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Fractional integral operators with homogeneous kernels on Morrey spaces with variable exponents

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Abstract. We establish the mapping properties of the fractional integral operators with homogeneous kernels on Morrey spaces with variable exponents.

1. Introduction.

The main theme of this paper is the mapping properties of the fractional integral operators with homogeneous kernels on Morrey spaces with variable exponents.

The fractional integral operators with homogeneous kernels are introduced by Muckenhoupt and Wheeden in [**35**]. We recall the definition of fractional integral operator with homogeneous kernel from [**35**]. Let $0 < \alpha < n$ and Ω be a homogeneous function on $\mathbb{R}^n$ with degree zero. That is, for any $x \in \mathbb{R}^n$ and $\lambda > 0$

$$\Omega(\lambda x) = \Omega(x).$$

The fractional integral operator with homogeneous kernel is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)dy.$$

For the mapping properties of $T_{\Omega,\alpha}$ on Lebesgue spaces, the reader is referred to [**31**, Theorem 3.3.1]. These mapping properties have been extended to the weighted Lebesgue spaces in [**14**].

In view of the definition of $T_{\Omega,\alpha}$, we see that $T_{\Omega,\alpha}$ is a generalization of the fractional integral operators (Riesz potentials). For some further generalizations of the fractional integral operators, such as the generalized fractional integral operators, the reader is referred to [**38**], [**42**], [**43**].

The mapping properties of the fractional integral operators had been extended to a number of function spaces, see [**39**]. The celebrated Adams inequalities, which is the mapping properties of the fractional integral operators on Morrey spaces, are given in [**1**]. The boundedness of the fractional integral operators on generalized Morrey spaces and Orlicz–Morrey spaces are given in [**37**], [**40**], [**45**].

In addition, the mapping properties of the fractional integral operators on Lebesgue spaces with variable exponents are established in [**2**], [**5**], [**8**], [**10**], [**15**], [**17**], [**28**], [**41**].

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These mapping properties have been further extended to Morrey spaces with variable exponents in [**3**], [**16**], [**17**], [**18**], [**19**], [**22**], [**26**], [**27**], [**32**], [**33**], [**34**].

Therefore, the above mentioned results give us the motivation to study the mapping properties of $T_{\Omega,\alpha}$ on Morrey spaces with variable exponents. Our main results consist of two theorems, Theorems 3.1 and 3.2.

Even though our main results are the mapping properties of $T_{\Omega,\alpha}$ on Morrey spaces with variable exponents. Some particular cases of the main results have theirs own independent interests such as the mapping properties of $T_{\Omega,\alpha}$ on the classical Morrey spaces and the Lebesgue spaces with variable exponents.

To establish our main results, several important notions and techniques from harmonic analysis, such as the weighted norm inequalities, the extrapolation theory and the block spaces, are involved.

To establish Theorem 3.1, we first need to have the mapping properties of $T_{\Omega,\alpha}$ on Lebesgue spaces with variable exponents. We obtain these results by using the weighted norm inequalities of $T_{\Omega,\alpha}$ on Lebesgue spaces [**14**]. Then, we apply the "off-diagonal" extrapolation [**20**] to these inequalities.

With the mapping properties on Lebesgue spaces with variable exponent, we have to use the idea of the lifting principle from [**21**] to obtain the mapping properties for the Morrey spaces with variable exponents.

To establish Theorem 3.2, we use the duality between the Morrey spaces with variable exponents and the block spaces with variable exponents.

This paper is organized as follows. The definition of Morrey space with variable exponent and some of its properties are given in Section 2. The main results are presented in Section 3. To obtain the proofs of our main theorems, we recall some supporting results in Section 4. The proofs of our main theorems are presented in Section 5.

2. Definitions.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the class of Lebesgue measurable functions on $\mathbb{R}^n$. For any Lebesgue measurable set E, the characteristic function of E is denoted by χ_E. For any $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x_0,r) : x_0 \in \mathbb{R}^n, r > 0\}$.

We recall the definition of the Lebesgue space with variable exponents from [**9**], [**12**]. For any Lebesgue measurable function $p : \mathbb{R}^n \to [1,\infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\} < \infty$$

where

$$\rho_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$. The reader is referred to [**9**], [**12**] for some basic properties of $L^{p(\cdot)}$. Particularly, $L^{p(\cdot)}$ is a Banach function space, see [**12**, Theorem 3.2.13].

We find that the associated space of $L^{p(\cdot)}$ is given by $L^{p'(\cdot)}$ where $(1/p(x)) + (1/p'(x)) = 1$ [**12**, Theorem 3.2.13]. The reader is referred to [**12**, Definition 2.7.1] for the definition of associate space.

Write

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\} \quad \text{and} \quad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}.$$

Throughout this paper, we assume that $p_- > 1$ and $p_+ < \infty$.

Let $0 \leq \alpha < n$. The fractional maximal operator M_α is given by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)| dy$$

where the supremum is taking over all balls $B \in \mathbb{B}$ containing x.

Obviously, when $\alpha = 0$, the fractional maximal operator is the Hardy–Littlewood maximal operator M.

Definition 2.1. For any exponent function $p(\cdot)$, we write $p(\cdot) \in \mathbb{M}$ if the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}$.

An important class of exponent function $p(\cdot)$ for which $p(\cdot) \in \mathbb{M}$ is the class of log-Hölder continuous functions. For the definition and details of this class, the reader is referred to [**9**, Chapter 3] and [**12**, Chapter 4].

By using Jensen's inequality, for any $\theta \geq 1$, we have $(Mf)^\theta \leq M(|f|^\theta)$. Therefore, whenever $p(\cdot) \in \mathbb{M}$, we find that

$$\|Mf\|^\theta_{L^{\theta p(\cdot)}} = \|(Mf)^\theta\|_{L^{p(\cdot)}} \leq \|M(|f|^\theta)\|_{L^{p(\cdot)}} \leq C\||f|^\theta\|_{L^{p(\cdot)}} = C\|f\|^\theta_{L^{\theta p(\cdot)}},$$

hence, $\theta p(\cdot) \in \mathbb{M}$.

We have the corresponding class of function spaces for fractional maximal operators [**25**, Definition 2.3].

Definition 2.2. Let $0 < \alpha < n$. A pair of exponent functions $(p(\cdot), q(\cdot))$ is said to be an α-Riesz pair if the fractional maximal operator $M_\alpha : L^{p(\cdot)} \to L^{q(\cdot)}$ is bounded.

Notice that α-Riesz pairs is defined for general Banach function spaces in [**25**, Definition 2.3].

In view of [**25**, Proposition 2.1], we have the following results for α-Riesz pairs.

Proposition 2.1. *Let $0 < \alpha < n$. If $(p(\cdot), q(\cdot))$ is an α-Riesz pair, then there exists a constant $C > 0$ such that for any $B \in \mathbb{B}$.*

$$\|\chi_B\|_{L^{p'(\cdot)}} \|\chi_B\|_{L^{q(\cdot)}} \leq C|B|^{1-\alpha/n}.$$

Notice that in the above result, we do not assume that either $p(\cdot) \in \mathbb{M}$ or $q(\cdot) \in \mathbb{M}$. The above result are one of the crucial supporting result to establish the vector-valued operators with singular kernels in [**25**].

The following example of α-Riesz pair is a straight forward consequence of the boundedness of fractional maximal operators on Lebesgue spaces with variable exponents [**8**, Corollary 2.12].

Lemma 2.2. *Let $0 < \alpha < n$, $p(\cdot), q(\cdot) : \mathbb{R}^n \to [1, \infty)$ with $p_+ < n/\alpha$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \mathbb{R}^n.$$

If there exists q_0 satisfying $n/(n-\alpha) < q_0$ and $q(\cdot)/q_0 \in \mathbb{M}$, then $(L^{p(\cdot)}, L^{q(\cdot)})$ is an α-Riesz pair.

Next, we have the definition of Morrey spaces with variable exponents.

Definition 2.3. Let $p : \mathbb{R}^n \to [1, \infty)$ and $u(x,r) : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be Lebesgue measurable functions. The Morrey space with variable exponent $M^u_{p(\cdot)}$ is the collection of all Lebesgue measurable functions f satisfying

$$\|f\|_{M^u_{p(\cdot)}} = \sup_{z \in \mathbb{R}^n, R > 0} \frac{1}{u(z,R)} \|\chi_{B(z,R)} f\|_{L^{p(\cdot)}} < \infty.$$

In the rest of the paper, we consider those Morrey spaces with variable exponents with the function u belonging to the following classes.

Definition 2.4. Let $q(\cdot) : \mathbb{R}^n \to [1, \infty)$ and $u(x,r) : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be Lebesgue measurable functions, we write $u \in \mathbb{W}_{q(\cdot)}$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}}} u(x, 2^{j+1}r) \le C u(x,r). \tag{2.1}$$

For any $1 \le \theta < \infty$, we write $u \in \mathcal{W}^\theta_{q(\cdot)}$ if

$$\sum_{k=0}^{\infty} \frac{\|\chi_{B(x,2^k r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}}} \frac{|B(x,r)|^\theta}{|B(x,2^k r)|^\theta} u(x, 2^k r) \le C u(x,r) \tag{2.2}$$

for some $C > 0$ independent of $x \in \mathbb{R}^n$ and $r > 0$.

The class $\mathbb{W}_{q(\cdot)}$ has been used in [**21**], [**22**], [**24**], [**25**] for the studies of the fractional integral operators, the vector-valued maximal inequalities and the vector-valued singular integral operators on Morrey spaces with variable exponents. For some examples of u such that $u \in \mathbb{W}_{q(\cdot)}$, the reader is referred to [**22**, pp. 366–368]. Additionally, the discussions given there also apply to the class $\mathcal{W}^\theta_{q(\cdot)}$.

3. Main results.

We now ready to present our main results on the boundedness of the fractional integral operators with homogeneous kernels on $M^u_{p(\cdot)}$. It consists of two theorems. We present them separately so that the conditions involved in each theorem can be clearly stated.

THEOREM 3.1. *Let $0 < \alpha < n$, $1 < s < \infty$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Suppose that $u \in \mathbb{W}_{q(\cdot)}$, $s' < p_- \leq p_+ < n/\alpha$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

If there exists q_0 with $ns'/(n - \alpha s') < q_0$ such that $q(\cdot)/q_0 \in \mathbb{M}$, then there exists a constant $C > 0$ such that for any $f \in M^u_{p(\cdot)}$,

$$\|T_{\Omega,\alpha} f\|_{M^u_{q(\cdot)}} \leq C\|f\|_{M^u_{p(\cdot)}}.$$

The following theorem is obtained by using duality through block spaces with variable exponents. The definition of block spaces with variable exponents is given in Section 4.

THEOREM 3.2. *Let $0 < \alpha < n$, $1 < s < \infty$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Suppose that $u \in \mathcal{W}^{(1/s')-(\alpha/n)}_{p'(\cdot)}$, $s' < (q')_- \leq (q')_+ < n/\alpha$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

If there exists p_0 with $ns'/(n - \alpha s') < p_0$ such that $p'(\cdot)/p_0 \in \mathbb{M}$, then there exists a constant $C > 0$ such that for any $f \in M^u_{p(\cdot)}$,

$$\|T_{\Omega,\alpha} f\|_{M^u_{q(\cdot)}} \leq C\|f\|_{M^u_{p(\cdot)}}.$$

Recall that in the Introduction, we proclaim that we use the weighted norm inequalities for $T_{\Omega,\alpha}$ to obtain our main results. Notice that there are two sets of condition so that the weighted norm inequalities for $T_{\Omega,\alpha}$ holds, see Theorem 4.3. Roughly speaking, the second one follows from the first one via duality. This is the main reason why we also have two theorems for our main results.

4. Weighted norm inequalities, extrapolation and block spaces.

In this section, we present some supporting materials for our main results, namely, the weighted norm inequalities for fractional integral operators, the "off-diagonal" extrapolation and the block spaces with variable exponents.

4.1. Weighted norm inequalities for $T_{\Omega,\alpha}$.

We first state the definition of the class $A(p, q)$ [**36**] which plays the same role of the Muckenhoupt A_p class for the study of fractional integral operators.

DEFINITION 4.1. Let $1 < p, q < \infty$. For any nonnegative locally integrable function ω, we write $\omega \in A(p, q)$ if there exists a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{1/q} \left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'} dx\right)^{1/p'} < C$$

where the supremum is taken over those cube Q in $\mathbb{R}^n$. When $1 \le q < \infty$, we write $\omega \in A(1,q)$, if there exists a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)^q dx\right)^{1/q} \left(\mathrm{esssup}_Q \frac{1}{\omega(x)}\right) < C.$$

Let A_p denote the class of Muckenhoupt weights. Then,

$$\omega \in A(p,p) \Leftrightarrow \omega^p \in A_p.$$

According to the definition of $A(p,q)$ for $p > 1$, we have

$$\omega \in A(p,q) \Leftrightarrow \omega^{-1} \in A(q',p'). \tag{4.1}$$

In addition, we have the following properties for the class $A(p,q)$.

PROPOSITION 4.1. *Let $1 < p < q < \infty$. We have*

$$\omega \in A(p,q) \Leftrightarrow \omega^q \in A_{1+(q/p')} \Leftrightarrow \omega^{-p'} \in A_{1+(p'/q)}.$$

The above proposition follows from Definition 4.1 and [**31**, Theorem 3.2.2] with $\alpha = (n/p) - (n/q)$. The following lemma is a supporting result for the boundedness of $T_{\Omega,\alpha}$ on $L^{p(\cdot)}$.

LEMMA 4.2. *Let $1 < p < q < \infty$ and $1 < s < \infty$. If $\omega \in A_1$, then $\omega^{s'/q} \in A(p/s', q/s')$.*

PROOF. We have $\omega \in A_1 \subset A_{1+((q/s')/(p/s')')}$. Therefore, Proposition 4.1 assures that $\omega^{s'/q} \in A(p/s', q/s')$. □

We state the weighted norm inequalities for $T_{\Omega,\alpha}$ from [**14**], [**31**].

THEOREM 4.3. *Let $0 < \alpha < n$, $1 < s < \infty$, $1/q = (1/p) - (\alpha/n)$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. If*

1. *$1 \le s' < p < n/\alpha$ and $\omega^{s'/q} \in A(p/s', q/s')$, or*
2. *$1 < p < n/\alpha$, $s > q$ and $\omega^{-s'/q} \in A(q'/s', p'/s')$.*

Then, there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)|^q \omega(x) dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)^{p/q} dx\right)^{1/p}. \tag{4.2}$$

We slightly rewrite the presentation of the weighted norm inequalities for $T_{\Omega,\alpha}$ from [**31**, Theorem 3.4.2] because the above presentation is adapted to the formulation of the extrapolation theory given in the following.

4.2. Extrapolation.

We state the "off-diagonal" extrapolation results for Lebesgue spaces with variable exponents from [**8**, Theorem 1.8].

Theorem 4.4. *Given a family $\mathcal{F}$, assume that for some p_0 and q_0, $0 < p_0 \le q_0 < \infty$ and every weight $\omega \in A_1$,*

$$\left(\int_{\mathbb{R}^n} f(x)^{q_0}\omega(x)dx\right)^{1/q_0} \le C_0\left(\int_{\mathbb{R}^n} g(x)^{p_0}\omega(x)^{p_0/q_0}dx\right)^{1/p_0}, \quad (f,g)\in\mathcal{F}. \tag{4.3}$$

Assume that $p(\cdot)$ satisfies $p_0 < p_- \le p_+ < p_0q_0/(q_0-p_0)$. Define $q(x)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad \forall x \in \mathbb{R}^n.$$

If $(q(x)/q_0)' \in \mathbb{M}$, then for all $(f,g)\in\mathcal{F}$ such that $f\in L^{q(\cdot)}$, we have

$$\|f\|_{L^{q(\cdot)}} \le C\|g\|_{L^{p(\cdot)}}.$$

Theorem 4.4 had been used in [**8**] to establish the mapping properties of the fractional integral operators on Lebesgue spaces with variable exponents [**8**, Corollary 2.12]. The reader is also referred to [**2**], [**10**], [**15**], [**17**], [**28**], [**41**] for some related results.

4.3. Block spaces with variable exponents.

The block spaces with variable exponents are introduced in [**6**]. For the studies of the vector-valued operators with singular kernels on block spaces with variable exponents, the reader is referred to [**25**]. In this paper, we use the mapping properties of the fractional integral operators on block spaces with variable exponents to establish Theorem 3.2.

We recall the definition of block spaces with variable exponents from [**6**, Definition 2.2].

Definition 4.2. Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ and $u:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions. A $b\in\mathcal{M}(\mathbb{R}^n)$ is an $(u,p(\cdot))$-block if it is supported in a ball $B(x_0,r)$, $x_0\in\mathbb{R}^n$, $r>0$, and

$$\|b\|_{L^{p(\cdot)}} \le \frac{1}{u(x_0,r)}. \tag{4.4}$$

We write $b\in b_{u,p(\cdot)}$ if b is an $(u,p(\cdot))$-block.

Define $\mathfrak{B}_{u,p(\cdot)}$ by

$$\mathfrak{B}_{u,p(\cdot)} = \left\{\sum_{k=1}^{\infty}\lambda_k b_k : \sum_{k=1}^{\infty}|\lambda_k| < \infty \text{ and } b_k \text{ is an } (u,p(\cdot))\text{-block}\right\}. \tag{4.5}$$

The space $\mathfrak{B}_{u,p(\cdot)}$ is endowed with the norm

$$\|f\|_{\mathfrak{B}_{u,p(\cdot)}} = \inf\left\{\sum_{k=1}^{\infty}|\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty}\lambda_k b_k\right\}. \tag{4.6}$$

We call $\mathfrak{B}_{u,p(\cdot)}$ the block space with variable exponent.

The family of block spaces for the Lebesgue spaces is introduced and studied in [**4**], [**29**], [**44**].

The reader is referred to [**6**] for some basic properties for $\mathfrak{B}_{u,p(\cdot)}$. In particular, the boundedness of the Hardy–Littlewood maximal operator is obtained in [**6**, Theorem 3.1]. Furthermore, the mapping properties for the vector-valued singular integral operators and the fractional integral operators on $\mathfrak{B}_{u,p(\cdot)}$ are established in [**25**].

We establish several duality results for $M^u_{p(\cdot)}$ and $\mathfrak{B}_{u,p(\cdot)}$ in the followings. The first one is the norm conjugate formula.

Proposition 4.5. *Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ and $u:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions. We have constants $C,D>0$ such that for any $f\in M^u_{p(\cdot)}$,*

$$C\|f\|_{M^u_{p(\cdot)}}\le\sup_{b\in b_{u,p'(\cdot)}}\left|\int_{\mathbb{R}^n}f(x)b(x)dx\right|\le D\|f\|_{M^u_{p(\cdot)}}. \tag{4.7}$$

Proof. Let b be an $(u,p'(\cdot))$-block with $\operatorname{supp} b\subset B(x_0,r)$. By using the Hölder inequality for $L^{p(\cdot)}$ [**12**, Lemma 3.2.20] and the definition of $M^u_{p(\cdot)}$, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n}f(x)b(x)dx\right|&\le D\|\chi_B f\|_{L^{p(\cdot)}}\|b\|_{L^{p'(\cdot)}}\\&\le Du(x_0,r)\|f\|_{M^u_{p(\cdot)}}\frac{1}{u(x_0,r)}\le D\|f\|_{M^u_{p(\cdot)}}.\end{aligned} \tag{4.8}$$

Therefore, the inequality on the right hand side of (4.7) follows.

Next, we establish the inequality on the left hand side of (4.7). According to the definition of $M^u_{p(\cdot)}$, there exists a $B(y,t)\in\mathbb{B}$ such that

$$\frac{1}{2}\|f\|_{M^u_{p(\cdot)}}\le\frac{1}{u(y,t)}\|\chi_{B(y,t)}f\|_{L^{p(\cdot)}}.$$

As $(L^{p(\cdot)})'=L^{p'(\cdot)}$ [**12**, Theorem 3.2.13], there exists a $g\in L^{p'(\cdot)}$ with $\|g\|_{L^{p'(\cdot)}}\le 1$ such that

$$\begin{aligned}\frac{1}{2}\|f\|_{M^u_{p(\cdot)}}&\le\frac{1}{u(y,t)}\|\chi_{B(y,t)}f\|_{L^{p(\cdot)}}\\&\le\frac{2}{u(y,t)}\left|\int_{\mathbb{R}^n}\chi_{B(y,t)}(x)f(x)g(x)dx\right|=2\left|\int_{\mathbb{R}^n}f(x)G(x)dx\right|\end{aligned}$$

where

$$G(x)=\frac{1}{u(y,t)}\chi_{B(y,t)}(x)g(x)\in b_{u,p'(\cdot)}.$$

Therefore, we establish (4.7). □

The above proposition extends the norm conjugate formula for $L^{p(\cdot)}$ [**12**, Corollary

3.2.14] to $M^u_{p(\cdot)}$. Next, we give a characterization of $M^u_{p(\cdot)}$ via blocks.

Proposition 4.6. *Let f be a Lebesgue measurable function. If*

$$\sup_{b\in b_{u,p'(\cdot)}} \left| \int_{\mathbb{R}^n} f(x)b(x)dx \right| < \infty,$$

then $f \in M^u_{p(\cdot)}$.

Proof. Let $g \in L^{p'(\cdot)}$ with $\|g\|_{L^{p'(\cdot)}} \le 1$. We find that for any $B(y,r) \in \mathbb{B}$,

$$G = \frac{1}{u(y,r)\|g\|_{L^{p'(\cdot)}}} \chi_{B(y,r)} g$$

is an $(u, p'(\cdot))$-block. Consequently,

$$\sup_{g\in L^{p'(\cdot)}, \|g\|_{L^{p'(\cdot)}}\le 1, B(y,r)\in\mathbb{B}} \frac{1}{u(y,r)} \left| \int_{\mathbb{R}^n} \chi_{B(y,r)}(x) f(x) g(x) dx \right| \tag{4.9}$$

$$\le \sup_{b\in b_{u,p'(\cdot)}} \left| \int_{\mathbb{R}^n} f(x)b(x)dx \right| < \infty. \tag{4.10}$$

The norm conjugate formula $L^{p(\cdot)}$ and $L^{p'(\cdot)}$ [**12**, Corollary 3.2.14] assures that

$$\sup_{g\in L^{p'(\cdot)}, \|g\|_{L^{p'(\cdot)}}\le 1} \left| \int_{\mathbb{R}^n} \chi_{B(y,r)}(x) f(x) g(x) dx \right| \ge \frac{1}{2} \|\chi_{B(y,r)} f\|_{L^{p(\cdot)}}. \tag{4.11}$$

Therefore, (4.9) and (4.11) yield

$$\sup_{B(y,r)\in\mathbb{B}} \frac{1}{u(y,r)} \|\chi_{B(y,r)} f\|_{L^{p(\cdot)}} < \infty.$$

That is, $f \in M^u_{p(\cdot)}$. □

We also have the Hölder inequality for $M^u_{p(\cdot)}$ and $\mathfrak{B}_{u,p'(\cdot)}$.

Proposition 4.7. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be Lebesgue measurable functions. We have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le C \|f\|_{M^u_{p(\cdot)}} \|g\|_{\mathfrak{B}_{u,p'(\cdot)}}. \tag{4.12}$$

Proof. Let $b \in \mathfrak{B}_{u,p'(\cdot)}$. For any $\epsilon > 0$, there exist a family of $(u, p'(\cdot))$-block $\{b_j\}_{j\in\mathbb{N}}$ and a sequence of scalars $\{\lambda_j\}_{j\in\mathbb{N}}$ such that $g = \sum_{j\in\mathbb{N}} \lambda_j b_j$ and

$$\sum_{j\in\mathbb{N}} |\lambda_j| \le (1+\epsilon)\|b\|_{\mathfrak{B}_{u,p'(\cdot)}}.$$

Thus, for any $f \in M^u_{p(\cdot)}$, (4.8) guarantees that

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \sum_{j\in\mathbb{N}} |\lambda_j| \int_{\mathbb{R}^n} |f(x)b_j(x)|dx \le C\sum_{j\in\mathbb{N}} |\lambda_j| \|f\|_{M^u_{p(\cdot)}}$$
$$\le C(1+\epsilon)\|f\|_{M^u_{p(\cdot)}} \|g\|_{\mathfrak{B}_{u,p'(\cdot)}}.$$

Since $\epsilon > 0$ is arbitrary, we obtain (4.12). □

5. Proofs of the main results.

In this section, we give the proofs of Theorems 3.1 and 3.2. We begin with the boundedness of $T_{\Omega,\alpha}$ on $L^{p(\cdot)}$. This is a supporting result for Theorems 3.1 and 3.2. On the other hand, it has its own independent interest.

Proposition 5.1. *Let $0 < \alpha < n$, $1 < s < \infty$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Suppose that $p(\cdot), q(\cdot) : \mathbb{R}^n \to [1,\infty)$ satisfy $s' < p_- \le p_+ < n/\alpha$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

If there exists q_0 with $ns'/(n-\alpha s') < q_0$ such that $q(\cdot)/q_0 \in \mathbb{M}$, then there exists a constant $C > 0$ such that for any $f \in L^{p(\cdot)}$,

$$\|T_{\Omega,\alpha} f\|_{L^{q(\cdot)}} \le C\|f\|_{L^{p(\cdot)}}. \tag{5.1}$$

Proof. As $s' < p_-$, we have

$$\frac{ns'}{n-\alpha s'} < \frac{np_-}{n-\alpha p_-}. \tag{5.2}$$

For the given q_0, in view of (5.2), we can assume that

$$q_0 < \frac{np_-}{n-\alpha p_-} \tag{5.3}$$

because $q(\cdot)/a \in \mathbb{M} \Rightarrow q(\cdot)/b \in \mathbb{M}$ provided that $b < a$.

Define p_0 by

$$\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}.$$

Since $ns'/(n-\alpha s') < q_0 < \infty$, we find that $\alpha/n < (1/q_0) + (\alpha/n) = 1/p_0 < 1/s'$. That is,

$$s' < p_0 < \frac{n}{\alpha}. \tag{5.4}$$

Moreover, the condition $p_+ < n/\alpha$ and (5.3) give

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n} < \frac{1}{p_+}, \quad \text{and} \quad p_0 < p_-, \tag{5.5}$$

respectively. Therefore, we have $p_0 < p_- \le p_+ < p_0 q_0/(q_0 - p_0)$.

In view of Theorem 4.3, for any ω with $\omega^{s'/q_0} \in A(p_0/s', q_0/s')$, we have

$$\left(\int_{\mathbb{R}^n} |T_{\Omega,\alpha} f(x)|^{q_0} \omega(x) dx\right)^{1/q_0} \le C \left(\int_{\mathbb{R}^n} |f(x)|^{p_0} \omega(x)^{p_0/q_0} dx\right)^{1/p_0}.$$

Lemma 4.2 guarantees that

$$\omega \in A_1 \Rightarrow \omega^{s'/q_0} \in A\left(\frac{p_0}{s'}, \frac{q_0}{s'}\right).$$

Therefore, we are allowed to apply Theorem 4.4 to $L^{p(\cdot)}$, $L^{q(\cdot)}$ and $T_{\Omega,\alpha}$ with respect to the set $\mathcal{F} = \left\{|T_{\Omega,\alpha} f|, |f| : f \in L^\infty_{comp}\right\}$ where L^∞_{comp} is the set of bounded function with compact support. Consequently, we have

$$\|T_{\Omega,\alpha} f\|_{L^{q(\cdot)}} \le C \|f\|_{L^{p(\cdot)}}, \quad \forall f \in L^\infty_{comp}.$$

As $p_+ < n/\alpha < \infty$, [**12**, Theorem 3.2.7] and [**12**, Theorem 3.4.12] assure that $L^{p(\cdot)}$ is a Banach space and L^∞_{comp} is dense in $L^{p(\cdot)}$. Therefore, $T_{\Omega,\alpha} f$ can be defined for all $f \in L^{p(\cdot)}$ via the density argument and, moreover, we establish (5.1). □

The subsequent result is an extension of Lemma 2.2.

LEMMA 5.2. *Let $0 < \alpha < n$, $1 < s < \infty$ and $\Omega \in L^s(\mathbb{S}^{n-1})$. Suppose that $p_+ < n/\alpha$ and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

If there exists q_0 with $ns'/(n - \alpha s') < q_0$ such that $q(\cdot)/q_0 \in \mathbb{M}$, then $(p(\cdot)/s', q(\cdot)/s')$ is a $(s'\alpha)$-Riesz pair.

PROOF. For any $1 < s < \infty$, we have

$$\frac{1}{p(x)/s'} - \frac{1}{q(x)/s'} = \frac{s'}{p(x)} - \frac{s'}{q(x)} = \frac{s'\alpha}{n}, \quad x \in \mathbb{R}^n.$$

Next, we have $(p(\cdot)/s')_+ \le p_+ < n/\alpha$.

Write $r_0 = q_0/s'$, we find that $(q(\cdot)/s')/r_0 = q(\cdot)/q_0 \in \mathbb{M}$. Furthermore, the inequality $ns'/(n - \alpha s') < q_0$ assures that

$$r_0 > \frac{n}{n - \alpha s'} > \frac{n}{n - \alpha}$$

because $s' > 1$. Therefore, Lemma 2.2 concludes that $(p(\cdot)/s', q(\cdot)/s')$ is a $(s'\alpha)$-Riesz pair. □

With the above preparation, we are now ready to present the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. Let $f \in M^u_{p(\cdot)}$. For any $z \in \mathbb{R}^n$ and $r > 0$, write $f(x) = f_0(x) + \sum_{j=1}^\infty f_j(x)$, where $f_0 = \chi_{B(z,2r)} f$ and $f_j = \chi_{B(z,2^{j+1}r) \setminus B(z,2^j r)} f$, $j \in \mathbb{N} \setminus \{0\}$.

According to Proposition 5.1, $T_{\Omega,\alpha}: L^{p(\cdot)} \to L^{q(\cdot)}$, we have $\|T_{\Omega,\alpha} f_0\|_{L^{q(\cdot)}} \le C\|f_0\|_{L^{p(\cdot)}}$.

We obtain

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}(T_{\Omega,\alpha} f_0)\|_{L^{q(\cdot)}} \le C\frac{1}{u(z,2r)}\|\chi_{B(z,2r)} f\|_{L^{p(\cdot)}}.$$

For any $x \in \mathbb{R}^n$ and $r > 0$, we have $\chi_{B(x,2r)} \le CM\chi_{B(x,r)}$ for some $C > 0$. Consequently, $q(\cdot)/q_0 \in \mathbb{M}$ ensures that

$$\|\chi_{B(x,2r)}\|_{L^{q(\cdot)}} = \|\chi_{B(x,2r)}\|_{L^{q(\cdot)/q_0}}^{1/q_0} \le C\|M\chi_{B(x,r)}\|_{L^{q(\cdot)/q_0}}^{1/q_0} \le C\|\chi_{B(x,r)}\|_{L^{q(\cdot)}}.$$

Hence, (2.1) implies that

$$u(z,2r) < Cu(z,r) \tag{5.6}$$

for some constant $C > 0$ independent of $z \in \mathbb{R}^n$ and $r > 0$. Thus, we have

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}(Tf_0)\|_{L^{q(\cdot)}} \le C \sup_{\substack{y\in\mathbb{R}^n \\ r>0}} \frac{1}{u(y,r)}\|\chi_{B(y,r)} f\|_{L^{p(\cdot)}}. \tag{5.7}$$

Furthermore, there is a constant $C > 0$ such that, for any $j \ge 1$

$$\begin{aligned}&\chi_{B(z,r)}(x)|(T_{\Omega,\alpha} f_j)(x)| \\ &\quad \le C\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)\backslash B(z,2^j r)} |\Omega(x-y)||x-y|^{-n+\alpha}|f(y)|dy. \end{aligned}\tag{5.8}$$

The Hölder inequality assures that

$$\begin{aligned}&\int_{B(z,2^{j+1}r)\backslash B(z,2^j r)} |\Omega(x-y)||x-y|^{-n+\alpha}|f(y)|dy \\ &\le C\left(\int_{B(z,2^{j+1}r)\backslash B(z,2^j r)} |\Omega(x-y)|^s|x-y|^{-s(n-\alpha)}dy\right)^{1/s}\|\chi_{B(z,2^{j+1}r)} f\|_{L^{s'}} \\ &= C\left(\int_{B(x-z,2^{j+1}r)\backslash B(x-z,2^j r)} |\Omega(y)|^s|y|^{-s(n-\alpha)}dy\right)^{1/s}\|\chi_{B(z,2^{j+1}r)} f\|_{L^{s'}}.\end{aligned}$$

As $x \in B(z,r)$, for any $y \in B(x-z,2^{j+1}r)\backslash B(x-z,2^j r)$, we have

$$|y| \le |y-(x-z)| + |x-z| \le 2^{j+1}r + r \le 2^{j+2}r$$

and

$$|y| \ge |y-(x-z)| - |x-z| \ge 2^j r - r \ge 2^{j-1}r.$$

That is,

$$B(x-z,2^{j+1}r)\backslash B(x-z,2^j r) \subseteq B(0,2^{j+2}r)\backslash B(0,2^{j-1}r).$$

Hence, for any $j \geq 1$, we obtain

$$\begin{aligned}
&\int_{B(z,2^{j+1}r)\setminus B(z,2^j r)} |\Omega(x-y)||x-y|^{-n+\alpha}|f(y)|dy \\
&\leq C\left(\int_{B(0,2^{j+2}r)\setminus B(0,2^{j-1}r)} |\Omega(y)|^s|y|^{-s(n-\alpha)}dy\right)^{1/s} \|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}} \\
&= C\left(\int_{2^{j-1}r}^{2^{j+2}r}\int_{\mathbb{S}^{n-1}} |\Omega(\theta)|^s t^{-s(n-\alpha)+n-1}d\theta dt\right)^{1/s} \|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}}.
\end{aligned}$$

Since $\Omega \in L^s(\mathbb{S}^{n-1})$, we obtain

$$\begin{aligned}
&\int_{B(z,2^{j+2}r)\setminus B(z,2^{j-1}r)} |\Omega(x-y)||x-y|^{-n}|f(y)|dy \\
&\leq C_0 2^{-(n-\alpha)(j-1)+n(j-1)/s} r^{-(n-\alpha)+n/s}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}} \\
&\leq C_1 2^{-(n-\alpha)(j+1)+n(j-1)/s} r^{-(n-\alpha)+n/s}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}}
\end{aligned}$$

for some $C_0, C_1 > 0$.

Thus, (5.8) becomes

$$\chi_{B(z,r)}(x)|(T_{\Omega,\alpha}f_j)(x)| \leq C\chi_{B(z,r)}(x)\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}}}{|B(z,2^{j+1}r)|^{(1/s')-(\alpha/n)}}. \tag{5.9}$$

In view of (5.4) and (5.5), we have $s' < p_0 < p_-$. The condition $ns'/(n-\alpha s') < q_0$ gives $s' < q_0$. Define $r(x) = p(x)/s'$ and $t(x) = q(x)/s'$. We have $r(\cdot), t(\cdot) : \mathbb{R}^n \to (1,\infty)$.

The Hölder inequality for $L^{r(\cdot)}$ yields

$$\begin{aligned}
\|\chi_{B(z,2^{j+1}r)}f\|_{L^{s'}} &\leq C\|\chi_{B(z,2^{j+1}r)}|f|^{s'}\|_{L^{r(\cdot)}}^{1/s'}\|\chi_{B(z,2^{j+1}r)}\|_{L^{r'(\cdot)}}^{1/s'} \\
&= C\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}}\|\chi_{B(z,2^{j+1}r)}\|_{L^{r'(\cdot)}}^{1/s'}.
\end{aligned} \tag{5.10}$$

According to Lemma 5.2, $(r(\cdot), t(\cdot))$ is a $(s'\alpha)$-Riesz pair, Proposition 2.1 guarantees that

$$\|\chi_B\|_{L^{r'(\cdot)}}\|\chi_B\|_{L^{t(\cdot)}} \leq C|B|^{1-s'\alpha/n}, \quad \forall B \in \mathbb{B} \tag{5.11}$$

for some $C > 0$.

Since $\|\chi_B\|_{L^{t(\cdot)}}^{1/s'} = \|\chi_B\|_{L^{q(\cdot)}}$, (5.9), (5.10) and (5.11) give

$$\chi_{B(z,r)}(x)|(T_{\Omega,\alpha}f_j)(x)| \leq C\chi_{B(z,r)}(x)\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}.$$

Applying the norm $\|\cdot\|_{L^{q(\cdot)}}$ on both sides of the above inequality, we have

$$\|\chi_{B(z,r)}T_{\Omega,\alpha}f_j\|_{L^{q(\cdot)}} \leq C\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}. \tag{5.12}$$

We find that

$$\begin{aligned}
&\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{L^{q(\cdot)}} \\
&\quad\le \frac{1}{u(z,r)}\|\chi_{B(z,r)}Tf_0\|_{L^{q(\cdot)}} + \frac{1}{u(z,r)}\sum_{j=1}^{\infty}\|\chi_{B(z,r)}Tf_j\|_{L^{q(\cdot)}} \\
&\quad\le \|Tf_0\|_{M^u_{q(\cdot)}} + \sum_{j=1}^{\infty}\frac{1}{u(z,r)}\|\chi_{B(z,r)}Tf_j\|_{L^{q(\cdot)}}.
\end{aligned}$$

Thus (5.7) and (5.12) assert that

$$\begin{aligned}
&\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{L^{q(\cdot)}} \\
&\le C\|f_0\|_{M^u_{q(\cdot)}} + C\sum_{j=1}^{\infty}\frac{u(z,2^{j+1}r)}{u(z,r)}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}\frac{1}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}} \\
&\le C\|f_0\|_{M^u_{q(\cdot)}} + C\sum_{j=1}^{\infty}\frac{u(z,2^{j+1}r)}{u(z,r)}\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{q(\cdot)}}}\|f\|_{M^u_{p(\cdot)}}.
\end{aligned}$$

Then, (2.1) yields

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{M^u_{q(\cdot)}} \le C\|f\|_{M^u_{p(\cdot)}}$$

for some $C>0$ independent of $z\in\mathbb{R}^n$ and $r>0$.

Finally, by taking supremum over $B(z,r)\in\mathbb{B}$ on both sides of the above inequality, we obtain our desired result. □

Before we give the proof of Theorem 3.2, we first establish the mapping properties of $T_{\Omega,\alpha}$ on block $b_{u,p(\cdot)}$.

PROPOSITION 5.3. *Let $0<\alpha<n$, $1<s<\infty$ and $\Omega\in L^s(\mathbb{S}^{n-1})$. Suppose that $s'<p_-\le p_+<\alpha/n$ and*

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}.$$

If there exists q_0 with $ns'/(n-\alpha s')<q_0$ such that $q(\cdot)/q_0\in\mathbb{M}$ and $u\in\mathcal{W}^{(1/s')-(\alpha/n)}_{q(\cdot)}$, then there exists a constant $C>0$ such that for any $b\in b_{u,p(\cdot)}$,

$$\|T_{\Omega,\alpha}b\|_{\mathfrak{B}_{u,q(\cdot)}}\le C.$$

PROOF. Let $z\in\mathbb{R}^n$, $r>0$ and b be an $(u,p(\cdot))$-block with support $B(z,r)$. For any $k\in\mathbb{N}$, write $b_0=\chi_{B(z,2r)}T_{\Omega,\alpha}b$, $B_k=B(z,2^kr)$ and

$$b_k=\chi_{B_k\backslash B_{k-1}}T_{\Omega,\alpha}b,\quad k\in\mathbb{N}\backslash\{0\}.$$

Since $T_{\Omega,\alpha}:L^{p(\cdot)}\to L^{q(\cdot)}$ is bounded, (5.6) assures that

$$\|b_0\|_{L^{q(\cdot)}} \le C\|T_{\Omega,\alpha}b\|_{L^{q(\cdot)}} \le C\|b\|_{L^{p(\cdot)}} \le \frac{C}{u(z,r)} \le \frac{C}{u(z,2r)}$$

for some $C>0$ independent of $b \in b_{u,p(\cdot)}$, $z \in \mathbb{R}^n$ and $r>0$. Thus, b_0 is a constant multiple of an $(u,q(\cdot))$-block.

Next, as $\operatorname{supp} b \subseteq B(z,r)$ and $T_{\Omega,\alpha}$ is a fractional integral operator with the homogeneous kernel, we find that

$$|b_k(x)| \le \chi_{B_k\backslash B_{k-1}}(x)|(T_{\Omega,\alpha}b)(x)| \le C\chi_{B_k\backslash B_{k-1}}(x)\int_{B(z,r)} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|b(y)|dy.$$

For any $k \in \mathbb{N}\backslash\{0\}$, $x \in B_k\backslash B_{k-1}$ and $y \in B(z,r)$, we have

$$|x-y| \ge |x-z| - |z-y| \ge 2^{k-1}r - r \ge 2^{k-2}r.$$

Consequently, the Hölder inequality yields

$$\begin{aligned}|b_k(x)| &\le C\chi_{B_k\backslash B_{k-1}}(x)2^{-(k-2)(n-\alpha)}r^{-n+\alpha}\int_{B(z,r)} |\Omega(x-y)||b(y)|dy\\ &\le C\chi_{B_k\backslash B_{k-1}}(x)2^{-(k-2)(n-\alpha)}r^{-n+\alpha}\left(\int_{B(x-z,r)} |\Omega(y)|^s dy\right)^{1/s}\|b\|_{L^{s'}}.\end{aligned}$$

For any $x \in B_k\backslash B_{k-1}$ and $y \in B(x-z,r)$, we have

$$|y| \le |y-(x-z)| + |x-z| \le r + 2^k r \le 2^{k+1}r$$

and

$$|y| \ge |y-(x-z)| - |x-z| \ge 2^{k-1}r - r \ge 2^{k-2}r.$$

Therefore, for any $x \in B_k\backslash B_{k-1}$, the belonging $\Omega \in L^s(\mathbb{R}^n)$ assures that

$$\begin{aligned}&|b_k(x)|\\ &\le C_1\chi_{B_k\backslash B_{k-1}}(x)2^{-(k-2)(n-\alpha)}r^{-n+\alpha}\left(\int_{B(0,2^{k+1}r)\backslash B(0,2^{k-2}r)} |\Omega(y)|^s dy\right)^{1/s}\|b\|_{L^{s'}}\\ &\le C_1\chi_{B_k\backslash B_{k-1}}(x)2^{-(k-2)(n-\alpha)}r^{-n+\alpha}\left(\int_{2^{k-2}r}^{2^{k+1}r} t^{n-1}dt\int_{\mathbb{S}^{n-1}} |\Omega(\theta)|^s d\theta\right)^{1/s}\|b\|_{L^{s'}}\\ &\le C\chi_{B_k\backslash B_{k-1}}(x)2^{-(k+1)((n/s')-\alpha)}r^{-(n/s')+\alpha}\|b\|_{L^{s'}} \le C\chi_{B_k}(x)\frac{\|b\|_{L^{s'}}}{|B_k|^{(1/s')-(\alpha/n)}}\end{aligned} \tag{5.13}$$

for some $C, C_1 > 0$.

Recall that $r(x) = p(x)/s'$ and $t(x) = q(x)/s'$. Since $b \in b_{u,p(\cdot)}$, by using the Hölder inequality for $L^{r(\cdot)}$, we have

$$\|b\|_{L^{s'}} \le 2\||b|^{s'}\|_{L^{r(\cdot)}}^{1/s'}\|\chi_{B(z,r)}\|_{L^{r'(\cdot)}}^{1/s'} = 2\|b\|_{L^{p(\cdot)}}\|\chi_{B(z,r)}\|_{L^{r'(\cdot)}}^{1/s'} \le 2\frac{\|\chi_{B(z,r)}\|_{L^{r'(\cdot)}}^{1/s'}}{u(z,r)}.$$

We apply the norm $\|\cdot\|_{L^{q(\cdot)}}$ on both sides of the inequality (5.13) and find that

$$\|b_k\|_{L^{q(\cdot)}} \le \frac{C}{u(z,r)}\|\chi_{B_k}\|_{L^{q(\cdot)}}\frac{\|\chi_{B(z,r)}\|_{L^{r'(\cdot)}}^{1/s'}}{|B_k|^{(1/s')-(\alpha/n)}}.$$

Thus, (5.11) gives

$$\|b_k\|_{L^{q(\cdot)}} \le \frac{C_2}{u(z,2^k r)}\frac{u(z,2^k r)}{u(z,r)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}\frac{|B(z,r)|^{(1/s')-(\alpha/n)}}{|B_k|^{(1/s')-(\alpha/n)}}$$

for some C_2 independent of b and k.

Write $b_k = \lambda_k\beta_k$ where

$$\lambda_k = C_2\frac{u(z,2^k r)}{u(z,r)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}\frac{|B(z,r)|^{(1/s')-(\alpha/n)}}{|B_k|^{(1/s')-(\alpha/n)}}.$$

Then, β_k is an $(u,q(\cdot))$-block with support $B(z,2^k r)$.

Furthermore, the condition $u \in \mathcal{W}_{q(\cdot)}^{(1/s')-(\alpha/n)}$ guarantees that

$$\sum_{k=1}^{\infty}|\lambda_k| = C_2\sum_{k=1}^{\infty}\frac{u(z,2^k r)}{u(z,r)}\frac{\|\chi_{B_k}\|_{L^{q(\cdot)}}}{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}}}\frac{|B(z,r)|^{(1/s')-(\alpha/n)}}{|B_k|^{(1/s')-(\alpha/n)}} < C_3$$

for some constant $C_3 > 0$ independent of b.

According to the definition of $\mathfrak{B}_{u,q(\cdot)}$, we find that

$$T_{\Omega,\alpha}b = \sum_{k=0}^{\infty}b_k = \sum_{k=0}^{\infty}\lambda_k\beta_k$$

belongs to $\mathfrak{B}_{u,q(\cdot)}$ with

$$\|T_{\Omega,\alpha}b\|_{\mathfrak{B}_{u,q(\cdot)}} \le \sum_{k=0}^{\infty}|\lambda_k| < C_3$$

for some $C_3 > 0$ independent of $b \in b_{u,p(\cdot)}$. □

We now combine the above result with the duality of $M_{p(\cdot)}^u$ and $\mathfrak{B}_{u,p'(\cdot)}$ to prove Theorem 3.2.

Proof of Theorem 3.2. We have $(q')_+ < \alpha/n$ and

$$\frac{1}{q'(x)} - \frac{1}{p'(x)} = \frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}.$$

Since $\tilde{\Omega}(x) = \overline{\Omega(-x)} \in L^s(\mathbb{S}^{n-1})$, Proposition 5.3 guarantees that for any $b \in b_{u,q'(\cdot)}$

$$\|T_{\tilde{\Omega},\alpha}b\|_{\mathfrak{B}_{u,p'(\cdot)}} \leq C. \tag{5.14}$$

Consequently, for any $f \in M^u_{p(\cdot)}$, (5.14) yields

$$\begin{aligned}\sup_{b\in b_{u,q'(\cdot)}} \left|\int_{\mathbb{R}^n} (T_{\Omega,\alpha}f)(x)b(x)dx\right| &= \sup_{b\in b_{u,q'(\cdot)}} \left|\int_{\mathbb{R}^n} f(x)T_{\tilde{\Omega},\alpha}b(x)dx\right| \\ &\leq C\|f\|_{M^u_{p(\cdot)}}\|T_{\tilde{\Omega},\alpha}b\|_{\mathfrak{B}_{u,p'(\cdot)}} \leq C\|f\|_{M^u_{p(\cdot)}}\end{aligned}$$

for some $C > 0$. Therefore, Proposition 4.6 assures that $T_{\Omega,\alpha}f \in M^u_{q(\cdot)}$. Moreover, Proposition 4.5 guarantees that for any $f \in M^u_{p(\cdot)}$

$$\|T_{\Omega,\alpha}f\|_{M^u_{q(\cdot)}} \leq C \sup_{b\in b_{u,q'(\cdot)}} \left|\int_{\mathbb{R}^n} (T_{\Omega,\alpha}f)(x)b(x)dx\right| \leq C\|f\|_{M^u_{p(\cdot)}}$$

for some $C > 0$. □

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