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Optimal problem for mixed p-capacities

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Abstract. In this paper, the optimal problem for mixed p-capacities is investigated. The Orlicz and L_q geominimal p-capacities are proposed and their properties, such as invariance under orthogonal matrices, isoperimetric type inequalities and cyclic type inequalities are provided as well. Moreover, the existence of the p-capacitary Orlicz–Petty bodies for multiple convex bodies is established, and the Orlicz and L_q mixed geominimal p-capacities for multiple convex bodies are introduced. The continuity of the Orlicz mixed geominimal p-capacities and some isoperimetric type inequalities of the L_q mixed geominimal p-capacities are proved.

1. Introduction.

The setting for this paper will be in the Euclidean space $\mathbb{R}^n$. A subset $K \subseteq \mathbb{R}^n$ is said to be convex if $\lambda x + (1-\lambda)y \in K$ for any $x, y \in K$ and any $\lambda \in [0,1]$. A convex compact subset $K \subseteq \mathbb{R}^n$ is called a convex body if $\mathrm{int}K \neq \phi$, where $\mathrm{int}K$ is the interior of K. Denote by $\mathcal{K}$ and $\mathcal{K}_0$ the set of all convex bodies and the set of all convex bodies with the origin o in their interiors, respectively. By $|K|$, we mean the volume of $K \in \mathcal{K}$ and, particularly, we use ω_n to denote the volume of the unit ball $B_2^n \subseteq \mathbb{R}^n$. We use S^{n-1} to denote the unit sphere in $\mathbb{R}^n$. For $K \in \mathcal{K}$, the volume radius of K, denoted by $\mathrm{vrad}(K)$, is defined by

$$\mathrm{vrad}(K) = \left(\frac{|K|}{\omega_n}\right)^{1/n}.$$

It is well known that the affine surface area is a very important concept in convex geometry. The study of the affine surface area can be traced back to Blaschke [**4**] (for $q = 1$), and later it was extended to L_q cases by Lutwak [**21**] (for $q > 1$), Schütt and Werner [**26**] (for $-n \neq q < 1$), Ludwig [**15**] (for Orlicz case). The affine surface area and its extensions have many applications, such as, in the theory of valuations, approximation of convex bodies by polytopes and the information theory of convex bodies (see e.g., [**2**], [**3**], [**11**], [**16**], [**17**], [**18**], [**26**], [**28**]). Geominimal surface area, which can be considered as a "dual" analogous concept of affine surface area, is also an important concept in convex geometry. The classical geominimal surface area was first introduced by Petty [**24**]. For a convex body $K \in \mathcal{K}_0$, the classical geominimal surface area $G(K)$ of K can

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be defined by the following optimal problem,

$$G(K) = \inf_{L \in \mathcal{K}_0} \left\{ \int_{S^{n-1}} h_L(u) dS(K,u) \ \text{with} \ |L^\circ| = \omega_n \right\}, \tag{1.1}$$

where L° denotes the polar body of L, h_L is the support function of L and $S(K,\cdot)$ is the surface area measure of K (see Section 2 for the detailed terminologies). Replacing the support function h_L by the reciprocal of the radial function ρ_L and $\mathcal{K}_0$ by $\mathcal{S}_0$ (the set of star bodies about the origin o) in (1.1), one gets the definition of affine surface area for $q = 1$.

Closely related to the affine and geominimal surface areas is another central concept in convex geometry, i.e., the mixed volumes. For two convex bodies K and L, the mixed volume $V_1(\cdot,\cdot)$ can be defined by:

$$V_1(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(K,u). \tag{1.2}$$

In view of (1.1) and (1.2), one gets

$$G(K) = \inf_{L \in \mathcal{K}_0} \left\{ nV_1(K,L) \ \text{with} \ |L^\circ| = \omega_n \right\}. \tag{1.3}$$

In [**24**], Petty proved that there existed a unique convex body M with $|M^\circ| = \omega_n$ solves the optimal problem in (1.3). This shows that one could define the classical geominimal surface area $G(\cdot)$ based on the mixed volume $V_1(\cdot,\cdot)$. Motivated by this definition (1.3), the classical geominimal surface area has been extended to L_q cases by Lutwak [**21**] (for $q > 1$) and Ye [**31**] (for $-n \neq q < 1$). Similarly, one can define the Orlicz geominimal surface area, please refer to [**32**], [**34**], [**35**]. Therefore, employing the relation between geominimal surface area and the corresponding mixed volume, one could define the Orlicz and L_q geominimal p-capacities ($1 < p < n$) with the help of the Orlicz and L_q mixed p-capacities.

Recall the definitions of the Orlicz and L_q mixed p-capacities for $1 < p < n$. Let $\mathcal{I}$ be the set of continuous functions $\varphi : (0,\infty) \to (0,\infty)$ such that φ is strictly increasing, $\lim_{t\to 0^+} \varphi(t) = 0$, $\varphi(1) = 1$ and $\lim_{t\to\infty} \varphi(t) = \infty$. For $K, L \in \mathcal{K}_0$, $p \in (1,n)$ and $\varphi : (0,\infty) \to (0,\infty)$, the nonhomogeneous and homogeneous Orlicz mixed p-capacities of K and L are given by

$$C_{p,\varphi}(K,L) = \frac{p-1}{n-p} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_p(K,u),$$

$$\int_{S^{n-1}} \varphi\left(\frac{C_p(K) \cdot h_L(u)}{\widehat{C}_{p,\varphi}(K,L) \cdot h_K(u)}\right) d\mu_p^*(K,u) = 1 \quad \text{for } \varphi \in \mathcal{I},$$

where $\mu_p(K,\cdot)$ is the p-capacitary measure on S^{n-1} given by (2.7), and $\mu_p^*(K,\cdot)$ is a probability measure on S^{n-1} given by (2.11). Here we would like to mention that the nonhomogeneous Orlicz mixed p-capacity $C_{p,\varphi}(\cdot,\cdot)$ was introduced in [**13**] and the homogeneous one in [**19**]. When $\varphi(t) = t$, the mixed p-capacity was provided in [**6**]. By letting $\varphi(t) = t^q$ for $-n \neq q \in \mathbb{R}$, one gets the L_q mixed capacities [**13**].

In Section 3, we define the Orlicz and L_q geominimal p-capacities with respect to $\mathcal{Q}_0$ which is a nonempty subset of $\mathcal{S}_0$. For instance, let $K \in \mathcal{K}_0$ be a convex body and $\varphi \in \mathcal{I}$, the nonhomogeneous Orlicz geominimal p-capacity $\mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0)$ of K can be formulated by the following optimal problem:

$$\mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0) = \inf_{L \in \mathcal{Q}_0} \Big\{ C_{p,\varphi}(K, L) \text{ with } |L^\circ| = \omega_n \Big\}.$$

Similarly, the homogeneous Orlicz geominimal p-capacity with respect to $\mathcal{Q}_0$, denoted by $\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0)$, can be defined with $C_{p,\varphi}(\cdot, \cdot)$ replaced by $\widehat{C}_{p,\varphi}(\cdot, \cdot)$. That is,

$$\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0) = \inf_{L \in \mathcal{Q}_0} \Big\{ \widehat{C}_{p,\varphi}(K, L) \text{ with } |L^\circ| = \omega_n \Big\}.$$

In this paper, we would focus on two special cases, which are $\mathcal{Q}_0 = \mathcal{K}_0$ and $\mathcal{Q}_0 = \mathcal{S}_0$. For convenience, we will write $\mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{K}_0), \mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{S}_0), \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K, \mathcal{K}_0)$ and $\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K, \mathcal{S}_0)$ by $\mathcal{G}^{orlicz}_{p,\varphi}(K), \mathcal{A}^{orlicz}_{p,\varphi}(K), \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)$ and $\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)$, respectively. For example,

$$\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K) = \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K, \mathcal{S}_0) = \inf_{L \in \mathcal{S}_0} \Big\{ \widehat{C}_{p,\varphi}(K, L) \text{ with } |L^\circ| = \omega_n \Big\}.$$

In [**19**], the authors showed that there was a convex body $M \in \mathcal{K}_0$ with $|M^\circ| = \omega_n$ such that $\mathcal{G}^{orlicz}_{p,\varphi}(K) = C_{p,\varphi}(K, M)$. Similarly, there is a convex body $\widehat{M} \in \mathcal{K}_0$ with $|\widehat{M}^\circ| = \omega_n$ such that $\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K) = \widehat{C}_{p,\varphi}(K, M)$.

We also provide a detailed study on the properties of the Orlicz geominimal p-capacity of K, such as the invariance under orthogonal matrices. In particular, we establish some isoperimetric type inequalities.

THEOREM 1.1. *Let $K \in \mathcal{K}_0$ be a convex body with its Santaló point or centroid at the origin o and $B_K = \mathrm{vrad}(K) B_2^n$.*
(i) *If $\varphi \in \mathcal{I}_0 \cup \mathcal{D}_0$, then*

$$\frac{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K)} \le \frac{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K)} \le \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball. Here $C_p(K)$ is the p-capacity of K.
(ii) *If $\varphi \in \mathcal{D}_1$, then there exists a universal constant $c > 0$ such that*

$$\frac{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K)} \ge \frac{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K)} \ge c \cdot \frac{C_p(K)}{C_p(B_K)}.$$

Special attention is also paid to the case when $\varphi(t) = t^q$ for $-n \ne q \in \mathbb{R}$ (see Proposition 3.4).

In section 4, we also investigate the existence of p-capacitary Orlicz–Petty bodies of $\boldsymbol{K} = (K_1, K_2, \cdots, K_m)$, a vector of convex bodies, and establish analogous isoperimetric inequalities for the L_q mixed geominimal p-capacity. These results are similar to those of the mixed L_q affine and geominimal surface areas in [**27**], [**29**], [**31**], [**33**], [**36**], which extended the L_q affine and geominimal surface areas.

2. Preliminaries and Notations.

In this section, we collect some basic notations and definitions in convex geometry. One can refer to [**10**], [**25**] for more details in the Brunn–Minkowski theory.

The Minkowski sum of two sets A and B in $\mathbb{R}^n$, denoted by $A+B$, is defined by $A+B=\{x+y : x\in A,\ y\in B\}$. The scalar product of $\lambda\in\mathbb{R}$ and $A\subseteq\mathbb{R}^n$, denoted by λA, is defined by $\lambda A=\{\lambda x : x\in A\}$. For a $n\times n$ matrix ϕ, we use $\det\phi$ and ϕ^t to denote the determinant of ϕ and the transpose of ϕ, respectively If $\det\phi\neq 0$, we say that ϕ is invertible and employ ϕ^{-1} to represent the inverse of ϕ. Denote by $O(n)$ the set of all $n\times n$ matrices such that $\phi\phi^t=\phi^t\phi=I_n$, where I_n is the identity matrix on $\mathbb{R}^n$.

The polar body of $K\in\mathcal{K}_0$, denoted by K°, is defined as follows:

$$K^\circ=\{x\in\mathbb{R}^n : \langle x,y\rangle\le 1 \text{ for any } y\in K\},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$. Denote by $K^{\circ\circ}$ the polar body of K°, and $K^{\circ\circ}=K$ for any $K\in\mathcal{K}_0$ (see e.g., [**25**, Theorem 1.6.1]). For $K\in\mathcal{K}$ and $z\in\mathrm{int}K$, one can define K^z, the polar body of K with respect to z, by $K^z=(K-z)^\circ+z$. For $K\in\mathcal{K}$, there exists a unique point $s(K)\in\mathrm{int}K$, which is called the Santaló point of K, (see e.g., [**22**]), such that, $|K^{s(K)}|=\inf\{|K^z| : z\in\mathrm{int}K\}$. The famous Blaschke–Santaló inequality states: for any $K\in\mathcal{K}$,

$$|K|\cdot|K^{s(K)}|\le\omega_n^2 \tag{2.4}$$

with equality if and only if K is an ellipsoid, i.e., $K=\phi B_2^n+x_0=\{\phi x+x_0 : x\in B_2^n\}$, where ϕ is some invertible $n\times n$ matrix on $\mathbb{R}^n$ and $x_0\in\mathbb{R}^n$ is some vector. On the other hand, there exists a universal constant $c>0$, such that, for any $K\in\mathcal{K}$,

$$|K|\cdot|K^{s(K)}|\ge c^n\omega_n^2. \tag{2.5}$$

This inequality is called the inverse Santaló inequality (see e.g., [**5**], [**14**], [**23**]).

The support function of a nonempty convex compact set $K\subseteq\mathbb{R}^n$, $h_K : S^{n-1}\to\mathbb{R}$, is defined by

$$h_K(u)=\max_{x\in K}\langle x,u\rangle \text{ for any } u\in S^{n-1}.$$

Clearly, for $K,L\in\mathcal{K}$ and any real number $\lambda\ge 0$,

$$h_{\lambda K}(u)=\lambda h_K(u) \text{ and } h_{K+L}(u)=h_K(u)+h_L(u) \text{ for any } u\in S^{n-1}.$$

A nonempty set $L\subseteq\mathbb{R}^n$ is said to be star-shaped about the origin o if for any $x\in L$, the line segment from the origin o to x is contained in L. For a compact star-shaped set L about the origin o, the radial function $\rho_L : S^{n-1}\to[0,\infty)$ is defined by

$$\rho_L(u)=\max\{r\ge 0 : ru\in L\} \quad \text{for any } u\in S^{n-1}.$$

A star body refers to a star-shaped set about the origin o with a positive and continuous radial function. Let $\mathcal{S}_0$ be the set of all star bodies, and clearly $\mathcal{K}_0 \subseteq \mathcal{S}_0$. It is well known that (see e.g., [**25**]) for any $K \in \mathcal{K}_0$ and any $u \in S^{n-1}$,

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)} \text{ and } h_{K^\circ}(u) = \frac{1}{\rho_K(u)}.$$

Moreover, for any $L \in \mathcal{S}_0$, there is an integral formula for volume

$$|L| = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n d\sigma(u),$$

where $\sigma(\cdot)$ is the spherical measure on S^{n-1}. For any $K \in \mathcal{K}$, the surface area measure $S(K, \cdot)$ (see e.g., [**1**], [**9**]), is defined as follows:

$$S(K, A) = \int_{\nu_K^{-1}(A)} d\mathcal{H}^{n-1}, \quad \text{for any measurable subset } A \subseteq S^{n-1},$$

where $\nu_K^{-1} : S^{n-1} \to \partial K$ is the inverse Gauss map and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on ∂K. A convex body $K \in \mathcal{K}$ is said to have a curvature function $f_K : S^{n-1} \to \mathbb{R}$, if its surface area measure $S(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure $\sigma(\cdot)$, and

$$f_K(u) = \frac{dS(K, u)}{d\sigma(u)},$$

almost everywhere, with respect to $\sigma(\cdot)$. Define $\mathcal{F}_0^+$, a subset of $\mathcal{K}_0$, by

$$\mathcal{F}_0^+ = \{K \in \mathcal{K}_0 : f_K \text{ is positive and continuous on } S^{n-1}\}.$$

For compact sets $E, F \subseteq \mathbb{R}^n$, the Hausdorff distance (see e.g., [**25**, (1.60)]) is defined by

$$d_H(E, F) = \min\{\lambda \geq 0 : E \subseteq F + \lambda B_2^n \text{ and } F \subseteq E + \lambda B_2^n\}.$$

For a sequence of compact sets $\{E_i\}_{i=1}^\infty$ and a compact set E, we say that $E_i \to E$ as $i \to \infty$ with respect to the Hausdorff metric if $d_H(E_i, E) \to 0$ as $i \to \infty$. The following lemma will be needed.

Lemma 2.1. (*see* [**19**]) *Let* $\{K_i\}_{i=1}^\infty \subseteq \mathcal{K}_0$ *be a uniformly bounded sequence such that the sequence* $\{|K_i^\circ|\}_{i=1}^\infty$ *is bounded. Then, there exists a subsequence* $\{K_{i_j}\}_{j=1}^\infty$ *of* $\{K_i\}_{i=1}^\infty$ *and a convex body* $K \in \mathcal{K}_0$ *such that* $K_{i_j} \to K$. *Moreover, if* $|K_i^\circ| = \omega_n$ *for all* $i = 1, 2, \cdots$, *then* $|K^\circ| = \omega_n$.

Let $C(S^{n-1})$ be the set of all continuous functions on S^{n-1}. For a sequence of measures $\{\mu_i\}_{i=1}^\infty$ on S^{n-1} and a measure μ on S^{n-1}, we say that μ_i converges weakly to μ if for any $f \in C(S^{n-1})$,

$$\lim_{i \to \infty} \int_{S^{n-1}} f d\mu_i = \int_{S^{n-1}} f d\mu.$$

For a function $f:\mathbb{R}^n\to\mathbb{R}$, the support set of f, denoted by $\mathrm{supp}(f)$, is defined by $\mathrm{supp}(f)=\overline{\{x\in\mathbb{R}^n : f(x)\neq 0\}}$. Let $C_c^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. Let's recall the definition of p-capacity. For a compact set $E\subseteq\mathbb{R}^n$ and $1\le p<n$, the p-capacity of E, denoted by $C_p(E)$, is defined by (see e.g., [**7**], [**8**])

$$C_p(E)=\inf\left\{\int_{\mathbb{R}^n}|\nabla f(x)|^p dx :\ f\in C_c^\infty(\mathbb{R}^n)\ \text{and}\ f(x)\ge 1\ \text{on}\ x\in E\right\},$$

where $|x|$ refers to the Euclidean norm of $x\in\mathbb{R}^n$. Clearly, $C_p(E)\le C_p(F)$ if $E\subseteq F$. We would like to mention that when $p=1$ and $K\in\mathcal{K}_0$, the 1-capacity of K is just the surface area of K.

The following lemma gives some basic properties of the p-capacity, and the results can also be found in [**8**, Chapter 4]. Here we provide the detailed proofs of these basic properties.

Lemma 2.2. *Let E be a compact set and $p\in[1,n)$.*
(i) *For any $\lambda>0$,*

$$C_p(\lambda E)=\lambda^{n-p}C_p(E).$$

(ii) *For any $x_0\in\mathbb{R}^n$,*

$$C_p(E+x_0)=C_p(E).$$

(iii) *For any $\phi\in O(n)$,*

$$C_p(\phi E)=C_p(E).$$

(iv) *The functional $C_p(\cdot)$ is continuous on $\mathcal{K}_0$ with respect to the Hausdorff metric.*

Proof. For convenience, we let $\mathcal{R}(E)=\{g\in C_c^\infty(\mathbb{R}^n)\ \text{and}\ g(x)\ge 1\ \text{on}\ x\in E\}$.

(i) For a differentiable function $f:\mathbb{R}^n\to\mathbb{R}$, let $f_\lambda(x)=f(\lambda x)$. Clearly, $f\in\mathcal{R}(\lambda E)$ if and only if $f_\lambda\in\mathcal{R}(E)$. By the definition of $C_p(\cdot)$, one has

$$\begin{aligned}C_p(\lambda E)&=\inf\left\{\int_{\mathbb{R}^n}|\nabla f(x)|^p dx :\ f\in\mathcal{R}(\lambda E)\right\}\\&=\inf\left\{\lambda^{n-p}\cdot\int_{\mathbb{R}^n}|\nabla f_\lambda(y)|^p dy :\ f_\lambda\in\mathcal{R}(E)\right\}\\&=\lambda^{n-p}\cdot C_p(E).\end{aligned}$$

(ii) Similarly, we can define $f_{x_0}(x)=f(x+x_0)$ for a differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ and hence $f\in\mathcal{R}(E+x_0)$ if and only if $f_{x_0}\in\mathcal{R}(E)$. By the definition of $C_p(\cdot)$, one has

$$C_p(E+x_0)=\inf\left\{\int_{\mathbb{R}^n}|\nabla f(x)|^p dx :\ f\in\mathcal{R}(E+x_0)\right\}$$

$$= \inf \left\{ \int_{\mathbb{R}^n} |\nabla f_{x_0}(y)|^p dy : f_{x_0} \in \mathcal{R}(E) \right\}$$
$$= C_p(E).$$

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $f_\phi(x) = f(\phi x)$ with $\phi \in O(n)$. Hence, $f \in \mathcal{R}(\phi E)$ if and only if $f_\phi \in \mathcal{R}(E)$. Moreover, if $x = \phi y$, then $|\nabla f(x)| = |\nabla f_\phi(y)|$. From the definition of $C_p(\cdot)$, one has

$$\begin{aligned} C_p(\phi E) &= \inf \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^p dx : f \in \mathcal{R}(\phi E) \right\} \\ &= \inf \left\{ \int_{\mathbb{R}^n} |\nabla f_\phi(y)|^p dy : f_\phi \in \mathcal{R}(E) \right\} \\ &= C_p(E). \end{aligned}$$

(iv) First of all, for $K \in \mathcal{K}_0$, $C_p(K) > 0$ (see e.g. [**7**]). For any $\epsilon > 0$, choose two positive constants $\lambda > 1$ and $\rho > 0$ such that $(\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K) < \epsilon$ and $\rho B_2^n \subseteq K$. It follows from [**25**, Lemma 1.8.18] that there exists a positive number $\delta > 0$ such that $\delta \le \rho(\lambda - 1)$ and $\rho B_2^n \subseteq \widetilde{K}$ when $d_H(K, \widetilde{K}) < \delta$. Thus,

$$K \subseteq \widetilde{K} + \delta B_2^n \subseteq \widetilde{K} + (\lambda - 1)\rho B_2^n \subseteq \widetilde{K} + (\lambda - 1)\widetilde{K} = \lambda \widetilde{K}.$$

This, together with the monotonicity and homogeneity of $C_p(\cdot)$, implies that

$$C_p(K) \le C_p(\lambda \widetilde{K}) = \lambda^{n-p} \cdot C_p(\widetilde{K}).$$

Similarly, one has $\widetilde{K} \subseteq \lambda K$ and $C_p(\widetilde{K}) \le \lambda^{n-p} \cdot C_p(K)$. Hence

$$\begin{aligned} C_p(K) - C_p(\widetilde{K}) &\le (\lambda^{n-p} - 1) \cdot C_p(\widetilde{K}) \le (\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K); \\ C_p(\widetilde{K}) - C_p(K) &\le (\lambda^{n-p} - 1) \cdot C_p(K) \le (\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K). \end{aligned}$$

Thus, one gets

$$|C_p(K) - C_p(\widetilde{K})| \le (\lambda^{n-p} - 1) \cdot \lambda^{n-p} \cdot C_p(K) < \epsilon. \qquad \square$$

For $1 < p < n$, the equation $\operatorname{div}(|\nabla U|^{p-2}\nabla U) = 0$ is called the p-Laplace equation. It can be easily checked that $U_0(x) = |x|^{(p-n)/(p-1)} (x \ne o)$ satisfies the p-Laplace equation except the origin o, and hence $U_0(x)$ is called the fundamental solution of the p-Laplace equation. The p-equilibrium potential of $K \in \mathcal{K}_0$ is a weak solution of the following boundary p-Laplace equation:

$$\begin{cases} \operatorname{div}(|\nabla U|^{p-2}\nabla U) = 0 & \text{in } \mathbb{R}^n \setminus K, \\ U(x) = 1 & \text{on } \partial K, \\ \lim_{|x| \to \infty} U(x) = 0. \end{cases} \tag{2.6}$$

For $K \in \mathcal{K}_0$, there exists a unique solution to (2.6) (see e.g., [**7**]). This implies that, for $K \in \mathcal{K}_0$, the p-equilibrium potential exists and is unique. In later context, we use

U_K to denote the p-equilibrium potential of $K \in \mathcal{K}_0$. Obviously, $U_{B_2^n}(x) = U_0(x) = |x|^{(p-n)/(p-1)}(x \neq o)$. Hereafter, we only consider $p \in (1, n)$.

Lemma 2.3. *Let $K \in \mathcal{K}_0$ and U_K be the p-equilibrium potential of K.*
(i) *The p-equilibrium potential of λK, for any $\lambda > 0$, is*

$$U_{\lambda K}(x) = U_K\big(x/\lambda\big).$$

(ii) *The p-equilibrium potential of $K + x_0$, for any $x_0 \in \mathbb{R}^n$, is*

$$U_{K+x_0}(x) = U_K(x - x_0).$$

(iii) *The p-equilibrium potential of ϕK, for any $\phi \in O(n)$, is*

$$U_{\phi K}(x) = U_K(\phi^t x).$$

Proof. The proofs of the assertions (i)–(iii) are similar, and we only provide the proof of (iii) which requires the most work. For convenience, let $U_\phi(x) = U_K(\phi^t x)$ for any $x \in \mathbb{R}^n$. Note that $U_K(x) = 1$ on ∂K and $\lim_{|x|\to\infty} U_K(x) = 0$. Along with $\phi \in O(n)$, one gets $U_\phi(x) = U_K(\phi^t x) = 1$ on $\partial(\phi K)$ and $\lim_{|x|\to\infty} U_\phi(x) = \lim_{|x|\to\infty} U_K(\phi^t x) = 0$. Moreover, for any $x \in \mathbb{R}^n \setminus \phi K$, one can get

$$\operatorname{div}(|\nabla U_\phi|^{p-2}\nabla U_\phi)(x) = \operatorname{div}(|\nabla U_K|^{p-2}\nabla U_K)(\phi^t x) = 0.$$

Thus U_ϕ is the p-equilibrium potential of ϕK, i.e., $U_{\phi K}(x) = U_\phi(x) = U_K(\phi^t x)$ for any $x \in \mathbb{R}^n$. ☐

For $K \in \mathcal{K}_0$, the p-capacitary measure $\mu_p(K, \cdot)$ on S^{n-1}, is defined by

$$\mu_p(K, A) = \int_{\nu_K^{-1}(A)} |\nabla U_K(x)|^p d\mathcal{H}^{n-1}, \text{ for any measurable subset } A \subseteq S^{n-1}. \tag{2.7}$$

One can easily get, for any $\lambda > 0$ and any $u \in S^{n-1}$,

$$\mu_p(\lambda K, u) = \lambda^{n-p-1}\mu_p(K, u) \text{ and } d\mu_p(K, u) = |\nabla U_K(\nu_K^{-1}(u))|^p dS(K, u). \tag{2.8}$$

Note that $U_{B_2^n}(x) = U_0(x) = |x|^{(p-n)/(p-1)}(x \notin B_2^n)$, one has

$$d\mu_p(B_2^n, u) = \left(\frac{n-p}{p-1}\right)^p d\sigma(u) \text{ for any } u \in S^{n-1}. \tag{2.9}$$

It has been proved in [**30**, Theorem 1] that $\mu_p(K, \cdot)$ is not concentrated on any hemisphere of S^{n-1}, i.e.,

$$\int_{S^{n-1}} \langle v, u\rangle_+ \, d\mu_p(K, u) > 0 \text{ for any } v \in S^{n-1},$$

where $\langle v, u\rangle_+ = \max\{\langle v, u\rangle, 0\}$. The famous Poincaré formula for p-capacity can be stated as follows: for any $K \in \mathcal{K}_0$,

$$C_p(K) = \frac{p-1}{n-p} \int_{S^{n-1}} h_K(u)\, d\mu_p(K,u).$$

In particular, one has

$$C_p(B_2^n) = \left(\frac{n-p}{p-1}\right)^{p-1} \cdot n\omega_n. \tag{2.10}$$

For $K \in \mathcal{K}_0$, $\mu_p^*(K,\cdot)$, a probability measure on S^{n-1}, is defined as follows:

$$\mu_p^*(K,A) = \frac{p-1}{n-p} \int_A \frac{h_K(u) \cdot d\mu_p(K,u)}{C_p(K)}, \text{ for any measurable subset } A \subseteq S^{n-1}. \tag{2.11}$$

3. The nonhomogeneous and homogeneous geominimal *p*-capacities.

In this section, the Orlicz and L_q geominimal p-capacities and their properties are provided. In particular, we establish a series of isoperimetric type inequalities related to these newly proposed geominimal p-capacities.

Firstly, let's recall some notations and the results in [**19**]. Let $\mathcal{D}$ be the set of continuous functions $\varphi : (0,\infty) \to (0,\infty)$ such that φ is strictly decreasing, $\lim_{t\to 0^+} \varphi(t) = \infty, \varphi(1) = 1$ and $\lim_{t\to\infty} \varphi(t) = 0$.

Definition 3.1. Let $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $K, L \in \mathcal{K}_0$. The nonhomogeneous Orlicz mixed p-capacity of K and L, $C_{p,\varphi}(K,L)$, is defined by

$$C_{p,\varphi}(K,L) = \frac{p-1}{n-p} \int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u) d\mu_p(K,u).$$

If $L \in \mathcal{S}_0$, we use $C_{p,\varphi}(K,L^\circ)$ for

$$C_{p,\varphi}(K,L^\circ) = \frac{p-1}{n-p} \int_{S^{n-1}} \varphi\left(\frac{1}{\rho_L(u) \cdot h_K(u)}\right) h_K(u) d\mu_p(K,u).$$

The homogeneous analogue is defined as follows.

Definition 3.2. Let $\varphi \in \mathcal{I} \cup \mathcal{D}$ and $K, L \in \mathcal{K}_0$. The homogeneous Orlicz mixed p-capacity of K and L, $\widehat{C}_{p,\varphi}(K,L)$, is defined by

$$\int_{S^{n-1}} \varphi\left(\frac{C_p(K) \cdot h_L(u)}{\widehat{C}_{p,\varphi}(K,L) \cdot h_K(u)}\right) d\mu_p^*(K,u) = 1.$$

If $L \in \mathcal{S}_0$, then we use $\widehat{C}_{p,\varphi}(K,L^\circ)$ for

$$\int_{S^{n-1}} \varphi\left(\frac{C_p(K)}{\widehat{C}_{p,\varphi}(K,L^\circ) \cdot \rho_L(u) \cdot h_K(u)}\right) d\mu_p^*(K,u) = 1.$$

Clearly, $\widehat{C}_{p,\varphi}(\cdot,\cdot)$ is homogeneous, i.e., if $K, L \in \mathcal{K}_0$ and $\varphi \in \mathcal{I} \cup \mathcal{D}$, then for $s, t > 0$

$$\widehat{C}_{p,\varphi}(sK, tL) = s^{n-p-1} \cdot t \cdot \widehat{C}_{p,\varphi}(K, L), \tag{3.12}$$

if $L \in \mathcal{S}_0$, then

$$\widehat{C}_{p,\varphi}(sK, (tL)^\circ) = s^{n-p-1} \cdot t^{-1} \cdot \widehat{C}_{p,\varphi}(K, L^\circ). \tag{3.13}$$

The existence theorem of the p-capacitary Orlicz–Petty bodies was provided as follows.

Theorem 3.1 ([**19**]). *Let $K \in \mathcal{K}_0$ be a convex body and $\varphi \in \mathcal{I}$.*
(i) *There exists a convex body $M \in \mathcal{K}_0$ such that $|M^\circ| = \omega_n$ and*

$$C_{p,\varphi}(K, M) = \inf\Big\{C_{p,\varphi}(K, L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\Big\}.$$

(ii) *There exists a convex body $\widehat{M} \in \mathcal{K}_0$ such that $|\widehat{M}^\circ| = \omega_n$ and*

$$\widehat{C}_{p,\varphi}(K, \widehat{M}) = \inf\Big\{\widehat{C}_{p,\varphi}(K, L) : L \in \mathcal{K}_0 \text{ and } |L^\circ| = \omega_n\Big\}.$$

In addition, if $\varphi \in \mathcal{I}$ is convex, then both M and $\widehat{M}$ are unique.

We use the set $\mathcal{T}_{p,\varphi}(K)$ to denote the collections of all convex bodies M, and the set $\widehat{\mathcal{T}}_{p,\varphi}(K)$ to denote the collection of all convex bodies $\widehat{M}$ in Theorem 3.1. A convex body $M \in \mathcal{T}_{p,\varphi}(K)$ is called a nonhomogeneous p-capacitary Orlicz–Petty body, and a convex body $\widehat{M} \in \widehat{\mathcal{T}}_{p,\varphi}(K)$ is called a homogeneous p-capacitary Orlicz–Petty body. Note when $\varphi \in \mathcal{I}$ is convex, $\mathcal{T}_{p,\varphi}(K)$ and $\widehat{\mathcal{T}}_{p,\varphi}(K)$ contain only one element.

3.1. The Orlicz geominimal p-capacity.

In this subsection, we provide a detailed study of the Orlicz geominimal p-capacities. Let

$$\begin{aligned}
\mathcal{I}_0 &= \mathcal{I} \cap \{\varphi : (0,\infty) \to (0,\infty) \,|\, \varphi(t^{-1/n}) \text{ is strictly convex on } (0,\infty)\};\\
\mathcal{D}_0 &= \mathcal{D} \cap \{\varphi : (0,\infty) \to (0,\infty) \,|\, \varphi(t^{-1/n}) \text{ is strictly concave on } (0,\infty)\};\\
\mathcal{D}_1 &= \mathcal{D} \cap \{\varphi : (0,\infty) \to (0,\infty) \,|\, \varphi(t^{-1/n}) \text{ is strictly convex on } (0,\infty)\}.
\end{aligned}$$

Let $\mathcal{Q}_0 \subseteq \mathcal{S}_0$ be a nonempty subset of $\mathcal{S}_0$. Since $\big|(\mathrm{vrad}(L^\circ)L)^\circ\big| = \big|[\mathrm{vrad}(L^\circ)]^{-1}L^\circ\big| = \omega_n$, and $h_{\mathrm{vrad}(L^\circ)L} = \mathrm{vrad}(L^\circ)h_L$ for any $L \in \mathcal{K}_0$ and $\rho_{\mathrm{vrad}(L^\circ)L} = \mathrm{vrad}(L^\circ)\rho_L$ for any $L \in \mathcal{S}_0$, one can define the geominimal p-capacity as follows.

Definition 3.3. For $K \in \mathcal{K}_0$, define $\mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0)$, the nonhomogeneous Orlicz geominimal p-capacity of K with respect to $\mathcal{Q}_0$, as follows:

$$\begin{aligned}
\mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0) &= \inf_{L \in \mathcal{Q}_0} \Big\{C_{p,\varphi}(K, \mathrm{vrad}(L)\, L^\circ)\Big\} \text{ for } \varphi \in \mathcal{I} \cup \mathcal{D}_1,\\
\mathcal{G}^{orlicz}_{p,\varphi}(K, \mathcal{Q}_0) &= \sup_{L \in \mathcal{Q}_0} \Big\{C_{p,\varphi}(K, \mathrm{vrad}(L)\, L^\circ)\Big\} \text{ for } \varphi \in \mathcal{D}_0.
\end{aligned}$$

Similarly, the homogeneous Orlicz geominimal p-capacity with respect to $\mathcal{Q}_0$, denoted by $\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K,\mathcal{Q}_0)$, can be defined with $C_{p,\varphi}(\cdot,\cdot)$ replaced by $\widehat{C}_{p,\varphi}(\cdot,\cdot)$ and $\mathcal{D}_1$ switching with $\mathcal{D}_0$.

Two special cases are important and we will focus on their properties in later context. The first one is the case when $\mathcal{Q}_0=\mathcal{K}_0$, then we use $\mathcal{G}_{p,\varphi}^{orlicz}(K)$ and $\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K)$ to denote $\mathcal{G}_{p,\varphi}^{orlicz}(K,\mathcal{K}_0)$ and $\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K,\mathcal{K}_0)$. The second case is $\mathcal{Q}_0=\mathcal{S}_0$ and we use $\mathcal{A}_{p,\varphi}^{orlicz}(K)$ and $\widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(K)$ for $\mathcal{G}_{p,\varphi}^{orlicz}(K,\mathcal{S}_0)$ and $\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K,\mathcal{S}_0)$. As $\mathcal{K}_0\subseteq\mathcal{S}_0$, then

$$\mathcal{A}_{p,\varphi}^{orlicz}(K)\le\mathcal{G}_{p,\varphi}^{orlicz}(K)\ \text{ for }\varphi\in\mathcal{I}\cup\mathcal{D}_1\ \text{ and }\ \mathcal{A}_{p,\varphi}^{orlicz}(K)\ge\mathcal{G}_{p,\varphi}^{orlicz}(K)\ \text{ for }\varphi\in\mathcal{D}_0;$$
$$\widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(K)\le\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K)\ \text{ for }\varphi\in\mathcal{I}\cup\mathcal{D}_0\ \text{ and }\ \widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(K)\ge\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K)\ \text{ for }\varphi\in\mathcal{D}_1.$$

From (3.12) and (3.13), one can easily get, for any $\lambda>0$,

$$\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(\lambda K)=\lambda^{n-p-1}\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K)\ \text{ and }\ \widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(\lambda K)=\lambda^{n-p-1}\widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(K).$$

The following results state that all the quantities above are $O(n)$-invariant. Moreover, when $\varphi\in\mathcal{I}$, it follows from Theorem 3.1 that $\mathcal{G}_{p,\varphi}^{orlicz}(K)=C_{p,\varphi}(K,M)$ for $M\in\mathcal{T}_{p,\varphi}(K)$ and $\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K)=\widehat{C}_{p,\varphi}(K,\widehat{M})$ for $\widehat{M}\in\widehat{\mathcal{T}}_{p,\varphi}(K)$.

COROLLARY 3.1. *If $\varphi\in\mathcal{I}\cup\mathcal{D}_0\cup\mathcal{D}_1$, then for any $\phi\in O(n)$ and for any $K\in\mathcal{K}_0$,*

$$\mathcal{G}_{p,\varphi}^{orlicz}(\phi K)=\mathcal{G}_{p,\varphi}^{orlicz}(K)\ \textit{ and }\ \widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(\phi K)=\widehat{\mathcal{G}}_{p,\varphi}^{orlicz}(K);$$
$$\mathcal{A}_{p,\varphi}^{orlicz}(\phi K)=\mathcal{A}_{p,\varphi}^{orlicz}(K)\ \textit{ and }\ \widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(\phi K)=\widehat{\mathcal{A}}_{p,\varphi}^{orlicz}(K).$$

PROOF. Here we only prove the equality of $\mathcal{G}_{p,\varphi}^{orlicz}(\phi K)=\mathcal{G}_{p,\varphi}^{orlicz}(K)$, and the other cases can be proved along a similar argument. Let $L\in\mathcal{K}_0$. Since $\phi\in O(n)$, then $|\phi L|=|L|$ and $\text{vrad}(\phi L)=\text{vrad}(L)$. Moreover, by Lemma 2.3 and (2.8), one has, for any $u\in S^{n-1}$,

$$\begin{aligned} d\mu_p(\phi K,u)&=|\nabla U_{\phi K}\big(\nu_{\phi K}^{-1}(u)\big)|^p dS(\phi K,u)\\ &=|\nabla U_K\big(\phi^t\cdot\phi\cdot\nu_K^{-1}(\phi^t u)\big)\cdot\phi^t|^p dS(K,\phi^t u)\\ &=|\nabla U_K\big(\nu_K^{-1}(\phi^t u)\big)|^p dS(K,\phi^t u)\\ &=d\mu_p(K,\phi^t u). \end{aligned}\tag{3.14}$$

For $u\in S^{n-1}$ and $\phi\in O(n)$, let $v=\phi^t u$. By (3.14) and $(\phi L)^\circ=\phi L^\circ$, one gets

$$\begin{aligned} C_{p,\varphi}(\phi K,\text{vrad}(\phi L)(\phi L)^\circ)&=C_{p,\varphi}(\phi K,\text{vrad}(L)\phi L^\circ)\\ &=\frac{p-1}{n-p}\int_{S^{n-1}}\varphi\left(\frac{h_{\text{vrad}(L)\phi L^\circ}(u)}{h_{\phi K}(u)}\right)h_{\phi K}(u)d\mu_p(\phi K,u)\\ &=\frac{p-1}{n-p}\int_{S^{n-1}}\varphi\left(\frac{h_{\text{vrad}(L)L^\circ}(\phi^t u)}{h_K(\phi^t u)}\right)h_K(\phi^t u)d\mu_p(K,\phi^t u)\\ &=\frac{p-1}{n-p}\int_{S^{n-1}}\varphi\left(\frac{h_{\text{vrad}(L)L^\circ}(v)}{h_K(v)}\right)h_K(v)d\mu_p(K,v)\\ &=C_{p,\varphi}(K,\text{vrad}(L)L^\circ). \end{aligned}$$

This, together with Definition 3.3, implies that if $\varphi \in \mathcal{I} \cup \mathcal{D}_1$,

$$\begin{aligned}\mathcal{G}^{orlicz}_{p,\varphi}(\phi K) &= \inf_{\phi L \in \mathcal{K}_0} \left\{ C_{p,\varphi}(\phi K, \operatorname{vrad}(\phi L)\,(\phi L)^\circ) \right\} \\ &= \inf_{L \in \mathcal{K}_0} \left\{ C_{p,\varphi}(K, \operatorname{vrad}(L)\, L^\circ) \right\} \\ &= \mathcal{G}^{orlicz}_{p,\varphi}(K).\end{aligned}$$

Replacing "inf" by "sup", one gets $\mathcal{G}^{orlicz}_{p,\varphi}(\phi K) = \mathcal{G}^{orlicz}_{p,\varphi}(K)$ when $\varphi \in \mathcal{D}_0$. □

In general, it is not easy to calculate $\mathcal{G}^{orlicz}_{p,\varphi}(K)$, $\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)$, $\mathcal{A}^{orlicz}_{p,\varphi}(K)$ and $\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)$. However, when $K = rB_2^n$ for some $r > 0$, we are able to calculate their precise values.

Proposition 3.1. *Let $\varphi \in \mathcal{I}_0 \cup \mathcal{D}_0 \cup \mathcal{D}_1$ and $r > 0$. Then*

$$\mathcal{A}^{orlicz}_{p,\varphi}(rB_2^n) = \mathcal{G}^{orlicz}_{p,\varphi}(rB_2^n) = \varphi\left(\frac{1}{r}\right) \cdot C_p(rB_2^n), \tag{3.15}$$

$$\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_2^n) = \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_2^n) = C_p(B_2^n). \tag{3.16}$$

Proof. The proofs of (3.15) and (3.16) are similar, and we only prove (3.16). For any $L \in \mathcal{S}_0$, let $\tilde{L} = L/\operatorname{vrad}(L)$. Thus $|\tilde{L}| = \omega_n$ and $\operatorname{vrad}(\tilde{L}) = 1$. If $\varphi \in \mathcal{I}_0$, with the help of (2.9), (2.10) and Jensen's inequality for the convex function $\varphi(t^{-1/n})$, one has

$$\begin{aligned}1 &= \int_{S^{n-1}} \varphi\left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ) \cdot \rho_{\tilde{L}}(u) \cdot h_{B_2^n}(u)} \right) d\mu_p^*(B_2^n, u) \\ &= \int_{S^{n-1}} \varphi\left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ) \cdot \rho_{\tilde{L}}(u)} \right) \frac{d\sigma(u)}{n\omega_n} \\ &\geq \varphi\left(\left(\int_{S^{n-1}} \left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ) \cdot \rho_{\tilde{L}}(u)} \right)^{-n} \frac{d\sigma(u)}{n\omega_n} \right)^{-1/n} \right) \\ &= \varphi\left(\frac{C_p(B_2^n)}{\widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ)} \right).\end{aligned}$$

Since φ is increasing and $\varphi(1) = 1$, one gets

$$C_p(B_2^n) \leq \widehat{C}_{p,\varphi}(B_2^n, \tilde{L}^\circ) = \widehat{C}_{p,\varphi}(B_2^n, \operatorname{vrad}(L)L^\circ).$$

Taking the infimum over $L \in \mathcal{S}_0$ and by Definition 3.3, one has

$$C_p(B_2^n) \leq \widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_2^n) \leq \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_2^n) = \inf_{L \in \mathcal{K}_0} \left\{ \widehat{C}_{p,\varphi}(B_2^n, \operatorname{vrad}(L)\, L^\circ) \right\} \leq C_p(B_2^n)$$

and hence $C_p(B_2^n) = \widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_2^n) = \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_2^n)$. The results for $\varphi \in \mathcal{D}_0 \cup \mathcal{D}_1$ follow from a similar argument. □

The isoperimetric type inequalities for $\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(\cdot)$, $\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(\cdot)$, $\mathcal{A}^{orlicz}_{p,\varphi}(\cdot)$ and $\mathcal{G}^{orlicz}_{p,\varphi}(\cdot)$ are established in the following theorems.

THEOREM 3.2. *Let $K \in \mathcal{K}_0$ be a convex body with its Santaló point or centroid at the origin o and B_K be an origin symmetric ball defined by $B_K = \mathrm{vrad}(K)B_2^n$.*
(i) *If $\varphi \in \mathcal{I}_0 \cup \mathcal{D}_0$, then*

$$\frac{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K)} \le \frac{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K)} \le \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.
(ii) *If $\varphi \in \mathcal{D}_1$, then there exists a universal constant $c > 0$ such that*

$$\frac{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K)} \ge \frac{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K)} \ge c \cdot \frac{C_p(K)}{C_p(B_K)}.$$

PROOF. (i) Let $\varphi \in \mathcal{I}_0 \cup \mathcal{D}_0$. It follows from the homogeneity of $\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(\cdot)$, $\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(\cdot)$ and $C_p(\cdot)$, and Proposition 3.1 that

$$\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K) = \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K) = \frac{C_p(B_K)}{\mathrm{vrad}(K)}. \tag{3.17}$$

By Definition 3.3 and (3.12), one has,

$$\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K) \le \widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K) \le \widehat{C}_{p,\varphi}(K, \mathrm{vrad}(K^\circ)\,K) = \mathrm{vrad}(K^\circ) \cdot C_p(K).$$

Together with (3.17) and the Blaschke–Santaló inequality (2.4), one has

$$\frac{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K)} \le \frac{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K)} \le \frac{C_p(K)}{C_p(B_K)}.$$

If K is an origin symmetric ball, say $K = rB_2^n$ for some $r > 0$, one can easily get $K = B_K$ and thus equality in part (i) holds.

(ii) If $\varphi \in \mathcal{D}_1$, by a similar argument and the inverse Santaló inequality (2.5), one has

$$\frac{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(B_K)} \ge \frac{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K)}{\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(B_K)} \ge \frac{\mathrm{vrad}(K) \cdot \mathrm{vrad}(K^\circ) \cdot C_p(K)}{C_p(B_K)} \ge c \cdot \frac{C_p(K)}{C_p(B_K)}. \qquad \square$$

Along the same lines, one can get the similar results for $\mathcal{G}^{orlicz}_{p,\varphi}(K)$ and $\mathcal{A}^{orlicz}_{p,\varphi}(K)$.

THEOREM 3.3. *Let $K \in \mathcal{K}_0$ be a convex body with its Santaló point or centroid at the origin o and $B_K = \mathrm{vrad}(K)B_2^n$.*
(i) *If $\varphi \in \mathcal{I}_0 \cup \mathcal{D}_1$, then*

$$\frac{\mathcal{A}^{orlicz}_{p,\varphi}(K)}{\mathcal{A}^{orlicz}_{p,\varphi}\big((B_{K^\circ})^\circ\big)} \le \frac{\mathcal{G}^{orlicz}_{p,\varphi}(K)}{\mathcal{G}^{orlicz}_{p,\varphi}\big((B_{K^\circ})^\circ\big)} \le \frac{C_p(K)}{C_p\big((B_{K^\circ})^\circ\big)}.$$

Moreover, if $\varphi \in \mathcal{I}_0$, then

$$\frac{\mathcal{A}_{p,\varphi}^{orlicz}(K)}{\mathcal{A}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{\mathcal{G}_{p,\varphi}^{orlicz}(K)}{\mathcal{G}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.

(ii) *If $\varphi \in \mathcal{D}_0$, then*

$$\frac{\mathcal{A}_{p,\varphi}^{orlicz}(K)}{\mathcal{A}_{p,\varphi}^{orlicz}(B_K)} \geq \frac{\mathcal{G}_{p,\varphi}^{orlicz}(K)}{\mathcal{G}_{p,\varphi}^{orlicz}(B_K)} \geq \frac{C_p(K)}{C_p(B_K)}.$$

Equality holds if K is an origin symmetric ball.

Proof. (i) It follows from Definition 3.3 that

$$\mathcal{A}_{p,\varphi}^{orlicz}(K) \leq \mathcal{G}_{p,\varphi}^{orlicz}(K) \leq C_{p,\varphi}(K, \mathrm{vrad}(K^\circ)\, K) = \varphi(\mathrm{vrad}(K^\circ)) \cdot C_p(K). \tag{3.18}$$

Note that $(B_{K^\circ})^\circ = (\mathrm{vrad}(K^\circ)B_2^n)^\circ = (1/\mathrm{vrad}(K^\circ))B_2^n$. By (3.15) in Proposition 3.1, one has

$$\mathcal{A}_{p,\varphi}^{orlicz}(B_K) = \mathcal{G}_{p,\varphi}^{orlicz}(B_K) = \varphi\left(\frac{1}{\mathrm{vrad}(K)}\right) \cdot C_p(B_K); \tag{3.19}$$

$$\mathcal{A}_{p,\varphi}^{orlicz}\big((B_{K^\circ})^\circ\big) = \mathcal{G}_{p,\varphi}^{orlicz}\big((B_{K^\circ})^\circ\big) = \varphi(\mathrm{vrad}(K^\circ)) \cdot C_p\big((B_{K^\circ})^\circ\big). \tag{3.20}$$

The desired result follows from (3.18) and (3.20).

If $\varphi \in \mathcal{I}_0$, by (3.18) and the Blaschke–Santaló inequality (2.4), one has

$$\mathcal{A}_{p,\varphi}^{orlicz}(K) \leq \mathcal{G}_{p,\varphi}^{orlicz}(K) \leq \varphi(\mathrm{vrad}(K^\circ)) \cdot C_p(K) \leq \varphi\left(\frac{1}{\mathrm{vrad}(K)}\right) \cdot C_p(K).$$

This along with (3.19) yields

$$\frac{\mathcal{A}_{p,\varphi}^{orlicz}(K)}{\mathcal{A}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{\mathcal{G}_{p,\varphi}^{orlicz}(K)}{\mathcal{G}_{p,\varphi}^{orlicz}(B_K)} \leq \frac{C_p(K)}{C_p(B_K)}.$$

If K is an origin symmetric ball, it can be easily checked that the equality holds. The case (ii) follows from the same lines as the proof of the case $\varphi \in \mathcal{I}_0$. □

3.2. The L_q geominimal p-capacity.

In this subsection, we let $\varphi(t) = t^q$ in Definition 3.1 and consider the L_q geominimal p-capacity of K with respect to $\mathcal{K}_0$ and $\mathcal{S}_0$. Let

$$C_{p,q}(K, L) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\frac{h_L(u)}{h_K(u)}\right)^q h_K(u) d\mu_p(K, u) \quad \text{for } L \in \mathcal{K}_0;$$

$$C_{p,q}(K, L^\circ) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\frac{1}{\rho_L(u) \cdot h_K(u)}\right)^q h_K(u) d\mu_p(K, u) \quad \text{for } L \in \mathcal{S}_0.$$

Definition 3.4. Let $-n \neq q \in \mathbb{R}$ and $K \in \mathcal{K}_0$. Define $\mathcal{G}_{p,q}(K)$, the L_q geominimal p-capacity with respect to $\mathcal{K}_0$, by

$$\mathcal{G}_{p,q}(K) = \inf_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,q}(K,L)\big)^{n/(n+q)} \cdot |L^\circ|^{q/(n+q)} \Big\}, \quad q \ge 0, \tag{3.21}$$

$$\mathcal{G}_{p,q}(K) = \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,q}(K,L)\big)^{n/(n+q)} \cdot |L^\circ|^{q/(n+q)} \Big\}, \quad -n \ne q < 0; \tag{3.22}$$

and define $\mathcal{A}_{p,q}(K)$, the L_q geominimal p-capacity with respect to $\mathcal{S}_0$, by

$$\mathcal{A}_{p,q}(K) = \inf_{L\in\mathcal{S}_0} \Big\{ \big(C_{p,q}(K,L^\circ)\big)^{n/(n+q)} \cdot |L|^{q/(n+q)} \Big\}, \quad q \ge 0, \tag{3.23}$$

$$\mathcal{A}_{p,q}(K) = \sup_{L\in\mathcal{S}_0} \Big\{ \big(C_{p,q}(K,L^\circ)\big)^{n/(n+q)} \cdot |L|^{q/(n+q)} \Big\}, \quad -n \ne q < 0. \tag{3.24}$$

Clearly, $\mathcal{G}_{p,0}(K) = \mathcal{A}_{p,0}(K) = C_p(K)$ for any $K \in \mathcal{K}_0$. Moreover, it can be easily checked that for $\varphi(t) = t^q$ $(q \ne -n)$ and any $K \in \mathcal{K}_0$,

$$\mathcal{G}_{p,q}(\lambda K) = \lambda^{n(n-p-q)/(n+q)}\mathcal{G}_{p,q}(K) \ \text{ and } \ \mathcal{A}_{p,q}(\lambda K) = \lambda^{n(n-p-q)/(n+q)}\mathcal{A}_{p,q}(K) \quad \text{for any } \ \lambda > 0;$$

$$\mathcal{G}_{p,q}(\phi K) = \mathcal{G}_{p,q}(K) \ \text{ and } \ \mathcal{A}_{p,q}(\phi K) = \mathcal{A}_{p,q}(K) \ \text{ for any } \ \phi \in O(n).$$

Moreover, if $q \ne 0, -n$, then with $\varphi(t) = t^q$, one has

$$\widehat{\mathcal{G}}^{orlicz}_{p,\varphi}(K) = \frac{C_p(K)^{1-(1/q)}}{\omega_n^{1/n}} \cdot \big(\mathcal{G}_{p,q}(K)\big)^{(n+q)/nq}; \tag{3.25}$$

$$\widehat{\mathcal{A}}^{orlicz}_{p,\varphi}(K) = \frac{C_p(K)^{1-(1/q)}}{\omega_n^{1/n}} \cdot \big(\mathcal{A}_{p,q}(K)\big)^{(n+q)/nq}. \tag{3.26}$$

Remark 3.1. By (3.16) and (3.21), one gets, for any $-n \ne q \in \mathbb{R}$,

$$\begin{aligned}\mathcal{G}_{p,q}(B_2^n) = \mathcal{A}_{p,q}(B_2^n) &= \big(C_p(B_2^n)\big)^{n/(n+q)} \cdot |B_2^n|^{q/(n+q)} \\ &= \big(C_{p,q}(B_2^n, B_2^n)\big)^{n/(n+q)} \cdot |B_2^n|^{q/(n+q)}.\end{aligned}$$

The following proposition provides a convenient formula to calculate $\mathcal{A}_{p,q}(K)$ for $q \ne -n$. For $K \in \mathcal{F}_0^+$, let

$$f_{\mu_p,q}(K,u) = h_K^{1-q}(u) \cdot |\nabla U_K\big(\nu_K^{-1}(u)\big)|^p \cdot f_K(u),$$

where U_K is the p-equilibrium potential of K and f_K is the curvature function of K. For $-n \ne q \in \mathbb{R}$, let

$$\xi_{\mu_p,q} = \big\{K \in \mathcal{F}_0^+ : \exists Q \in \mathcal{S}_0 \ \text{ s.t. } f_{\mu_p,q}(K,u) = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1}\big\}.$$

Clearly, $B_2^n \in \xi_{\mu_p,q}$ as one can let $Q_0 = ((n-p)/(p-1))^{p/(n+q)} \cdot B_2^n \in \mathcal{S}_0$ and thus for any $u \in S^{n-1}$,

$$f_{\mu_p,q}(B_2^n, u) = \left(\frac{n-p}{p-1}\right)^p = \big(\rho_{Q_0}(u)\big)^{n+q}.$$

Proposition 3.2. *If $K \in \xi_{\mu_p,q}$, then for $-n \ne q \in \mathbb{R}$,*

$$\mathcal{A}_{p,q}(K)=\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}f_{\mu_p,q}(K,u)^{n/(n+q)}d\sigma(u). \tag{3.27}$$

Proof. Let $L\in\mathcal{S}_0$. It can be easily checked that (3.27) is true for $q=0$, i.e.,

$$\mathcal{A}_{p,0}(K)=\frac{p-1}{n-p}\cdot\int_{S^{n-1}}h_K(u)\cdot d\mu_p(K,u)=C_p(K).$$

If $q>0$, by Hölder inequality [**12**], one has

$$\begin{aligned}&\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}f_{\mu_p,q}(K,u)^{n/(n+q)}d\sigma(u)\\&=\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}\left[\rho_L^{-q}(u)f_{\mu_p,q}(K,u)\rho_L^{q}(u)\right]^{n/(n+q)}d\sigma(u)\\&\le\left(\frac{p-1}{n-p}\cdot\int_{S^{n-1}}\rho_L^{-q}(u)f_{\mu_p,q}(K,u)d\sigma(u)\right)^{n/(n+q)}\left(\frac{1}{n}\int_{S^{n-1}}\rho_L^{n}(u)d\sigma(u)\right)^{q/(n+q)}\\&=C_{p,q}(K,L^\circ)^{n/(n+q)}\cdot|L|^{q/(n+q)}.\end{aligned}$$

Equality holds if and only if $\rho_L^{n+q}(u)=f_{\mu_p,q}(K,u)$ for any $u\in S^{n-1}$. Taking the infimum over $L\in\mathcal{S}_0$, one gets

$$\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}f_{\mu_p,q}(K,u)^{n/(n+q)}d\sigma(u)\le\mathcal{A}_{p,q}(K). \tag{3.28}$$

On the other hand, since $K\in\xi_{\mu_p,q}$, there exists a star body $Q\in\mathcal{S}_0$ such that

$$\rho_Q(u)=\left(f_{\mu_p,q}(K,u)\right)^{1/(n+q)}\quad\text{for any }u\in S^{n-1}.$$

Then

$$\begin{aligned}&\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}f_{\mu_p,q}(K,u)^{n/(n+q)}d\sigma(u)\\&\qquad=C_{p,q}(K,Q^\circ)^{n/(n+q)}\cdot|Q|^{q/(n+q)}\ge\mathcal{A}_{p,q}(K).\end{aligned}$$

This together with (3.28) yields

$$\mathcal{A}_{p,q}(K)=\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}f_{\mu_p,q}(K,u)^{n/(n+q)}d\sigma(u).$$

Along the same lines, one can prove (3.27) when $-n\ne q<0$. □

Remark 3.2. Motivated by the definition of the p-curvature image of $K\in\mathcal{F}_0^+$ in [**21**], [**31**], for any $K\in\xi_{\mu_p,q}$ and $-n\ne q\in\mathbb{R}$, we can define $\Lambda_{\mu_p,q}K\in\mathcal{S}_0$, the p-capacity q-curvature image of K, by

$$f_{\mu_p,q}(K,u)=\frac{n-p}{n(p-1)|\Lambda_{\mu_p,q}K|}\cdot\left(\rho_{\Lambda_{\mu_p,q}K}(u)\right)^{n+q}\quad\text{for any }u\in S^{n-1}.$$

By the proof of Proposition 3.2, one also gets

$$\mathcal{A}_{p,q}(K) = \big(C_{p,q}\big(K, (\Lambda_{\mu_p,q}K)^\circ\big)\big)^{n/(n+q)} \cdot |\Lambda_{\mu_p,q}K|^{q/(n+q)} = |\Lambda_{\mu_p,q}K|^{q/(n+q)}.$$

For $-n \neq q \in \mathbb{R}$, let

$$\nu_{\mu_p,q} = \Big\{K \in \mathcal{F}_0^+ : \exists Q \in \mathcal{K}_0 \ \text{ s.t. } f_{\mu_p,q}(K,u) = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1}\Big\}.$$

Clearly, $\nu_{\mu_p,q} \subseteq \xi_{\mu_p,q}$ and $B_2^n \in \nu_{\mu_p,q}$, which yields $\nu_{\mu_p,q} \neq \phi$. The following results provide a convenient formula to calculate $\mathcal{G}_{p,q}(K)$ when $K \in \nu_{\mu_p,q}$.

Proposition 3.3. *If $-n \neq q \in \mathbb{R}$ and $K \in \nu_{\mu_p,q}$, then $\mathcal{G}_{p,q}(K) = \mathcal{A}_{p,q}(K)$.*

Proof. First of all, we prove that $\Lambda_{\mu_p,q}K \in \mathcal{K}_0$ if $K \in \nu_{\mu_p,q}$. As $K \in \nu_{\mu_p,q}$, there is a convex body $Q \in \mathcal{K}_0$ such that $f_{\mu_p,q}(K,u) = (\rho_Q(u))^{n+q}$ for any $u \in S^{n-1}$. Together with Remark 3.2, one gets, for any $u \in S^{n-1}$,

$$\frac{n-p}{n(p-1)|\Lambda_{\mu_p,q}K|} \cdot \big(\rho_{\Lambda_{\mu_p,q}K}(u)\big)^{n+q} = \big(\rho_Q(u)\big)^{n+q},$$

and hence

$$\Lambda_{\mu_p,q}K = \left(\frac{n(p-1)|\Lambda_{\mu_p,q}K|}{n-p}\right)^{1/(n+q)} Q \in \mathcal{K}_0.$$

Next we shall prove $\mathcal{G}_{p,q}(K) = \mathcal{A}_{p,q}(K)$ under three different cases: $q = 0$, $q > 0$ and $-n \neq q < 0$. The case $q = 0$ is trivial as $\mathcal{G}_{p,0}(K) = \mathcal{A}_{p,0}(K) = C_p(K)$.

If $q > 0$, by (3.21) and (3.23), one gets $\mathcal{G}_{p,q}(K) \geq \mathcal{A}_{p,q}(K)$. On the other hand, by Remark 3.2, $\Lambda_{\mu_p,q}K \in \mathcal{K}_0$ and Definition 3.4, one has

$$\mathcal{A}_{p,q}(K) = \big(C_{p,q}\big(K, (\Lambda_{\mu_p,q}K)^\circ\big)\big)^{n/(n+q)} \cdot |\Lambda_{(\mu_p,q)}K|^{q/(n+q)} \geq \mathcal{G}_{p,q}(K).$$

These imply $\mathcal{G}_{p,q}(K) = \mathcal{A}_{p,q}(K)$.

If $-n \neq q < 0$, similarly, employing (3.22) and (3.24), Remark 3.2, $\Lambda_{\mu_p,q}K \in \mathcal{K}_0$ and Definition 3.4, one gets

$$\mathcal{G}_{p,q}(K) \leq \mathcal{A}_{p,q}(K) = \big(C_{p,q}(K, \big(\Lambda_{\mu_p,q}K\big)^\circ)\big)^{n/(n+q)} \cdot |\Lambda_{\mu_p,q}K|^{q/(n+q)} \leq \mathcal{G}_{p,q}(K).$$

Thus $\mathcal{G}_{p,q}(K) = \mathcal{A}_{p,q}(K)$ when $-n \neq q < 0$. □

The following isoperimetric type inequalities for $\mathcal{G}_{p,q}(K)$ and $\mathcal{A}_{p,q}(K)$ can be easily obtained from Theorem 3.2, Theorem 3.3, (3.25) and (3.26), and $\mathcal{G}_{p,0}(K) = \mathcal{A}_{p,0}(K) = C_p(K)$ for any $K \in \mathcal{K}_0$.

Proposition 3.4. *Let $K \in \mathcal{K}_0$ be a convex body with its Santaló point or centroid at the origin o and $B_K = \mathrm{vrad}(K)B_2^n$.*
(i) *For $q \geq 0$,*

$$\frac{\mathcal{A}_{p,q}(K)}{\mathcal{A}_{p,q}(B_K)} \le \frac{\mathcal{G}_{p,q}(K)}{\mathcal{G}_{p,q}(B_K)} \le \left(\frac{C_p(K)}{C_p(B_K)}\right)^{n/(n+q)}.$$

(ii) *For* $-n < q < 0$,

$$\frac{\mathcal{A}_{p,q}(K)}{\mathcal{A}_{p,q}(B_K)} \ge \frac{\mathcal{G}_{p,q}(K)}{\mathcal{G}_{p,q}(B_K)} \ge \left(\frac{C_p(K)}{C_p(B_K)}\right)^{n/(n+q)}.$$

(iii) *For* $q < -n$, *there exists a universal constant* $c > 0$ *such that*

$$\frac{\mathcal{A}_{p,q}(K)}{\mathcal{A}_{p,q}(B_K)} \ge \frac{\mathcal{G}_{p,q}(K)}{\mathcal{G}_{p,q}(B_K)} \ge c^{nq/(n+q)} \left(\frac{C_p(K)}{C_p(B_K)}\right)^{n/(n+q)}.$$

The cyclic inequality for $\mathcal{G}_{p,r}(K)$ is given by the following theorem.

Theorem 3.4. *Let* $K \in \mathcal{K}_0$.

(i) *If* $-n < t < 0 < r < s$ *or* $-n < s < 0 < r < t$, *then*

$$\mathcal{G}_{p,r}(K) \le \big(\mathcal{G}_{p,t}(K)\big)^{(r-s)(n+t)/(t-s)(n+r)} \cdot \big(\mathcal{G}_{p,s}(K)\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

(ii) *If* $-n < t < r < s < 0$ *or* $-n < s < r < t < 0$, *then*

$$\mathcal{G}_{p,r}(K) \le \big(\mathcal{G}_{p,t}(K)\big)^{(r-s)(n+t)/(t-s)(n+r)} \cdot \big(\mathcal{G}_{p,s}(K)\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

(iii) *If* $t < r < -n < s < 0$ *or* $s < r < -n < t < 0$, *then*

$$\mathcal{G}_{p,r}(K) \ge \big(\mathcal{G}_{p,t}(K)\big)^{(r-s)(n+t)/(t-s)(n+r)} \cdot \big(\mathcal{G}_{p,s}(K)\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

Proof. Let $K, L \in \mathcal{K}_0$ and s, r, t be three real numbers such that $0 < (t-r)/(t-s) < 1$. By Hölder inequality, one has

$$\begin{aligned} C_{p,r}(K,L) &= \frac{p-1}{n-p}\int_{S^{n-1}} h_L^r(u)\cdot h_K^{1-r}(u)\,d\mu_p(K,u) \\ &\le \frac{p-1}{n-p}\left(\int_{S^{n-1}} h_L^t(u)\cdot h_K^{1-t}(u)\,d\mu_p(K,u)\right)^{(r-s)/(t-s)} \\ &\quad \cdot\left(\int_{S^{n-1}} h_L^s(u)\cdot h_K^{1-s}(u)\,d\mu_p(K,u)\right)^{(r-t)/(s-t)} \\ &= \big(C_{p,t}(K,L)\big)^{(r-s)/(t-s)}\cdot\big(C_{p,s}(K,L)\big)^{(r-t)/(s-t)}. \end{aligned} \tag{3.29}$$

(i) Assume that $-n < t < 0 < r < s$. Then

$$0 < \frac{t-r}{t-s} < 1, \quad \frac{n}{n+r} > 0, \quad \frac{(r-s)(n+t)}{(t-s)(n+r)} > 0 \ \text{ and } \ \frac{(r-t)(n+s)}{(s-t)(n+r)} > 0.$$

Together with (3.29) and Definition 3.4, one has

$$\mathcal{G}_{p,r}(K) = \inf_{L\in\mathcal{K}_0}\left\{\big(C_{p,r}(K,L)\big)^{n/(n+r)}\cdot|L^\circ|^{r/(n+r)}\right\}$$

$$\leq \inf_{L\in\mathcal{K}_0} \Big\{ \Big[\big(C_{p,t}(K,L)\big)^{n/(n+t)} \cdot |L^\circ|^{t/(n+t)} \Big]^{(r-s)(n+t)/(t-s)(n+r)} \cdot \Big[\big(C_{p,s}(K,L)\big)^{n/(n+s)} \cdot |L^\circ|^{s/(n+s)} \Big]^{(r-t)(n+s)/(s-t)(n+r)} \Big\}$$
$$\leq \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,t}(K,L)\big)^{n/(n+t)} \cdot |L^\circ|^{t/(n+t)} \Big\}^{(r-s)(n+t)/(t-s)(n+r)} \cdot \inf_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,s}(K,L)\big)^{n/(n+s)} \cdot |L^\circ|^{s/(n+s)} \Big\}^{(r-t)(n+s)/(s-t)(n+r)}$$
$$= \big(\mathcal{G}_{p,t}(K)\big)^{(r-s)(n+t)/(t-s)(n+r)} \cdot \big(\mathcal{G}_{p,s}(K)\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

By switching the roles of s and t, one gets the case $-n < s < 0 < r < t$.

(ii) It's enough to prove the case $-n < t < r < s < 0$, since the case $-n < s < r < t < 0$ can be proved by switching the roles of s and t. In this case, one has

$$0 < \frac{t-r}{t-s} < 1, \quad \frac{n}{n+r} > 0, \quad \frac{(r-s)(n+t)}{(t-s)(n+r)} > 0 \ \text{ and } \ \frac{(r-t)(n+s)}{(s-t)(n+r)} > 0.$$

Together with (3.29) and Definition 3.4, one has

$$\mathcal{G}_{p,r}(K) = \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,r}(K,L)\big)^{n/(n+r)} \cdot |L^\circ|^{r/(n+r)} \Big\}$$
$$\leq \sup_{L\in\mathcal{K}_0} \Big\{ \Big[\big(C_{p,t}(K,L)\big)^{(r-s)/(t-s)} \cdot \big(C_{p,s}(K,L)\big)^{(r-t)/(s-t)} \Big]^{n/(n+r)} \cdot |L^\circ|^{r/(n+r)} \Big\}$$
$$\leq \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,t}(K,L)\big)^{n/(n+t)} \cdot |L^\circ|^{t/(n+t)} \Big\}^{(r-s)(n+t)/(t-s)(n+r)} \cdot \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,s}(K,L)\big)^{n/(n+s)} \cdot |L^\circ|^{s/(n+s)} \Big\}^{(r-t)(n+s)/(s-t)(n+r)}$$
$$= \big(\mathcal{G}_{p,t}(K)\big)^{(r-s)(n+t)/(t-s)(n+r)} \cdot \big(\mathcal{G}_{p,s}(K)\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

(iii) Let $t < r < -n < s < 0$. Thus

$$0 < \frac{t-r}{t-s} < 1, \quad \frac{n}{n+r} < 0, \quad \frac{(r-s)(n+t)}{(t-s)(n+r)} > 0 \ \text{ and } \ \frac{(r-t)(n+s)}{(s-t)(n+r)} < 0.$$

Together with (3.29) and Definition 3.4, one has

$$\mathcal{G}_{p,r}(K) = \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,r}(K,L)\big)^{n/(n+r)} \cdot |L^\circ|^{r/(n+r)} \Big\}$$
$$\geq \sup_{L\in\mathcal{K}_0} \Big\{ \Big[\big(C_{p,t}(K,L)\big)^{(r-s)/(t-s)} \cdot \big(C_{p,s}(K,L)\big)^{(r-t)/(s-t)} \Big]^{n/(n+r)} \cdot |L^\circ|^{r/(n+r)} \Big\}$$
$$\geq \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,t}(K,L)\big)^{n/(n+t)} \cdot |L^\circ|^{t/(n+t)} \Big\}^{(r-s)(n+t)/(t-s)(n+r)} \cdot \sup_{L\in\mathcal{K}_0} \Big\{ \big(C_{p,s}(K,L)\big)^{n/(n+s)} \cdot |L^\circ|^{s/(n+s)} \Big\}^{(r-t)(n+s)/(s-t)(n+r)}$$

$$= \big(\mathcal{G}_{p,t}(K)\big)^{(r-s)(n+t)/(t-s)(n+r)} \cdot \big(\mathcal{G}_{p,s}(K)\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

The case $s < r < -n < t < 0$ follows by switching the roles of s and t. □

In fact, one can check that the cyclic inequalities also hold only if one of r, s, t equals 0, and hence the following results regarding the monotonicity of $\mathcal{G}_{p,s}(K)$ on $s \in \mathbb{R}$ can be obtained.

THEOREM 3.5. *Let $K \in \mathcal{K}_0$ and $t, s \neq 0$.*
(i) *If $-n < s < t$ or $s < t < -n$, then*

$$\left(\frac{\mathcal{G}_{p,s}(K)}{C_p(K)}\right)^{(n+s)/s} \leq \left(\frac{\mathcal{G}_{p,t}(K)}{C_p(K)}\right)^{(n+t)/t}.$$

(ii) *If $s < -n < t$, then*

$$\left(\frac{\mathcal{G}_{p,s}(K)}{C_p(K)}\right)^{(n+s)/s} \geq \left(\frac{\mathcal{G}_{p,t}(K)}{C_p(K)}\right)^{(n+t)/t}.$$

4. The mixed geominimal p-capacities for multiple convex bodies.

4.1. The Orlicz mixed geominimal p-capacities.

Let m be a positive integer and $\mathcal{Q}_0$ be a nonempty subset of $\mathcal{S}_0$. In the following, denote the cartesian product $\underbrace{\mathcal{Q}_0 \times \cdots \times \mathcal{Q}_0}_{m}$ by $(\mathcal{Q}_0)^m$. By $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathcal{Q}_0)^m$, we mean that, for any $1 \leq i \leq m$, $L_i \in \mathcal{Q}_0$. Let $\boldsymbol{L}^\circ$ refer to the vector $(L_1^\circ, L_2^\circ, \cdots, L_m^\circ)$. Let $\boldsymbol{K_i} = (K_{i1}, K_{i2}, \cdots, K_{im})$ for any $i \geq 1$ and $\boldsymbol{K} = (K_1, K_2, \cdots, K_m)$. By $\boldsymbol{K_i} \to \boldsymbol{K}$ as $i \to \infty$ we mean that, for any $1 \leq j \leq m$, $K_{ij} \to K_j$ as $i \to \infty$. By $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_m) \in (\mathcal{I})^m$, we mean that each $\varphi_i \in \mathcal{I}$ for $i = 1, 2, \cdots, m$, similarly, $\boldsymbol{\varphi} \in (\mathcal{D})^m$ means $\varphi_i \in \mathcal{D}$ for $i = 1, 2, \cdots, m$.

DEFINITION 4.1. Let $\boldsymbol{\varphi} \in (\mathcal{I})^m$ or $\boldsymbol{\varphi} \in (\mathcal{D})^m$, $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathcal{F}_0^+)^m$ and $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathcal{K}_0)^m$. The Orlicz mixed p-capacity of $\boldsymbol{K}$ and $\boldsymbol{L}$, denoted by $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L})$, is defined by

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L}) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{i=1}^m \varphi_i\left(\frac{h_{L_i}(u)}{h_{K_i}(u)}\right) f^*_{K_i}(u)\right)^{1/m} d\sigma(u),$$

where $f^*_{K_i}(u) = h_{K_i}(u) \cdot |\nabla U_{K_i}(\nu^{-1}_{K_i}(u))|^p \cdot f_{K_i}(u)$ for any $1 \leq i \leq m$. If $\boldsymbol{L} \in (\mathcal{S}_0)^m$, then define $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L}^\circ)$ by

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L}^\circ) = \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{i=1}^m \varphi_i\left(\frac{1}{\rho_{L_i}(u) h_{K_i}(u)}\right) f^*_{K_i}(u)\right)^{1/m} d\sigma(u).$$

The continuity of $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\cdot, \cdot)$ is stated as follows.

Proposition 4.1. *Let $\{\boldsymbol{K_i}\}_{i=1}^{\infty} \subseteq (\mathcal{F}_0^+)^m$ and $\{\boldsymbol{L_i}\}_{i=1}^{\infty} \subseteq (\mathcal{K}_0)^m$ be such that $\boldsymbol{K_i} \to \boldsymbol{K} \in (\mathcal{F}_0^+)^m$ and $\boldsymbol{L_i} \to \boldsymbol{L} \in (\mathcal{K}_0)^m$ as $i \to \infty$. If $\boldsymbol{\varphi} \in (\mathcal{I})^m$ or $\boldsymbol{\varphi} \in (\mathcal{D})^m$, $(\prod_{j=1}^m f_{K_{ij}})^{1/m}$ converges uniformly to $(\prod_{j=1}^m f_{K_j})^{1/m}$ on S^{n-1}, then $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}, \boldsymbol{L_i}) \to \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \boldsymbol{L})$ as $i \to \infty$.*

Proof. For any $u \in S^{n-1}$, any $i \geq 1$ and any $1 \leq k \leq m$, let

$$a_i(u) = \left(\prod_{j=1}^m \varphi_j\left(\frac{h_{L_{ij}}(u)}{h_{K_{ij}}(u)}\right) \cdot h_{K_{ij}}(u) \cdot f_{K_{ij}}(u)\right)^{1/m},$$

$$b_{i,k}(u) = \left(\prod_{j=1}^k |\nabla U_{K_{ij}}(\nu_{K_{ij}}{}^{-1}(u))|^p\right)^{1/m},$$

$$a(u) = \left(\prod_{j=1}^m \varphi_j\left(\frac{h_{L_j}(u)}{h_{K_j}(u)}\right) \cdot h_{K_j}(u) \cdot f_{K_j}(u)\right)^{1/m},$$

$$b_k(u) = \left(\prod_{j=1}^k |\nabla U_{K_j}(\nu_{K_j}{}^{-1}(u))|^p\right)^{1/m}.$$

The convergences of $\boldsymbol{K_i} \to \boldsymbol{K}$ and $\boldsymbol{L_i} \to \boldsymbol{L}$ imply that, for any $1 \leq j \leq m$, $h_{K_{ij}} \to h_{K_j}$ and $h_{L_{ij}} \to h_{L_j}$ uniformly on S^{n-1}. Thus, there are two constants $c, C > 0$ such that

$$c \cdot B_2^n \subseteq K_{ij}, K_j, L_{ij}, L_j \subseteq C \cdot B_2^n, \text{ for any } \ i \geq 1 \text{ and any } \ 1 \leq j \leq m.$$

and hence for any $u \in S^{n-1}$,

$$\frac{c}{C} \leq \frac{h_{L_{ij}}(u)}{h_{K_{ij}}(u)}, \frac{h_{L_j}(u)}{h_{K_j}(u)} \leq \frac{C}{c}.$$

Since φ_j is continuous on the interval $[c/C, C/c]$, then one has

$$\left[\prod_{j=1}^m \varphi_j\left(\frac{h_{L_{ij}}(u)}{h_{K_{ij}}(u)}\right) \cdot h_{K_{ij}}(u)\right]^{1/m} \to \left[\prod_{j=1}^m \varphi_j\left(\frac{h_{L_j}(u)}{h_{K_j}(u)}\right) \cdot h_{K_j}(u)\right]^{1/m} \text{ uniformly on } S^{n-1}.$$

Combining with the assumption that $(\prod_{j=1}^m f_{K_{ij}})^{1/m} \to (\prod_{j=1}^m f_{K_j})^{1/m}$ uniformly on S^{n-1}, one gets $a_i(u) \to a(u)$ uniformly on S^{n-1} and hence there exists a positive constant C_1, such that, $|a_i(u)| \leq C_1$ for any $i \geq 1$ and any $u \in S^{n-1}$. By [**6**, Lemma 2.10, Lemma 4.6], one has, for any $1 \leq j \leq m$,

$$\int_{S^{n-1}} \left| \left|\nabla U_{K_{ij}}(\nu_{K_{ij}}{}^{-1}(u))\right|^p - \left|\nabla U_{K_j}(\nu_{K_j}{}^{-1}(u))\right|^p \right| d\sigma(u) \to 0. \tag{4.30}$$

Moreover, there exist two positive constants C_2 (only dependent on $\boldsymbol{K}, n$ and p) and i_0 such that when $i \geq i_0$, for any $1 \leq j \leq m$,

$$\int_{S^{n-1}} |\nabla U_{K_{ij}}\big({\nu_{K_{ij}}}^{-1}(u)\big)|^p d\sigma(u) \le C_2 \quad \text{and} \quad \int_{S^{n-1}} |\nabla U_{K_j}\big({\nu_{K_j}}^{-1}(u)\big)|^p d\sigma(u) \le C_2. \tag{4.31}$$

Note that $a_i(u)\cdot b_{i,m}(u) - a(u)\cdot b_m(u) = (a_i(u)-a(u))\cdot b_m(u) + a_i(u)\cdot(b_{i,m}(u)-b_m(u))$. Hence, to prove $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i},\boldsymbol{L_i}) \to \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K},\boldsymbol{L})$, it is enough to prove

$$\int_{S^{n-1}} (a_i(u)-a(u))\cdot b_m(u)d\sigma(u) \to 0; \tag{4.32}$$

$$\int_{S^{n-1}} a_i(u)\cdot(b_{i,m}(u)-b_m(u))d\sigma(u) \to 0. \tag{4.33}$$

By the uniform convergence of $a_i(u)\to a(u)$, together with (4.31) and Hölder inequality [**12**], one can easily get (4.32). As

$$\begin{aligned} &a_i(u)\cdot(b_{i,m}(u)-b_m(u)) \\ &= a_i(u)\cdot b_{i,m-1}(u)\left(\left|\nabla U_{K_{im}}\big({\nu_{K_{im}}}^{-1}(u)\big)\right|^{p/m} - \left|\nabla U_{K_m}\big({\nu_{K_m}}^{-1}(u)\big)\right|^{p/m}\right) \\ &\quad + a_i(u)\cdot(b_{i,m-1}(u)-b_{m-1}(u))\left|\nabla U_{K_m}\big({\nu_{K_m}}^{-1}(u)\big)\right|^{p/m}, \end{aligned}$$

by the triangle inequality, $|a_i(u)|\le C_1$, inequality $|\sqrt[m]{a}-\sqrt[m]{b}|\le\sqrt[m]{|a-b|}$ for $a,b\ge 0$, Hölder inequality [**12**], and (4.30)–(4.31), one gets, for any $i\ge i_0$,

$$\begin{aligned} &\left|\int_{S^{n-1}} a_i(u)\cdot(b_{i,m}(u)-b_m(u))d\sigma(u)\right| \\ &\le C_1\cdot C_2^{(m-1)/m}\cdot\left(\int_{S^{n-1}}\left|\left|\nabla U_{K_{im}}\big({\nu_{K_{im}}}^{-1}(u)\big)\right|^p - \left|\nabla U_{K_m}\big({\nu_{K_m}}^{-1}(u)\big)\right|^p\right|d\sigma(u)\right)^{1/m} \\ &\quad + \left|\int_{S^{n-1}} a_i(u)\cdot(b_{i,m-1}(u)-b_{m-1}(u))\left|\nabla U_{K_m}\big({\nu_{K_m}}^{-1}(u)\big)\right|^{p/m} d\sigma(u)\right|. \end{aligned}$$

Repeating the process above, one gets

$$\begin{aligned} &\left|\int_{S^{n-1}} a_i(u)\cdot(b_{i,m}(u)-b_m(u))d\sigma(u)\right| \\ &\le \sum_{j=1}^m C_1\cdot C_2^{(m-1)/m}\cdot\left(\int_{S^{n-1}}\left|\left|\nabla U_{K_{ij}}\big({\nu_{K_{ij}}}^{-1}(u)\big)\right|^p - \left|\nabla U_{K_j}\big({\nu_{K_j}}^{-1}(u)\big)\right|^p\right|d\sigma(u)\right)^{1/m} \\ &\to 0. \end{aligned}$$

Hence, (4.33) is also true and then $\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i},\boldsymbol{L_i}) \to \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K},\boldsymbol{L})$ as $i\to\infty$. □

The following theorem shows the existence of the p-capacitary Orlicz–Petty bodies for multiple convex bodies.

THEOREM 4.1. *Let $\boldsymbol{K}\in(\mathcal{F}_0^+)^m$ and $\boldsymbol{\varphi}\in(\mathcal{I})^m$. There exists a convex body $M\in\mathcal{K}_0$ such that $|M^\circ|=\omega_n$ and*

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M) = \inf\Big\{\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, L, \cdots, L) : L \in \mathcal{K}_0 \quad and \quad |L^\circ| = \omega_n\Big\}.$$

Proof. For convenience, let

$$\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \inf\Big\{\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, L, \cdots, L) : L \in \mathcal{K}_0 \;\text{ and }\; |L^\circ| = \omega_n\Big\}.$$

Clearly, $\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) \leq \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, B^n_2, \cdots, B^n_2) < \infty$. Let $\{M_i\}_{i=1}^\infty \subseteq \mathcal{K}_0$ be a sequence of convex bodies such that

$$\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M_i, \cdots, M_i) \to \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) \;\text{ and }\; |M_i^\circ| = \omega_n \;\text{ for any }\; i \geq 1.$$

As $\boldsymbol{K} \in (\mathcal{F}_0^+)^m$, there exist two positive constants $R_0 > 0$ and $C_1 > 0$, such that, $h_{K_j}(u) \leq R_0$ and $f_{K_j}(u) \cdot h_{K_j}(u) \geq C_1$ for any $1 \leq j \leq m$ and any $u \in S^{n-1}$. By [**6**, Lemma 2.18], there is a positive constant C_2, such that, $|\nabla U_{K_j}(\nu^{-1}_{K_j}(u))|^p \geq C_2$ almost everywhere on S^{n-1} for any $1 \leq j \leq m$.

For any $i \geq 1$, let $R_i = \rho_{M_i}(u_i) = \max_{u\in S^{n-1}}\{\rho_{M_i}(u)\}$ and hence $h_{M_i}(u) \geq R_i \cdot \langle u, u_i\rangle_+$ for any $u \in S^{n-1}$. Again, suppose that u_i converges to $v \in S^{n-1}$. Since the spherical measure $\sigma(\cdot)$ is not concentrated on any hemisphere of S^{n-1}, there exists an integer j_0 such that

$$\int_{\{u\in S^{n-1}:\, \langle u,v\rangle_+ \geq 1/j_0\}} \langle u, v\rangle_+ \, d\sigma(u) > 0.$$

Assume that M_i is not bounded uniformly, i.e., $\sup_{i\geq 1} R_i = \infty$. Without loss of generality, let $R_i \to \infty$ as $i \to \infty$. Thus, for any positive constant $C > 0$,

$$\begin{aligned}
\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) &= \lim_{i\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M_i, M_i, \cdots, M_i) \\
&= \liminf_{i\to\infty} \frac{p-1}{n-p} \int_{S^{n-1}} \left(\prod_{j=1}^m \varphi_j\left(\frac{h_{M_i}(u)}{h_{K_j}(u)}\right) f^*_{K_j}(u)\right)^{1/m} d\sigma(u) \\
&\geq \frac{C_1\cdot C_2\cdot(p-1)}{n-p} \cdot \liminf_{i\to\infty} \int_{S^{n-1}} \left[\prod_{j=1}^m \varphi_j\left(\frac{R_i\cdot\langle u, u_i\rangle_+}{R_0}\right)\right]^{1/m} d\sigma(u) \\
&\geq \frac{C_1\cdot C_2\cdot(p-1)}{n-p} \cdot \liminf_{i\to\infty} \int_{S^{n-1}} \left[\prod_{j=1}^m \varphi_j\left(\frac{C\cdot\langle u, u_i\rangle_+}{R_0}\right)\right]^{1/m} d\sigma(u) \\
&= \frac{C_1\cdot C_2\cdot(p-1)}{n-p} \cdot \int_{S^{n-1}} \left[\prod_{j=1}^m \varphi_j\left(\frac{C\cdot\langle u, v\rangle_+}{R_0}\right)\right]^{1/m} d\sigma(u) \\
&\geq \frac{C_1\cdot C_2\cdot(p-1)}{n-p} \cdot \left[\prod_{j=1}^m \varphi_j\left(\frac{C}{R_0\cdot j_0}\right)\right]^{1/m} \\
&\qquad \cdot \int_{\{u\in S^{n-1}:\, \langle u,v\rangle_+ \geq 1/j_0\}} \langle u, v\rangle_+ \, d\sigma(u).
\end{aligned} \tag{4.34}$$

Letting $C \to \infty$, one gets a contradiction $\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) \geq \infty$. Thus, $\sup_{i\geq 1} R_i < \infty$ and $\{M_i\}_{i=1}^\infty$ is bounded. By Lemma 2.1, one gets a convergent subsequence of $\{M_i\}_{i=1}^\infty$ which converges to some convex body $M \in \mathcal{K}_0$ with $|M^\circ| = \omega_n$. Without loss of generality, let $M_i \to M$ as $i \to \infty$. Thus, by Proposition 4.1, one has

$$\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \lim_{i\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M_i, \cdots, M_i) = \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M), \tag{4.35}$$

as desired. □

The convex body $M \in \mathcal{K}_0$ in (4.35) can be called a p-capacitary Orlicz–Petty bodies of $\boldsymbol{K}$, and if $\boldsymbol{\varphi} \in (\mathcal{I})^m$, such a convex body M exists for $\boldsymbol{K} \in (\mathcal{F}_0^+)^m$. The following theorem deals with the continuity of the functional $\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\cdot)$ on $(\mathcal{F}_0^+)^m$ for the case $\boldsymbol{\varphi} \in (\mathcal{I})^m$.

THEOREM 4.2. *Let $\{\boldsymbol{K_i}\}_{i=1}^\infty \subseteq (\mathcal{F}_0^+)^m$ and $\boldsymbol{K} \in (\mathcal{F}_0^+)^m$ be such that $\boldsymbol{K_i} \to \boldsymbol{K}$ as $i \to \infty$ and $\boldsymbol{\varphi} \in (\mathcal{I})^m$. If $(\prod_{j=1}^m f_{K_{ij}})^{1/m}$ converges uniformly to $(\prod_{j=1}^m f_{K_j})^{1/m}$ on S^{n-1}, then $\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}) \to \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K})$ as $i \to \infty$.*

PROOF. Let $M \in \mathcal{K}_0$ and $M_i \in \mathcal{K}_0$ be such that $|M^\circ| = |M_i^\circ| = \omega_n$,

$$\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M) \text{ and } \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}) = \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}, M_i, \cdots, M_i) \text{ for any } i \geq 1.$$

Then Proposition 4.1 yields

$$\begin{aligned}
\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) &= \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M, \cdots, M) \\
&= \lim_{i\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}_i, M, \cdots, M) \\
&= \limsup_{i\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}_i, M, \cdots, M) \\
&\geq \limsup_{i\to\infty} \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}).
\end{aligned} \tag{4.36}$$

By [**6**, (4.19)], there exist two positive constants C_3 (only dependent on $\boldsymbol{K}, n$ and p) and i_0, such that, $|\nabla U_{K_{ij}}(\nu_{K_{ij}}^{-1}(u))|^p \geq C_3$ and $|\nabla U_{K_j}(\nu_{K_j}^{-1}(u))|^p \geq C_3$ almost everywhere on S^{n-1} for any $i \geq i_0$ and $1 \leq j \leq m$. With a modification of (4.34), one gets that $\{M_i\}_{i=1}^\infty$ is bounded. Let $\{\boldsymbol{K_{i_k}}\}_{k=1}^\infty \subseteq \{\boldsymbol{K_i}\}_{i=1}^\infty$ be a subsequence, such that,

$$\lim_{k\to\infty} \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_{i_k}}) = \liminf_{i\to\infty} \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}).$$

It follows from the boundedness of $\{M_{i_k}\}_{k=1}^\infty$ and Lemma 2.1 that there exist a subsequence $\{M_{i_{k_j}}\}_{j=1}^\infty$ of $\{M_{i_k}\}_{k=1}^\infty$ and $M' \in \mathcal{K}_0$ such that $M_{i_{k_j}} \to M'$ as $j \to \infty$ and $|(M')^\circ| = \omega_n$. By Proposition 4.1, one has

$$\begin{aligned}
\liminf_{i\to\infty} \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i}) &= \lim_{j\to\infty} \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_{i_{k_j}}}) \\
&= \lim_{j\to\infty} \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_{i_{k_j}}}, M_{i_{k_j}}, \cdots, M_{i_{k_j}}) \\
&= \boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, M', \cdots, M') \\
&\geq \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}).
\end{aligned}$$

Together with (4.36), one gets $\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}) = \lim_{i\to\infty} \boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K_i})$ as desired. □

4.2. The L_q mixed geominimal p-capacity.

In this subsection we will discuss the L_q mixed geominimal p-capacity for multiple convex bodies. Firstly, we introduce the Orlicz mixed geominimal p-capacity.

DEFINITION 4.2. Let $\boldsymbol{K} \in (\mathcal{F}_0^+)^m$.
(i) If $\boldsymbol{\varphi} \in (\mathcal{I})^m$ or $\boldsymbol{\varphi} \in (\mathcal{D}_1)^m$, define $\boldsymbol{\mathcal{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K})$, the Orlicz mixed geominimal p-capacity with respect to $\mathcal{K}_0$, by

$$\boldsymbol{\mathcal{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K}) = \inf\Big\{\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \underbrace{L, \cdots, L}_{m}) : L \in \mathcal{K}_0 \ \text{ and } \ |L^\circ| = \omega_n\Big\}.$$

(ii) If $\boldsymbol{\varphi} \in (\mathcal{D}_0)^m$, define $\boldsymbol{\mathcal{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K})$, the Orlicz mixed geominimal p-capacity with respect to $\mathcal{K}_0$, by

$$\boldsymbol{\mathcal{G}}_{p,\boldsymbol{\varphi}}^{orlicz}(\boldsymbol{K}) = \sup\Big\{\boldsymbol{C}_{p,\boldsymbol{\varphi}}(\boldsymbol{K}, \underbrace{L, \cdots, L}_{m}) : L \in \mathcal{K}_0 \ \text{ and } \ |L^\circ| = \omega_n\Big\}.$$

Let $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathcal{S}_0)^m$. Define the dual mixed volume of L by [**20**]

$$\tilde{V}(\boldsymbol{L}) = \tilde{V}(L_1, L_2, \cdots, L_m) = \frac{1}{n}\int_{S^{n-1}} \Big(\prod_{i=1}^m \rho_{L_i}(u)\Big)^{n/m} d\sigma(u).$$

Clearly, for any $L \in \mathcal{S}_0$, $\tilde{V}(\underbrace{L, L, \cdots, L}_{m}) = |L|$. Moreover, by Hölder inequality, one has

$$\tilde{V}(\boldsymbol{L}) \le \prod_{i=1}^m |L_i|^{1/m} \quad \text{for any} \quad \boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathcal{S}_0)^m,$$

and equality holds if and only if L_i $(1 \le i \le m)$ are dilates of each other. For $\phi \in O(n)$ and $\boldsymbol{L} = (L_1, L_2, \cdots, L_m) \in (\mathcal{S}_0)^m$, define $\phi\boldsymbol{L}$ by $\phi\boldsymbol{L} = (\phi L_1, \phi L_2, \cdots, \phi L_m)$. It can be checked that $\tilde{V}(\phi\boldsymbol{L}) = \tilde{V}(\boldsymbol{L})$. When $\varphi_i(t) = t^q$ for any $1 \le i \le m$, $\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L})$, the Orlicz mixed p-capacity of $\boldsymbol{K}$ and $\boldsymbol{L}$, is given by

$$\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}) = \frac{p-1}{n-p}\int_{S^{n-1}} \bigg(\prod_{i=1}^m \big(h_{L_i}(u)\big)^q f_{\mu_{p,q}}(K_i, u)\bigg)^{1/m} d\sigma(u).$$

If $\boldsymbol{L} \in (\mathcal{S}_0)^m$, we let

$$\boldsymbol{C}_{p,q}(\boldsymbol{K}, \boldsymbol{L}^\circ) = \frac{p-1}{n-p}\int_{S^{n-1}} \bigg(\prod_{i=1}^m \big(\rho_{L_i}(u)\big)^{-q} f_{\mu_{p,q}}(K_i, u)\bigg)^{1/m} d\sigma(u).$$

Let $\mathcal{Q}_0$ be a nonempty subset of $\mathcal{S}_0$.

DEFINITION 4.3. Let $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathcal{F}_0^+)^m$ and $-n \ne q \in \mathbb{R}$.
(i) For $q \ge 0$, the L_q mixed geominimal p-capacity with respect to $\mathcal{Q}_0$, is defined by

$$\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K},\mathcal{Q}_0)=\inf_{L\in\mathcal{Q}_0}\Big\{\big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\underbrace{L^\circ,\cdots,L^\circ}_{m})\big)^{n/(n+q)}\cdot|L|^{q/(n+q)}\Big\}.$$

(ii) For $-n\neq q<0$, the L_q mixed geominimal p-capacity with respect to $\mathcal{Q}_0$, is defined by

$$\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K},\mathcal{Q}_0)=\sup_{L\in\mathcal{Q}_0}\Big\{\big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\underbrace{L^\circ,\cdots,L^\circ}_{m})\big)^{n/(n+q)}\cdot|L|^{q/(n+q)}\Big\}.$$

There are many ways to extend/modify Definition 4.3 and to define different L_q mixed geominimal p-capacities. For instance, one can replace $|L|^{q/(n+q)}$ by $\prod_{i=1}^m|L_i|^{q/m(n+q)}$ or $\tilde{V}(\boldsymbol{L})^{q/(n+q)}$. However, their properties are similar to those for $\boldsymbol{\mathcal{G}}_{p,q}(\cdot)$ defined in Definition 4.3 and hence they will not be discussed here.

Again, we will focus on the case $\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})=\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K},\mathcal{K}_0)$ and $\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})=\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K},\mathcal{S}_0)$. Clearly, for any $K\in\mathcal{K}_0$,

$$\boldsymbol{\mathcal{G}}_{p,q}(\underbrace{K,\cdots,K}_{m})=\mathcal{G}_{p,q}(K)\quad\text{and}\quad\boldsymbol{\mathcal{A}}_{p,q}(\underbrace{K,\cdots,K}_{m})=\mathcal{A}_{p,q}(K).$$

Moreover, if $\varphi_i(t)=t^q$ $(1\le i\le m,-n\neq q\in\mathbb{R})$, then, for any $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in(\mathcal{F}_0^+)^m$,

$$\boldsymbol{\mathcal{G}}^{orlicz}_{p,\boldsymbol{\varphi}}(\boldsymbol{K})=\omega_n^{-q/n}\cdot\boldsymbol{\mathcal{G}}^{(n+q)/n}_{p,q}(\boldsymbol{K}).$$

The following proposition states $\boldsymbol{\mathcal{G}}_{p,q}(\cdot)$ and $\boldsymbol{\mathcal{A}}_{p,q}(\cdot)$ are $O(n)$-invariant.

Proposition 4.2. *Let $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in(\mathcal{F}_0^+)^m$ and $-n\neq q\in\mathbb{R}$. Then for any $\phi\in O(n)$, one has*

$$\boldsymbol{\mathcal{G}}_{p,q}(\phi\boldsymbol{K})=\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})\quad\textit{and}\quad\boldsymbol{\mathcal{A}}_{p,q}(\phi\boldsymbol{K})=\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}).$$

Proof. We only prove $\boldsymbol{\mathcal{G}}_{p,q}(\phi\boldsymbol{K})=\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})$, and $\boldsymbol{\mathcal{A}}_{p,q}(\phi\boldsymbol{K})=\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})$ follows along a similar argument. For any $1\le i\le m$ and any $u\in S^{n-1}$, let $v=\phi^tu$ and then

$$\begin{aligned}f_{\mu_p,q}(\phi K_i,u)&=h^{1-q}_{\phi K_i}(u)\cdot|\nabla U_{\phi K_i}\big(\nu^{-1}_{\phi K_i}(u)\big)|^p\cdot f_{\phi K_i}(u)\\&=h^{1-q}_{K_i}(\phi^tu)\cdot|\nabla U_{K_i}\big(\nu^{-1}_{K_i}(\phi^tu)\big)|^p\cdot f_{K_i}(\phi^tu)\\&=f_{\mu_p,q}(K_i,v).\end{aligned}$$

Hence for any $\boldsymbol{L}\in(\mathcal{S}_0)^m$,

$$\begin{aligned}\boldsymbol{C}_{p,q}(\phi\boldsymbol{K},(\phi\boldsymbol{L})^\circ)&=\frac{p-1}{n-p}\int_{S^{n-1}}\bigg(\prod_{i=1}^m\big(\rho_{\phi L_i}(u)\big)^{-q}f_{\mu_p,q}(\phi K_i,u)\bigg)^{1/m}d\sigma(u)\\&=\frac{p-1}{n-p}\int_{S^{n-1}}\bigg(\prod_{i=1}^m\big(\rho_{L_i}(\phi^tu)\big)^{-q}f_{\mu_p,q}(K_i,\phi^tu)\bigg)^{1/m}d\sigma(u)\end{aligned}$$

$$
\begin{aligned}
&= \frac{p-1}{n-p}\int_{S^{n-1}} \left(\prod_{i=1}^{m} \big(\rho_{L_i}(v)\big)^{-q} f_{\mu_p,q}(K_i,v)\right)^{1/m} d\sigma(v) \\
&= \boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ}).
\end{aligned}
$$

Together with $(\phi L)^{\circ} = \phi L^{\circ}$ and $|\phi L| = |L|$ for any $L \in \mathcal{S}_0$, one has, for $q \geq 0$,

$$
\begin{aligned}
\boldsymbol{\mathcal{G}}_{p,q}(\phi\boldsymbol{K}) &= \inf_{\phi L\in\mathcal{K}_0} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{\phi K},(\phi L)^{\circ},(\phi L)^{\circ},\cdots,(\phi L)^{\circ})\big)^{n/(n+q)} \cdot |\phi L|^{q/(n+q)} \right\} \\
&= \inf_{L\in\mathcal{K}_0} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},L^{\circ},L^{\circ},\cdots,L^{\circ})\big)^{n/(n+q)} \cdot |L|^{q/(n+q)} \right\} \\
&= \boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K}).
\end{aligned}
$$

The case $-n \neq q < 0$ follows along the same lines. □

For $\boldsymbol{\mathcal{A}}_{p,q}(\cdot)$, we have the following result.

PROPOSITION 4.3. *Let* $\boldsymbol{K} = (K_1, K_2, \cdots, K_m) \in (\mathcal{F}_0^{+})^m$.
(i) *If* $q \geq 0$, *then*

$$
\begin{aligned}
\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}) &= \inf_{\boldsymbol{L}\in(\mathcal{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{n/(n+q)} \cdot \prod_{i=1}^{m} |L_i|^{q/m(n+q)} \right\} \\
&= \inf_{\boldsymbol{L}\in(\mathcal{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{n/(n+q)} \cdot \tilde{V}(\boldsymbol{L})^{q/(n+q)} \right\}.
\end{aligned}
$$

(ii) *If* $-n < q < 0$, *then*

$$
\begin{aligned}
\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}) &= \sup_{\boldsymbol{L}\in(\mathcal{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{n/(n+q)} \cdot \prod_{i=1}^{m} |L_i|^{q/m(n+q)} \right\} \\
&= \sup_{\boldsymbol{L}\in(\mathcal{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{n/(n+q)} \cdot \tilde{V}(\boldsymbol{L})^{q/(n+q)} \right\}.
\end{aligned}
$$

(iii) *If* $q < -n$, *then*

$$
\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}) = \sup_{\boldsymbol{L}\in(\mathcal{S}_0)^m} \left\{ \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},\boldsymbol{L}^{\circ})\big)^{n/(n+q)} \cdot \tilde{V}(\boldsymbol{L})^{q/(n+q)} \right\}.
$$

For $-n \neq q \in \mathbb{R}$, let

$$
\begin{aligned}
&\boldsymbol{\xi}_{\mu_p,q} \\
&= \left\{ \boldsymbol{K} \in (\mathcal{F}_0^{+})^m : \exists Q \in \mathcal{S}_0 \text{ s.t. } \left(\prod_{i=1}^{m} f_{\mu_p,q}(K_i,u)\right)^{1/m} = \big(\rho_Q(u)\big)^{n+q} \text{ for any } u \in S^{n-1} \right\}.
\end{aligned}
$$

One can easily check that $(B_2^n, \cdots, B_2^n) \in \boldsymbol{\xi}_{\mu_p,q}$, and hence $\boldsymbol{\xi}_{\mu_p,q} \neq \phi$. In general, it is difficult to get the precise value of $\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})$. However, the following proposition provides a convenient formula to calculate $\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})$ if $\boldsymbol{K} \in \boldsymbol{\xi}_{\mu_p,q}$. The proof of this proposition is similar to the ones of Proposition 3.2, so we omit it.

PROPOSITION 4.4. *Let $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in\boldsymbol{\xi}_{\mu_p,q}$. Then, for any $-n\neq q\in\mathbb{R}$,*

$$\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})=\left(\frac{1}{n}\right)^{q/(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\cdot\int_{S^{n-1}}\left(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\right)^{n/m(n+q)}d\sigma(u).$$

The following result can be obtained.

COROLLARY 4.1. *Let $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in\boldsymbol{\xi}_{\mu_p,q}$ and $-n\neq q\in\mathbb{R}$. Then*

$$|\Lambda_{\mu_p,q}K_1|^n\cdots|\Lambda_{\mu_p,q}K_m|^n\cdot\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})^{m(n+q)}=\tilde{V}\big(\Lambda_{\mu_p,q}K_1,\cdots,\Lambda_{\mu_p,q}K_m\big)^{m(n+q)}.$$

PROOF. By Remark 3.2 and Proposition 4.4, one has

$$\begin{aligned}
&\tilde{V}\big(\Lambda_{\mu_p,q}K_1,\cdots,\Lambda_{\mu_p,q}K_m\big)\\
&=\frac{1}{n}\int_{S^{n-1}}\left(\prod_{i=1}^m\rho_{\Lambda_{\mu_p,q}K_i}(u)\right)^{n/m}d\sigma(u)\\
&=\frac{1}{n}\int_{S^{n-1}}\left(\prod_{i=1}^m\frac{n(p-1)|\Lambda_{\mu_p,q}K_i|}{n-p}\cdot f_{\mu_p,q}(K_i,u)\right)^{n/m(n+q)}d\sigma(u)\\
&=\frac{1}{n}\cdot\left(\prod_{i=1}^m n|\Lambda_{\mu_p,q}K_i|\right)^{n/m(n+q)}\cdot\left(\frac{p-1}{n-p}\right)^{n/(n+q)}\int_{S^{n-1}}\left(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\right)^{n/m(n+q)}d\sigma(u)\\
&=\left(\prod_{i=1}^m|\Lambda_{\mu_p,q}K_i|\right)^{n/m(n+q)}\cdot\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}).
\end{aligned}$$

This yields the desired result. □

Let $-n\neq q\in\mathbb{R}$. We define $\boldsymbol{\nu}_{\mu_p,q}$, a subset of $(\mathcal{F}_0^+)^m$, as follows:

$$\begin{aligned}
&\boldsymbol{\nu}_{\mu_p,q}\\
&=\left\{\boldsymbol{K}\in(\mathcal{F}_0^+)^m:\exists Q\in\mathcal{K}_0\text{ s.t. }\left(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\right)^{1/m}=\big(\rho_Q(u)\big)^{n+q}\text{ for any }u\in S^{n-1}\right\}.
\end{aligned}$$

The following proposition provides a convenient formula to calculate $\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})$ for $\boldsymbol{K}\in\boldsymbol{\nu}_{\mu_p,q}$. In particular,

$$\boldsymbol{\mathcal{G}}_{p,q}(B_2^n,\cdots,B_2^n)=\boldsymbol{\mathcal{A}}_{p,q}(B_2^n,\cdots,B_2^n)=\mathcal{A}_{p,q}(B_2^n)=\big(C_p(B_2^n)\big)^{n/(n+q)}\cdot|B_2^n|^{q/(n+q)}.$$

PROPOSITION 4.5. *Let $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in\boldsymbol{\nu}_{\mu_p,q}$ and $-n\neq q\in\mathbb{R}$. Then*

$$\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})=\boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}).$$

PROOF. Due to $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in\boldsymbol{\nu}_{\mu_p,q}$, we can define $L\in\mathcal{K}_0$ by its radial function:

$$\big(\rho_L(u)\big)^{n+q} = \bigg(\prod_{i=1}^m f_{\mu_p,q}(K_i,u)\bigg)^{1/m} \quad \text{for any } u\in S^{n-1}.$$

When $q=0$, the desired formula follows trivially, i.e.,

$$\boldsymbol{\mathcal{G}}_{p,0}(\boldsymbol{K}) = \boldsymbol{\mathcal{A}}_{p,0}(\boldsymbol{K}) = \frac{p-1}{n-p}\int_{S^{n-1}}\bigg(\prod_{i=1}^m f_{\mu_p,0}(K_i,u)\bigg)^{1/m} d\sigma(u).$$

If $q>0$, it follows from the proof of Proposition 4.4 and $L\in\mathcal{K}_0$ that

$$\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K}) \ge \boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K}) = \big(\boldsymbol{C}_{p,q}(\boldsymbol{K},L^\circ,\cdots,L^\circ)\big)^{n/(n+q)}\cdot|L|^{q/(n+q)} \ge \boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K}).$$

Hence $\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K}) = \boldsymbol{\mathcal{A}}_{p,q}(\boldsymbol{K})$. The case $-n\ne q<0$ follows from a similar argument. □

Similar to Theorem 3.4, we have the following cyclic inequalities for $\boldsymbol{\mathcal{G}}_{p,q}(\cdot)$. Similar results hold for $\boldsymbol{\mathcal{A}}_{p,q}(\cdot)$.

THEOREM 4.3. *Let $\boldsymbol{K}\in(\mathcal{F}_0^+)^m$.*
(i) *If $-n<t<0<r<s$ or $-n<s<0<r<t$, then*

$$\boldsymbol{\mathcal{G}}_{p,r}(\boldsymbol{K}) \le \big(\boldsymbol{\mathcal{G}}_{p,t}(\boldsymbol{K})\big)^{(r-s)(n+t)/(t-s)(n+r)}\cdot\big(\boldsymbol{\mathcal{G}}_{p,s}(\boldsymbol{K})\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

(ii) *If $-n<t<r<s<0$ or $-n<s<r<t<0$, then*

$$\boldsymbol{\mathcal{G}}_{p,r}(\boldsymbol{K}) \le \big(\boldsymbol{\mathcal{G}}_{p,t}(\boldsymbol{K})\big)^{(r-s)(n+t)/(t-s)(n+r)}\cdot\big(\boldsymbol{\mathcal{G}}_{p,s}(\boldsymbol{K})\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

(iii) *If $t<r<-n<s<0$ or $s<r<-n<t<0$, then*

$$\boldsymbol{\mathcal{G}}_{p,r}(\boldsymbol{K}) \ge \big(\boldsymbol{\mathcal{G}}_{p,t}(\boldsymbol{K})\big)^{(r-s)(n+t)/(t-s)(n+r)}\cdot\big(\boldsymbol{\mathcal{G}}_{p,s}(\boldsymbol{K})\big)^{(r-t)(n+s)/(s-t)(n+r)}.$$

From Definition 4.3 and Hölder inequality, one can get the Aleksandrov–Fenchel inequality for $\boldsymbol{\mathcal{G}}_{p,q}(\cdot)$. Similar results can be obtained for $\boldsymbol{\mathcal{A}}_{p,q}(\cdot)$.

THEOREM 4.4. *Let $\boldsymbol{K}\in(\mathcal{F}_0^+)^m$. For $1\le j\le m$ and $-n<q<0$, one has*

$$\big(\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})\big)^j \le \prod_{i=1}^j \boldsymbol{\mathcal{G}}_{p,q}(K_1,K_2,\cdots,K_{m-j};\underbrace{K_{m-j+i},K_{m-j+i},\cdots,K_{m-j+i}}_{j}).$$

Moreover, if $j=m$, one has

$$\big(\boldsymbol{\mathcal{G}}_{p,q}(\boldsymbol{K})\big)^m \le \prod_{i=1}^m \mathcal{G}_{p,q}(K_i).$$

Furthermore, Definition 4.3 yields the isoperimetric type inequality as follows.

COROLLARY 4.2. *Let $\boldsymbol{K}=(K_1,K_2,\cdots,K_m)\in(\mathcal{F}_0^+)^m$.*
(i) *If $q\ge 0$, then*

$$\frac{\mathcal{G}_{p,q}(K_1, K_2, \cdots, K_m)}{\mathcal{G}_{p,q}(\underbrace{B_2^n, B_2^n, \cdots, B_2^n}_{m})} \leq \prod_{i=1}^{m} \left(\frac{C_{p,q}(K_i, B_2^n)}{C_{p,q}(B_2^n, B_2^n)} \right)^{n/m(n+q)}.$$

Equality holds if $K_i = r_i B_2^n$ *with* $r_i > 0$ *for any* $1 \leq i \leq m$ *and* $\prod_{i=1}^m r_i = 1$.

(ii) *If* $q < -n$, *then*

$$\frac{\mathcal{G}_{p,q}(K_1, K_2, \cdots, K_m)}{\mathcal{G}_{p,q}(\underbrace{B_2^n, B_2^n, \cdots, B_2^n}_{m})} \geq \prod_{i=1}^{m} \left(\frac{C_{p,q}(K_i, B_2^n)}{C_{p,q}(B_2^n, B_2^n)} \right)^{n/m(n+q)}.$$

Equality holds if $K_i = r_i B_2^n$ *with* $r_i > 0$ *for any* $1 \leq i \leq m$ *and* $\prod_{i=1}^m r_i = 1$.

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