$-move when n is even.

Lemma 3.2. *Let n be a positive integer. An ends exchange move is realized by a sequence of $A(n)$-moves and arrow moves.*

Proof. By Lemma 2.5, it suffices to show that an H-move is realized by a sequence of $A(n)$-moves and arrow moves for any n. Figure 3.5 (resp. 3.6) indicates the proof for the case $n = 3$ (resp. $n = 2$); the case where n is odd (resp. even) is strictly similar. □

Lemma 3.3. *An arrow presentation for a $V(n)$-move is realized by a sequence of $A(n)$-moves and arrow moves. Conversely, surgery along an $A(n)$-move is realized by a sequence of $V(n)$-moves and welded Reidemeister moves.*

Proof. It is not hard to see that the right-hand side move in Figure 3.3 (resp. Figure 3.4) is realized by a sequence of the left-hand side moves in Figure 3.3 (resp. Figure 3.4) and arrow moves. See, for example, Figure 3.7 in the case where n is odd.

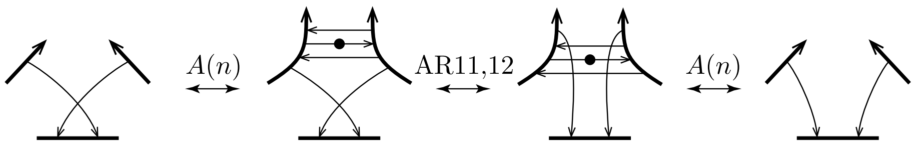

Figure 3.5. The case where n is odd.

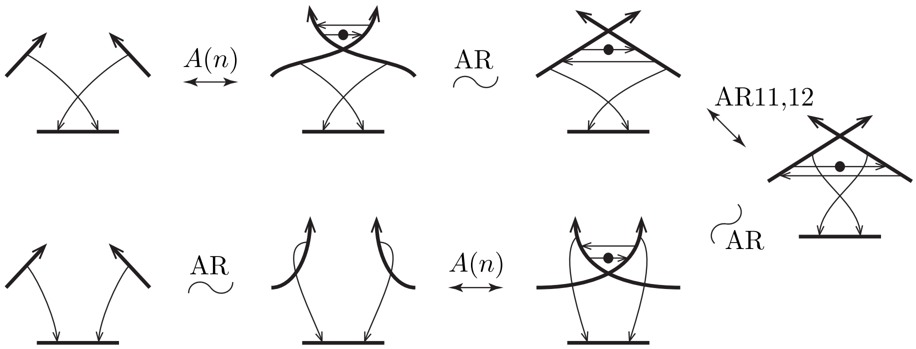

Figure 3.6. The case where n is even.

Furthermore, an arrow presentation for a $V(n)$-move is realized by a sequence of the left-hand side moves in Figure 3.3 (resp. Figure 3.4) and arrow moves when n is odd (resp. even). Conversely, it is obvious that surgery along the left-hand side move in Figure 3.3 (resp. Figure 3.4) is realized by a sequence of $V(n)$-moves and welded Reidemeister moves when n is odd (resp. even). Therefore, we have the conclusion. □

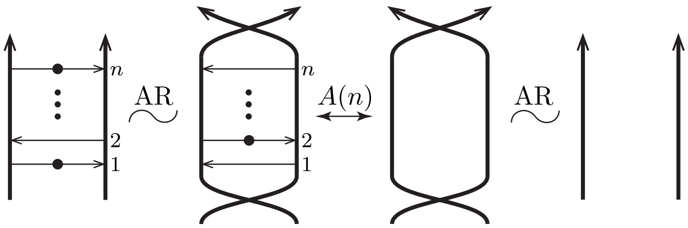

Figure 3.7.

As a consequence of Lemmas 3.1, 3.2 and 3.3, we have the following.

Proposition 3.4. *Let n be a positive integer. A* UC*-move is realized by a sequence of $V(n)$-moves and welded Reidemeister moves. Hence, the $V(n)$-move is an unknotting operation for welded knots.*

Here, we define the A^n*-move* as a local move on arrow presentations illustrated in Figure 3.8.

Figure 3.8. A^n-move.

Lemma 3.5. *Let n be an odd integer. An A^n-move is realized by a sequence of $A(n)$-moves and arrow moves.*

Proof. We consider the *head-tail reversal move* illustrated in Figure 3.9, which is realized by a sequence of AR9 and $A(n)$-moves. (Figure 3.10 shows that one of the head-tail reversal moves is realized by a sequence of AR9 and $A(n)$-moves. The other case is shown similarly.) Combining an $A(n)$-move with head-tail reversal moves, we can realize an A^n-move. $\square$

Figure 3.9. Head-tail reversal move.

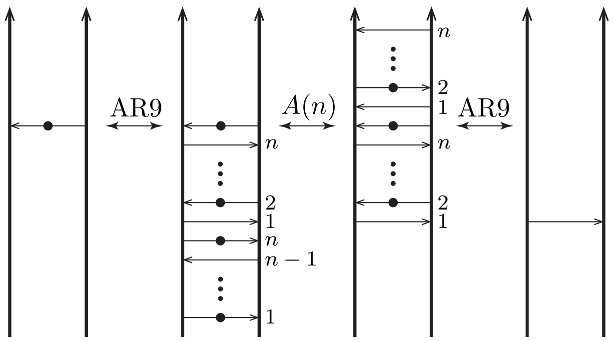

Figure 3.10.

By arguments similar to those in the proof of Lemma 3.3 we have the following.

Lemma 3.6. *An arrow presentation for a V^n-move is realized by a sequence of A^n-moves and arrow moves. Conversely, surgery along an A^n-move is realized by a sequence of V^n-moves and welded Reidemeister moves.*

4. Proofs of theorems.

In this section, we will prove Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1. If n is even, then any welded link can be deformed into some welded knot by $V(n)$-moves, since $V(n)$-moves can change the number of components of a welded link. Therefore, Theorem 1.1 follows from Proposition 3.4. □

Fix μ distinct points $0 < x_1 < \cdots < x_\mu < 1$ in the unit interval $[0,1]$. Let $I_1, \ldots, I_\mu$ be μ copies of $[0,1]$. A *μ-component virtual string link diagram* is the image of an immersion

$$\bigsqcup_{i=1}^{\mu} I_i \longrightarrow [0,1] \times [0,1]$$

such that the image of each I_i runs from $(x_i, 0)$ to $(x_i, 1)$, and the singularities are only transverse double points that are either classical or virtual. Note that each ith strand of a virtual string link diagram is oriented from $(x_i, 0)$ to $(x_i, 1)$. The μ-component virtual string link diagram $\{x_1, \ldots, x_\mu\} \times [0,1]$ in $[0,1] \times [0,1]$ is called the *μ-component trivial string link diagram*.

Let $\mathbf{1}$ be the μ-component trivial string link diagram. For an integer a, let $(\mathbf{1}, H_{ij}(a))$ denote the arrow presentation of Figure 4.1, that is, $H_{ij}(a)$ consists of $|a|$ horizontal w-arrows whose tails (resp. heads) are attached to the ith (resp. jth) strand of $\mathbf{1}$ $(1 \leq i < j \leq \mu)$ such that each w-arrow has exactly one twist if $a \geq 0$, and no twist otherwise. Note that, for arrow presentations $(\mathbf{1}, H_{ij}(a))$ and $(\mathbf{1}, H_{kl}(a'))$, the stacking products $(\mathbf{1}, H_{ij}(a)) * (\mathbf{1}, H_{kl}(a'))$ and $(\mathbf{1}, H_{kl}(a')) * (\mathbf{1}, H_{ij}(a))$ are related by ends exchange moves and arrow moves, hence, by $A(n)$-moves and arrow moves. Here, the *stacking product* $(\mathbf{1}, A) * (\mathbf{1}, B)$ of arrow presentations $(\mathbf{1}, A)$ and $(\mathbf{1}, B)$ is the arrow presentation corresponding to the diagram $\mathbf{1}_A * \mathbf{1}_B$. Let $\prod_{1 \leq i < j \leq \mu} (\mathbf{1}, H_{ij}(a_{ij}))$ denote the stacking products of $(\mathbf{1}, H_{ij}(a_{ij}))$ for integers a_{ij}. We remark that the ordered linking numbers λ_{ij} and λ_{ji} of the closure of the string link diagram $\prod_{1 \leq i < j \leq \mu} \mathbf{1}_{H_{ij}(a_{ij})}$ are equal to a_{ij} and 0, respectively.

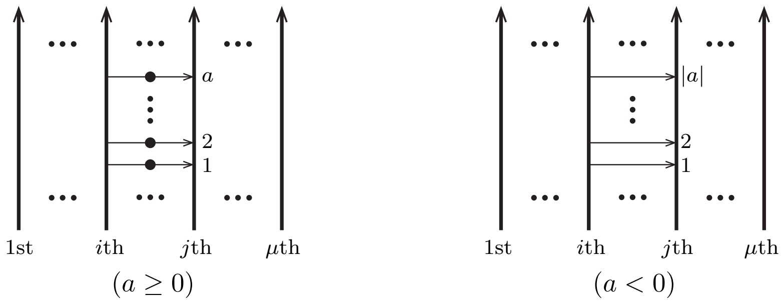

Figure 4.1. Arrow presentation $(\mathbf{1}, H_{ij}(a))$.

Lemma 4.1. *Let n be an odd integer. For any μ-component virtual string link diagram D, there are integers a_{ij} with $0 \leq a_{ij} < n$ $(1 \leq i < j \leq \mu)$ such that an arrow presentation for D can be related to $\prod_{1 \leq i < j \leq \mu} (\mathbf{1}, H_{ij}(a_{ij}))$ by $A(n)$-moves and arrow moves.*

Proof. Let $(\mathbf{1}, \bigcup_{1\le i,j\le\mu} W_{ij})$ be an arrow presentation for a μ-component virtual string link diagram, where W_{ij} is a set of w-arrows for $\mathbf{1}$ whose tails (resp. heads) are attached to the ith (resp. jth) strand ($1 \le i,j \le \mu$, possibly $i = j$). We show that $(\mathbf{1}, \bigcup_{1\le i,j\le\mu} W_{ij})$ can be deformed into the desired form by $A(n)$-moves and arrow moves (including ends exchange moves, head-tail reversal moves and A^n-moves).

First, each w-arrow in W_{ii} ($1 \le i \le \mu$) can be moved into position to be removed by a single AR8. Hence, all w-arrows in W_{ii} are removed for any i. Next, $(\mathbf{1}, \bigcup_{1\le i\ne j\le\mu} W_{ij})$ can be deformed into $\prod_{1\le i<j\le\mu}(\mathbf{1}, H_{ij}(a_{ij}))$ for some integers a_{ij} by combining head-tail reversal moves, ends exchange moves and AR9. Finally, we obtain the desired form by performing A^n-moves and AR9. □

Proof of Theorem 1.2. It suffices to show the "if" part. Let D and D' be virtual link diagrams of L and L', respectively. For any virtual link diagram, there is a virtual string link diagram whose closure is welded isotopic to the virtual link diagram.[5] Hence, by Lemma 4.1, two arrow presentations (T, A) for D and (T', A') for D' are related to the closures of $\prod_{1\le i<j\le\mu}(\mathbf{1}, H_{ij}(a_{ij}))$ and $\prod_{1\le i<j\le\mu}(\mathbf{1}, H_{ij}(a'_{ij}))$, respectively, for some non-negative integers a_{ij}, a'_{ij} ($< n$), by $A(n)$-moves and arrow moves. Then, for any i, j ($1 \le i < j \le \mu$), we have

$$a_{ij} \equiv \lambda_{ij}(D) + \lambda_{ji}(D) \equiv \lambda_{ij}(D') + \lambda_{ji}(D') \equiv a'_{ij} \pmod{n}.$$

Since $0 \le a_{ij}, a'_{ij} < n$, it follows that $a_{ij} = a'_{ij}$. Therefore, (T, A) and (T', A') are related by $A(n)$-moves and arrow moves. Consequently, $D(= T_A)$ and $D'(= T'_{A'})$ are related by $V(n)$-moves and welded Reidemeister moves. □

For an integer b, let $(\mathbf{1}, \overline{H}_{ij}(b))$ denote the arrow presentation of Figure 4.2, that is, $\overline{H}_{ij}(b)$ consists of $|b|$ horizontal w-arrows whose heads (resp. tails) are attached to the ith (resp. jth) strand of $\mathbf{1}$ ($1 \le i < j \le \mu$) such that each w-arrow has no twist if $b \ge 0$, and exactly one twist otherwise. We remark that, for integers a_{ij} and b_{ij}, the ordered linking numbers λ_{ij} and λ_{ji} of the closure of the string link diagram $\prod_{1\le i<j\le\mu}(\mathbf{1}_{H_{ij}(a_{ij})} * \mathbf{1}_{\overline{H}_{ij}(b_{ij})})$ are equal to a_{ij} and b_{ij}, respectively.

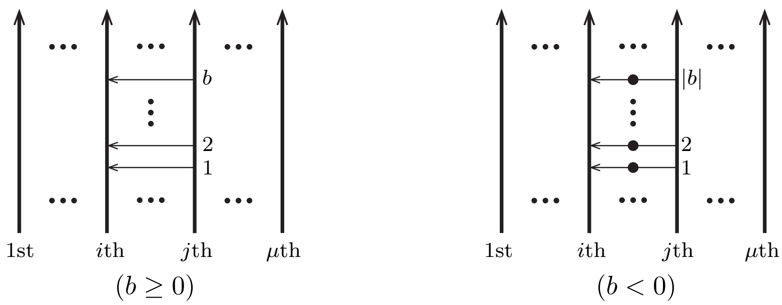

Figure 4.2. Arrow presentation $(\mathbf{1}, \overline{H}_{ij}(b))$.

[5]In fact, for a small disk which is disjoint from a given μ-component virtual link diagram, by using VR2 we can deform the diagram so that the intersection between the deformed diagram and the disk is the μ-component trivial string link diagram.

LEMMA 4.2. *Let n be a positive integer. For any μ-component virtual string link diagram D, there are integers a_{ij}, b_{ij} with $0 \leq a_{ij}, b_{ij} < n$ $(1 \leq i < j \leq \mu)$ such that an arrow presentation for D can be related to $\prod_{1\leq i<j\leq\mu}((\mathbf{1}, H_{ij}(a_{ij})) * (\mathbf{1}, \overline{H}_{ij}(b_{ij})))$ by A^n-moves, ends exchange moves and arrow moves.*

This lemma is proved by a similar way to Lemma 4.1. Note that we are not permitted to use the head-tail reversal move. This is the reason why we need not only $H_{ij}(a)$ but also $\overline{H}_{ij}(b)$.

PROOF OF THEOREM 1.3. It suffices to show the "if" part. Let D and D' be virtual link diagrams of L and L', respectively. By Lemma 4.2, two arrow presentations (T, A) for D and (T', A') for D' are related to the closures of $\prod_{1\leq i<j\leq\mu}((\mathbf{1}, H_{ij}(a_{ij})) * (\mathbf{1}, \overline{H}_{ij}(b_{ij})))$ and $\prod_{1\leq i<j\leq\mu}((\mathbf{1}, H_{ij}(a'_{ij})) * (\mathbf{1}, \overline{H}_{ij}(b'_{ij})))$, respectively, for some non-negative integers $a_{ij}, b_{ij}, a'_{ij}, b'_{ij}$ $(< n)$, by A^n-moves, ends exchange moves and arrow moves. Then, for any i, j $(1 \leq i < j \leq \mu)$, we have

$$a_{ij} \equiv \lambda_{ij}(D) \equiv \lambda_{ij}(D') \equiv a'_{ij} \pmod n$$

and

$$b_{ij} \equiv \lambda_{ji}(D) \equiv \lambda_{ji}(D') \equiv b'_{ij} \pmod n.$$

Since $0 \leq a_{ij}, b_{ij}, a'_{ij} b'_{ij} < n$, it follows that $a_{ij} = a'_{ij}$ and $b_{ij} = b'_{ij}$. Therefore, (T, A) and (T', A') are related by A^n-moves, ends exchange moves and arrow moves. Lemmas 3.1 and 3.6 imply that $D(= T_A)$ and $D'(= T'_{A'})$ are related by V^n-moves, UC-moves and welded Reidemeister moves. □

5. V^n-moves and UC-moves.

As mentioned in Section 1, the V^n-move is not an unknotting operation except for $n = 1$. To prove this, we will use the Alexander polynomials, which are obtained from the group of welded links using Fox free derivatives. Here, the *group* of a virtual link diagram is known to be a welded link invariant [**11**, Section 4], and hence (the elementary ideals in the sense of [**4**] and) the Alexander polynomials extend naturally to welded link invariants. By arguments similar to those in the proof of Theorem 1 in [**13**], we can show the following.

PROPOSITION 5.1. *Let n be a positive integer. If two welded links L and L' are V^n-equivalent, then for a non-negative integer k and for the kth elementary ideals $E^k_L(t)$ and $E^k_{L'}(t)$ of L and L', respectively, we have*

$$E^k_L(t) \equiv E^k_{L'}(t) \mod I(1 - t^n),$$

where $I(1-t^n)$ is the ideal generated by $1-t^n$ in $\mathbb{Z}[t^{\pm 1}]$. In particular, for the (1-variable) kth Alexander polynomials $\Delta^k_L(t)$ and $\Delta^k_{L'}(t)$ of L and L', respectively, we have

$$\Delta^k_L(t) \equiv \varepsilon t^r \Delta^k_{L'}(t) \mod I(1 - t^n)$$

for some $\varepsilon \in \{\pm 1\}$ *and* $r \in \mathbb{Z}$.

Proof. Let D and D' be virtual link diagrams of L and L', respectively. It suffices to show that if D and D' are related by a single V^n-move, then for properly chosen Alexander matrices $A_D(t)$ and $A_{D'}(t)$,

$$A_D(t) \equiv A_{D'}(t) \mod I(1-t^n).$$

Suppose that D' is obtained from D by a single R1 and a single V^n-move, and put labels x_1, x_2 and x_3 on arcs of D and D' as illustrated in Figure 5.1 and labels $x_4, \dots, x_l$ on the other arcs outside the figure.

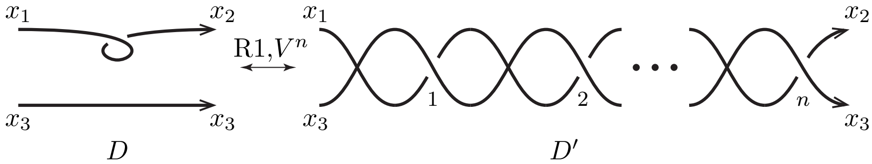

Figure 5.1.

Then, we obtain presentations of the groups $G(D)$ and $G(D')$ of D and D', respectively, as follows:

$$\begin{aligned} G(D) &= \langle x_1, x_2, x_3, x_4, \dots, x_l \mid x_1 x_2^{-1}, \{r_i\} \rangle, \\ G(D') &= \langle x_1, x_2, x_3, x_4, \dots, x_l \mid x_1 x_3^n x_2^{-1} x_3^{-n}, \{r_i\} \rangle, \end{aligned}$$

where $\{r_i\}$ is the set of relations corresponding to the other crossings. Using Fox free derivatives [**4**], we have the Alexander matrices $A_D(t)$ and $A_{D'}(t)$ of D and D', respectively, as follows:

$$A_D(t) = \left(\begin{array}{cccccc} 1 & -1 & 0 & 0 & \cdots & 0 \\ \hdashline \multicolumn{6}{c}{\mathfrak{a}\gamma \left(\dfrac{r_i}{x_j} \right)} \end{array} \right), \quad A_{D'}(t) = \left(\begin{array}{cccccc} 1 & -t^n & t^n - 1 & 0 & \cdots & 0 \\ \hdashline \multicolumn{6}{c}{\mathfrak{a}\gamma \left(\dfrac{r_i}{x_j} \right)} \end{array} \right).$$

Therefore, $A_D(t) - A_{D'}(t)$ is a zero matrix modulo $I(1-t^n)$. $\square$

Proposition 5.2. *The V^n-move is not an unknotting operation on welded knots for $n \geq 2$.*

Proof. We show that the trefoil knot is not V^n-equivalent to the trivial knot for $n \geq 2$. The first Alexander polynomial of the trefoil knot is $1 - t + t^2$, and that of the trivial knot is 1. It suffices to show that $1 - t + t^2 - \varepsilon t^r \notin I(1-t^n)$ for any $n \geq 2$ by Proposition 5.1 ($\varepsilon \in \{\pm 1\}, r \in \mathbb{Z}$).

Suppose that $n \geq 2$. We define a map $f_n : \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ by $f_n(\sum_i a_i t^i) = \sum_{i \equiv 0,2 \pmod n} a_i$, where $a_i \in \mathbb{Z}$. Since $f_n(\delta t^s(1-t^n)) = 0$ ($\delta \in \{\pm 1\}, s \in \mathbb{Z}$), it follows that $f(b) = 0$ for any element $b \in I(1-t^n)$. On the other hand, $f_n(1 - t + t^2 - \varepsilon t^r)$ is not equal to 0 for any $n \geq 2$. This completes the proof. $\square$

The proposition above immediately implies the following corollary.

Corollary 5.3. *A* UC*-move is realized by a sequence of V^n-moves and welded Reidemeister moves if and only if $n = 1$.*

6. Unoriented V^n-moves and associated core groups.

In this section, we will discuss relations among *unoriented* V^n-moves, associated core groups and the multiplexing of crossings.

For an unoriented classical link diagram D, the *associated core group* $\Pi^{(2)}_D$ is defined as follows. Each arc of D yields a generator, and each classical crossing gives a relation $yx^{-1}yz^{-1}$, where x and z correspond to the underpasses and y corresponds to the overpass at the crossing. This group $\Pi^{(2)}_D$ is known as a classical link invariant [**6**], [**9**], [**12**], [**26**].

Remark 6.1. Let L be an unoriented classical link in the 3-sphere and D a classical diagram of L. Wada [**26**] proved that $\Pi^{(2)}_D$ is isomorphic to the free product of the fundamental group of the double branched cover $M^{(2)}_L$ of the 3-sphere branched along L and the infinite cyclic group $\mathbb{Z}$. That is, $\Pi^{(2)}_D \cong \pi_1(M^{(2)}_L) * \mathbb{Z}$.

We similarly define the associated core group $\Pi^{(2)}_D$ of an unoriented *virtual* link diagram D by generators and relations as described above. (Note that virtual crossings do not produce any generator or relation.) It is not hard to see that $\Pi^{(2)}_D$ is preserved by welded Reidemeister moves, and hence we define the *associated core group* $\Pi^{(2)}_L$ of an unoriented welded link L to be the associated core group $\Pi^{(2)}_D$ of a diagram D of L. Moreover we have the following.

Proposition 6.2. *If n is even, then $\Pi^{(2)}_L$ is preserved by unoriented V^n-moves.*

Proof. $\Pi^{(2)}_L$ is preserved by unoriented V^2-moves as illustrated in Figure 6.1, and furthermore, an unoriented V^n-move is realized by unoriented V^2-moves for any even integer n. ☐

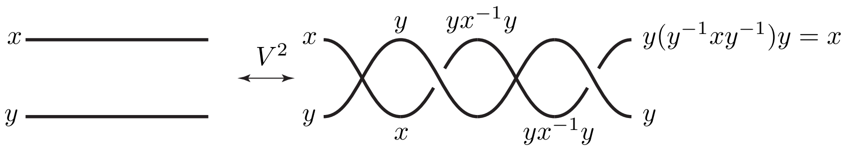

Figure 6.1.

There are welded knots whose associated core groups are nontrivial, for example, all knots having nontrivial Fox colorings; see [**24**, Section 4.1]. Therefore, the proposition above gives an alternative proof of Proposition 5.2 for even integers n.

In [**16**], the authors introduced the *multiplexing* of crossings for an unoriented virtual link diagram, which yields a new unoriented virtual link diagram. Let $(m_1, \ldots, m_\mu)$ be a μ-tuple of integers, and let $D = \bigcup_{i=1}^{\mu} D_i$ be an unoriented μ-component virtual link diagram. For a classical crossing of D whose overpass belongs to D_j, we define the *multiplexing* of the crossing associated with m_j as a local modification illustrated in

Figure 6.2. When $m_j = 0$, the multiplexing of the crossing is the virtualization of this crossing. The number of classical crossings that appear in the multiplexing of the crossing is the absolute value of m_j. Let $D(m_1, \ldots, m_\mu)$ denote the virtual link diagram obtained from D by the multiplexing of all classical crossings of D associated with $(m_1, \ldots, m_\mu)$. For welded isotopic virtual link diagrams D and D', $D(m_1, \ldots, m_\mu)$ and $D'(m_1, \ldots, m_\mu)$ are also welded isotopic for any $(m_1, \ldots, m_\mu) \in \mathbb{Z}^\mu$ [**16**, Theorem 2.1]. For an unoriented μ-component welded link L, we define $L(m_1, \ldots, m_\mu)$ to be $D(m_1, \ldots, m_\mu)$ of a diagram D of L.

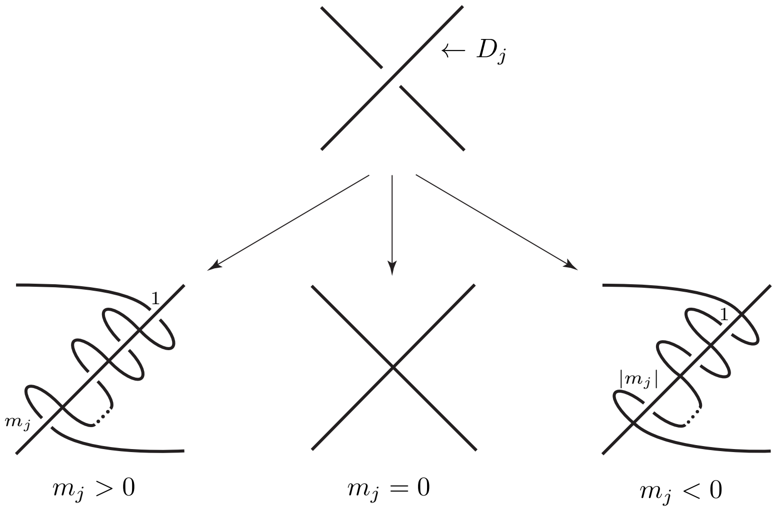

Figure 6.2. Multiplexing of a crossing.

It is not hard to see that $L(m_1, \ldots, m_\mu)$ can be deformed into $L(0, \ldots, 0)$ by unoriented V^2-moves for any μ-tuple $(m_1, \ldots, m_\mu)$ of even integers. Since $L(0, \ldots, 0)$ is trivial, we have the following.

Proposition 6.3. *Let $(m_1, \ldots, m_\mu)$ be a μ-tuple of even integers. For any unoriented μ-component welded link L, $L(m_1, \ldots, m_\mu)$ is deformed into the μ-component trivial link by unoriented V^2-moves.*

In [**16**, Theorem 3.2], the authors proved that two unoriented classical knots K and K' are equivalent up to mirror image if and only if $K(m)$ and $K'(m)$ are welded isotopic up to mirror image for any fixed non-zero integer m. Hence, it follows that if a classical knot K is nontrivial, then $K(m)$ is also nontrivial. By Propositions 6.2 and 6.3, if m is even, then $\Pi^{(2)}_{K(m)}$ is isomorphic to the associated core group of the trivial knot, that is, $\Pi^{(2)}_{K(m)} \cong \mathbb{Z}$. Therefore, we have the following theorem although, by Remark 6.1, the associated core groups seem to be very strong invariants.

Theorem 6.4. *Let m $(\neq 0)$ be an even integer. For any nontrivial unoriented welded knot K, $K(m)$ is nontrivial and $\Pi^{(2)}_{K(m)} \cong \mathbb{Z}$.*

7. $\overline{V}(n)$-moves and $\overline{V}^n$-moves.

7.1. $\overline{V}(n)$-moves.

When n is odd, one may consider the $V(n)$-move involving two strands that are oriented antiparallel. We call such a move the $\overline{V}(n)$*-move*. In this subsection, we will show that the $V(n)$- and $\overline{V}(n)$-moves are equivalent.

For an odd integer n, we define the $\overline{A}(n)$*-move* as a local move on arrow presentations illustrated in Figure 7.1. By arguments similar to those in the proof of Lemma 3.3, we have the following.

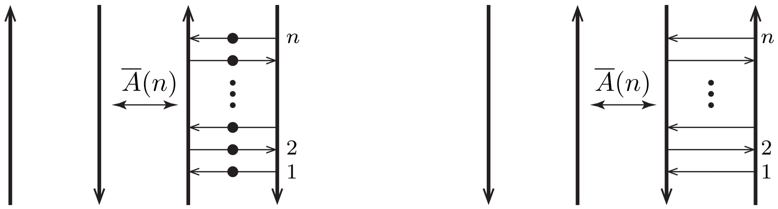

Figure 7.1. $\overline{A}(n)$-move.

Lemma 7.1. *Let n be an odd integer. An arrow presentation for a $\overline{V}(n)$-move is realized by a sequence of $\overline{A}(n)$-moves and arrow moves. Conversely, surgery along an $\overline{A}(n)$-move is realized by a sequence of $\overline{V}(n)$-moves and welded Reidemeister moves.*

By deformations similar to those in Figure 2.12, we have the following.

Lemma 7.2. *An* H*-move is realized by a sequence of* H′*-moves and arrow moves, where the* H′*-move is a local move on arrow presentations illustrated in Figure* 7.2.

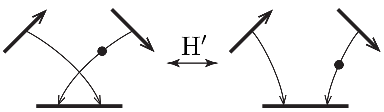

Figure 7.2. H′-move.

Here, we consider two *allowable moves* AR11′ and AR12′ illustrated in Figure 7.3. Each of the moves is realized by a sequence of arrow moves similar to that in Figure 2.9. Using the moves AR11′ and AR12′, we have the following.

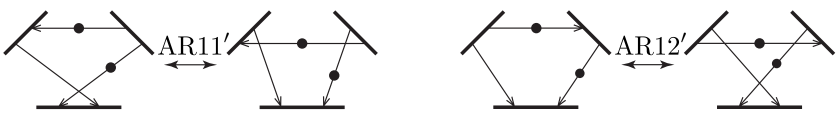

Figure 7.3. Allowable moves AR11′ and AR12′.

Lemma 7.3. *Let n be an odd integer. An ends exchange move is realized by a sequence of $\overline{A}(n)$-moves and arrow moves.*

Proof. By Lemmas 2.5 and 7.2, it suffices to show that an H′-move is realized by a sequence of $\overline{A}(n)$-moves and arrow moves. Figure 7.4 indicates the proof. While the figure describes only the case $n = 3$, the proof is essentially the same in all cases. □

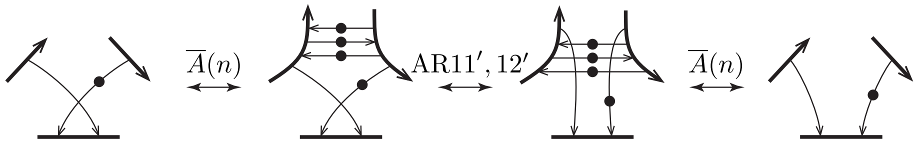

Figure 7.4.

Now we can show the following.

Proposition 7.4. *Let n be an odd integer. The $V(n)$- and $\overline{V}(n)$-moves are equivalent.*

Proof. By Lemmas 3.3 and 7.1, it is enough to show that the $A(n)$- and $\overline{A}(n)$-moves are equivalent. Figure 7.5 shows that an $\overline{A}(n)$-move is realized by a sequence of $A(n)$-moves and arrow moves. (Note that we can use ends exchange moves by Lemma 3.2.) While the figure describes only the case $n = 3$, the other cases are shown similarly.

Conversely, by deformations similar to those in the figure, it is not hard to see that an $A(n)$-move is realized by a sequence of $\overline{A}(n)$-moves and arrow moves. This completes the proof. □

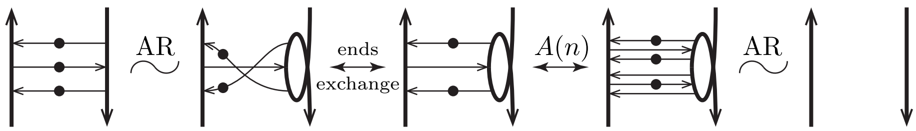

Figure 7.5.

7.2. $\overline{V}^n$-moves.

For a positive integer n, we define the $\overline{V}^n$*-move* as a V^n-move involving two strands that are oriented antiparallel. Also, we define the $\overline{A}^n$*-move* as a local move on arrow presentations illustrated in Figure 7.6. By arguments similar to those in the proof of Lemma 3.3, we have the following.

Lemma 7.5. *Let n be a positive integer. An arrow presentation for a $\overline{V}^n$-move is realized by a sequence of $\overline{A}^n$-moves and arrow moves. Conversely, surgery along an $\overline{A}^n$-move is realized by a sequence of $\overline{V}^n$-moves and welded Reidemeister moves.*

Figure 7.6. $\overline{A}^n$-move.

It is not hard to see that the A^n- and $\overline{A}^n$-moves are equivalent. This together with Lemmas 3.6 and 7.5 implies the following.

PROPOSITION 7.6. *Let n be a positive integer. The V^n- and $\overline{V}^n$-moves are equivalent.*

References

[1] H. Aida, Unknotting operations of polygonal type, Tokyo J. Math., **15** (1992), 111–121.

[2] B. Audoux, P. Bellingeri, J.-B. Meilhan and E. Wagner, On usual, virtual and welded knotted objects up to homotopy, J. Math. Soc. Japan, **69** (2017), 1079–1097.

[3] B. Audoux, P. Bellingeri, J.-B. Meilhan and E. Wagner, Extensions of some classical local moves on knot diagrams, Michigan Math. J., **67** (2018), 647–672.

[4] R. H. Crowell and R. H. Fox, Introduction to Knot Theory, Reprint of the 1963 original, Grad. Texts in Math., **57**, Springer-Verlag, New York-Heidelberg, 1977.

[5] R. Fenn, R. Rimányi and C. Rourke, The braid-permutation group, Topology, **36** (1997), 123–135.

[6] R. Fenn and C. Rourke, Racks and links in codimension two, J. Knot Theory Ramifications, **1** (1992), 343–406.

[7] M. Goussarov, M. Polyak and O. Viro, Finite-type invariants of classical and virtual knots, Topology, **39** (2000), 1045–1068.

[8] J. Hoste, Y. Nakanishi and K. Taniyama, Unknotting operations involving trivial tangles, Osaka J. Math., **27** (1990), 555–566.

[9] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra, **23** (1982), 37–65.

[10] T. Kanenobu, Forbidden moves unknot a virtual knot, J. Knot Theory Ramifications, **10** (2001), 89–96.

[11] L. H. Kauffman, Virtual knot theory, European J. Combin., **20** (1999), 663–691.

[12] A. J. Kelly, Groups from link diagrams, Ph.D. Thesis, Univ. of Warwick (1990). Available at http://wrap.warwick.ac.uk/63616/.

[13] S. Kinoshita, On the distribution of Alexander polynomials of alternating knots and links, Proc. Amer. Math. Soc., **79** (1980), 644–648.

[14] J.-B. Meilhan and A. Yasuhara, Arrow calculus for welded and classical links, Algebr. Geom. Topol., **19** (2019), 397–456.

[15] K. Miyazaki and A. Yasuhara, Generalized #-unknotting operations, J. Math. Soc. Japan, **49** (1997), 107–123.

[16] H. A. Miyazawa, K. Wada and A. Yasuhara, Link invariants derived from multiplexing of crossings, Algebr. Geom. Topol., **18** (2018), 2497–2507.

[17] H. Murakami and Y. Nakanishi, On a certain move generating link-homology, Math. Ann., **284** (1989), 75–89.

[18] H. Murakami, Some metrics on classical knots, Math. Ann., **270** (1985), 35–45.

[19] T. Nakamura, Y. Nakanishi, S. Satoh and A. Yasuhara, The pass move is an unknotting operation for welded knots, Topology Appl., **247** (2018), 9–19.

[20] T. Nasybullov, Classification of fused links, J. Knot Theory Ramifications, **25** (2016), 1650076, 21 pp.

[21] S. Nelson, Unknotting virtual knots with Gauss diagram forbidden moves, J. Knot Theory Ramifications, **10** (2001), 931–935.

[22] T. Okabayashi, Forbidden moves for virtual links, Kobe J. Math., **22** (2005), 49–63.

[23] J. H. Przytycki, t_k moves on links, In: Braids, Santa Cruz, CA, 1986, Contemp. Math., **78**, Amer. Math. Soc., Providence, RI, 1988, 615–656.

[24] J. H. Przytycki, On Slavik Jablan's work on 4-moves, J. Knot Theory Ramifications, **25** (2016), 1641014, 26 pp.

[25] S. Satoh, Crossing changes, Delta moves and sharp moves on welded knots, Rocky Mountain J. Math., **48** (2018), 967–979.

[26] M. Wada, Group invariants of links, Topology, **31** (1992), 399–406.

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