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Topology of mixed hypersurfaces of cyclic type

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Abstract. Let $f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}}) = z_1^{a_1+b_1}\bar{z}_1^{b_1}z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}}\bar{z}_{n-1}^{b_{n-1}}z_n + z_n^{a_n+b_n}\bar{z}_n^{b_n}z_1$ be a mixed weighted homogeneous polynomial of cyclic type and $g_{II}(\boldsymbol{z}) = z_1^{a_1}z_2 + \cdots + z_{n-1}^{a_{n-1}}z_n + z_n^{a_n}z_1$ be the associated weighted homogeneous polynomial where $a_j \geq 1$ and $b_j \geq 0$ for $j = 1, \dots, n$. We show that two links $S_\varepsilon^{2n-1} \cap f_{II}^{-1}(0)$ and $S_\varepsilon^{2n-1} \cap g_{II}^{-1}(0)$ are diffeomorphic and their Milnor fibrations are isomorphic.

1. Introduction.

Let $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ be a mixed polynomial of complex variables $\boldsymbol{z} = (z_1, \dots, z_n)$ given as

$$f(\boldsymbol{z},\bar{\boldsymbol{z}}) := \sum_{i=1}^{m} c_i \boldsymbol{z}^{\nu_i}\bar{\boldsymbol{z}}^{\mu_i},$$

where $c_i \in \mathbb{C}^*$ and $\boldsymbol{z}^{\nu_i} = z_1^{\nu_{i,1}} \cdots z_n^{\nu_{i,n}}$ for $\nu_i = (\nu_{i,1}, \dots, \nu_{i,n})$ (respectively $\bar{\boldsymbol{z}}^{\mu_i} = \bar{z}_1^{\mu_{i,1}} \cdots \bar{z}_n^{\mu_{i,n}}$ for $\mu_i = (\mu_{i,1}, \dots, \mu_{i,n})$). Here $\bar{z}_j$ represents the complex conjugate of z_j.

A point $\boldsymbol{w} \in \mathbb{C}^n$ is called *a mixed singular point of* $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ if the gradient vectors of $\Re f$ and $\Im f$ are linearly dependent at $\boldsymbol{w}$. Certain restricted classes of mixed polynomials of the variables $\boldsymbol{z}$ which admit Milnor fibrations had been considered by Seade, see for instance [**7**], [**8**]. The last author introduced the notion of the Newton boundary and the concept of non-degeneracy for a mixed polynomial and he showed the existence of Milnor fibration for the class of strongly non-degenerate mixed polynomials [**3**].

We consider the classes of mixed polynomials which was first introduced by Ruas–Seade–Verjovsky [**6**] and Cisneros-Molina [**1**]. Let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be integers such that $\gcd(p_1, \dots, p_n) = \gcd(q_1, \dots, q_n) = 1$. We define the S^1-action and the $\mathbb{R}^*$-action on $\mathbb{C}^n$ as follows:

$$c \circ \boldsymbol{z} = (c^{p_1}z_1, \dots, c^{p_n}z_n), \quad c \in S^1,$$
$$r \circ \boldsymbol{z} = (r^{q_1}z_1, \dots, r^{q_n}z_n), \quad r \in \mathbb{R}^*.$$

If there exists a positive integer d_p such that $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ satisfies

$$f(c^{p_1}z_1, \dots, c^{p_n}z_n, \bar{c}^{p_1}\bar{z}_1, \dots, \bar{c}^{p_1}\bar{z}_n) = c^{d_p}f(\boldsymbol{z},\bar{\boldsymbol{z}}), \quad c \in S^1,$$

we say that $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ is *a polar weighted homogeneous polynomial*. Similarly $f(\boldsymbol{z},\bar{\boldsymbol{z}})$ is called *a radial weighted homogeneous polynomial* if there exists a positive integer d_r such

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that

$$f(r^{q_1}z_1,\dots,r^{q_n}z_n,r^{q_1}\bar z_1,\dots,r^{q_n}\bar z_n)=r^{d_r}f(\boldsymbol{z},\bar{\boldsymbol{z}}),\quad r\in\mathbb{R}^*.$$

Let f be a polar and radial weighted homogeneous polynomial. Then f admits the global Milnor fibration $f:\mathbb{C}^n\setminus f^{-1}(0)\to\mathbb{C}^*$, see for instance [**6**], [**1**], [**2**], [**3**].

Let $f(\boldsymbol{z},\bar{\boldsymbol{z}})=\sum_{i=1}^m c_i\boldsymbol{z}^{\nu_i}\bar{\boldsymbol{z}}^{\mu_i}$ be a mixed polynomial with $c_j\neq 0, j=1,\dots,m$. Put

$$g(\boldsymbol{z}):=\sum_{i=1}^m c_i\boldsymbol{z}^{\nu_i-\mu_i}.$$

We call g *the associated Laurent polynomial of* f. A mixed polynomial f is called *simplicial* if $m\le n$ and the ranks of the matrices $N\pm M$ are m where $N=(\nu_1,\dots,\nu_n)$ and $M=(\mu_1,\dots,\mu_n)$. Here ν_i and μ_i are considered as column vectors $\nu_i={}^t(\nu_{i1},\dots,\nu_{in})$, $\mu_i={}^t(\mu_{i1},\dots,\mu_{in})$. f is called *full* if $m=n$. A full simplicial mixed polynomial f and its associated Laurent polynomial g admit a unique polar weight and a unique radial weight in the above sense [**2**]. It is useful to consider a graph Γ associated to f. First we associate a vertex v_i if z_i or $\bar z_i$ appears in f. We join v_i and v_j by an edge if there is a monomial $\boldsymbol{z}^{\nu_k}\bar{\boldsymbol{z}}^{\mu_k}$ which contains both variables z_i,z_j. That is $\nu_{k,a}+\mu_{k,a}>0$ for $a=i,j$. Most important graphs are a bamboo graph

$$B_n:\ \overset{v_1}{\bullet}\!\!-\!\!-\!\!-\overset{v_2}{\bullet}\!\!-\!\!-\!\!-\cdots-\!\!-\!\!-\overset{v_{n-1}}{\bullet}\!\!-\!\!-\!\!-\overset{v_n}{\bullet}$$

and a cyclic graph C_n which is obtained from B_n adding an edge between v_n and v_1.

We restrict the Milnor fibrations defined by f and g on the complex torus $\mathbb{C}^{*n}$ where $\mathbb{C}^{*n}=(\mathbb{C}^*)^n$. In [**2**, Theorem 10], it is shown that there exists a canonical diffeomorphism $\varphi:\mathbb{C}^{*n}\to\mathbb{C}^{*n}$ which gives an isomorphism of the Milnor fibrations defined by f and g:

$$\begin{array}{ccc}\mathbb{C}^{*n}\setminus f^{-1}(0) & \xrightarrow{\varphi} & \mathbb{C}^{*n}\setminus g^{-1}(0)\\ \downarrow f & & \downarrow g\\ \mathbb{C}^* & = & \mathbb{C}^*\end{array}.$$

However the canonical diffeomorphism φ does not extend to $\mathbb{C}^n\setminus\{O\}$ in general. Here O is the origin of $\mathbb{C}^n$. The exceptional case is a mixed Brieskorn polynomial, for which this canonical diffeomorphism extends as a continuous homeomorphism [**6**]. In [**4**], the last author studied the following simplicial polar weighted homogeneous polynomials:

$$\begin{cases} f_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z},\bar{\boldsymbol{z}}) &= z_1^{a_1+b_1}\bar z_1^{b_1}+\cdots+z_{n-1}^{a_{n-1}+b_{n-1}}\bar z_{n-1}^{b_{n-1}}+z_n^{a_n+b_n}\bar z_n^{b_n}\\ f_I(\boldsymbol{z},\bar{\boldsymbol{z}}) &= z_1^{a_1+b_1}\bar z_1^{b_1}z_2+\cdots+z_{n-1}^{a_{n-1}+b_{n-1}}\bar z_{n-1}^{b_{n-1}}z_n+z_n^{a_n+b_n}\bar z_n^{b_n}\\ f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}}) &= z_1^{a_1+b_1}\bar z_1^{b_1}z_2+\cdots+z_{n-1}^{a_{n-1}+b_{n-1}}\bar z_{n-1}^{b_{n-1}}z_n+z_n^{a_n+b_n}\bar z_n^{b_n}z_1,\end{cases}$$

where $a_j\ge 1$ and $b_j\ge 0$ for $j=1,\dots,n$. Here the notation is the same as in [**4**]. Note that the graph of f_I is a bamboo and that of f_{II} is a cyclic graph. The graph of $f_{\boldsymbol{a},\boldsymbol{b}}$ is n disjoint vertices without any edges. A polar weighted homogeneous polynomial $f_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z},\bar{\boldsymbol{z}})$ and a weighted homogeneous polynomial $g_{\boldsymbol{a}}(\boldsymbol{z})$ are called *a mixed Brieskorn polynomial* and *a Brieskorn polynomial* respectively. A mixed polynomial $f_I(\boldsymbol{z},\bar{\boldsymbol{z}})$ is called *a simplicial mixed polynomial of bamboo type* and $f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}})$ is called *a simplicial mixed polynomial of cyclic type* respectively. He showed that two links of f_ι and the associated polynomial $g_\iota(\boldsymbol{z})$ in a small sphere are isotopic and their Milnor fibrations are isomorphic for $\iota = (\boldsymbol{a},\boldsymbol{b})$ and I. He conjectured the assertion will be also true for the case f_{II}.

2. Statement of the result.

The purpose of this paper is to give a positive answer to the above conjecture. Thus we study the following simplicial polynomial $f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}})$ and its associated weighted homogeneous polynomial $g_{II}(\boldsymbol{z})$:

$$\begin{cases} f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}}) & = z_1^{a_1+b_1}\bar{z}_1^{b_1}z_2+\cdots+z_{n-1}^{a_{n-1}+b_{n-1}}\bar{z}_{n-1}^{b_{n-1}}z_n+z_n^{a_n+b_n}\bar{z}_n^{b_n}z_1, \\ g_{II}(\boldsymbol{z}) & = z_1^{a_1}z_2+\cdots+z_{n-1}^{a_{n-1}}z_n+z_n^{a_n}z_1, \end{cases}$$

where $a_j \geq 1$ and $b_j \geq 0$ for $j = 1,\ldots,n$. We assume that f_{II} contains a conjugate $\bar{z}_j$ for some j. This implies

(a) there exists $j \in \{1,\ldots,n\}$ such that $b_j \geq 1$.

We also assume that f_{II} is simplicial. As the determinant of $N-M$ is given by $a_1\cdots a_n + (-1)^{n+1}$, we assume also that

(b) there exists $k \in \{1,\ldots,n\}$ such that $a_k \geq 2$.

Though this assumption is not necessary if n is odd, we assume (b) anyway. Since $f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}})$ is a polar and radial weighted homogeneous, $f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}})$ admits a global fibration

$$f_{II} : \mathbb{C}^n \setminus f_{II}^{-1}(0) \to \mathbb{C}^*$$

[**6**], [**1**], [**2**], [**3**]. The complex polynomial $g_{II}(\boldsymbol{z})$ is a weighted homogeneous polynomial with respect to the same polar weight of f_{II} and g_{II} with $n=3$ is listed in the classification of weighted homogeneous surfaces in $\mathbb{C}^3$ with isolated singularity [**5**]. We consider the hypersurfaces

$$V_f := f_{II}^{-1}(0), \quad V_g := g_{II}^{-1}(0)$$

and respective links

$$K_{f,\varepsilon} = V_f \cap S_\varepsilon^{2n-1}, \quad K_{g,\varepsilon} = V_g \cap S_\varepsilon^{2n-1}$$

where S_ε^{2n-1} is the $(2n-1)$-dimensional sphere centered at the origin O with radius ε. Then the two links $K_{f,\varepsilon}$ and $K_{g,\varepsilon}$ are smooth for any $\varepsilon > 0$ ([**2**]). We consider the following family of mixed polynomials:

$$\begin{aligned} f_{II,t}(\boldsymbol{z},\bar{\boldsymbol{z}}) &:= (1-t)f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}}) + tg_{II}(\boldsymbol{z}) \\ &= (1-t)(z_1^{a_1+b_1}\bar{z}_1^{b_1}z_2 + \cdots + z_{n-1}^{a_{n-1}+b_{n-1}}\bar{z}_{n-1}^{b_{n-1}}z_n + z_n^{a_n+b_n}\bar{z}_n^{b_n}z_1) \\ &\quad + t(z_1^{a_1}z_2 + \cdots + z_{n-1}^{a_{n-1}}z_n + z_n^{a_n}z_1) \\ &= \sum_{j=1}^n z_j^{a_j}z_{j+1}\{(1-t)|z_j|^{2b_j} + t\} \end{aligned}$$

where $0 \le t \le 1$. Here the numbering is modulo n, so $z_{n+1} = z_1$. Though the mixed polynomial $f_{II,t}$ is not radial weighted homogeneous for $t \ne 0, 1$, $f_{II,t}$ is polar weighted homogeneous for $0 \le t \le 1$ with the same weight $P = (p_1, \ldots, p_n)$ which is characterized by $a_j p_j + p_{j+1} = d_p$, $j = 1, \ldots, n$. Put

$$V_t = f_{II,t}^{-1}(0),\ K_{t,\varepsilon} = S_\varepsilon^{2n-1} \cap V_t, \quad 0 \le t \le 1.$$

Note that

$$\begin{aligned} &f_{II,0} = f_{II},\ f_{II,1} = g_{II} \\ &V_f = V_0,\ K_{f,\varepsilon} = K_{0,\varepsilon},\ V_g = V_1,\ K_{g,\varepsilon} = K_{1,\varepsilon}. \end{aligned}$$

First recall that V_t has an isolated mixed singularity at the origin O and $V_t \setminus \{O\}$ is non-singular for any $0 \le t \le 1$ by [**4**, Lemma 9]. Our main result is:

Transversality Theorem 1. *Let V_t be as above. For any fixed $\varepsilon > 0$, the sphere S_ε^{2n-1} and the family of hypersurfaces V_t are transversal for $0 \le t \le 1$.*

3. Proof of Transversality Theorem 1.

3.1. Strategy of the proof.

We follow the recipe of [**4**]. First recall that

$$\begin{aligned} f_{II,t}(\boldsymbol{z},\bar{\boldsymbol{z}}) &:= (1-t)f_{II}(\boldsymbol{z},\bar{\boldsymbol{z}}) + tg_{II}(\boldsymbol{z}) \\ &= \sum_{j=1}^n z_j^{a_j}z_{j+1}\{(1-t)|z_j|^{2b_j} + t\}. \end{aligned}$$

Recall that V_t is non-singular off the origin by [**4**]. To show the transversality of the sphere $S_{\varepsilon_0}^{2n-1}$ and V_t, we have to show that the Jacobian matrix of $\Re f_{II,t}, \Im f_{II,t}$ and $\rho(\boldsymbol{z})$ has rank 3 at every intersection $\boldsymbol{w} \in S_{\varepsilon_0}^{2n-1} \cap V_t$. Here $\rho(\boldsymbol{z}) = \|\boldsymbol{z}\|^2$, the square of the radius $\|\boldsymbol{z}\|$. However this computation is extremely complicated. Instead, we follow the recipe of [**4**]. We will show *the existence of a tangent vector $\boldsymbol{v} \in T_{\boldsymbol{w}}V_t$ which is not tangent to the sphere $S_{\varepsilon_0}^{2n-1}$.*

Take a point $\boldsymbol{w} = (w_1, \ldots, w_n) \in V_t \cap S_{\varepsilon_0}^{2n-1}$ and fix it hereafter. To find such a vector $\boldsymbol{v}$, we will construct a real analytic path

$$\boldsymbol{w}(s) = (r_1(s)w_1, \ldots, r_n(s)w_n)$$

on a neighborhood of $s = 0$ so that $\boldsymbol{w}(0) = \boldsymbol{w}$ and

$$f_{II,t}(\boldsymbol{w}(s),\bar{\boldsymbol{w}}(s)) = (s+1)f_{II,t}(\boldsymbol{w},\bar{\boldsymbol{w}}) \tag{1}$$

where $r_j(s)$, $j = 1,\dots,n$ are real-valued functions on $|s| \ll 1$ which satisfy certain functional equalities. The equality (1) implies that the curve $\boldsymbol{w}(s)$ is an embedded curve in V_t with $\boldsymbol{w}(0) = \boldsymbol{w}$. Then we define the vector as the tangent vector of this curve at $s = 0$:

$$\boldsymbol{v} = \frac{d\boldsymbol{w}}{ds}(0). \tag{2}$$

To find such a path $\boldsymbol{w}(s) = (r_1(s)w_1,\dots,r_n(s)w_n)$, we solve a certain functional equation, using the inverse mapping theorem.

3.2. Construction of $\boldsymbol{w}(s)$.

First, for $\boldsymbol{w} \in V_t$ with $\boldsymbol{w} \neq O$, we consider the following map:

$$\begin{aligned} \Phi_{\boldsymbol{w}} &: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \\ (r_1,\dots,r_n,s) &\mapsto (h_1,\dots,h_n,s), \end{aligned}$$

where h_j is a polynomial function of variables $r_1,\dots,r_n$ and s defined by

$$h_j = r_j^{a_j} r_{j+1}\{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\} - (s+1)\{(1-t)|w_j|^{2b_j} + t\},\ j = 1,\dots,n \tag{3}$$

where t is fixed on $0 \le t \le 1$. Here the numbering is modulo n, so $r_{n+1} = r_1$. We want to solve the equations $h_1 = \dots = h_n = 0$ in $r_1,\dots,r_n$ expressing r_j as a function of s so that we get the system of equations

$$h_j(r_1(s),\dots,r_n(s),s) \equiv 0,\ j = 1,\dots,n. \tag{4}$$

This equality is equivalent to (1) which is more explicitly written as

$$\begin{aligned} f_{II,t}(\boldsymbol{w}(s),\bar{\boldsymbol{w}}(s)) &= (s+1)f_{II,t}(\boldsymbol{w},\bar{\boldsymbol{w}}) \quad \text{where} \\ f_{II,t}(\boldsymbol{w}(s),\bar{\boldsymbol{w}}(s)) &= \sum_{j=1}^{n} (r_j(s)w_j)^{a_j}(r_{j+1}(s)w_{j+1})\{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\}, \\ (s+1)f_{II,t}(\boldsymbol{w},\bar{\boldsymbol{w}}) &= (s+1)\sum_{j=1}^{n} w_j^{a_j} w_{j+1}\{(1-t)|w_j|^{2b_j} + t\}. \end{aligned}$$

We will solve the functional equality (1) using the inverse mapping theorem.

Lemma 1. *Let $\boldsymbol{w} \in V_t$ with $\boldsymbol{w} \neq O$ and $0 < t < 1$. Then the Jacobian matrix $J(\Phi_{\boldsymbol{w}})$ has rank $n+1$ at $(r_1,\dots,r_n,s) = (1,\dots,1,0)$, where*

$$J(\Phi_{\boldsymbol{w}}) = \begin{pmatrix} \dfrac{\partial h_1}{\partial r_1} \cdots \dfrac{\partial h_1}{\partial r_n} & \dfrac{\partial h_1}{\partial s} \\ \vdots & \\ \dfrac{\partial h_n}{\partial r_1} \cdots \dfrac{\partial h_n}{\partial r_n} & \dfrac{\partial h_n}{\partial s} \\ \dfrac{\partial s}{\partial r_1} \cdots \dfrac{\partial s}{\partial r_n} & \dfrac{\partial s}{\partial s} \end{pmatrix}.$$

PROOF. By a direct computation, the Jacobian matrix of $\Phi_{\boldsymbol{w}}$ is given as

$$J(\Phi_{\boldsymbol{w}}) = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & 0 & \dots & 0 & -\beta_1 \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_{n-1,n-1} & \alpha_{n-1,n} & \vdots \\ \alpha_{n,1} & 0 & \dots & 0 & \alpha_{n,n} & -\beta_n \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_{j,j} &= r_j^{a_j-1} r_{j+1}\{(1-t)|w_j|^{2b_j}(a_j+2b_j)r_j^{2b_j} + a_j t\}, \\ \alpha_{j,j+1} &= r_j^{a_j}\{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\}, \\ \beta_j &= \{(1-t)|w_j|^{2b_j} + t\}, \qquad j = 1, \dots, n. \end{aligned}$$

Since $0 < t < 1$, $\alpha_{j,j}$ and $\alpha_{j,j+1}$ at $(1, \dots, 1, 0)$ are positive real numbers for each $j = 1, \dots, n$. If $\alpha_{j,j}$ and $\alpha_{j,j+1}$ are not evaluated at $(1, \dots, 1, 0)$, they may be negative, for instance $\alpha_{j,j}$ is negative if r_j is positive and r_{j+1} is negative. The determinant $\det J(\Phi_{\boldsymbol{w}})$ is given as

$$\begin{aligned} \det J(\Phi_{\boldsymbol{w}}) &= \alpha_{1,1} \dots \alpha_{n,n} + (-1)^{n+1} \alpha_{1,2} \cdots \alpha_{n-1,n} \alpha_{n,1} \\ &= \prod_{j=1}^{n} r_j^{a_j-1} r_{j+1}\{(1-t)|w_j|^{2b_j}(a_j+2b_j) r_j^{2b_j} + a_j t\} \\ &\quad + (-1)^{n+1} \prod_{j=1}^{n} r_j^{a_j}\{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\}. \end{aligned} \tag{5}$$

The proof of Lemma 1 is reduced to the following assertion.

ASSERTION 1. $\det J(\Phi_{\boldsymbol{w}}) > 0$.

PROOF. (i) If n is an odd number, $\det J(\Phi_{\boldsymbol{w}})$ at $(1, \dots, 1, 0)$ is obviously positive. (ii) Suppose that n is a positive even number. Consider

$$\alpha'_{j,j} := r_j^{a_j}\{(1-t)|w_j|^{2b_j}(a_j+2b_j) r_j^{2b_j} + a_j t\}.$$

Note that $\prod_{j=1}^{n} \alpha_{j,j} = \prod_{j=1}^{n} \alpha'_{j,j}$. We have the following.

$$\det J(\Phi_{\boldsymbol{w}}) = \prod_{j=1}^{n} \alpha_{j,j} - \prod_{j=1}^{n} \alpha_{j,j+1} = \prod_{j=1}^{n} \alpha'_{j,j} - \prod_{j=1}^{n} \alpha_{j,j+1}$$
$$\det J(\Phi_{\boldsymbol{w}}) \geq 0 \iff \prod_{j=1}^{n} \frac{\alpha'_{j,j}}{\alpha_{j,j+1}} \geq 1.$$

As $\alpha'_{j,j} \geq \alpha_{j,j+1}$, the equality takes place if $\alpha'_{j,j} = \alpha_{j,j+1}$ for $j = 1, \dots, n$. We assume that $\alpha'_{j,j} = \alpha_{j,j+1}$ at $(1, \dots, 1, 0)$ for any $j, 1 \leq j \leq n$. Then

$$(1-t)|w_j|^{2b_j}(a_j + 2b_j) + a_j t = (1-t)|w_j|^{2b_j} + t,\ j = 1, \dots, n$$

and this is the case if and only if $(w_j, a_j) = (0, 1)$ or $(a_j, b_j) = (1, 0)$. Thus the Jacobian of $\Phi_{\boldsymbol{w}}$ at $(1, \dots, 1, 0)$ is equal to 0 if and only if $(w_j, a_j) = (0, 1)$ or $(a_j, b_j) = (1, 0)$ for $j = 1, \dots, n$. However these cases do not happen, since by assumption (b) there exists j with $a_j \geq 2$. Thus the assertion is proved. This completes also the proof of Lemma 1. □

Now we are ready to prove the transversality of $S^{2n-1}_{\varepsilon_0}$ and V_t for any $\varepsilon_0 > 0$ and $0 \leq t \leq 1$.

3.3. Proof of Transversality Theorem.

The assertion is known for $t = 0, 1$ by [**3**]. Thus we assume that $0 < t < 1$. Recall that $f_{II,t} : \mathbb{C}^n \to \mathbb{C}$ has a unique singularity at the origin O for any $0 \leq t \leq 1$ by [**4**, Lemma 9]. As the codimension of $T_{\boldsymbol{w}} S^{2n-1}_{\varepsilon_0}$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is 1, to show the transversality, it suffices to show the existence of a vector $\boldsymbol{v} \in T_{\boldsymbol{w}} V_t$ with $\boldsymbol{v} \notin T_{\boldsymbol{w}} S^{2n-1}_{\varepsilon_0}$.

For a given $\boldsymbol{w} = (w_1, \dots, w_n) \in V_t$, we consider the nullity set $I_{\boldsymbol{w}} = \{i \mid w_i = 0\}$.

Case 1: $I_{\boldsymbol{w}} = \emptyset$. This is the most essential case and does not appear for the mixed polynomials $f_{\boldsymbol{a},\boldsymbol{b}}$ and f_I. The corresponding graph is cyclic.

By Lemma 1, the Jacobian of $\Phi_{\boldsymbol{w}}$ at $(1, \dots, 1, 0)$ is non-zero. By the Inverse mapping theorem, there exist a neighborhood $U \subset \mathbb{R}^{n+1}$ of $(1, \dots, 1, 0)$ and a neighborhood $W \subset \mathbb{R}^{n+1}$ of $\Phi_{\boldsymbol{w}}(1, \dots, 1, 0) = (0, \dots, 0)$ and a real analytic mapping $\Psi_{\boldsymbol{w}} = (\psi_1, \dots, \psi_n, \mathrm{id}) : \mathrm{W} \to \mathrm{U}$ so that

$$\Phi_{\boldsymbol{w}} \circ \Psi_{\boldsymbol{w}} = \mathrm{id}_{\mathrm{W}} \quad \text{and} \quad \Psi_{\boldsymbol{w}} \circ \Phi_{\boldsymbol{w}} = \mathrm{id}_{\mathrm{U}}.$$

Put $\mathbf{0} := (0, \dots, 0) \in \mathbb{R}^n$ and consider $V \subset \mathbb{R} := W \cap (\{\mathbf{0}\} \times \mathbb{R})$, a neighborhood of $0 \in \mathbb{R}$ and define smooth functions $r_j : V \to \mathbb{R}$ of the variable s by $r_j(s) := \psi_j(0, \dots, 0, s)$. Note that $r_j(0) = 1$. We have the equalities:

$$h_j(r_1(s), \dots, r_n(s), s) \equiv 0,\ s \in V,\ j = 1, \dots, n.$$

As we have seen in the above discussion, this implies

$$r_j^{a_j}(s) r_{j+1}(s)\{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\} - (s+1)\{(1-t)|w_j|^{2b_j} + t\} \equiv 0,$$

which implies

$$\begin{aligned} f_{II,t}(\boldsymbol{w}(s), \bar{\boldsymbol{w}}(s)) &= \sum_{j=1}^{n} r_j(s)^{a_j} r_{j+1}(s) w_j^{a_j} w_{j+1} \{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\} \\ &= (s+1) f_{II,t}(\boldsymbol{w}, \bar{\boldsymbol{w}}). \end{aligned}$$

Thus $f_{II,t}(\boldsymbol{w}(s), \bar{\boldsymbol{w}}(s)) \equiv 0$. Put $\boldsymbol{v} = d\boldsymbol{w}/ds(0)$. We have $\boldsymbol{v} \in T_{\boldsymbol{w}} V_t$ by the definition. Now to finish the proof of the transversality assertion, we need only to show

Assertion 2. *$\boldsymbol{v} \neq \boldsymbol{0}$ and $\boldsymbol{v} \notin T_{\boldsymbol{w}} S_{\varepsilon_0}^{2n-1}$.*

To prove the assertion, we consider the differential in s of

$$h_j(r_1(s), \ldots, r_n(s), s) = r_j(s)^{a_j} r_{j+1}(s) \{(1-t)|w_j|^{2b_j} r_j(s)^{2b_j} + t\} - (s+1)\{(1-t)|w_j|^{2b_j} + t\}.$$

By a direct computation, we get the equality

$$\begin{aligned} \frac{d}{ds} h_j(r_1(s), \ldots, r_n(s), s) &= \left(\sum_{k=1}^{n} \frac{\partial h_j}{\partial r_k} \frac{dr_k}{ds} \right) - \beta_j \\ &= \alpha_{j,j} \frac{dr_j}{ds} + \alpha_{j,j+1} \frac{dr_{j+1}}{ds} - \beta_j \equiv 0 \end{aligned}$$

where

$$\begin{aligned} \alpha_{j,j} &= r_j^{a_j - 1} r_{j+1} \{(1-t)|w_j|^{2b_j} (a_j + 2b_j) r_j^{2b_j} + a_j t\}, \\ \alpha_{j,j+1} &= r_j^{a_j} \{(1-t)|w_j|^{2b_j} r_j^{2b_j} + t\}, \\ \beta_j &= \{(1-t)|w_j|^{2b_j} + t\} \end{aligned}$$

for $j = 1, \ldots, n$. The above equality can be written as

$$A \begin{pmatrix} \frac{dr_1}{ds} \\ \vdots \\ \vdots \\ \vdots \\ \frac{dr_n}{ds} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \vdots \\ \vdots \\ \beta_n \end{pmatrix} \quad \text{where}$$

$$A := \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & 0 & \ldots & 0 \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \alpha_{n-1,n-1} & \alpha_{n-1,n} \\ \alpha_{n,1} & 0 & \ldots & 0 & \alpha_{n,n} \end{pmatrix}.$$

Observe that the above equality says $dr_j/ds(s)$ is independent of s. By Lemma 1, the determinant of A is positive. We first consider the differential dr_1/ds and will show that $dr_1/ds(0) \geq 0$. Put $m = [n/2]$, the largest integer such that $m \leq n/2$. By the Cramer's formula, the differential dr_1/ds of r_1 is equal to

$$\begin{aligned}
\frac{dr_1}{ds} =& \frac{1}{\det A}\det\begin{pmatrix} \beta_1 & \alpha_{1,2} & 0 & \dots & 0 \\ \vdots & \alpha_{2,2} & \alpha_{2,3} & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_{n-1,n} \\ \beta_n & 0 & \dots & 0 & \alpha_{n,n} \end{pmatrix} \\
=& \frac{1}{\det A}\sum_{j=1}^{n}(-1)^{j-1}A_{j-1}\beta_j A'_{j+1} \\
=& \begin{cases} \frac{1}{\det A}\sum_{k=1}^{m}(A_{2k-2}\beta_{2k-1}A'_{2k} - A_{2k-1}\beta_{2k}A'_{2k+1}), & n = 2m \\ \frac{1}{\det A}\sum_{k=1}^{m}(A_{2k-2}\beta_{2k-1}A'_{2k} - A_{2k-1}\beta_{2k}A'_{2k+1}) + A_{n-1}\beta_n A'_{n+1}, & n = 2m+1 \end{cases}
\end{aligned}$$

where

$$A_{j-1} = \begin{cases} 1 & j = 1 \\ \prod_{\ell=1}^{j-1}\alpha_{\ell,\ell+1} & j \geq 2 \end{cases}, \quad A'_{j+1} = \begin{cases} \prod_{\ell=j}^{n-1}\alpha_{\ell+1,\ell+1} & j \leq n-1 \\ 1 & j = n \end{cases}.$$

We have

$$A_{2k-2}\beta_{2k-1}A'_{2k} - A_{2k-1}\beta_{2k}A'_{2k+1} = A_{2k-2}A'_{2k+1}(\beta_{2k-1}\alpha_{2k,2k} - \alpha_{2k-1,2k}\beta_{2k}).$$

As $\alpha_{j,j+1}(1,\dots,1) = \beta_j$ for $j = 1,\dots,n$, we observe that

$$\begin{aligned}
&\beta_{2k-1}\alpha_{2k,2k}(1,\dots,1) - \alpha_{2k-1,2k}(1,\dots,1)\beta_{2k} \\
&= \beta_{2k-1}\{\alpha_{2k,2k}(1,\dots,1) - \beta_{2k}\} \\
&= \beta_{2k-1}\{(1-t)|w_{2k}|^{2b_{2k}}(a_{2k}+2b_{2k}) + a_{2k}t - (1-t)|w_{2k}|^{2b_{2k}} - t\} \\
&= \beta_{2k-1}\{(1-t)|w_{2k}|^{2b_{2k}}(a_{2k}+2b_{2k}-1) + (a_{2k}-1)t\} \geq 0.
\end{aligned}$$

The equality holds only if $a_{2k} = 1$ and $b_{2k} = 0$. Note that $w_i \neq 0$ for any $i = 1,\dots,n$ by the assumption. Anyway we have

$$\frac{dr_1}{ds}(0) \geq 0.$$

If n is an odd integer, we see that $dr_1/ds(0) > 0$ by the last unpaired term: $dr_1/ds(0) \geq A_{n-1}\beta_n A'_{n+1} > 0$. If there exists some k such that $a_{2k} \geq 2$, we have also the strict inequality: $dr_1/ds(0) > 0$.

Next we consider dr_k/ds for $k \geq 2$. First observe that our polynomial $f_{II,t}$ has a symmetry for the cyclic permutation of the coordinates $\sigma = (1,2,\dots,n)$. Secondly after cyclic change of coordinates, say $\boldsymbol{z}' = (z'_1,\dots,z'_n) = (z_{\sigma^i(1)},\dots,z_{\sigma^i(n)})$, the equality (3) does not change. That is, $\boldsymbol{w}'(s) = (r_{\sigma^i(1)}w_{\sigma^i(1)},\dots,r_{\sigma^i(n)}w_{\sigma^i(n)})$ is the obtained solution curve. The tangent vector $\boldsymbol{v}' = d\boldsymbol{w}'/ds(0)$ is also equal to $\boldsymbol{v}$ after the corresponding cyclic permutation of coordinates. Therefore we can apply the above argument to have the inequality $(dr_{\sigma^i(1)}/ds)(0) \geq 0$ for any i. As we have some j with $a_j \geq 2$, this implies

$$\frac{dr_{j-1}}{ds}(0) > 0.$$

Now we are ready to show that $\boldsymbol{v} \neq 0$ and $\boldsymbol{v} \notin T_{\boldsymbol{w}} S^{2n-1}_{\varepsilon_0}$. By the assumption of $\boldsymbol{w}$, the path $\boldsymbol{w}(s)$ satisfies

$$f_{II,t}(\boldsymbol{w}(s), \bar{\boldsymbol{w}}(s)) = (s+1) f_{II,t}(\boldsymbol{w}, \bar{\boldsymbol{w}}) \equiv 0,$$
$$\frac{d\|\boldsymbol{w}(s)\|^2}{ds}|_{s=0} = 2\sum_{j=1}^{n} r_j(0)\frac{dr_j}{ds}(0)|w_j|^2 = 2\sum_{j=1}^{n} \frac{dr_j}{ds}(0)|w_j|^2 > 0.$$

This implies that $\boldsymbol{v} \neq 0$ and $\boldsymbol{v} \notin T_{\boldsymbol{w}} S^{2n-1}_{\varepsilon_0}$.

Case 2: Now we consider the case $I_{\boldsymbol{w}} \neq \emptyset$. Put $I^c_{\boldsymbol{w}}$ be the complement of $I_{\boldsymbol{w}}$ and $\mathbb{C}^{*I^c_{\boldsymbol{w}}} = \{\boldsymbol{z} \in \mathbb{C}^n \mid z_i = 0, i \in I_{\boldsymbol{w}}\}$. We consider the mixed polynomial $f'(\boldsymbol{z}, \bar{\boldsymbol{z}}) = f_{II,t}|_{\mathbb{C}^{*I^c_{\boldsymbol{w}}}}$. Let J be the set of indices j for which z_j or $\bar{z}_j$ appears in f'. Note that $J \subset I^c_{\boldsymbol{w}}$ but it can be a proper subset.

Case 2-1. Assume that $f' \equiv 0$, i.e., $J = \emptyset$. We take simply a real analytic path as follows:

$$\boldsymbol{w}(s) = (s+1)\boldsymbol{w}$$

for $s \in \mathbb{R}$. Since $\boldsymbol{w} \in V_t \setminus \{O\}$, we observe that

$$f_{II,t}(\boldsymbol{w}(s), \bar{\boldsymbol{w}}(s)) \equiv 0, \quad \frac{d\|\boldsymbol{w}(s)\|^2}{ds}|_{s=0} = 2\|\boldsymbol{w}\|^2 > 0.$$

Case 2-2. Assume that $f' \not\equiv 0$. Then using the connected components of the graph of f', we can express f' uniquely as follows.

$$f'(\boldsymbol{z}, \bar{\boldsymbol{z}}) = f_1(\boldsymbol{z}_{I_1}) + \cdots + f_k(\boldsymbol{z}_{I_k})$$

where the graph of f_i is a bamboo and the variables of f_i, f_j, $i \neq j$ are disjoint and the above expression is a join type expression. Here I_i be the set of indices of variables of f_i and $\boldsymbol{z}_{I_i} = (z_j)_{j \in I_i}$ are the variables of f_i for $i = 1, \ldots, k$. We have the equality $\cup_{i=1}^k I_i = J$ and $I_i \cap I_j = \emptyset$ for $i \neq j$. Put $\mathbb{C}^{I_i} = \{\boldsymbol{z} \in \mathbb{C}^n \mid z_j = 0, j \notin I_i\}$. Fixing i, we will construct a curve $\boldsymbol{w}_{I_i}(s)$ on $\mathbb{C}^{*I_i}$ so that

$$f_i(\boldsymbol{w}_{I_i}(s)) = (s+1) f_i(\boldsymbol{w}_{I_i}).$$

The construction of the curve $\boldsymbol{w}_{I_i}(s)$ can be reduced to the argument of [**4**, Lemma 10]. We will give briefly the proof which is based on the argument of [**4**].

For $j \notin J$, we put $w_j(s) = w_j$ and $\boldsymbol{w}_{J^c}(s) = \boldsymbol{w}_{J^c} \in \mathbb{C}^{J^c}$ where $\mathbb{C}^{J^c} = \{\boldsymbol{z} \in \mathbb{C}^n \mid z_{j'} = 0, j' \in J\}$. Here $\boldsymbol{w}_{J^c}$ is the projection of $\boldsymbol{w}$ to $\mathbb{C}^{J^c}$. For each $i = 1, \ldots, k$, we will construct a curve $\boldsymbol{w}_{I_i}(s)$ on $\mathbb{C}^{I_i}$ and define $\boldsymbol{w}_J(s) = \boldsymbol{w}_{I_1}(s) + \cdots + \boldsymbol{w}_{I_k}(s)$. Finally we define a curve $\boldsymbol{w}(s) = \boldsymbol{w}_{J^c} + \boldsymbol{w}_J(s) \in \mathbb{C}^n$ so that

$$\begin{aligned} f_i(\boldsymbol{w}_{I_i}(s)) &= (s+1) f_i(\boldsymbol{w}_{I_i}), \\ f_{II,t}(\boldsymbol{w}(s)) &= f'(\boldsymbol{w}(s)) \\ &= f_1(\boldsymbol{w}_{I_1}(s)) + \cdots + f_k(\boldsymbol{w}_{I_k}(s)) \\ &= (s+1)\{f_1(\boldsymbol{w}_{I_1}) + \cdots + f_k(\boldsymbol{w}_{I_k})\} \\ &= (s+1) f_{II,t}(\boldsymbol{w}) \equiv 0. \end{aligned}$$

So we fix i. For simplicity's sake, we assume $I_i = \{j \mid \upsilon_i \le j \le \tau_i\}$ with $\tau_i \le n$. The last assumption $\tau_i \le n$ is for the simplicity of the indices. This implies that

$$f_i(\boldsymbol{z}_{I_i}) = \sum_{j=\upsilon_i}^{\tau_i - 1} z_j^{a_j} z_{j+1} \{|z_j|^{2b_j}(1-t) + t\}.$$

We will show that there exists a differentiable positive real-valued function solution $(r_{\upsilon_i}(s), \dots, r_{\tau_i}(s))$ of the following equation so that $w_j(s) = r_j(s) w_j$, $j \in I_i$ and

$$w_j^{a_j}(s) w_{j+1}(s) \{|w_j(s)|^{2b_j}(1-t) + t\} = (s+1) w_j^{a_j} w_{j+1} \{|w_j|^{2b_j}(1-t) + t\}$$

for $j = \upsilon_i, \dots, \tau_i - 1$. We first consider the equality

$$(E_j''): \quad r_j^{a_j} \{|w_j|^{2b_j} r_j^{2b_j}(1-t) + t\} = s_j \{|w_j|^{2b_j}(1-t) + t\}$$
$$\text{where } s_j := (s+1)/r_{j+1}, \quad \upsilon_i \le j \le \tau_i - 1.$$

First we define $r_{\tau_i} = 1$ to start with. The left side of (E_j'') is a monotone increasing function of $r_j > 0$. Thus assuming $s_j > 0$ and considering s_j as an independent variable, we can solve (E_j'') in r_j as a function of s_j. Thus we put $r_j = \psi_j(s_j)$. We claim

Assertion 3.

$$\psi_j(1) = 1, \quad \frac{d\psi_j}{ds_j}(s_j) > 0, \tag{6}$$
$$\psi_j(s_j)^{a_j} \le s_j, \ j = \upsilon_i, \dots, \tau_i - 1. \tag{7}$$

Proof. For $j = \tau_i - 1$, the assertion is obvious. Assume that $j < \tau_i - 1$. The assertion (6) is obvious. The assertion (7) follows from (6), as $\psi(s_j)$ is monotone increasing on s_j and

$$|w_j|^{2b_j} r_j^{2b_j}(1-t) + t \ge |w_j|^{2b_j}(1-t) + t, \ r_j \ge 1. \qquad \square$$

Now we define $s_j(s)$ and $r_j(s)$ inductively from $j = \tau_i$ downward (more precisely from the right end vertex of the graph to the left) as follows:

$$r_{\mu_i}(s) = 1, \quad s_j(s) = (s+1)/r_{j+1}(s), \, r_j(s) = \psi_j(s_j(s))$$

for $\upsilon_i \le j \le \tau_i - 1$.

Assertion 4. *$s_j(s) \geq 1$ and $r_j(s) \geq 1$ for $j = \upsilon_i, \ldots, \tau_i - 1$ and $s \geq 0$.*

Proof. We show the assertion by a downward induction. For $j = \tau_i - 1$, the assertion is obvious. By the inequality (7), we have for $j < \tau_i - 1$

$$\begin{aligned} s_j(s)^{a_{j+1}} &= \left(\frac{s+1}{r_{j+1}(s)}\right)^{a_{j+1}} = \frac{(s+1)^{a_{j+1}}}{\psi_{j+1}(s_{j+1}(s))^{a_j+1}} \\ &\geq \frac{(s+1)^{a_{j+1}}}{s_{j+1}(s)} = (s+1)^{a_{j+1}-1} r_{j+2}(s) \end{aligned}$$

for $s \geq 0$. By the definition of $r_j(s)$ and Assertion 3, $s_j(s) \geq 1$ and $r_j(s) \geq 1$ for $j = \upsilon_i, \ldots, \tau_i - 1$ and $s \geq 0$. □

By Assertion 3 and Assertion 4, we see easily that

$$\begin{cases} \frac{dr_{\tau_i - 1}}{ds}(0) > 0 \\ \frac{dr_j}{ds}(0) \geq 0, \ j = \upsilon_i, \ldots, \tau_i - 2. \end{cases}$$

Now we define the curve $\boldsymbol{w}_{I_i}(s)$ on $\mathbb{C}^{I_i}$ by

$$w_j(s) = r_j(s) w_j, \quad j \in I_i.$$

As a vector in $\mathbb{C}^n$, the other coefficients of $\boldsymbol{w}_{I_i}(s)$ are defined to be zero. Then by the construction we have

$$\boldsymbol{w}_{I_i}(0) = \boldsymbol{w}_{I_i}, \quad f_i(\boldsymbol{w}_{I_i}(s), \bar{\boldsymbol{w}}_{I_i}(s)) = (s+1) f_i(\boldsymbol{w}_{I_i}, \bar{\boldsymbol{w}}_{I_i}),$$
$$\frac{d\|\boldsymbol{w}_{I_i}\|^2}{ds}|_{s=0} \geq 2 \frac{dr_{\tau_i - 1}}{ds}(0)|w_{\tau_i - 1}|^2 > 0$$

where $|s| \ll 1$ and $1 \leq i \leq k$. After constructing $\boldsymbol{w}_{I_i}(s)$ for each $i = 1, \ldots, k$, we define a smooth curve $\boldsymbol{w}(s) = (w_1(s), \ldots, w_n(s))$ by the summation

$$\begin{aligned} \boldsymbol{w}(s) &= \boldsymbol{w}_{J^c}(s) + \boldsymbol{w}_J(s), \\ \boldsymbol{w}_J(s) &= \boldsymbol{w}_{I_1}(s) + \cdots + \boldsymbol{w}_{I_k}(s). \end{aligned}$$

Then $\boldsymbol{w}(s)$ satisfies

$$\begin{aligned} f_{II,t}(\boldsymbol{w}(s)) = f'(\boldsymbol{w}(s), \bar{\boldsymbol{w}}(s)) &= \sum_{i=1}^{k} f_i(\boldsymbol{w}_{I_i}(s), \bar{\boldsymbol{w}}_{I_i}(s)) \\ &= (s+1) \sum_{i=1}^{k} f_i(\boldsymbol{w}_{I_i}, \bar{\boldsymbol{w}}_{I_i}) = (s+1) f'(\boldsymbol{w}, \bar{\boldsymbol{w}}) \\ &= (s+1) f_{II,t}(\boldsymbol{w}, \bar{\boldsymbol{w}}) \equiv 0, \end{aligned}$$
$$\frac{d\|\boldsymbol{w}(s)\|^2}{ds}|_{s=0} = \sum_{i=1}^{k} \frac{d\|\boldsymbol{w}_{I_i}(s)\|^2}{ds}|_{s=0} > 0.$$

Thus defining $\boldsymbol{v} := d\boldsymbol{w}/ds(0)$, we conclude $\boldsymbol{v} \in T_{\boldsymbol{w}}V_t \setminus T_{\boldsymbol{w}}S^{2n-1}_{\varepsilon_0}$. This completes the proof of the transversality.

REMARK 1. In the above argument, if v_n is a vertex of the graph of f_i and it is not the right end vertex, we use the expression $I_i = \{j \mod n \mid v_i \le j \le \tau_i\}$ with $\tau_i > n$. This implies that

$$f_i(\boldsymbol{z}_{I_i}) = \sum_{j=v_i}^{\tau_i-1} z_j^{a_j} z_{j+1}\{|z_j|^{2b_j}(1-t)+t\}$$

where $z_{j+n} = z_j, a_{j+n} = a_j, b_{j+n} = b_j$. We do the same argument as above starting the right end variable $z_{\tau_i} = z_{\tau_i - n}$.

3.4. Applications.

COROLLARY 1. *Let V_t be the hypersurface defined by $f_{II,t}$ and let $K_{t,\varepsilon}$ be its link. Then there exists an isotopy $\psi_t : (S^{2n-1}_\varepsilon, K_{0,\varepsilon}) \to (S^{2n-1}_\varepsilon, K_{t,\varepsilon})$ for $0 \le t \le 1$ with $\psi_0 = \mathrm{id}$.*

This is immediate from Ehresmann's fibration theorem ([**9**]). As for the Milnor fibration of the second type, we have:

COROLLARY 2. *For a fixed $\varepsilon > 0$, there exists a positive real number η_0 so that $f^{-1}_{II,t}(\eta)$ and S^{2n-1}_ε intersect transversely for any η, $|\eta| \le \eta_0$ and $0 \le t \le 1$. In particular this implies that there exists a family of diffeomorphisms $\psi_t : \partial E_0(\eta_0, \varepsilon) \to \partial E_t(\eta_0, \varepsilon)$ such that the following diagram is commutative*:

$$\begin{array}{ccc} \partial E_0(\eta_0,\varepsilon) & \xrightarrow{\psi_t} & \partial E_t(\eta_0,\varepsilon) \\ \downarrow f_{II,0} & & \downarrow f_{II,t} \\ S^1_{\eta_0} & = & S^1_{\eta_0} \end{array}$$

where $\partial E_t(\eta_0, \varepsilon) = \{\boldsymbol{z} \in \mathbb{C}^n \mid |f_{II,t}(\boldsymbol{z})| = \eta_0, \|\boldsymbol{z}\| \le \varepsilon\}$.

PROOF. Fix a positive real number ε. Let

$$\partial\mathcal{E}(\eta_0,\varepsilon) := \{(\boldsymbol{z},t) \in \mathbb{C}^n \times [0,1] \mid |f_{II,t}(\boldsymbol{z})| = \eta_0, \|\boldsymbol{z}\| \le \varepsilon\}$$
$$\partial^2\mathcal{E}(\eta_0,\varepsilon) := \{(\boldsymbol{z},t) \in \mathbb{C}^n \times [0,1] \mid |f_{II,t}(\boldsymbol{z})| = \eta_0, \|\boldsymbol{z}\| = \varepsilon\}.$$

Since S^{2n-1}_ε intersects with V_t transversely and $S^{2n-1}_\varepsilon \cap V_t$ is compact for any $0 \le t \le 1$, there exists a positive real number η_0 such that $f^{-1}_{II,t}(\eta)$ and S^{2n-1}_ε intersect transversely for any $\eta, |\eta| \le \eta_0$ and $0 \le t \le 1$. Thus the projection $\pi' : (\partial\mathcal{E}(\eta_0,\varepsilon), \partial^2\mathcal{E}(\eta_0,\varepsilon)) \to [0,1]$ is a proper submersion. By the Ehresmann's fibration theorem [**9**], π' is a locally trivial fibration over $[0,1]$. So the projection π' induces a family of isomorphisms $\psi_t : \partial E_0(\eta_0,\varepsilon) \to \partial E_t(\eta_0,\varepsilon)$ of fibrations for any $\boldsymbol{z}$ with $|f_{II,t}(\boldsymbol{z})| \le \eta_0$ and $0 \le t \le 1$. □

Now, we consider again the Milnor fibration of the link complement. Consider the mapping

$$f_{II,t}/|f_{II,t}|:\ S_\varepsilon^{2n-1}\setminus K_{t,\varepsilon}\to S^1. \tag{8}$$

As $f_{II,t}(\boldsymbol{z},\bar{\boldsymbol{z}})$ is polar weighted homogeneous polynomial, the S^1-action gives non-vanishing vector field, denoted as $\partial/\partial\theta$ on $S_\varepsilon^{2n-1}\setminus K_{t,\varepsilon}$ so that $f_{II,t}(c\circ\boldsymbol{z})=c^{d_p}f_{II,t}(\boldsymbol{z})$ for $c\in S^1$, this gives a fibration structure for (8) for any $\varepsilon>0$ and we call it *a spherical Milnor fibration or a Milnor fibration of the first description.* The isomorphism class of the fibration does not depend on ε. Consider two fibrations

$$f_{II,t}:\partial E_t(\eta_0,\varepsilon)\to S^1_{\eta_0},\quad f_{II,t}/|f_{II,t}|:S_\varepsilon^{2n-1}\setminus K_{t,\varepsilon}\to S^1.$$

The first fibration is called *a Milnor fibration of the second description or a tubular Milnor fibration.* The isomorphism class of the tubular fibration does not depend on the choice of ε and $\eta_0\ll\varepsilon$. As we know that two fibrations are isomorphic for sufficiently small $\varepsilon>0$ and any t ([**3**, Theorem 36]), they are isomorphic for any ε. Combining this and Corollary 2, we can sharpen Corollary 1 as follows.

Corollary 3. *Let $\psi_t:(S_\varepsilon^{2n-1},K_{0,\varepsilon})\to(S_\varepsilon^{2n-1},K_{t,\varepsilon})$ be an isotopy in Corollary 1. ψ_t can be constructed so that the following diagram is commutative.*

$$\begin{array}{ccc} S_\varepsilon^{2n-1}\setminus K_{0,\varepsilon} & \xrightarrow{\psi_t} & S_\varepsilon^{2n-1}\setminus K_{t,\varepsilon}\\ \downarrow f_{II,0}/|f_{II,0}| & & \downarrow f_{II,t}/|f_{II,t}|\\ S^1 & \xrightarrow{id} & S^1\end{array}$$

Taking $t=1$, we get a positive answer to the conjecture in [**4**].

Proof. Choose a positive real number η_0 as in Corollary 2. Consider the cobordism variety $\mathcal{V}_\varepsilon:=\{(\boldsymbol{z},t)\in S_\varepsilon^{2n-1}\times[0,1]\,|\,f_{II,t}(\boldsymbol{z},\bar{\boldsymbol{z}})=0\}$ and its open neighborhood $\mathcal{W}_\eta:=\{(\boldsymbol{z},t)\in S_\varepsilon^{2n-1}\times[0,1]\,|\,|f_{II,t}(\boldsymbol{z})|<\eta\}$ of $\mathcal{V}_\varepsilon$. Consider the projection mapping

$$\pi:S_\varepsilon^{2n-1}\times[0,1]\to[0,1],\quad(\boldsymbol{z},t)\mapsto t.$$

Let $(\partial/\partial\theta)'$ be the projection of the gradient vector of $\Im\log f_{II,t}(\boldsymbol{z},\bar{\boldsymbol{z}})$ to the tangent space of $S_\varepsilon^{2n-1}\times[0,1]\setminus\mathcal{V}_\varepsilon$. Using the vector field $\partial/\partial\theta$ on $S_\varepsilon^{2n-1}\times[0,1]$, we see easily that $(\partial/\partial\theta)'$ is a non-vanishing vector on $S_\varepsilon^{2n-1}\times[0,1]\setminus\mathcal{V}_\varepsilon$ which is linearly independent with $\partial/\partial t$ over $\mathbb{R}$. Now we construct a vector filed $\mathcal{X}$ on $S_\varepsilon^{2n-1}\times[0,1]\setminus\mathcal{V}_\varepsilon$ such that

1. $d\pi_*(\mathcal{X}(\boldsymbol{z},t))=\partial/\partial u$ and $\{\mathcal{X}(\boldsymbol{z},t),(\partial/\partial\theta)'(\boldsymbol{z},t)\}$ are orthogonal.
2. For $(\boldsymbol{z},t)\in\mathcal{W}_{\eta_0/2}$, $\{\mathcal{X}(\boldsymbol{z},t),\ \mathrm{grad}\,|f_{II,t}|(\boldsymbol{z},t)\}$ are also orthogonal.

Here $\partial/\partial u$ is a tangent vector on $[0,1]$. The condition (1) implies the argument of $f_{II,t}$ does not change along the integral curve of $\mathcal{X}$. The conditions (1) and (2) implies the integral curve of $\mathcal{X}$ keeps the level $f_{II,t}=\eta$ for any η with $|\eta|\le\eta_0/2$. Thus integral curves of vector field $\mathcal{X}$ exists over $[0,1]$ and we construct the isotopy ψ_t using the integration curves of $\mathcal{X}$. □

Remark 2. Let $f(\boldsymbol{z},\bar{\boldsymbol{z}})=\sum_{i=1}^m c_i\boldsymbol{z}^{\nu_i}\bar{\boldsymbol{z}}^{\mu_i}$ be a full simplicial mixed polynomial and $g(\boldsymbol{z})$ be the associated Laurent polynomial of f. The last author defined a canonical diffeomorphism of $\varphi:\mathbb{C}^{*n}\to\mathbb{C}^{*n}$ as follows ([**2**]):

$$\varphi : \mathbb{C}^{*n} \to \mathbb{C}^{*n},$$
$$\boldsymbol{z} = (\rho_1 \exp(i\theta_1), \dots, \rho_n \exp(i\theta_n)) \mapsto \boldsymbol{w} = (\xi_1 \exp(i\theta_1), \dots, \xi_n \exp(i\theta_n))$$

where $(\rho_1, \dots, \rho_n)$ and $(\xi_1, \dots, \xi_n)$ satisfy

$$(N+M)\begin{pmatrix} \log \rho_1 \\ \vdots \\ \log \rho_n \end{pmatrix} = (N-M)\begin{pmatrix} \log \xi_1 \\ \vdots \\ \log \xi_n \end{pmatrix}$$

where $N = (\nu_1, \dots, \nu_n)$ and $M = (\mu_1, \dots, \mu_n)$. Then φ satisfies that $\varphi(\mathbb{C}^{*n} \cap f^{-1}(c)) = \mathbb{C}^{*n} \cap g^{-1}(c)$ for any $c \in \mathbb{C}$ ([**2**, Theorem 10]). However φ cannot be extended to a homeomorphism of $\mathbb{C}^n \setminus \{O\}$ to itself in general, except the case of mixed Brieskorn polynomial.

Example 1. We will give an example of the above remark. Let $f(\boldsymbol{z}, \bar{\boldsymbol{z}})$ be a simplicial polynomial defined by

$$f(\boldsymbol{z}, \bar{\boldsymbol{z}}) = z_1^3 \bar{z}_1 z_2 + z_2^3 \bar{z}_2 z_3 + z_3^3 \bar{z}_3 z_1.$$

Then the diffeomorphism of $\varphi : \mathbb{C}^{*3} \to \mathbb{C}^{*3}, \boldsymbol{z} = (z_1, z_2, z_3) \mapsto \boldsymbol{w} = (w_1, w_2, w_3)$ is given by

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} |z_1|^{17/9}|z_2|^{-4/9}|z_3|^{2/9} \exp(i\theta_1) \\ |z_1|^{2/9}|z_2|^{17/9}|z_3|^{-4/9} \exp(i\theta_2) \\ |z_1|^{-4/9}|z_2|^{2/9}|z_3|^{17/9} \exp(i\theta_3) \end{pmatrix}.$$

The above map cannot extend to a continuous map on the coordinate planes $\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 z_3 = 0\}$ as the negative exponents in the above description. So the map φ cannot extend to a homeomorphism of $\mathbb{C}^3 \setminus \{O\}$ to itself.

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