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A filtration for isoparametric hypersurfaces in Riemannian manifolds

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Abstract. This paper introduces the notion of k-isoparametric hypersurface in an $(n+1)$-dimensional Riemannian manifold for $k = 0, 1, \dots, n$. Many fundamental and interesting results (towards the classification of homogeneous hypersurfaces among other things) are given in complex projective spaces, complex hyperbolic spaces, and even in locally rank one symmetric spaces.

1. Introduction.

A smooth non-constant function $f : N \to \mathbb{R}$ defined on a Riemannian manifold N is called *transnormal* if there is a smooth function $b : \mathbb{R} \to \mathbb{R}$ such that

$$|\nabla f|^2 = b(f), \tag{1}$$

where ∇f is the gradient of f. If in addition there is a continuous function $a : \mathbb{R} \to \mathbb{R}$ such that

$$\triangle f = a(f), \tag{2}$$

where $\triangle f$ is the Laplacian of f, then the function f is called *isoparametric* (cf. [**Wan87**], [**GT13**], [**GT14**]). Élie Cartan (cf. [**Car38**], [**Car39**], [**Car39'**], [**Car40**]) pointed out: equation (1) means that the level hypersurfaces $M_t := f^{-1}(t)$ (where t are regular values of f) are parallel and equation (2) further implies that these parallel hypersurfaces have constant mean curvatures.

The preimage of the global maximum (resp. minimum) of an isoparametric (or transnormal) function f is called the *focal variety* of f, denoted by M_+ (resp. M_-), if nonempty. A fundamental structural result established by [**Wan87**] asserts that each focal variety of a transnormal function is a smooth submanifold (may be disconnected and have different dimensions), and each connected component P_t of M_t is a tube (tubular hypersurface) or a "half-tube" (when $\mathrm{codim}(P) = 1$) of the same radius around a

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connected component P of $M_{\pm}$. A hypersurface M in N is called *isoparametric* if it is a level hypersurface of some locally defined isoparametric function f on N. Therefore, a hypersurface M in N is an isoparametric hypersurface if and only if all its nearby parallel hypersurfaces have constant mean curvatures, which is a local property with respect to the ambient space N in general, while isoparametric functions are global objects of N that may restrict strongly the geometry and topology of the ambient space N, as stated in [**GT13**].

The theory of isoparametric functions (hypersurfaces) originated from studies of hypersurfaces in real space forms with constant principal curvatures (see [**Tho00**], [**Cec08**] for excellent surveys). As Élie Cartan (cf. [**Car38**], [**Car39**], [**Car39'**], [**Car40**]) asserted, a hypersurface in a real space form has constant principal curvatures if and only if all its nearby parallel hypersurfaces have constant mean curvatures, thus it is an isoparametric hypersurface defined as before. Isoparametric hypersurfaces in Euclidean or hyperbolic space were easily classified due to the celebrated Cartan identity. However, it turns out that isoparametric hypersurfaces in the unit spheres are more complicated and plentiful, and thus have not been completely classified up to now (for the newest progress, please see [**GH87**], [**CCJ07**], [**Imm08**], [**Chi11**], [**DN85**], [**Miy13**], [**QT14**], [**QT15**], [**TY13**], [**TXY14**], etc.). In the early 1970's, Münzner [**Mün80**] produced a far-reaching generalization of Cartan's work. He showed that an isoparametric hypersurface in a sphere S^{n+1} is an open part of a level hypersurface, say M, of an isoparametric function f which is the restriction to S^{n+1} of a Cartan polynomial F. By a *Cartan polynomial* (or *isoparametric polynomial*), we mean a homogeneous polynomial F on $\mathbb{R}^{n+2}$ satisfying the Cartan–Münzner equations

$$|\nabla F|^2 = g^2|x|^{2g-2}, \quad x \in \mathbb{R}^{n+2}, \tag{3}$$

$$\Delta F = \frac{g^2}{2}(m_2 - m_1)|x|^{g-2}, \tag{4}$$

where ∇F, ΔF denote the gradient and Laplacian of F on $\mathbb{R}^{n+2}$ respectively, and m_1, m_2 the multiplicities of the maximal and minimal principal curvatures of M, $g = \deg(F)$ the number of distinct principal curvatures of M. Further, using an elegant topological method Münzner proved the remarkable result that the number g must be 1, 2, 3, 4, or 6.

Note that the Cartan–Münzner equations (3)–(4) of the isoparametric polynomial F on $\mathbb{R}^{n+2}$ correspond to the equations (1)–(2) of the isoparametric function f on S^{n+1} with the following equalities:

$$b(f) = g^2(1 - f^2), \qquad a(f) = \frac{g^2}{2}(m_2 - m_1) - g(n+g)f, \tag{5}$$

which only mean that the level hypersurfaces $M_t := f^{-1}(t)$ have constant mean curvatures. On the other hand, due to Cartan's result, the level hypersurfaces M_t essentially have constant principal curvatures and hence constant mean curvatures of each order (which are elementary symmetric polynomials of principal curvatures). This fantastic phenomenon suggests that there are hidden $n-1$ more equations describing the constancy of higher order mean curvatures of an isoparametric hypersurface in a sphere for the isoparametric function f (resp. isoparametric polynomial F) besides the equations (1)–(2) (resp. Cartan–Münzner equations (3)–(4)). It is this observation that stimulates us to exhibit these hidden equations (see Theorem 2.1) which should possibly be helpful to provide a geometric or an algebraic proof of Münzner's remarkable result on g mentioned above.

Observing that there do exist isoparametric hypersurfaces in complex projective spaces with non-constant principal curvatures (cf. [**Wan82**]), we will be concerned with the isoparametric functions (resp. hypersurfaces) on Riemannian manifolds satisfying these hidden equations (resp. more constant higher order mean curvatures). This treatment will filter isoparametric functions (resp. isoparametric hypersurface) by *k-isoparametric functions* (resp. *k-isoparametric hypersurfaces*) on a Riemannian manifold N^{n+1} for $k = 1, \ldots, n$.

We now set up some notations. First of all, for an n by n real symmetric matrix (or self-dual operator) A with n real eigenvalues $(\mu_1, \ldots, \mu_n) =: \mu$ and $k = 1, \ldots, n$, we denote by $\sigma_k(A) = \sigma_k(\mu)$ the k-th elementary symmetric polynomial of μ, i.e.,

$$\sigma_k(A) = \sigma_k(\mu) = \sum_{i_1 < \cdots < i_k} \mu_{i_1} \cdots \mu_{i_k} = \sum_{i_1 < \cdots < i_k} A\begin{pmatrix} i_1 \cdots i_k \\ i_1 \cdots i_k \end{pmatrix}, \quad \sigma_0(A) = \sigma_0(\mu) \equiv 1, \quad (6)$$

where $A\binom{i_1 \cdots i_k}{i_1 \cdots i_k}$'s are the principal k-minors of A; and denote by $\rho_k(A) = \rho_k(\mu)$ the k-th power sum, i.e.,

$$\rho_k(A) = \rho_k(\mu) = \sum_{i=1}^{n} \mu_i^k = tr(A^k), \quad \rho_0(A) = \rho_0(\mu) \equiv n. \quad (7)$$

In these notations, the Newton's identities can be stated as

$$k\sigma_k = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i} \rho_i, \quad \text{for} \quad k = 1, \ldots, n, \quad (8)$$

which show in particular that for $k = 1, \ldots, n$,

$$\{\sigma_1, \ldots, \sigma_k\} \text{ are constant if and only if } \{\rho_1, \ldots, \rho_k\} \text{ are constant.} \quad (9)$$

Next, on a Riemannian manifold N^{n+1}, we define a sequence of partial differential operators $\{\triangle_1, \ldots, \triangle_{n+1}\}$ over $C^\infty(N^{n+1})$ by

$$\triangle_k f := \sigma_k(H_f), \quad \text{for} \quad k = 1, \ldots, n+1, \quad (10)$$

where H_f is the Hessian of f on N^{n+1}. It is respectively the Laplacian and the Monge–Ampère operator when $k = 1$ and $k = n + 1$. Note that $\triangle_k$ is nonlinear when $k \geq 2$.

Definition 1.1. For $1 \leq k \leq n$, a non-constant smooth function f on a Riemannian manifold N^{n+1} is called *k-isoparametric*, if f is a transnormal function satisfying equation (1), and in addition there exist continuous functions $a_1, \dots, a_k \in C(\mathbb{R})$, such that

$$\triangle_i f = a_i(f), \quad \text{for} \ \ i = 1, \dots, k. \tag{11}$$

We denote by $\mathscr{I}_k(N^{n+1})$ the set consisting of k-isoparametric functions on N^{n+1}. A hypersurface M^n in N^{n+1} is called *k-isoparametric* if it is a level hypersurface of some locally defined k-isoparametric function f on N^{n+1}.

Remark 1.1. For simplicity, we will call a transnormal function f a *0-isoparametric function*, denoted by $f \in \mathscr{I}_0(N^{n+1})$ (and by $\triangle_0 f := |\nabla f|^2$). Note that a 1-isoparametric function is exactly the usual isoparametric function introduced at the beginning of this paper. Generally, in a Riemannian manifold N^{n+1}, a k-isoparametric hypersurface can not determine a corresponding global k-isoparametric function. However, as we stated before, a 1-isoparametric hypersurface in a sphere does determine a corresponding global isoparametric function according to Cartan–Münzner's construction of isoparametric polynomial. Furthermore, in a compact symmetric space N^{n+1}, a 1-isoparametric hypersurface M^n also determines a corresponding global isoparametric function f on N^{n+1}. To show this assertion, first we know that by [**HLO06**] M^n must be an equifocal hypersurface (cf. [**TT95**], [**Tan98**]). Next, it follows from Terng and Thorbergsson [**TT95**] that M^n determines a transnormal system on N^{n+1} with t-regular foils of codimension one, which then by Miyaoka [**Miy13'**] corresponds to a global transnormal function $\bar{f}$ on N^{n+1} whose regular level hypersurfaces are parallel to M^n and have constant mean curvatures. Finally we get a desired global isoparametric function f on N^{n+1} via $\bar{f}$ with the same level sets by some regularization.

It follows directly from the definition that the sets of 1-, 2-, $\dots$, n-isoparametric functions (hypersurfaces) induce a filtration of isoparametric functions (hypersurfaces) on a Riemannian manifold N^{n+1} as:

$$\big(\mathscr{I}_0(N^{n+1}) \supset \big)\mathscr{I}_1(N^{n+1}) \supset \cdots \supset \mathscr{I}_n(N^{n+1}). \tag{12}$$

By a straightforward verification, we will see in the next section that, a hypersurface M^n in N^{n+1} is k-isoparametric if and only if all its nearby parallel hypersurfaces, say M_t ($M_{t_0}{=}M$), have constant i-th mean curvatures $H_i(t)$ for $i = 1, \dots, k$, where $H_i(t) := \sigma_i(S(t)) = \sigma_i(\mu(t))$ is the i-th elementary symmetric polynomial of the shape operator $S(t)$ or principal curvatures $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$ of M_t. In particular, M^n is an n-isoparametric hypersurface if and only if all its nearby parallel hypersurfaces have constant principal curvatures. In this case, M^n is called a *totally isoparametric hypersurface* and the corresponding (local) function a *totally isoparametric function.* In this way, Cartan's rigidity result can be restated as:

A 1-isoparametric hypersurface in a real space form is totally isoparametric. (13)

Although Cartan's rigidity result can hardly hold in a general Riemannian manifold, we will be able to extend it to those Riemannian manifolds with some symmetries other than real space forms, as stated in the following theorems.

THEOREM 1.1. *Let $\mathbb{C}P^m$ be the complex projective space equipped with Fubini–Study metric of constant holomorphic sectional curvature* 4. *Then*

(i) *A* 1*-isoparametric hypersurface in a complex even-dimensional projective space $\mathbb{C}P^{2n}$ is totally isoparametric. In fact, it is homogeneous.*
(ii) *Each complex odd-dimensional projective space $\mathbb{C}P^{2n+1}$ admits a* 2*-isoparametric hypersurface which is not* 3*-isoparametric.*

The homogeneity conclusion in (i) follows from the following sequence of equivalent conditions for an *isoparametric hypersurface* $\widetilde{M}^{2n-1}$ in $\mathbb{C}P^n$ by putting results of [**Wan82**], [**Kim86**], [**Par89**][1] and [**Xia00**] together:

$$\begin{aligned}
&\widetilde{M} \text{ has constant principal curvatures}\\
&\quad \Leftrightarrow \widetilde{M} \text{ is Hopf, i.e., } J\tilde{\nu} \text{ is principal}\\
&\quad \Leftrightarrow \text{one of the focal submanifolds is complex}\\
&\quad \Leftrightarrow \widetilde{M} \text{ is homogeneous (i.e. an open part of a homogeneous hypersurface)}\\
&\quad \Leftrightarrow l \equiv 2 \Leftrightarrow l \equiv \text{const} \Leftrightarrow \tilde{g} \equiv \text{const}\\
&\quad \Leftrightarrow \widetilde{M} \text{ is totally isoparametric,}
\end{aligned} \tag{14}$$

where $\tilde{\nu}$ is a unit normal vector field on $\widetilde{M}$, J the canonical complex structure of $\mathbb{C}P^n$, $\tilde{g}$ the number of distinct principal curvatures of $\widetilde{M}$ and l the number of non-horizontal eigenspaces of the shape operator on $M := \pi^{-1}(\widetilde{M})$ by the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$.

It is worth to point out that hypersurfaces with constant principal curvatures in any Riemannian manifold other than a real space form are far from being classified; and even the set of g, the number of distinct principal curvatures, has not been determined so well as Münzner did for such hypersurfaces in spheres (see [**Ber10**] for a detailed survey). Combining with the remarkable classification of homogeneous hypersurfaces in complex projective spaces by Takagi [**Tak73**, Theorem 1.1 (i)] classifies completely isoparametric hypersurfaces in $\mathbb{C}P^{2n}$ indeed. Our classification should be compared with the case of complex hyperbolic space $\mathbb{C}H^n$ in which [**DD12**] recently constructed inhomogeneous examples of isoparametric hypersurfaces for each $n \geq 3$.

Examples in Theorem 1.1(ii) are constructed explicitly by projecting certain OT-FKM-type isoparametric hypersurfaces in spheres by the Hopf fibration. Here, the deduction that 1-isoparametric is sufficient for 2-isoparametric in $\mathbb{C}P^n$ (or more generally in an Einstein manifold) can be easily seen from relations of the shape operators of M and $\widetilde{M}$ by the Hopf fibration (or from the Riccati equation).

[1]It was pointed out by [**Xia00**] that there exist some mistakes in [**Par89**]. However, the conclusions we cited are correct.

A deeper exploration of these relations by the Hopf fibration and the equivalence sequence (14) will lead us to a classification of isoparametric hypersurfaces in $\mathbb{C}P^n$ with constant 3rd mean curvatures as follows:

Theorem 1.2. *A 1-isoparametric hypersurface in $\mathbb{C}P^n$ with constant 3rd mean curvature, i.e., $H_3 \equiv \mathrm{const}$, is totally isoparametric and hence homogeneous.*

We expect this theorem to play a special role in solving Chern conjecture, which asserts that a closed hypersurface in a sphere with constant 1st and 2nd mean curvatures must be an isoparametric hypersurface (cf. [**SW08**], [**GT12**]). This conjecture has been proved only for the case of 3-dimensional closed hypersurfaces in S^4, while remains open for higher dimensional cases. On the other hand, one has not any example of inhomogeneous hypersurface with constant principal curvatures in Riemannian symmetric spaces other than real space forms (cf. [**Ber10**]). It turns out that there are some relations between these two questions as stated in the following.

Corollary 1.1. *Suppose that $\widetilde{M}$ is an inhomogeneous hypersurface in $\mathbb{C}P^n$ with constant mean curvatures H_1, H_2, H_3. Then the inverse image $M = \pi^{-1}(\widetilde{M})$ in S^{2n+1} under the Hopf fibration is a non-isoparametric hypersurface with constant first mean curvature H_1 and constant second mean curvature $H_2 - 1$, giving a counterexample to Chern conjecture.*

In general, the ambient space N is lack of such "satisfied structures" (e.g., Hopf fibration, complex structure, explicit representation of curvature tenser, etc.) as $\mathbb{C}P^n$, resulting in obstructions for us to get rigidity results as Theorems 1.1, 1.2 for CP^n. However, when N is a complex space form or more generally a locally rank one symmetric space, there still exist certain symmetries of the curvature tensor, which make the Riccati equation more useful in dealing with parallel hypersurfaces in such spaces than in general Riemannian manifolds as in [**GT14**]. For example, by making use of the Riccati equation, we obtain the following rigidity result (compared with Theorem 1.2 where $H_3 \equiv \mathrm{const}$ is an assumption weaker than 3-isoparametric):

Theorem 1.3. *A 3-isoparametric hypersurface M^n in a locally rank one symmetric space N^{n+1} is 5-isoparametric. If in addition N^{n+1} is locally a complex space form, then M^n is totally isoparametric.*

Remark 1.2. The key point in the proof of this theorem is that the normal Jacobi operator $K_\nu : \mathcal{T}M \to \mathcal{T}M$ defined by $K_\nu(X) := R(\nu, X)\nu = (\nabla_{[\nu,X]} - [\nabla_\nu, \nabla_X])\nu$, for $X \in \mathcal{T}M$, where ν is a unit normal vector field on M, has constant eigenvalues and is parallel along the normal geodesics. In fact, if both $tr(K_\nu)$ and $tr(K_\nu^2)$ are constant, in the same way we find that 3-isoparametric is sufficient for 4-isoparametric in a locally symmetric space. Fortunately, there are many locally symmetric spaces with constant $tr(K_\xi)$ and constant $tr(K_\xi^2)$, independent of the choice of the unit tangent vector ξ. Such locally symmetric spaces are involved in the Lichnerowicz conjecture and have been classified in [**CGW82**].

The following rigidity result is another application of the Riccati equation. To state it, we need to introduce the concept of *curvature-adapted* or *compatible* hypersurfaces (resp. submanifolds), namely, whose normal Jacobi operator K_ν and shape operator S_ν (resp. $S_\nu \oplus I$) commute, or equivalently, are simultaneously diagonalizable for each unit normal vector ν (cf. [**Ber91**], [**Gra04**]).

THEOREM 1.4. *Let M^n be a curvature-adapted hypersurface in a locally rank one symmetric space N^{n+1}. If either*

(i) *M^n has constant principal curvatures, or*
(ii) *M^n is a* 1*-isoparametric hypersurface,*

then M^n is totally isoparametric.

REMARK 1.3. The proof of Theorem 1.4(i) yields also that a tube (tubular hypersurface) M^n around a curvature-adapted submanifold of constant principal curvatures in a locally rank one symmetric space N^{n+1} is a curvature-adapted hypersurface of constant principal curvatures and thus totally isoparametric.

REMARK 1.4. It is clear to see that (cf. [**Gra04**]), given a curvature-adapted hypersurface M in a locally symmetric space N, each nearby parallel hypersurface M_t is automatically curvature-adapted. Theorem 1.4 holds also for a hypersurface M in an Osserman manifold whose nearby parallel hypersurfaces M_t are all curvature-adapted. In fact, the proof of Theorem 1.4 depends mainly on the constancy of eigenvalues of the Jacobi operator, while an Osserman manifold is exactly a Riemannian manifold N whose Jacobi operator has constant eigenvalues including multiplicities, independent of the choice of the unit tangent vector and the point on N. Essentially, Osserman conjectured that an Osserman manifold (named later) is a locally rank one symmetric space. This conjecture has been verified to be true except for the case when $\dim N = 16$ (cf. [**Chi88**], [**Nik05**], [**BGN09**]).

In a locally rank one symmetric space N^{n+1} with non-constant sectional curvatures, all known examples of totally isoparametric hypersurfaces are homogeneous. Recall that for a hypersurface in $\mathbb{C}P^n$, totally isoparametric is equivalent to homogeneous by the equivalence sequence (14). In all probability, this equivalence still holds, at least, in each compact case ($\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{O}P^2$). On the other hand, curvature-adapted hypersurfaces in complex space forms ($\mathbb{C}P^n$, $\mathbb{C}H^n$, $\mathbb{C}^n$) are just Hopf hypersurfaces. Similar to Kimura's work in $\mathbb{C}P^n$ (cf. [**Kim86**]), Berndt [**Ber89**] proved that a Hopf hypersurface with constant principal curvatures in $\mathbb{C}H^n$ is necessarily homogeneous. Based on this remarkable result, Theorem 1.4 (ii) yields

COROLLARY 1.2. *A* 1*-isoparametric Hopf hypersurface in $\mathbb{C}H^n$ is homogeneous.*

We conclude this section with some remarks. In virtue of the classification of homogeneous hypersurfaces in $\mathbb{C}H^n$ by [**BT07**], 1-isoparametric Hopf hypersurfaces in $\mathbb{C}H^n$ are consequently classified. An interesting phenomenon appeared in $\mathbb{C}H^n$ that there exist many non-Hopf homogeneous hypersurfaces, which is quite different from that in $\mathbb{C}P^n$ (cf. [**Ber10**]). As is well known, a hypersurface in a non-flat complex space form is curvature-adapted if and only if it is Hopf. The concept of curvature-adapted hypersurface gives a natural generalization of Hopf hypersurfaces in Hermitian manifolds to more general Riemannian manifolds. Surprisingly, Berndt ([**Ber91**]) proved that a hypersurface in quaternionic projective space $\mathbb{H}P^n$ is curvature-adapted if and only if it is homogeneous. While in quaternionic hyperbolic space $\mathbb{H}H^n$, the classification of curvature-adapted hypersurfaces is still an open problem (cf. [**Mur14**], [**Ber91**]). Moreover, the classification of curvature-adapted hypersurfaces in octonionic space forms is still elusive.

2. Hidden Cartan–Münzner equations.

This section will be devoted to the establishment of an inductive formula for those $(n-1)$ equations implied by Cartan–Münzner equations (3)–(4) for isoparametric functions (polynomials) on S^{n+1}. We first show the following geometric characterization of a k-isoparametric hypersurface M^n defined by a (local) k-isoparametric function f on a Riemannian manifold N^{n+1}:

Lemma 2.1. *A hypersurface is k-isoparametric if and only if each of its nearby parallel hypersurfaces has constant i-th mean curvatures for $i = 1, \ldots, k$.*

Proof. Let $M^n := f^{-1}(t_0)$ be a k-isoparametric hypersurface in a Riemannian manifold N^{n+1}, where t_0 is a regular value of the (local) k-isoparametric function f satisfying equations (1) and (11). For $\varepsilon > 0$ sufficiently small, $t \in (t_0-\varepsilon, t_0+\varepsilon)$, $M_t := f^{-1}(t)$ is still a hypersurface that is parallel to M by equation (1), since now $\nabla f/|\nabla f|$ is the tangent vector field along the normal geodesic of M at each point. It is well known that the shape operator, say $S(t)$, of M_t with respect to the unit normal vector field $\nu = \nabla f/|\nabla f|$ is characterized by (cf. [**CR85**]):

$$\langle S(t)X, Y\rangle = \frac{-H_f(X,Y)}{|\nabla f|}, \tag{15}$$

where X, Y are tangent vectors to M_t and H_f the Hessian of f. Now let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of M_t and e_i the eigenvector of $S(t)$ with respect to principal curvature μ_i for $i = 1, \ldots, n$. Then it follows from equation (1) that $H_f(e_i, \nu) = 0$ and $H_f(\nu,\nu) = b'(f)/2$. Thus under the orthonormal frame $\{e_1, \ldots, e_n, \nu\}$ of N^{n+1}, the Hessian H_f is expressed as the diagonal matrix

$$H_f = \mathrm{diag}\big(-\sqrt{b(f)}\mu_1, \ldots, -\sqrt{b(f)}\mu_n, b'(f)/2\big). \tag{16}$$

Therefore, a straightforward calculation using (16) shows that $\triangle_j f := \sigma_j(H_f)$ can be expressed in terms of the mean curvatures $H_i := H_i(t) = \sigma_i(S(t))$ for $i \le j$ as:

$$\triangle_j f = \big(-\sqrt{b(f)}\big)^j H_j + \big(-\sqrt{b(f)}\big)^{j-1}\frac{b'(f)}{2}H_{j-1}, \tag{17}$$

and conversely,

$$H_j = \frac{1}{(2\sqrt{b(f)})^j}\bigg(\sum_{i=1}^{j}(-1)^i 2^i (b'(f))^{j-i}\triangle_i f + (b'(f))^j\bigg). \tag{18}$$

Preparing these equations, we are now in a position to complete the proof of the lemma. First, for a given k-isoparametric function f, the equations (11) and (18) yield that $H_1, \ldots, H_k$ are functions of f, thus constant on each M_t. Consequently, the nearby parallel hypersurfaces of a k-isoparametric hypersurface have constant mean curvatures $H_1, \ldots, H_k$. Conversely, if each nearby parallel hypersurface $M'_t := \exp_M(t\nu)$ of M^n (the image of M^n under the normal exponential map at distance $t \in (-\varepsilon, \varepsilon)$) has constant mean curvatures $H_1, \ldots, H_k$ (continuously depend on t), we can define a function f on the local neighborhood $\bigcup_{t\in(-\varepsilon,\varepsilon)} M'_t \subset N^{n+1}$ of M^n in N^{n+1} by $f|_{M'_t} := t$ for $t \in (-\varepsilon, \varepsilon)$. Clearly $|\nabla f|^2 = 1$ and by (17), $\triangle_1 f, \ldots, \triangle_k f$ are constant on each M'_t and continuously depend on $t = f$. Namely, f is a local k-isoparametric function on N^{n+1} and thus M^n is k-isoparametric. $\square$

In particular, an n-isoparametric hypersurface M^n has constant mean curvatures $H_1, \ldots, H_n$, thus constant principal curvatures. This justifies the notion of *totally isoparametric*. As mentioned in the introduction, by Cartan's rigidity result (13), we know that for an isoparametric function $f = F|_{S^{n+1}}$ satisfying (5) on S^{n+1}, the restriction of a Cartan polynomial F satisfying (3) and (4), the mean curvatures $H_1, \ldots, H_n$, or equivalently $Q_1 := \rho_1(S(t)), \ldots, Q_n := \rho_n(S(t))$, where $S(t)$ is the shape operator of the level hypersurface $M_t := f^{-1}(t)$, are constant on M_t and continuously (smoothly, in fact) depend on $f = t \in (-1, 1)$. This argument together with (17) allows us to construct $(n-1)$ smooth functions $a_2, \ldots, a_n \in C^\infty(\mathbb{R})$ other than the functions b and a in (5) by

$$\triangle_2 f = a_2(f), \ldots, \triangle_n f = a_n(f); \tag{19}$$

or equivalently, $(n-1)$ smooth functions $p_2, \ldots, p_n \in C^\infty(\mathbb{R})$ ($p_1 = a_1 = a$) by

$$\rho_2(H_f) = p_2(f), \ldots, \rho_n(H_f) = p_n(f). \tag{20}$$

Correspondingly, we find out the $(n-1)$ *hidden Cartan–Münzner equations* for polynomial F involving $\triangle_i F := \sigma_i(H_F)$, or equivalently, involving $\rho_i(H_F)$. So once we have formulae for one of the sets $\{H_i\}$, $\{Q_i\}$, $\{a_i\}$, $\{p_i\}$, $\{\sigma_i(H_F)|_{S^{n+1}} =: \bar{\sigma}_i\}$, and $\{\rho_i(H_F)|_{S^{n+1}} =: \bar{\rho}_i\}$, the others can be obtained by Newton's identities (8), the equalities (17), (18), and the relation between the Hessian H_f of f on S^{n+1} and the Hessian H_F of F on $\mathbb{R}^{n+2}$. In this way, the Münzner's geometric construction of f and the formulae for principal curvatures of M_t lead us to an inductive formula for the set $\{Q_i\}$, and then for the set $\{\bar{\rho}_i\}$ as follows (Q_i, $\bar{\rho}_i$ are regarded as functions of $t = f \in (-1, 1)$):

THEOREM 2.1. *In the same notations as above, for $k = 1, \ldots, n-1$, the following equalities are valid.*

$$Q_{k+1} = \frac{g}{k}\sqrt{1-t^2}\,\frac{dQ_k}{dt} - Q_{k-1},$$

$$Q_0 = n, \quad Q_1 = \frac{m_1 g}{2}\sqrt{\frac{1+t}{1-t}} - \frac{m_2 g}{2}\sqrt{\frac{1-t}{1+t}}.$$

$$\bar{\rho}_{k+1} = \begin{cases} -\dfrac{g^2}{k}(1-t^2)\dfrac{d\bar{\rho}_k}{dt} - g(g-2)t\bar{\rho}_k + g^2(g-1)\bar{\rho}_{k-1} & \\ +2g^{k+1}(g-1)^k(g-2), & \textit{for } k \textit{ odd};\\ -\dfrac{g^2}{k}(1-t^2)\dfrac{d\bar{\rho}_k}{dt} - g(g-2)t\bar{\rho}_k + g^2(g-1)\bar{\rho}_{k-1} & \\ +2g^{k+1}(g-1)^k(g-2)t, & \textit{for } k \textit{ even}, \end{cases}$$

$$\bar{\rho}_0 = n+2, \quad \bar{\rho}_1 = \frac{g^2}{2}(m_2 - m_1).$$

REMARK 2.1. Since F is a homogeneous polynomial of degree g on $\mathbb{R}^{n+2}$, we could homogenize the expressions of $\bar{\rho}_k$ so as to extend the Cartan–Münzner equations (3), (4) on $\mathbb{R}^{n+2}$. For instance, by the inductive formula, we list ($n \geq 4$):

$$\rho_2(H_F) = -\frac{g^3}{2}(g-2)(m_2-m_1)F|x|^{g-4} + g^2(g-1)(n+2g-2)|x|^{2g-4},$$

$$\begin{aligned}\rho_3(H_F) = {}& \frac{g^4}{4}(g-2)(g-4)(m_2-m_1)F^2|x|^{g-6} - ng^3(g-1)(g-2)F|x|^{2g-6}\\ &+ \frac{g^4}{4}(g^2-2)(m_2-m_1)|x|^{3g-6},\end{aligned}$$

$$\begin{aligned}\rho_4(H_F) = {}& -\frac{g^5}{12}(g-2)(g-4)(g-6)(m_2-m_1)F^3|x|^{g-8}\\ &+ \frac{2n}{3}g^4(g-1)(g-2)(g-3)F^2|x|^{2g-8}\\ &- \frac{g^5}{12}(g-2)(5g^2-2g-12)(m_2-m_1)F|x|^{3g-8}\\ &+ \left(\frac{n}{3}g^4(g-1)(g^2+g-3) + 2g^4(g-1)^4\right)|x|^{4g-8}.\end{aligned}$$

PROOF. According to Münzner [**Mün80**], the level hypersurface $M_t^n := f^{-1}(t)$ ($t \in (-1,1)$) of f has g distinct principal curvatures $\{\lambda_i = \cot(\tau + ((i-1)\pi/g)) | i = 1, \ldots, g\}$ with multiplicities m_i satisfying $m_i = m_{i+2}$ (subscripts $\bmod g$), where $f = t = \cos(g\tau)$ on M_t and $\tau \in (0, \pi/g)$ is in fact the oriented (with respect to the unit normal vector field $\nu := \nabla f/|\nabla f|$) distance from M_t to the focal submanifold $M_+ := f^{-1}(1)$ (cf. [**CR85**]). As a consequent result we have

$$Q_k = m_1 \sum_{i=1}^{[(g+1)/2]} \left(\cot\left(\tau + \frac{2(i-1)\pi}{g}\right)\right)^k + m_2 \sum_{i=1}^{[g/2]} \left(\cot\left(\tau + \frac{(2i-1)\pi}{g}\right)\right)^k. \quad (21)$$

Since it is difficult to give a general formula for high order power sum of the cotangent functions, we turn to give an inductive formula instead of a general formula for Q_k. Observe that

$$\frac{d}{d\tau}(\cot(\tau+\theta))^k = -k\big((\cot(\tau+\theta))^{k-1} + (\cot(\tau+\theta))^{k+1}\big),$$
$$\frac{dt}{d\tau} = -g\sin(g\tau) = -g\sqrt{1-t^2},$$

and thus

$$(\cot(\tau+\theta))^{k-1} + (\cot(\tau+\theta))^{k+1} = \frac{g}{k}\sqrt{1-t^2}\frac{d}{dt}(\cot(\tau+\theta))^k$$

which implies immediately the first inductive formula of the theorem by taking sum in (21).

To distinguish the notations, we denote by ∇ and D the Levi-Civita connections on S^{n+1} and $\mathbb{R}^{n+2}$, respectively. By definition, we have for $X, Y \in \mathcal{T}S^{n+1}$,

$$\begin{aligned} H_F(X,Y) &= X(YF) - (D_XY)F = X(YF) - (\nabla_XY)F + \langle X,Y\rangle\frac{\partial F}{\partial r} \\ &= H_f(X,Y) + \langle X,Y\rangle gf, \end{aligned} \tag{22}$$

where $\partial F/\partial r$ is the partial derivative of F with respect to the radial direction.

Let $\{e_1,\ldots,e_n\}$ be the principal orthonormal frame of M_t as in the proof of Lemma 2.1 and be arranged such that under this frame the shape operator

$$S(t) = \mathrm{diag}(\mu_1,\ldots,\mu_n) = \mathrm{diag}(\lambda_1 I_{m_1},\ldots,\lambda_g I_{m_g}).$$

Then $\{e_1,\ldots,e_n,\nu_x := \nabla f(x)/|\nabla f(x)|\}$ is an orthonormal frame of $\mathcal{T}_xS^{n+1}$ and under this frame the Hessian H_f can be expressed as in formula (16), while $\{e_1,\ldots,e_n,\nu_x,x\}$ is an orthonormal frame of $\mathbb{R}^{n+2}$ at $x \in S^{n+1}$. It is easily seen that

$$H_F(e_i,x) = 0, \quad H_F(\nu_x,x) = (g-1)|\nabla f|, \quad H_F(x,x) = g(g-1)f,$$

and thus by (16), (22), the Hessian H_F at $x \in S^{n+1}$ can be expressed as

$$H_F = \begin{pmatrix} -\sqrt{b(f)}\mu_1 + gf & & & & \\ & \ddots & & & \\ & & -\sqrt{b(f)}\mu_n + gf & & \\ & & & \frac{b'(f)}{2} + gf & (g-1)\sqrt{b(f)} \\ & & & (g-1)\sqrt{b(f)} & g(g-1)f \end{pmatrix} \tag{23}$$

which has eigenvalues $\{-g\sqrt{1-f^2}\mu_1 + gf, \ldots, -g\sqrt{1-f^2}\mu_n + gf, g(g-1), -g(g-1)\}$ by using formula (5). Consequently, we have on $M_t = f^{-1}(t)$,

$$\begin{aligned}\bar{\rho}_k = m_1 &\sum_{i=1}^{[(g+1)/2]}\Big(-g\sqrt{1-t^2}\cot\Big(\tau+\frac{2(i-1)\pi}{g}\Big)+gt\Big)^k\\ &+m_2\sum_{i=1}^{[g/2]}\Big(-g\sqrt{1-t^2}\cot\Big(\tau+\frac{(2i-1)\pi}{g}\Big)+gt\Big)^k\\ &+g^k(g-1)^k(1+(-1)^k). \end{aligned}\tag{24}$$

At last, by taking derivative of $\bar{\rho}_k$ with respect to t in (24) and using the relation $t = \cos(g\tau)$, we arrive at the second inductive formula of the theorem. □

Example. As is well known, Cartan's polynomial $F : \mathbb{R}^{3m+2} \longrightarrow \mathbb{R}$ for isoparametric hypersurfaces in spheres with $g = 3$ distinct principal curvatures can be written as

$$F(x) = u^3 - 3uv^2 + \frac{3}{2}u(X\overline{X} + Y\overline{Y} - 2Z\overline{Z}) + \frac{3\sqrt{3}}{2}v(X\overline{X} - Y\overline{Y}) + \frac{3\sqrt{3}}{2}(XYZ + \overline{XYZ}).$$

In this formula, $x = (u, v, X, Y, Z) \in \mathbb{R}^{3m+2}$, u and v are real parameters, while X, Y, Z are coordinates in the algebra $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (Quaternions) or $\mathbb{O}$ (Cayley numbers), for the case $m_1 = m_2 = m = 1, 2, 4$, or 8, respectively. For example, we compute the Hessian of F in the case of $m = 1$:

$$H_F = 3\begin{pmatrix} 2u & -2v & X & Y & -2Z\\ -2v & -2u & \sqrt{3}X & -\sqrt{3}Y & 0\\ X & \sqrt{3}X & u+\sqrt{3}v & \sqrt{3}Z & \sqrt{3}Y\\ Y & -\sqrt{3}Y & \sqrt{3}Z & u-\sqrt{3}v & \sqrt{3}X\\ -2Z & 0 & \sqrt{3}Y & \sqrt{3}X & -2u \end{pmatrix}.$$

Then direct calculations lead to

$$\begin{gathered}\triangle_1(F) = 0, \quad \triangle_2(F) = -63|x|^2, \quad \triangle_3(F) = -54F,\\ \triangle_4(F) = 3^5\cdot 4|x|^4, \quad \triangle_5(F) = 2^3\cdot 3^5|x|^2F.\end{gathered}$$

3. Isoparametric hypersurfaces in complex projective spaces.

In this section we begin by establishing an equivalence condition for an isoparametric hypersurface $\widetilde{M}^{2n-1}$ in $\mathbb{C}P^n$ to have constant 3rd mean curvature H_3, that is the constancy of an S^1-invariant function, say α, on $M^{2n} := \pi^{-1}(\widetilde{M}^{2n-1}) \subset S^{2n+1}$ (π is the Hopf fibration). As a consequence, $\widetilde{M}^{2n-1}$ is 3-isoparametric if and only if α is constant on each nearby parallel hypersurface M_t of M^{2n}. Next, we turn to prove Theorem 1.1 and Theorem 1.2. In particular, we construct explicitly some S^1-invariant OT-FKM-type isoparametric polynomials on $\mathbb{R}^{4n+4}$, calculating the function α which turns out non-constant on some level hypersurface. In this way, we get finally the examples in (ii) of Theorem 1.1, as desired.

Let $\widetilde{M}^{2n-1}$ be a hypersurface in $\mathbb{C}P^n$ with unit normal vector field $\tilde{\nu}$. Observe that the unit normal vector field ν of $M^{2n} := \pi^{-1}(\widetilde{M}^{2n-1}) \subset S^{2n+1}$ (π is the Hopf fibration) is just the horizontal lift of $\tilde{\nu}$, i.e., $\pi_*\nu = \tilde{\nu}$. For simplicity, we will use the same symbols for Levi-Civita connections, shape operators, Hessians, etc., on spheres and Euclidean spaces as last section and only add a tilde to the corresponding symbols on $\mathbb{C}P^n$. Let $\tilde{J}$ be the complex structure on $\mathbb{C}P^n$ induced from the canonical complex structure J on $\mathbb{R}^{2n+2}$ by the Hopf fibration, so that Jx is the tangent vector field of the S^1-fibre through $x \in S^{2n+1}$. It follows that $J\nu$ is the horizontal lift of $\tilde{J}\tilde{\nu}$, thus a global unit tangent vector field of M^{2n} perpendicular with Jx. These arguments allow us to choose a local orthonormal basis $\{\tilde{e}_1, \ldots, \tilde{e}_{2n-2}, \tilde{J}\tilde{\nu}\}$ on $\widetilde{M}^{2n-1}$, so that the shape operator $\widetilde{S}_{\tilde{\nu}}$ of $\widetilde{M}^{2n-1}$ is expressed by a symmetric matrix $\widetilde{S}$. Following [**Wan82**], the shape operator S_ν of M^{2n} under the orthonormal basis $\{e_1, \ldots, e_{2n-2}, J\nu, Jx\}$ ($\pi_* e_i = \tilde{e}_i$) can be expressed by a symmetric matrix S:

$$S = \left(\begin{array}{ccc|c} & & & 0 \\ & \widetilde{S} & & \vdots \\ & & & -1 \\ \hline 0 & \cdots & -1 & 0 \end{array}\right) \tag{25}$$

To see the relation of S and $\widetilde{S}$ above, we remark that the Hopf fibration is a Riemannian submersion with totally geodesic S^1-fibres, and it follows that $[Jx, \nu] = 0$, and $\langle \widetilde{S}_{\tilde{\nu}}(\widetilde{X}), \widetilde{Y} \rangle = \langle S_\nu(X), Y \rangle$ for $\widetilde{X}, \widetilde{Y} \in \mathcal{T}\widetilde{M}^{2n-1}$ and their horizontal lift $X, Y \in \mathcal{T}M^{2n}$. Hence

$$-S_\nu(Jx) = \nabla_{Jx}\nu = \nabla_\nu Jx = JD_\nu x = J\nu. \tag{26}$$

Now define the S^1-invariant function α on M^{2n} (it will play an important role in this section) by

$$\alpha := \langle S_\nu(J\nu), J\nu \rangle = \langle \nu, \nabla_{J\nu} J\nu \rangle = \langle \widetilde{S}_{\tilde{\nu}}(\tilde{J}\tilde{\nu}), \tilde{J}\tilde{\nu} \rangle \circ \pi. \tag{27}$$

Therefore, using (25), we derive that

$$\begin{gathered} \sigma_1(S_\nu) = \sigma_1(\widetilde{S}_{\tilde{\nu}}) \circ \pi, \quad \sigma_2(S_\nu) = \sigma_2(\widetilde{S}_{\tilde{\nu}}) \circ \pi - 1, \\ \sigma_3(S_\nu) = \sigma_3(\widetilde{S}_{\tilde{\nu}}) \circ \pi - (\sigma_1(\widetilde{S}_{\tilde{\nu}}) \circ \pi - \alpha). \end{gathered} \tag{28}$$

Note that the inverse images under the Hopf fibration of parallel hypersurfaces in $\mathbb{C}P^n$ are still parallel hypersurfaces in S^{2n+1}. Now by (28) we obtain the following.

Proposition 3.1 (cf. [**Wan82**]). *A hypersurface $\widetilde{M}^{2n-1}$ in $\mathbb{C}P^n$ is isoparametric if and only if its inverse image $M^{2n} := \pi^{-1}(\widetilde{M}^{2n-1})$ under the Hopf fibration π is an isoparametric hypersurface in S^{2n+1}.*

Corollary 3.1. *Let $\widetilde{M}^{2n-1}$ be a 1-isoparametric hypersurface in $\mathbb{C}P^n$. Then*

a) *It must be 2-isoparametric*;
b) *It has constant 3rd mean curvature H_3 if and only if the function α defined by* (27) *is constant on $M^{2n} := \pi^{-1}(\widetilde{M})$*;
c) *It is 3-isoparametric if and only if the function α is constant on each (nearby) parallel hypersurface M_t of $M^{2n} := \pi^{-1}(\widetilde{M})$.*

Proof. It follows immediately from Proposition 3.1, Cartan's rigidity result (13), identities in (28), as well as Lemma 2.1. □

Suppose we are now given an isoparametric hypersurface $\widetilde{M}$ in $\mathbb{C}P^n$. Let $F : \mathbb{R}^{2n+2} \to \mathbb{R}$ be the isoparametric polynomial (satisfying Cartan–Münzner equations (3)–(4)) corresponding to the isoparametric hypersurface $M = \pi^{-1}(\widetilde{M}) \subset S^{2n+1}$, the inverse image of $\widetilde{M}$, and $f = F|_{S^{2n+1}}$. Denote by the same symbol J the matrix representation of the corresponding complex structure J in terms of the Euclidean coordinates[2] $x = (x_1, \dots, x_{2n+2})^t = \sum_{k=1}^{2n+2} x_k \partial x_k = Dx^t \cdot x$, namely,

$$J(Dx^t) := J(\partial x_1, \dots, \partial x_{2n+2}) = (\partial x_1, \dots, \partial x_{2n+2}) \cdot J = Dx^t \cdot J,$$

hence $J(V) = J \cdot V$, for any vector V on $\mathbb{R}^{2n+2}$.

In these notations, we are ready to give an explicit formula for the function α defined by (27) on M in terms of F and J. Indeed, α can be regarded as a function on $S^{2n+1} \backslash \{M_\pm\}$ as follows.

Proposition 3.2. *The function α on each parallel hypersurface $M_t := f^{-1}(t) = F^{-1}(t) \cap S^{2n+1}$ of M can be described as a sum*

$$\alpha = \frac{1}{g^3(1-F^2)^{3/2}} \{ g^3 F(3 - 2F^2) + \Omega_F \}, \tag{29}$$

where $\Omega_F := DF^t \cdot J \cdot D^2F \cdot J \cdot DF|_{S^{2n+1}}$ and $D^2F = D(DF^t)$ is the matrix of the Hessian H_F.

Proof. At any point $p \in M_t \subset S^{2n+1}$, it is clear that the spherical gradient of f is expressed by

$$\nabla f(p) = DF - \langle DF, p \rangle p = DF - gfp,$$

and we can define the unit normal vector field of M_t by

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[2]Throughout this paper, by using the congruence $\mathbb{R}^N \cong \mathcal{T}_x\mathbb{R}^N$, we identify $\partial x_k = \partial/\partial x_k = Dx_k$ with the k-th coordinate vector field $(0, \dots, 1, \dots, 0)^t$ for $1 \le k \le N$, and $Dx^t = (\partial x_1, \dots, \partial x_{2n+2})$ with the identity matrix I. The superscript t means transposition, vectors are written in columns as points and also regarded as $(N \times 1)$-matrices. The derivative D gives a column vector when it acts on a function as gradient, and thus gives a matrix when it acts on a row vector of functions. A dot "$\cdot$" between matrices means standard matrix product, and $\langle A, B \rangle := tr(A^t \cdot B)$ denotes the inner product of A, B in the $(m \times n)$ matrix space $M(m, n)$.

$$\nu = \nabla f/|\nabla f| = \nabla f/\sqrt{b}.$$

Thus $J(\nu)_p = (1/\sqrt{b})J(DF - gfp) = (1/\sqrt{b})J \cdot (DF - gfp)$, and by definition,

$$\begin{aligned}
\alpha|_p = \langle \nu_p, \nabla_{J\nu_p} J\nu \rangle &= \frac{1}{b\sqrt{b}} \langle DF - gfp, \nabla_{J(DF-gfp)} J(DF - gfx) \rangle \\
&= \frac{1}{b\sqrt{b}} \langle DF - gfp, D_{J(DF-gfp)} J(DF - gFx) \rangle \\
&= \frac{1}{b\sqrt{b}} \langle DF - gfp, J(D_{J(DF-gfp)} (DF - gFx)) \rangle \\
&= \frac{1}{b\sqrt{b}} \langle DF - gfp, J(D(DF - gFx)^t|_{x=p} \cdot J(DF - gfp)) \rangle \\
&= \frac{1}{b\sqrt{b}} \langle DF - gfp, J((D^2F - gDF \cdot x^t - gFDx^t)|_{x=p} \cdot J(DF - gfp)) \rangle \\
&= \frac{1}{b\sqrt{b}} \langle DF - gfp, J \cdot (D^2F - gDF \cdot p^t - gFI) \cdot J \cdot (DF - gfp) \rangle \\
&= \frac{1}{b\sqrt{b}} (DF^t - gfp^t) \cdot J \cdot (D^2F - gp \cdot DF^t - gFI) \cdot J \cdot (DF - gfp). \qquad (30)
\end{aligned}$$

where x is the position vector field extending p. Note that M_t is S^1-invariant and thus $Jp \in \mathcal{T}_p M_t$, which implies $DF^t Jp = \langle \nabla f, Jp \rangle = 0$ and thus $D^2F \cdot Jp = JDF$. In addition, we have on hand several simple equalities:

$$J^2 = -I, \quad p^t Jp = 0, \quad |\nabla f|^2 = b = g^2(1 - f^2),$$
$$|DF|^2|_p = g^2, \quad p^t \cdot DF = gF.$$

Applying these equalities, we conclude

$$\begin{aligned}
(DF^t - gFp^t) \cdot Jp \cdot DF^t &= 0, \\
(DF^t - gFp^t) \cdot J \cdot gFI \cdot J \cdot (DF - gFp) &= -g^3 f(1 - f^2), \\
(DF^t - gFp^t) \cdot J \cdot D^2F \cdot J \cdot (DF - gFp) &= DF^t \cdot J \cdot D^2F \cdot J \cdot DF + 2g^3F - g^3F^3.
\end{aligned}$$

Substituting all these equalities in (30), we get immediately the desired formula (29). □

Now we investigate the function α on the OT-FKM-type isoparametric hypersurfaces in spheres which almost cover all isoparametric hypersurfaces with four distinct principal curvatures (cf. [**CCJ07**]). For a symmetric Clifford system $\{A_0, \ldots, A_m\}$ on $\mathbb{R}^{2r}$, i.e., A_i's are symmetric matrices satisfying $A_iA_j + A_jA_i = 2\delta_{ij} I_{2r}$, the OT-FKM-type isoparametric polynomial F on $\mathbb{R}^{2r}$ is then defined as (cf. [**FKM81**]):

$$F(z) = |z|^4 - 2\sum_{p=0}^{m} \langle A_p z, z \rangle^2, \qquad (31)$$

where we take the coordinate system $z = (x^t, y^t)^t = (x_1, \ldots, x_r, y_1, \ldots, y_r)^t \in \mathbb{R}^{2r}$. By orthogonal transformations, we can write

$$A_0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right), \qquad A_1 = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right), \tag{32}$$

$$A_j = \left(\begin{array}{c|c} 0 & -E_j \\ \hline E_j & 0 \end{array}\right), \qquad j = 2, \ldots, m,$$

where $\{E_2, \ldots, E_m\}$ is a skew-symmetric Clifford system on $\mathbb{R}^r$, i.e., E_i's are skew-symmetric matrices satisfying $E_iE_j + E_jE_i = -2\delta_{ij}I_r$. It can be verified that the level hypersurfaces of this polynomial restricted to the unit sphere have 4 distinct constant principal curvatures with multiplicities $m_1 = m$ and $m_2 = r-m-1$, provided $r-m-1 > 0$. Now fixing a complex structure J on $\mathbb{R}^{2r}$ under the coordinate system as $J = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$, we define the corresponding S^1-action on $\mathbb{R}^{2r}$ by $e^{i\theta} \cdot z = (\cos\theta + \sqrt{-1}\sin\theta)z = \cos\theta z + \sin\theta Jz$. We prepare in advance the following equalities which will be useful later.

$$A_0J = -JA_0 = -A_1, \quad A_1J = -JA_1 = A_0, \quad A_0A_1 = -A_1A_0 = -J, \tag{33}$$
$$A_jJ = JA_j, \quad \text{for } j = 2, \ldots, m.$$

PROPOSITION 3.3. *The OT-FKM-type isoparametric polynomial F defined by* (31) *and* (32) *is S^1-invariant under the fixed complex structure J and thus induces an isoparametric function $\tilde{f}$ on $\mathbb{C}P^{r-1}$ through the Hopf fibration. Moreover, the function Ω_F defined in* (29) *at the point $z \in S^{2r-1}$ can be expressed as a sum*

$$\begin{aligned} \frac{1}{64}\Omega_F &= 2F^2 - F - 2 + 8(1+F)\big(\langle A_0z, z\rangle^2 + \langle A_1z, z\rangle^2\big) \\ &\quad + 16\sum_{q=2}^{m}\left(\sum_{p=2}^{m}\langle A_pz, z\rangle\langle A_qz, JA_pz\rangle\right)^2. \end{aligned} \tag{34}$$

REMARK 3.1. When $m = 1$, $\Omega_F = 64(-2F^2 - F + 2)$ and thus α is constant on each level hypersurface of $f = F|_{S^{2r-1}}$. Consequently, it follows from Proposition 3.1 that the isoparametric function $\tilde{f}$ on $\mathbb{C}P^{r-1}$ induced from f is now 3-isoparametric.

PROOF. By a direct calculation using (33), we have for $j \geq 2$ that

$$\langle A_jz, z\rangle = \langle A_jJz, Jz\rangle, \quad \langle A_jJz, z\rangle = 0,$$

which imply

$$\langle A_je^{i\theta}z, e^{i\theta}z\rangle = \langle A_j(\cos\theta z + \sin\theta Jz), \cos\theta z + \sin\theta Jz\rangle = \langle A_jz, z\rangle.$$

Using

$$\langle A_0 e^{i\theta} z, e^{i\theta} z\rangle^2 + \langle A_1 e^{i\theta} z, e^{i\theta} z\rangle^2 = \langle A_0 z, z\rangle^2 + \langle A_1 z, z\rangle^2,$$

we verify the S^1-invariance of F, i.e., $F(e^{i\theta} z) = F(z)$ for any $e^{i\theta} \in S^1$.

To compute Ω_F, first we observe that

$$\begin{aligned}
\frac{1}{4} DF &= |z|^2 z - 2\sum_{p=0}^{m} \langle A_p z, z\rangle A_p z,\\
\frac{1}{4} D^2 F &= |z|^2 I + 2zz^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle A_p - 4\sum_{p=0}^{m} A_p z z^t A_p.
\end{aligned}$$

Then by definition,

$$\begin{aligned}
\frac{1}{64}\Omega_F = &\left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot J \cdot \left(I + 2zz^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle A_p - 4\sum_{p=0}^{m} A_p z z^t A_p\right)\\
&\cdot J \cdot \left(z - 2\sum_{p=0}^{m} \langle A_p z, z\rangle A_p z\right),
\end{aligned}$$

which will be calculated by 4 parts as follows:

$$\begin{aligned}
\text{(i)}\ &\left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot J \cdot I \cdot J \cdot \left(z - 2\sum_{q=0}^{m} \langle A_q z, z\rangle A_q z\right)\\
&= \frac{1}{4} DF^t \cdot J \cdot I \cdot J \cdot \frac{1}{4} DF = -\frac{1}{16}|DF|^2 = -1;\\
\text{(ii)}\ &\left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot J \cdot 2zz^t \cdot J \cdot \left(z - 2\sum_{q=0}^{m} \langle A_q z, z\rangle A_q z\right)\\
&= 8\sum_{p,q=0}^{m} \langle A_p z, z\rangle \langle A_q z, z\rangle z^t A_p J z \cdot z^t J A_q z\\
&= -8\left(\sum_{p=0}^{1} \langle A_p z, z\rangle \langle J A_p z, z\rangle\right)^2 = 0;\\
\text{(iii)}\ &\left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot J \cdot \left(-2\sum_{q=0}^{m} \langle A_q z, z\rangle A_q\right) \cdot J \cdot \left(z - 2\sum_{j=0}^{m} \langle A_j z, z\rangle A_j z\right)\\
&= -2\sum_{q=0}^{m} \langle A_q z, z\rangle \left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot J A_q J \cdot \left(z - 2\sum_{j=0}^{m} \langle A_j z, z\rangle A_j z\right)\\
&= -2\sum_{q=0}^{1} \langle A_q z, z\rangle \left\{\left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot A_q \cdot \left(z - 2\sum_{j=0}^{m} \langle A_j z, z\rangle A_j z\right)\right\}\\
&\quad +2\sum_{q=2}^{m} \langle A_q z, z\rangle \left\{\left(z^t - 2\sum_{p=0}^{m} \langle A_p z, z\rangle z^t A_p\right) \cdot A_q \cdot \left(z - 2\sum_{j=0}^{m} \langle A_j z, z\rangle A_j z\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= 6\big(\langle A_0z,z\rangle^2+\langle A_1z,z\rangle^2\big)-6\sum_{p=2}^{m}\langle A_pz,z\rangle^2\\
&\quad -8\sum_{q=0}^{1}\langle A_qz,z\rangle\Big(\langle A_qz,z\rangle^3+\sum_{p\neq q}\langle A_pz,z\rangle^2\langle A_qz,z\rangle\Big)\\
&\quad +8\sum_{q=2}^{m}\langle A_qz,z\rangle\Big(\langle A_qz,z\rangle^3+\sum_{p\neq q}\langle A_pz,z\rangle^2\langle A_qz,z\rangle\Big)\\
&= -3(1-F)+12\big(\langle A_0z,z\rangle^2+\langle A_1z,z\rangle^2\big)+2(1-F)^2\\
&\quad -8\big(\langle A_0z,z\rangle^2+\langle A_1z,z\rangle^2\big)(1-F);
\end{aligned}
$$

(iv) $$\Big(z^t-2\sum_{p=0}^{m}\langle A_pz,z\rangle z^tA_p\Big)\cdot J\cdot\Big(-4\sum_{q=0}^{m}A_qzz^tA_q\Big)\cdot J\cdot\Big(z-2\sum_{j=0}^{m}\langle A_jz,z\rangle A_jz\Big)$$

$$
\begin{aligned}
&= -4\sum_{q=0}^{1}z^tJA_qz\cdot z^tA_qJz+16\sum_{p,q}\langle A_pz,z\rangle z^tA_pJA_qz\cdot z^tA_qJz\\
&\quad -16\sum_{p,q,j}\langle A_pz,z\rangle\langle A_jz,z\rangle z^tA_pJA_qz\cdot z^tA_qJA_jz\\
&= 4\big(\langle A_0z,z\rangle^2+\langle A_1z,z\rangle^2\big)+16\sum_{p,q=0}^{1}\langle A_pz,z\rangle\langle A_pz,JA_qz\rangle\langle A_qJz,z\rangle\\
&\quad +16\sum_{q=0}^{m}\Big(\sum_{p=0}^{m}\langle A_pz,z\rangle\langle A_qz,JA_pz\rangle\Big)^2\\
&= 4\big(\langle A_0z,z\rangle^2+\langle A_1z,z\rangle^2\big)+16\sum_{q=2}^{m}\Big(\sum_{p=2}^{m}\langle A_pz,z\rangle\langle A_qz,JA_pz\rangle\Big)^2.
\end{aligned}
$$

Finally, taking sum of (i), (ii), (iii), (iv), we complete the proof of the proposition. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 (i). We prefer to prove this assertion by making use of several known results although there might be some direct approaches. The notations remain the same as before.

First, a result of Park [**Par89**] asserts that: If $M^{2N}=\pi^{-1}(\widetilde{M}^{2N-1})$ is the inverse image of an isoparametric hypersurface $\widetilde{M}^{2N-1}$ in $\mathbb{C}P^N$, then the number g of distinct principal curvatures of the isoparametric hypersurface M in S^{2N+1} must be 2, 4 or 6; and if $g=6$, then the two multiplicities satisfy $m_1=m_2=1$ and thus $N=3$ in this case. So when $N=2n$ is even, g must be only 2, or 4. Hence it suffices to analyze these two cases for our aim.

When $g=2$, Proposition 2.1 in [**Xia00**] stated that $\widetilde{M}$ has 2 or 3 constant principal curvatures, thus is totally isoparametric and homogeneous by equivalence sequence (14).

When $g=4$, firstly, according to Abresch [**Abr83**], we can show that either one of the multiplicities $\{m_1,m_2\}$ equals 1, or $m_1=m_2=2$. In fact, since $\dim_R\mathbb{C}P^{2n}=$

$4n$, it follows that the corresponding isoparametric hypersurface in the sphere is of $4n$ dimension, that is, $m_1 + m_2 = 2n$. Hence in the main theorem of [**Abr83**], the case $4A$ is excluded; the case $4B_1$ occurs only when $\min\{m_1, m_2\} = 1$; and the case $4B_2$ occurs only when $m_1 = m_2 = 2$, as we claimed.

In the first case, i.e., $\min\{m_1, m_2\} = 1$, by virtue of [**Tak76**], the isoparametric hypersurface M^{4n} in the sphere S^{4n+1} must be homogeneous, and corresponds to the isotropy representation of the rank two symmetric space $W := SO(2n+3)/S(O(2) \times O(2n+1))$, where $2n = m_1 + m_2$. Let $\mathfrak{o}(2n+3) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, where $\mathfrak{k}$ is the Lie algebra of $O(2) \times O(2n+1)$. Then the isoparametric hypersurface M^{4n} is congruent to a principal orbit of the adjoint action of $S(O(2) \times O(2n+1))$ on the vector space $\mathfrak{p} \cong \mathbb{R}^{4n+2}$. In this representation, it is not difficult to show that there is a unique complex structure (up to a sign, the standard complex structure) on $\mathfrak{p}$ such that M^{4n} is S^1-invariant with respect to this complex structure, as [**Xia00**] claimed. This property helps us deduce that the number l of non-horizontal principal eigenspaces of M^{4n} equals 2 identically. Hence by the equivalence sequence (14), $\widetilde{M}^{4n-1}$ is totally isoparametric and homogeneous.

At last, we need to prove that the second case, i.e., $m_1 = m_2 = 2$, is impossible. Without loss of generality, we can assume that M^{4n} is compact. Recall that a topological theorem of Münzner [**Mün80**] determines the cohomology rings of a compact isoparametric hypersurface in a sphere. By applying it, we have $H^q(M^{4n}, \mathbb{Z}_2) = 0$ for any odd number q, $H^0(M^{4n}, \mathbb{Z}_2) = H^{4n}(M^{4n}, \mathbb{Z}_2) = \mathbb{Z}_2$, and $H^{2k}(M^{4n}, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, for $k = 1, \ldots, 2n-1$. It follows from Poincaré duality that the Euler characteristic $\chi(M)$ of M^{4n} is equal to $2g = 8 > 0$. On the other hand, since M^{4n} is the inverse image of a hypersurface $\widetilde{M}^{4n-1}$ in $\mathbb{C}P^{2n}$, Jx (x is the position vector of M) is a globally defined tangent vector field without singularities on M and thus by the Hopf index theorem, the Euler characteristic $\chi(M) = 0$, a contradiction which completes the proof of Theorem 1.1(i). $\square$

Proof of Theorem 1.1 (ii). By Proposition 3.1 and Corollary 3.1, it suffices to construct a required S^1-invariant isoparametric hypersurface M (resp. isoparametric polynomial F) in S^{4n+3} (resp. on $\mathbb{R}^{4n+4}$) such that the function α is non-constant on M. More specifically, because of Proposition 3.3 we will look for symmetric Clifford systems $\{A_0, A_1, \ldots, A_m\}$ on $\mathbb{R}^{4n+4}$ in the form (32), such that the function Ω_F for the corresponding OT-FKM-type isoparametric polynomial F defined by (31), could be not only computed explicitly by formula (34), but also non-constant on some level hypersurface M of $F|_{S^{4n+3}}$. Towards the aim, the first non-trivial case is when $m = 2$, where we successfully find an example for each $n \geq 1$.

Note that when $m = 2$, the formula (34) can be deduced to

$$\Omega_F = 64\big(-2F^2 - F + 2 - 8(1+F)\langle A_2 z, z\rangle^2\big). \tag{35}$$

Let $E_2 = \left(\begin{array}{c|c} 0 & -I_{n+1} \\ \hline I_{n+1} & 0 \end{array}\right)$ be the sub-matrix of A_2 in (32) with $2r = 4n+4$. Then it is easily verified that $\{A_0, A_1, A_2\}$ is now a symmetric Clifford system.

Now for $z = (x^t, y^t)^t = (x_1, \ldots, x_{2n+2}, y_1, \ldots, y_{2n+2})^t \in \mathbb{R}^{4n+4}$ $(n \geq 1)$, the OT-FKM-type isoparametric polynomial F can be written as

$$F(z) = |z|^4 - 2\{(|x|^2 - |y|^2)^2 + 4\langle x, y\rangle^2 + 4\langle E_2 x, y\rangle^2\}. \tag{36}$$

Let $M = F^{-1}(0) \bigcap S^{4n+3}$, and $z = (x^t, y^t)^t$, $\check{z} = (\check{x}^t, \check{y}^t)^t$ be two points in M with $x_1 = \check{x}_1 = 1/\sqrt{2}$, $y_1 = y_2 = \check{y}_{n+2} = \check{y}_{n+3} = 1/2$ and the other coordinates vanishing. Then it is easily calculated that $\Omega_F(z) = 128$ and $\Omega_F(\check{z}) = -128$. Equivalently, α is non-constant on M. Therefore, the isoparametric hypersurface $\widetilde{M} = \pi(M)$ in $\mathbb{C}P^{2n+1}$ is not 3-isoparametric, as desired. $\square$

It is worthy remarking that the isoparametric polynomial F in (36) also induces a homogeneous hypersurface in $\mathbb{C}P^{2n+1}$ under some other complex structure, by comparing Takagi's ([**Tak73**]) classification. Next, we calculate the function α (or equivalently Ω_F defined in (29)) explicitly for the inhomogeneous example of Ozeki–Takeuchi [**OT75**] with $g = 4$ and multiplicities $(m_1, m_2) = (3, 4r)$ under two different complex structures. Both functions turn out non-constant on a level hypersurface in the sphere. As a result, these two induced isoparametric hypersurfaces in $\mathbb{C}P^{4r+3}$ are not 3-isoparametric. From another point of view, by comparing Takagi's classification, we know that these two induced isoparametric hypersurfaces in $\mathbb{C}P^{4r+3}$ are not homogeneous, and hence by Theorem 1.2 they are not 3-isoparametric.

More Examples. First we decompose the quaternionic space $\mathbb{H}^{2r+2} \cong \mathbb{R}^{8r+8}$ $(r \geq 1)$ as $\mathbb{H}^{2r+2} = (\mathbb{H} \times \mathbb{H}^r) \times (\mathbb{H} \times \mathbb{H}^r)$, i.e., for

$$z = (x_1, \ldots, x_{4r+4}, y_1, \ldots, y_{4r+4})^t \in \mathbb{R}^{8r+8} \cong \mathbb{H}^{2r+2},$$

we write $z = (u^t, v^t)^t$, where $u = (u_0, \hat{u}) \in \mathbb{H} \times \mathbb{H}^r$, $v = (v_0, \hat{v}) \in \mathbb{H} \times \mathbb{H}^r$, $\hat{u} = (u_1, \ldots, u_r)$, $\hat{v} = (v_1, \ldots, v_r)$, and

$$u_i = x_{4i+1} + x_{4i+2}\boldsymbol{i} + x_{4i+3}\boldsymbol{j} + x_{4i+4}\boldsymbol{k} \in \mathbb{H} \cong \mathbb{R}^4,$$
$$v_i = y_{4i+1} + y_{4i+2}\boldsymbol{i} + y_{4i+3}\boldsymbol{j} + y_{4i+4}\boldsymbol{k} \in \mathbb{H} \cong \mathbb{R}^4.$$

Then the isoparametric polynomial F of the inhomogeneous example of Ozeki–Takeuchi [**OT75**] with $g = 4$ and multiplicities $(m_1, m_2) = (3, 4r)$ is defined by

$$F(z) = |z|^4 - 2\{4(|u \cdot \bar{v}^t|^2 - \langle u, v\rangle^2) + (|\hat{u}|^2 - |\hat{v}|^2 + 2\langle u_0, v_0\rangle)^2\}, \tag{37}$$

where the canonical involution $\bar{v}_i = y_{4i+1} - y_{4i+2}\boldsymbol{i} - y_{4i+3}\boldsymbol{j} - y_{4i+4}\boldsymbol{k}$ and the quaternionic multiplication in $\mathbb{H}$ are used.

Let

$$A_0 = \begin{pmatrix} 0 & & I_4 & \\ & I_{4r} & & 0 \\ I_4 & & 0 & \\ & 0 & & -I_{4r} \end{pmatrix}, \quad A_p = \begin{pmatrix} & & D_p & & \\ & & & \ddots & \\ & & & & D_p \\ -D_p & & & & \\ & \ddots & & & \\ & & -D_p & & \end{pmatrix} \quad \text{for } p = 1, 2, 3,$$

where $D_1 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$, $D_2 = \begin{pmatrix} & & -1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & -1 & & \end{pmatrix}$, and $D_3 = \begin{pmatrix} & & 0 & -1 \\ & & -1 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{pmatrix}$. Then a straightforward calculation shows that $\{A_0, A_1, A_2, A_3\}$ is a symmetric Clifford system (though not in the form (32)) and the polynomial F defined by (37) can also be expressed as the OT-FKM-type isoparametric polynomial:

$$F = |z|^4 - 2\sum_{p=0}^{3} \langle A_p z, z\rangle^2. \tag{38}$$

In the following, we will calculate the function α under two different complex structures:

(i) Let J be the complex structure on $\mathbb{R}^{8r+8} \cong \mathbb{H}^{2r+2}$ as the orthogonal transformation induced by the right multiplication of $\boldsymbol{i}$ whose matrix representation is

$$J = \begin{pmatrix} D_0 & & & \\ & D_0 & & \\ & & \ddots & \\ & & & D_0 \end{pmatrix} \quad \text{where} \quad D_0 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}. \tag{39}$$

Evidently, F is S^1-invariant under this complex structure and $f = F|_{S^{8r+7}}$ thus induces an isoparametric function $\tilde{f}$ on $\mathbb{C}P^{4r+3}$ through the corresponding Hopf fibration.

By direct calculations , we have the following relations

$$D_0 D_p = D_p D_0, \quad JA_p = A_p J, \quad \text{for} \quad p = 0, 1, 2, 3,$$
$$D_1 D_2 = -D_2 D_1 = D_3, \quad D_2 D_3 = -D_3 D_2 = D_1, \quad D_3 D_1 = -D_1 D_3 = D_2.$$

Therefore,

$$(JA_p)^t = -JA_p, \quad (A_p J A_q)^t = A_p J A_q, \quad \text{for} \quad p, q = 0, 1, 2, 3, \quad p \neq q,$$

which help us deduce the formula for Ω_F (defined in (29)) under J in (39) as

$$\Omega_F = 64\left\{2F^2 - F - 2 + 16\sum_{q=0}^{3}\left(\sum_{p=0}^{3} \langle A_p z, z\rangle \langle JA_q z, A_p z\rangle\right)^2\right\}.$$

(ii) Let J' be another complex structure on $\mathbb{R}^{8r+8} \cong \mathbb{H}^{2r+2}$ as the orthogonal transformation induced by the left multiplication of $\boldsymbol{i}$ whose matrix representation is

$$J' = \begin{pmatrix} D_1 & & & \\ & D_1 & & \\ & & \ddots & \\ & & & D_1 \end{pmatrix}, \quad \text{where } D_1 \text{ was given before.} \tag{40}$$

Similarly, F is also S^1-invariant under J' and thus induces an isoparametric function $\tilde{f}'$ on $\mathbb{C}P^{4r+3}$ through the corresponding Hopf fibration.

Again, by direct calculations, we have the following relations

$$\begin{aligned} J'A_0 &= A_0J', & J'A_1 &= A_1J' \\ J'A_2 &= -A_2J' = A_3, & J'A_3 &= -A_3J' = -A_2, \end{aligned}$$

and hence $A_pJ'A_q$ is skew-symmetric for almost all $p \neq q$ except for $A_0J'A_1$, $A_1J'A_0$, $A_2J'A_3 = -I$, $A_3J'A_2 = I$ which are symmetric. Then we can deduce the formula for Ω_F (defined in (29)) under J' in (40) as

$$\begin{aligned} \frac{1}{64}\Omega_F = 2F^2 - F - 2 + 8(1+F)\big(\langle A_2z, z\rangle^2 + \langle A_3z, z\rangle^2\big) \\ + 16\big(\langle A_0z, z\rangle^2 + \langle A_1z, z\rangle^2\big)\langle A_0J'A_1z, z\rangle^2. \end{aligned}$$

In conclusion, let $M = F^{-1}(0)\bigcap S^{8r+7}$, and $z = (x^t, y^t)^t$, $\check{z} = (\check{x}^t, \check{y}^t)^t$ be two points in M with $x_1 = (1/2)\sqrt{2+\sqrt{2}}$, $y_1 = \check{y}_5 = (1/2)\sqrt{2-\sqrt{2}}$, $\check{x}_1 = \check{y}_1 = (1/2\sqrt{2})\sqrt{2+\sqrt{2}}$, and the other coordinates vanishing. Then it is easily calculated that, under both complex structures J, J' defined in (i), (ii) above, $\Omega_F(z) = 128$ and $\Omega_F(\check{z}) = -128$. Therefore, α is non-constant on M under both J and J'. This means that the isoparametric hypersurfaces $\widetilde{M} = \pi(M)$, $\widetilde{M}' = \pi'(M)$ in $\mathbb{C}P^{4r+3}$ are not 3-isoparametric, where π and π' are the corresponding Hopf fibrations $S^{8r+7} \longrightarrow \mathbb{C}P^{4r+3}$ induced by J and J', respectively.

To conclude this section, we come to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Suppose that $\widetilde{M}^{2n-1}$ is an isoparametric hypersurface in $\mathbb{C}P^n$ of constant 3rd mean curvature H_3 with unit normal vector field $\tilde{\nu}$. Then $M^{2n} = \pi^{-1}(\widetilde{M}^{2n-1})$ is an isoparametric hypersurface in S^{2n+1}. As we pointed out before, it has $g = 2, 4$, or 6 distinct constant principal curvatures $\lambda_1 > \cdots > \lambda_g$. Let T_{λ_i} be the principal distribution on M corresponding to λ_i and thus $\mathcal{T}M = T_{\lambda_1} \oplus \cdots \oplus T_{\lambda_g}$. By Corollary 3.1, $\alpha := \langle S_\nu J\nu, J\nu\rangle$ is now constant on M. We will use the same notations as those at the beginning of this section.

Let x be the position vector field of M. Then Jx is the vertical vector field tangent to the S^1-fibres of the Hopf fibration. Represent Jx as

$$Jx = \phi_1\epsilon_1 + \cdots + \phi_g\epsilon_g, \tag{41}$$

where $\epsilon_i \in T_{\lambda_i}$ is a unit vector and $\phi_i \geq 0$ is the length of the component of Jx in T_{λ_i} for $i = 1, \ldots, g$. It follows from (26) and (41) that

$$S_\nu Jx = \lambda_1\phi_1\epsilon_1 + \cdots + \lambda_g\phi_g\epsilon_g = -J\nu, \tag{42}$$

which together with the fact that $Jx, J\nu$ are orthogonal unit vectors implies

$$\phi_1^2 + \cdots + \phi_g^2 = 1, \quad \lambda_1\phi_1^2 + \cdots + \lambda_g\phi_g^2 = 0, \quad \lambda_1^2\phi_1^2 + \cdots + \lambda_g^2\phi_g^2 = 1. \tag{43}$$

Similarly, by (42) we have

$$\alpha = \langle S_\nu J\nu, J\nu\rangle = \lambda_1^3\phi_1^2 + \cdots + \lambda_g^3\phi_g^2. \tag{44}$$

Note that the number l of non-horizontal eigenspaces of S_ν equals the number of non-zero ϕ_i's. By the equivalence sequence (14), it suffices to prove $l \equiv$ const case by case with respect to $g = 2, 4$, or 6.

(i) When $g = 2$, it follows immediately from (43) that

$$\phi_1 = \sqrt{\frac{-\lambda_2}{\lambda_1 - \lambda_2}}, \quad \phi_2 = \sqrt{\frac{\lambda_1}{\lambda_1 - \lambda_2}}, \quad \lambda_1 = -\frac{1}{\lambda_2} > 0,$$

which imply $\alpha = \lambda_1 + \lambda_2 \equiv$ const, $l \equiv 2$ and thus $\widetilde{M}$ is homogeneous by (14) (see also [**Xia00**]).

(ii) When $g = 4$, it follows immediately from (43) and (44) that

$$\begin{pmatrix}\phi_1^2\\ \phi_2^2\\ \phi_3^2\\ \phi_4^2\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & 1\\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2\\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3\end{pmatrix}^{-1} \begin{pmatrix}1\\ 0\\ 1\\ \alpha\end{pmatrix},$$

which implies that $l \equiv$ const provided $\alpha \equiv$ const and thus $\widetilde{M}$ is homogeneous by (14).

(iii) When $g = 6$, as mentioned before, [**Par89**] proved that $m_1 = m_2 = m$ must be 1. We need only to prove that α is always non-constant in this case. By virtue of [**Xia00**], for any c_1, c_2, c_3 satisfying $c_1^2 + c_2^2 + c_3^2 = 1$, there is a point $x \in M$ such that

$$\phi_i(x)^2 = \frac{c_i^2}{1 + \lambda_i^2}, \quad \phi_{i+3}(x)^2 = \frac{c_i^2}{1 + \lambda_{i+3}^2}, \quad \text{for } i = 1, 2, 3.$$

Then

$$\alpha(x) = (\lambda_1 + \lambda_4)c_1^2 + (\lambda_2 + \lambda_5)c_2^2 + (\lambda_3 + \lambda_6)c_3^2.$$

Now let $c_1 = 1,\ c_2 = c_3 = 0$ and $x' \in M$ be the corresponding point. Then $\alpha(x') = \lambda_1 + \lambda_4$. Similarly, let $c_1 = c_3 = 0$, $c_2 = 1$ and $x'' \in M$ be the corresponding point. Then $\alpha(x'') = \lambda_2 + \lambda_5$. Therefore, α is always non-constant on M as we required.

The proof is now complete. □

Proof of Corollary 1.1. Assume that $\widetilde{M}$ is an inhomogeneous hypersurface in $\mathbb{C}P^n$ with constant 1st, 2nd, and 3rd mean curvatures H_1, H_2, H_3. Then by equalities (28), the inverse image $M = \pi^{-1}(\widetilde{M})$ in S^{2n+1} under the Hopf fibration has constant 1st mean curvature H_1 and constant 2nd mean curvature $H_2 - 1$. It suffices to show that M is not an isoparametric hypersurface. We will prove this by contradiction.

Suppose M is isoparametric. It follows from Proposition 3.1 that $\widetilde{M}$ is also an isoparametric hypersurface in $\mathbb{C}P^n$. Since $\widetilde{M}$ has constant 3rd mean curvature by assumption, it is homogeneous by Theorem 1.2, which contradicts the assumption that $\widetilde{M}$ is an inhomogeneous hypersurface. $\square$

4. Isoparametric hypersurfaces in rank one symmetric spaces.

In this section, by using the Riccati equation we first derive some "weakly" inductive formulae for $Q_k := \rho_k(S(t))$ on parallel hypersurfaces M_t in a general Riemannian manifold in the spirit of Theorem 2.1. Next, by using further symmetries of the Jacobi operator on a complex space form, more generally, on a locally rank one symmetric space, we will prove Theorem 1.3 and Theorem 1.4.

For our purpose, let $\{M_t : t \in (-\varepsilon, \varepsilon)\}$ be a family of parallel hypersurfaces in a Riemannian manifold N^{n+1}, ν_t the unit normal vector field on M_t, $S(t) = S_{\nu_t}$ the shape operator of M_t, and $R(t) = K_{\nu_t}$ the normal Jacobi operator (see definition in Remark 1.2) on M_t. It is convenient to denote the covariant derivatives of the operators $S(t)$, $R(t)$ along normal geodesics of M_t by $S'(t) := \nabla_{\nu_t} S(t)$, $R'(t) := \nabla_{\nu_t} R(t)$, respectively. In this way, the well known Riccati equation can be given by (cf. [**Gra04**]):

$$S'(t) = S(t)^2 + R(t). \tag{45}$$

By taking trace with respect to a parallel orthonormal frame $\{E_1, \ldots, E_n\}$ along a normal geodesic of M_t, we get the Riccati equation for the mean curvature $H(t)$ of M_t:

$$H'(t) = \|S(t)\|^2 + Ric(t), \tag{46}$$

where $Ric(t) = Ric(\nu_t, \nu_t)$ denotes the Ricci curvature of N in the normal direction of M_t.

Proposition 4.1. *A 1-isoparametric hypersurface in an Einstein manifold N^{n+1} must be 2-isoparametric.*

Proof. Observe that $H(t)$ is now a function depending only on t, and hence $H'(t)$ is constant on M_t. Since $Ric(t) \equiv \rho$ the Einstein constant, the conclusion follows immediately from the equality (46), Lemma 2.1 and Newton's identities (8). $\square$

For $i \geq 0,\ j \geq 0$, $t \in (-\varepsilon, \varepsilon)$, we introduce a function $\Gamma_{ij}(t)$ on M_t by

$$\Gamma_{ij}(t) := tr\big(S(t)^i R(t)^j\big). \tag{47}$$

Clearly, $\Gamma_{i0}(t) = tr(S(t)^i) = \rho_i(S(t)) =: Q_i(t)$. As discussed before, applying Lemma 2.1 and Newton's identities (8), once we find some inductive formulae for the Q_i's on M_t as those in Theorem 2.1, we would establish similar rigidity results with Cartan's rigidity result (13), Theorem 1.1 (i) and Theorem 1.2. Towards this aim, we simply take a derivative of Q_i with respect to t to obtain:

Lemma 4.1. *With notations as above,*

$$Q_{i+1}(t) = \frac{1}{i} Q_i'(t) - \Gamma_{i-1,1}(t). \tag{48}$$

Proof. It follows directly from the definitions and the Riccati equation (45) that

$$\begin{aligned}
Q_i'(t) = tr\big(\nabla_{\nu_t} S(t)^i\big) &= \sum_{j=0}^{i-1} tr\big(S(t)^j \big(\nabla_{\nu_t} S(t)\big) S(t)^{i-1-j}\big) \\
&= i\ tr\big(S(t)^{i-1} S'(t)\big) = i\big(tr(S(t)^{i+1}) + tr(S(t)^{i-1} R(t))\big) \\
&= i\ \big(Q_{i+1}(t) + \Gamma_{i-1,1}(t)\big)
\end{aligned}$$

as required. □

However, the inductive formula (48) does not work effectively unless the functions $\Gamma_{i-1,1}(t)$ are constant on M_t. So we need to investigate some inductive properties of the functions $\Gamma_{i-1,1}(t)$ as the following.

Lemma 4.2. *With notations as above,*

$$\Gamma_{i+1,1}(t) = \frac{1}{i}\left\{\Gamma_{i1}'(t) - \sum_{j=0}^{i-1} tr\big(S(t)^j R(t) S(t)^{i-1-j} R(t)\big) - tr\big(S(t)^i R'(t)\big)\right\}. \tag{49}$$

Proof. Similarly, it follows directly from the definitions and the Riccati equation (45) that

$$\begin{aligned}
\Gamma_{i1}'(t) &= \sum_{j=0}^{i-1} tr\big(S(t)^j S'(t) S(t)^{i-1-j} R(t)\big) + tr\big(S(t)^i R'(t)\big) \\
&= i\ \Gamma_{i+1,1}(t) + \sum_{j=0}^{i-1} tr\big(S(t)^j R(t) S(t)^{i-1-j} R(t)\big) + tr\big(S(t)^i R'(t)\big).
\end{aligned}$$ □

In order to make these inductive formulae work effectively, we put some restrictions on the ambient manifold so as to control the last two terms on the right side of (49). To be more precise, we have the following result stated in Remark 1.2.

Corollary 4.1. *Let N^{n+1} be a locally symmetric space with the property that, the first and second elementary symmetric polynomials on eigenvalues of its Jacobi operator K_ξ are constant and independent of the choice of the unit tangent vector $\xi \in \mathcal{T}N$. Then any 1-isoparametric hypersurface in N must be 2-isoparametric, additionally, any 3-isoparametric hypersurface in N must be 4-isoparametric.*

Proof. The first assertion is a consequence of Proposition 4.1, since N is now an Einstein manifold. To prove the second assertion, it suffices to show that

$$Q_4(t) = \Gamma_{40}(t) = tr(S(t)^4)$$

is constant on each nearby parallel hypersurface M_t of a 3-isoparametric hypersurface M in N for $t \in (-\varepsilon, \varepsilon)$. Since M is 3-isoparametric, $Q_1(t)$, $Q_2(t)$ and $Q_3(t)$ are all constant on M_t and thus smooth functions depending only on t. Then by (48),

$$\Gamma_{11}(t) = \frac{1}{2}Q_2'(t) - Q_3(t)$$

is a smooth function depending only on t. Consequently, by (49),

$$\Gamma_{21}(t) = \Gamma_{11}'(t) - tr(R(t)^2)$$

is a smooth function depending only on t, since by assumptions we have

$$tr(R(t)^2) \equiv \text{const} \quad \text{and} \quad R'(t) \equiv 0.$$

Again by (48),

$$Q_4(t) = \frac{1}{3}Q_3'(t) - \Gamma_{21}(t)$$

is a smooth function depending only on t, which completes the proof. □

As mentioned in Remark 1.2, the restrictions we put in this corollary is not so strong that, there exist many locally symmetric spaces with rank greater than one satisfying these conditions. Now we will be concerned with the locally rank one symmetric spaces N^{n+1}. Obviously, the Jacobi operator K_ξ has constant eigenvalues independent of the choice of the unit tangent vector $\xi \in \mathcal{T}N$. This property will be useful in the establishment of further rigidity results. To warm up before the proof of the theorems, we deal with the case when K_ξ has only one constant eigenvalue c besides the trivial 0-eigenvalue, i.e., when N^{n+1} is a real space form with constant sectional curvature c. Now, we derive Cartan's rigidity result (13) as a simple application of the inductive formulae (48)–(49).

Proof of (13). With notations as before, $R(t) \equiv cI$, $\Gamma_{i1}(t) = cQ_i(t)$ and $R'(t) \equiv 0$ under any orthonormal frame of M_t. Then either of (48) and (49) can be deduced to

$$Q_{i+1}(t) = \frac{1}{i}Q_i'(t) - cQ_{i-1}(t),$$

which immediately implies (13) by induction (note that the preceding formula differs from that in Theorem 2.1 as the parameter t has different meanings). □

Proof of Theorem 1.3. Obviously we need only to consider the case when N^{n+1} is a locally rank one symmetric space with non-constant sectional curvatures. In this case the Jacobi operator K_ξ has two distinct non-zero constant eigenvalues κ_1, κ_2 independent of ξ. Let M^n be a 3-isoparametric hypersurface in N^{n+1} and M_t $(t \in (-\varepsilon, \varepsilon))$

nearby parallel hypersurfaces with unit normal vector fields ν_t. Then there exists a local orthonormal frame $\{e_1(t),\dots,e_n(t)\}$ of M_t parallel along normal geodesics of M, i.e., $\nabla_{\nu_t}e_i(t)=0$, such that under this frame,

$$R(t)=K_{\nu_t}|_{TM_t}=\operatorname{diag}(\kappa_1 I_{n-m},\ \kappa_2 I_m), \tag{50}$$

where $m=1,3$, or 7 corresponding to the case when N is locally a complex space form, a quaternionic space form, or an octonionic space form, respectively. Therefore, as in the proof of Corollary 4.1, $Q_1(t),Q_2(t),Q_3(t),\Gamma_{11}(t),\Gamma_{21}(t),Q_4(t)$ are all smooth functions depending only on t. Decompose $S(t)^i$ into the same blocks as $R(t)$ above for $i\geq 0$, namely,

$$S(t)^i=\left(\begin{array}{c|c} A_i & B_i \\ \hline B_i^t & C_i \end{array}\right),\quad A_0=I_{n-m},\quad C_0=I_m,\quad B_0=0, \tag{51}$$

where A_i, C_i are symmetric matrices of order $n-m$ and m, respectively. These arguments yield

$$Q_i(t)=tr(A_i)+tr(C_i),\quad \Gamma_{ij}(t)=\kappa_1^j\ tr(A_i)+\kappa_2^j\ tr(C_i). \tag{52}$$

Notice that $\kappa_1,\ \kappa_2$ are two distinct constants. It follows that if for some $i,j\geq 1$, $Q_i(t),\ \Gamma_{ij}(t)$ are smooth functions depending only on t, then by (52), $tr(A_i),\ tr(C_i)$ and hence $\Gamma_{ik}(t)$ are also smooth functions depending only on t for any $k\geq 1$. Now since $Q_1(t),Q_2(t),\Gamma_{11}(t),\Gamma_{21}(t)$ are such functions, $\Gamma_{1k}(t),\ \Gamma_{2k}(t)$ are smooth functions depending only on t for any $k\geq 1$. Then by (48), (49) again,

$$\Gamma_{31}(t)=\frac{1}{2}\Gamma_{21}'(t)-\Gamma_{12}(t),\quad Q_5(t)=\frac{1}{4}Q_4'(t)-\Gamma_{31}(t),$$

are smooth functions depending only on t, which means that M is 5-isoparametric. It completes the proof of the first part of the theorem.

Now assume N^{n+1} is locally a complex space form. Then $m=1$ in the diagonalization (50) of $R(t)$, and C_i in the block decomposition (51) of $S(t)^i$ is a real number (function) for each $i\geq 1$. To prove the second part of the theorem, i.e., a 3-isoparametric hypersurface M in N^{n+1} must be totally isoparametric, it suffices to prove the following:

Lemma 4.3. *With notations and assumptions as above. If for some $i\geq 1$, $Q_1(t),\dots,Q_{i+2}(t)$, $\Gamma_{11}(t),\dots,\Gamma_{i1}(t)$ are smooth functions depending only on t, then so are $Q_{i+3}(t)$ and $\Gamma_{i+1,1}(t)$.*

Since then by the inductive formula (48), the assumption that M is 3-isoparametric will imply $Q_1(t),Q_2(t),Q_3(t)$ and $\Gamma_{11}(t)$ are smooth functions depending only on t. So by this lemma we can show inductively that $\Gamma_{k1}(t)$ and $Q_k(t)$ are smooth functions depending only on t for each $k\geq 1$. Therefore, M is totally isoparametric as required.

Proof of Lemma 4.3. Under the assumptions, it follows from (52) that

$tr(A_1), \ldots, tr(A_i)$, $C_1, \ldots, C_i$ are smooth functions depending only on t. Note that $S(t)^j S(t)^{i-1-j} = S(t)^{i-1}$ for each $0 \leq j \leq i-1$. Substituting this into the block decomposition (51), we get

$$A_j A_{i-1-j} + B_j B_{i-1-j}^t = A_{i-1}, \quad C_j C_{i-1-j} + B_j^t B_{i-1-j} = C_{i-1}.$$

Hence,

$$tr(B_j B_{i-1-j}^t) = tr(B_j^t B_{i-1-j}) = C_{i-1} - C_j C_{i-1-j}$$

is a smooth function depending only on t, and so is

$$tr(A_j A_{i-1-j}) = tr(A_{i-1}) - tr(B_j B_{i-1-j}^t).$$

On the other hand, a direct calculation shows that for each $0 \leq j \leq i-1$,

$$\begin{aligned} &tr\big(S(t)^j R(t) S(t)^{i-1-j} R(t)\big) \\ &\quad = \kappa_1^2 \; tr(A_j A_{i-1-j}) + 2\kappa_1\kappa_2 \; tr(B_j B_{i-1-j}^t) + \kappa_2^2 \; C_j C_{i-1-j} \end{aligned}$$

is then a smooth function depending only on t. Therefore, by (49),

$$\Gamma_{i+1,1}(t) = \frac{1}{i}\left\{\Gamma_{i1}'(t) - \sum_{j=0}^{i-1} tr\big(S(t)^j R(t) S(t)^{i-1-j} R(t)\big)\right\}$$

is a smooth function depending only on t, and so is, by (48),

$$Q_{i+3}(t) = \frac{1}{i+2} Q_{i+2}'(t) - \Gamma_{i+1,1}(t). \qquad \square$$

The proof of Theorem 1.3 is now complete. $\square$

Proof of Theorem 1.4. Use the same notations as before. Let M^n be a curvature-adapted hypersurface in a locally rank one symmetric space N^{n+1} of non-constant sectional curvatures. Denote by M_t, $t \in (-\varepsilon, \varepsilon)$ nearby parallel hypersurfaces of $M_0 = M$. Evidently, each M_t is curvature-adapted and the principal orthonormal eigenvectors $\{e_i(t) | i = 1, \ldots, n\}$ of M_t can be chosen to be parallel along normal geodesics such that under this frame, the normal Jacobi operator $R(t)$ can be diagonalized as in (50), and the shape operator $S(t)$ can be diagonalized as

$$S(t) = \mathrm{diag}(\mu_1(t), \ldots, \mu_n(t)),$$

where $\mu_i(t)$'s are principal curvature functions of M_t (cf. [**Gra04**]). Moreover,

$$S'(t) = \mathrm{diag}(\mu_1'(t), \ldots, \mu_n'(t)),$$

and thus the Riccati equation (45) can be written as

$$\mu_i'(t) = \mu_i(t)^2 + \kappa_i, \quad \text{for} \ \ i = 1, \ldots, n, \tag{53}$$

where $\kappa_i = \kappa_1$ for $i \leq n - m$ and $\kappa_i = \kappa_2$ for $i > n - m$. Therefore, since κ_i's are constant, the principal curvatures $\mu_i(t)$'s of M_t, $t \in (-\varepsilon, \varepsilon)$, are uniquely determined by initial values $\mu_i(0)$'s, the principal curvatures of M. So when $\mu_i(0)$'s are constant on M, $\mu_i(t)$'s are constant on M_t, which completes the proof of the first part (i).

As for the second part (ii), the assumption that M is 1-isoparametric implies that $Q_1(t) = \sum_{i=1}^n \mu_i(t)$ is a smooth function depending only on t. Introduce two functions by

$$\Phi_i(t) := \sum_{p=1}^{n-m} \mu_p(t)^i, \quad \Psi_i(t) := \sum_{p=n-m+1}^{n} \mu_p(t)^i.$$

Then we have

$$Q_i(t) = \Phi_i(t) + \Psi_i(t), \quad \Gamma_{i1}(t) = \kappa_1 \cdot \Phi_i(t) + \kappa_2 \cdot \Psi_i(t), \tag{54}$$

and by (53),

$$\Phi_i'(t) = i(\Phi_{i+1}(t) + \kappa_1 \Phi_{i-1}(t)), \quad \Psi_i'(t) = i(\Psi_{i+1}(t) + \kappa_2 \Psi_{i-1}(t)), \tag{55}$$

and meanwhile, formula (48) can be rewritten as

$$\frac{1}{i} Q_i'(t) = Q_{i+1}(t) + \kappa_1 Q_{i-1}(t) + (\kappa_2 - \kappa_1)\Psi_{i-1}(t). \tag{56}$$

Taking the k-th derivative of $Q_1(t)$ with respective to t by (55), (56) inductively, we obtain

$$\frac{1}{k!} Q_1^{(k)}(t) = Q_{k+1}(t) + \sum_{j=0}^{k-1} \big(c_{kj} Q_j(t) + d_{kj} \Psi_j(t)\big), \tag{57}$$

where c_{kj}, d_{kj} are some constants depending only on the indices and κ_1, κ_2. As $Q_1(t)$ is a smooth function depending only on t, so is $Q_1^{(k)}(t)$, i.e., $Q_1^{(k)}(t)$ is constant on M_t for each $k \geq 0$. Fixing $t \in (-\varepsilon, \varepsilon)$ in (57), then it follows that the principal curvatures $\mu_1(t), \ldots, \mu_n(t)$ of M_t are solutions of the algebraic equations

$$P_{k+1}(x_1, \ldots, x_n) := \rho_{k+1}(x_1, \ldots, x_n) + \widehat{P}_k(x_1, \ldots, x_n) = 0, \quad \text{for} \ \ k = 0, \ldots, n-1, \tag{58}$$

where $\rho_j(x_1, \ldots, x_n) := \sum_{i=1}^n x_i^j$ is the j-th power sum over the variables $(x_1, \ldots, x_n)$ as in (7), while

$$\widehat{P}_k(x_1,\ldots,x_n):=\sum_{j=0}^{k-1}\big(c_{kj}\ \rho_j(x_1,\ldots,x_n)+d_{kj}\ \rho_j(x_{n-m+1},\ldots,x_n)\big)-\frac{1}{k!}Q_1^{(k)}(t)$$

is a polynomial of degree less than k with constant coefficients for $k\geq 1$ and $\widehat{P}_0:=-Q_1(t)$ is a constant.

Finally, the case $n\leq 2$ is not possible, since N^{n+1} is a locally rank one symmetric space of non-constant sectional curvature. For $n\geq 3$, we can not derive directly from (56) that $Q_i(t)$'s or $\mu_i(t)$'s are constant on M_t. However, making use of the following lemma and (58), we know that $(\mu_1(t),\ldots,\mu_n(t))$ belongs to a finite subset of $\mathbb{C}^n$ and thus $\mu_i(t)$'s are constant on M_t since M_t is connected. It means that M is totally isoparametric.

The proof is now complete. □

Now, we have to state explicitly the lemma on algebraic geometry used above.

Lemma 4.4. *For each $n\geq 1$, define polynomials $P_k\in\mathbb{C}[x_1,\ldots,x_n]$ by*

$$P_k:=\rho_k(x_1,\ldots,x_n)+\widetilde{P}_{k-1}(x_1,\ldots,x_n),\quad \textit{for}\ \ k=1,\ldots,n,$$

where ρ_k is the k-th power sum polynomial as before, $\widetilde{P}_{k-1}$ is an arbitrary polynomial of degree less than k. Then $P_1,\ldots,P_n$ form a regular sequence in $\mathbb{C}[x_1,\ldots,x_n]$. Consequently, the dimension of each variety V_k in $\mathbb{C}^n$ defined by $P_1=\cdots=P_k=0$ is less than or equal to $n-k$ for $k=1,\ldots,n$. In particular, V_n is a finite subset of $\mathbb{C}^n$.

Proof. First recall (cf. [**Eis95**], [**Mat80**]) that a sequence $r_1,\ldots,r_k$ in a commutative ring $\mathcal{R}$ with identity is called a *regular sequence* if (1) the ideal $(r_1,\ldots,r_k)\neq\mathcal{R}$; (2) r_1 is not a zero divisor in $\mathcal{R}$; and (3) r_{i+1} is not a zero divisor in the quotient ring $\mathcal{R}/(r_1,\ldots,r_i)$ for $i=1,\ldots,k-1$.

Now we will work on the polynomial ring $\mathcal{R}=\mathbb{C}[x_1,\ldots,x_n]$. Obviously, it is a *Cohen-Macaulay* ring, possessing the property that $\dim(\mathcal{R}/(P_1,\ldots,P_k))=n-k$ for a regular sequence $P_1,\ldots,P_k$ in $\mathcal{R}$. Meanwhile, we know that $\dim(V_k)=\dim(\mathcal{R}/I(V_k))$, where $I(V_k)\supset(P_1,\ldots,P_k)$ is the ideal of the variety V_k. Therefore, when $P_1,\ldots,P_n$ form a regular sequence, $\dim(V_k)\leq n-k$ for $k=1,\ldots,n$. In particular, $\dim(V_n)=0$. The last assertion is due to the facts that every variety in $\mathbb{C}^n$ can be expressed as a union of finite irreducible varieties and that a zero-dimensional irreducible variety in $\mathbb{C}^n$ is just a point. To complete the proof of the lemma, it suffices to show that the polynomials $P_1,\ldots,P_n$ form a regular sequence in $\mathcal{R}$.

Obviously, P_1 forms a regular sequence in $\mathcal{R}$. Suppose that $P_1,\ldots,P_n$ do not form a regular sequence, there exists some k with $1\leq k<n$ such that P_{k+1} is a zero divisor modulo $(P_1,\ldots,P_k)$ in $\mathcal{R}$. Then we may choose a relation of minimal degree of the form

$$f_1P_1+\cdots+f_{k+1}P_{k+1}=0,\tag{59}$$

where $f_1,\ldots,f_{k+1}$ are polynomials of minimal degrees modulo $(P_1,\ldots,P_k)$. Denote by $D(>0)$ the maximal degree of f_iP_i's. Let $f_{i_1}P_{i_1},\ldots,f_{i_r}P_{i_r}$ be those of maximal degree D for some $1\leq i_1<\cdots<i_r\leq k+1$. Then one can pick out the homogeneous components

$\tilde{f}_{i_1}\rho_{i_1}, \ldots, \tilde{f}_{i_r}\rho_{i_r}$ of maximal degree from them in equation (59) such that

$$\tilde{f}_{i_1}\rho_{i_1} + \cdots + \tilde{f}_{i_r}\rho_{i_r} = 0, \tag{60}$$

where $\tilde{f}_{i_1}, \ldots, \tilde{f}_{i_r}$ are the homogeneous components of maximal degrees of $f_{i_1}, \ldots, f_{i_r}$, respectively. Recall a well known fact that the power sum polynomials $\rho_1, \ldots, \rho_n$ form a regular sequence in $\mathcal{R}$ ([**Smi95**]). Then by (60), $r > 1$ and $\tilde{f}_{i_r} \in (\rho_1, \ldots, \rho_{i_r-1})$, which imply that there exist homogeneous polynomials $a_1, \ldots, a_{i_r-1}$ such that

$$\tilde{f}_{i_r} = a_1\rho_1 + \cdots + a_{i_r-1}\rho_{i_r-1},$$

and therefore,

$$f_{i_r} = a_1P_1 + \cdots + a_{i_r-1}P_{i_r-1} + \hat{f}_{i_r} \equiv \hat{f}_{i_r}, \quad \mathrm{mod}\ (P_1, \ldots, P_k),$$

where $\hat{f}_{i_r}$ is a polynomial of degree less than $D - i_r = \deg(f_{i_r})$, which contradicts the original choice of minimal relation (59).

The proof is now complete. □

We conclude this section with a brief proof of Remark 1.4.

Proof of Remark 1.4. Suppose that the ambient manifold N^{n+1} is an Osserman manifold. Its Jacobi operator K_ξ, by definition, has constant eigenvalues independent of ξ and points all over N. This property guarantees that the normal Jacobi operator $R(t) = K_{\nu_t}$ of the parallel hypersurface M_t in N^{n+1} has constant eigenvalues $\kappa_1, \ldots, \kappa_n$ for any $t \in (-\varepsilon,\ \varepsilon)$, though the covariant derivative $R'(t) := \nabla_{\nu_t}R(t)$ along normal direction ν_t might not vanish, different from the case in a locally rank one symmetric space. Further, we suppose that each M_t in N is curvature-adapted, that is, the shape operator $S(t)$ and the normal Jacobi operator $R(t)$ are simultaneously diagonalizable, which is automatically satisfied for a curvature-adapted hypersurface in a locally symmetric space. Therefore, the assertion in Remark 1.4 actually does nothing but abandon the assumption $R'(t) = 0$ in Theorem 1.4.

Let $\epsilon_1(t), \ldots, \epsilon_n(t)$ be a local orthonormal frame of M_t smoothly depending on t such that they are eigenvectors of $R(t)$ and $S(t)$ at the same time, corresponding to eigenvalues $\kappa_1, \ldots, \kappa_n$ and $\mu_1(t), \ldots, \mu_n(t)$, respectively. Then under this frame,

$$\langle S'(t)\epsilon_i(t), \epsilon_i(t)\rangle = \mu_i(t)^2 + \kappa_i.$$

The left side of the preceding equation can be deduced to

$$\begin{aligned}\langle S'(t)\epsilon_i(t), \epsilon_i(t)\rangle &= \big\langle \nabla_{\nu_t}(S(t)\epsilon_i(t)) - S(t)\nabla_{\nu_t}\epsilon_i(t),\ \epsilon_i(t)\big\rangle \\ &= \mu_i'(t) + \big\langle \big(\mu_i(t)I - S(t)\big)\ \nabla_{\nu_t}\epsilon_i(t),\ \epsilon_i(t)\big\rangle \\ &= \mu_i'(t).\end{aligned}$$

Hence, we still have the Riccati equation (53) in this case:

$$\mu_i'(t) = \mu_i(t)^2 + \kappa_i.$$

With this equality, we are able to complete the proof of Remark (1.4). For this purpose, we distinguish two cases. First, if M has constant principal curvatures, it follows directly from the identity above that M is totally isoparametric; Next, if M is 1-isoparametric, we can also take the k-th derivative of $Q_1(t)$ to obtain a sequence of algebraic equations $P_k = 0$, $k = 0, 1, \ldots, n-1$, for $(\mu_1(t), \ldots, \mu_n(t))$ similar to (58). By means of Lemma 4.4, we know that $\mu_1(t), \ldots, \mu_n(t)$ are constant on M_t and thus M is totally isoparametric, as desired. □

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