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From an Itô type calculus for Gaussian processes to integrals of log-normal processes increasing in the convex order

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Abstract. We present an Itô type formula for a Gaussian process, in which only the one-marginals of the Gaussian process are involved. Thus, this formula is well adapted to the study of processes increasing in the convex order, in a Gaussian framework. In particular, we give conditions ensuring that processes defined as integrals, with respect to one parameter, of exponentials of two-parameter Gaussian processes, are increasing in the convex order with respect to the other parameter. Finally, we construct Gaussian sheets allowing to exhibit martingales with the same one-marginals as the previously defined processes.

1. Introduction.

The following notation will be used throughout our paper:

- If X and Y are two real valued random variables,

$$X \stackrel{\mathrm{d}}{=} Y$$

  means that these variables have the same law.
- If $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two real valued processes,

$$(X_t, t \geq 0) \stackrel{(\mathrm{d})}{=} (Y_t, t \geq 0)$$

  means that the two processes are identical in law.
- $\boldsymbol{S}_n$ denotes the space of $n \times n$ symmetric matrices with real entries, whereas $\boldsymbol{S}_n^+$ denotes the convex cone in $\boldsymbol{S}_n$ consisting of positive matrices. Thus, a matrix $M = (m_{j,k})_{1 \leq j,k \leq n}$ belongs to $\boldsymbol{S}_n^+$ if M belongs to $\boldsymbol{S}_n$ and

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$$\forall \alpha_1, \ldots, \alpha_n \in \boldsymbol{R} \qquad \sum_{1 \leq j,k \leq n} \alpha_j \, \alpha_k \, m_{j,k} \geq 0.$$

In the sequel, $\boldsymbol{S}_n$ is assumed to be equipped with the following partial order, induced by the convex cone $\boldsymbol{S}_n^+$:

$$\forall M, N \in \boldsymbol{S}_n \qquad M \leq N \Longleftrightarrow (N - M) \in \boldsymbol{S}_n^+.$$

1.1. PCOC's and 1-martingales.

An $\boldsymbol{R}$-valued process $(X_t, t \geq 0)$ is said to *increase in the convex order* if

$$\forall t \geq 0 \quad \boldsymbol{E}[|X_t|] < \infty,$$

and for every convex function $\psi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$,

$$t \in \boldsymbol{R}_+ \longrightarrow \boldsymbol{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

We call such a process $(X_t, t \geq 0)$ a PCOC, this acronym being derived from the French name: Processus Croissant pour l'Ordre Convexe.

A process $(X_t, t \geq 0)$ is called a 1-*martingale* if there exists (on a suitable filtered probability space) a martingale $(M_t, t \geq 0)$ which has the same one-dimensional marginals as $(X_t, t \geq 0)$, that is, for each $t \geq 0$,

$$X_t \overset{\mathrm{d}}{=} M_t.$$

Such a martingale $(M_t, t \geq 0)$ is said to be *associated* with this process $(X_t, t \geq 0)$. Note that several different martingales may be associated with a given process.

It is an easy consequence of Jensen's inequality that an $\boldsymbol{R}$-valued process $(X_t, t \geq 0)$ which is a 1-martingale, is a PCOC. A remarkable result due to Kellerer [**Ke**] states that, conversely, any $\boldsymbol{R}$-valued process $(X_t, t \geq 0)$ which is a PCOC, is a 1-martingale.

A few comments about the history of Kellerer's theorem may be of interest here. Three steps may be singled out:

i) If X_1 and X_2 are two random variables such that

$$\boldsymbol{E}[g(X_1)] \leq \boldsymbol{E}[g(X_2)]$$

for every convex function g, then there exists a Markovian kernel Q such that:

$\nu_1 Q = \nu_2$, where ν_i $(i = 1, 2)$ is the law of X_i.

Many works have been devoted to this result, by several famous authors among whom Hardy-Littlewood, Polya, Blackwell, Stein, Cartier-Fell-Meyer, Strassen, Rotschild-Stiglitz, to name but a few. The main tool used in these works is the Hahn-Banach theorem.

More modern proofs related to Skorokhod embedding theorem, were then given for this problem, along the lines of Rost [**R**] and Falkner-Fitzsimmons [**FF**].

ii) A second step consists in dealing with a sequence $X_1, X_2, \dots, X_n, \dots$ of random variables which are increasing in the convex order. This is done, e.g. by Strassen [**St**].

iii) Finally, Kellerer's work (following those of Strassen [**St**] and Doob [**D**]) deals with a continuous time process $(U_t, t \geq 0)$ assumed to be increasing in the convex order, to show that it is a 1-martingale. Kellerer's result hinges on an "integral representation of dilations"; see [**Ke**, Theorem 3].

In any case, the proofs offered by Strassen, Doob and especially Kellerer are not constructive, and generally, it is a difficult problem to give a concrete description of a martingale which is associated to a PCOC; this problem has been the aim of several recent papers ([**BY**], [**HY1**], [**HY2**], [**HY3**], [**HRY**]).

1.2. Our guiding example.

Our interest in the study of PCOC's and associated martingales originated from the result by Carr, Ewald and Xiao [**CEX**] that the process:

$$A_t = \frac{1}{t}\int_0^t \exp\left(B_s - \frac{s}{2}\right)\mathrm{d}s, \quad t \geq 0,$$

where $(B_s, s \geq 0)$ is a standard Brownian motion, is a PCOC. It has been shown later by Baker and Yor [**BY**] that a martingale associated with this process $(A_t, t \geq 0)$ is

$$M_t = \int_0^1 \exp\left(W_{s,t} - \frac{s\,t}{2}\right)\mathrm{d}s, \quad t \geq 0$$

where W denotes the standard Brownian sheet.

1.3. Generalizations of our guiding example.

The ubiquity of Brownian motion stems, for a large part, from the fact that it belongs to the intersection of important classes of stochastic processes, e.g.: martingales, Lévy processes, Gaussian processes. Thus, the solution of a given problem involving Brownian motion often generalizes into one involving either of these classes of stochastic processes. The above result by Carr, Ewald and Xiao is no exception to this rule, as it has already been generalized as follows:

(i) If $(N_t, t \geq 0)$ is a martingale, then the process

$$\frac{1}{t}\int_0^t N_s \mathrm{d}s, \quad t \geq 0$$

is a PCOC (see [**HPRY**]).

(ii) An interesting particular case of (i) is: if $(L_t, t \geq 0)$ is a Lévy process such that $\boldsymbol{E}[\exp(L_1)] < \infty$, then the process

$$\frac{1}{t}\int_0^t \frac{\exp(L_s)}{\boldsymbol{E}[\exp(L_s)]}\mathrm{d}s, \quad t \geq 0$$

is a PCOC and an associated martingale may be expressed, using a Lévy sheet (see [**BY**], [**HY2**], [**HRY**]).

In this paper, we are concerned with generalizations of the result by Carr, Ewald and Xiao, in a Gaussian framework. Thus, we consider a family $(G_{\bullet,t}, t \geq 0)$ of real valued, centered, Gaussian processes:

$$G_{\bullet,t} := (G_{\lambda,t}, \lambda \in \Lambda),$$

where Λ denotes a measure space. For any signed finite measure σ on Λ, we set, for $t \geq 0$,

$$A_t^{(\sigma)} = \int_\Lambda \exp\left(G_{\lambda,t} - \frac{1}{2}\boldsymbol{E}[(G_{\lambda,t})^2]\right)\sigma(\mathrm{d}\lambda)$$

and we give conditions ensuring that $(A_t^{(\sigma)}, t \geq 0)$ is a PCOC (Theorem 3.1). In some cases, we also exhibit a martingale which is associated to $(A_t^{(\sigma)}, t \geq 0)$, and which is constructed in terms of a Gaussian sheet (see Theorem 4.1).

1.4. Organisation of the paper.

The remainder of this paper is organised as follows:

- In Section 2, we develop an Itô type calculus for Gaussian processes. In fact, we provide two proofs for our Itô type formula; the first one hinges on Gaussian characteristic functions arguments, whereas the second one, given in Subsection 2.4, rests on Gaussian integration by parts.
- In Section 3, we use the previous calculus to prove that, under certain conditions, processes $(A_t^{(\sigma)}, t \geq 0)$ as defined in Subsection 1.3 are PCOC's.

- In Section 4, we construct Gaussian sheets allowing, for some processes $(A_t^{(\sigma)}, t \geq 0)$, to exhibit martingales having the same one-dimensional marginals. This yields, in these cases, another proof that they are PCOC's.

2. An Itô type calculus for centered Gaussian processes.

2.1. An Itô type formula.

In this subsection, we consider a family of $\boldsymbol{R}^n$-valued centered Gaussian variables:

$$(G_t, t \in [a,b])$$

where $[a,b]$ denotes a compact interval of $\boldsymbol{R}$. We denote, for $t \in [a,b]$, by $(G_t^{(1)}, \dots, G_t^{(n)})$ the components of the vector G_t, and by

$$C(t) = (c_{j,k}(t))_{1 \leq j,k \leq n}$$

the covariance matrix of G_t.

We assume that, for all $1 \leq j,k \leq n$,

$$t \in [a,b] \longrightarrow c_{j,k}(t) \in \boldsymbol{R}$$

is a continuous function with finite variation.

The main result of this subsection is the following weak form of an Itô type formula.

THEOREM 2.1. *Let*

$$F : (x,t) \in \boldsymbol{R}^n \times [a,b] \longrightarrow F(x,t) \in \boldsymbol{R}$$

be a $C^{2,1}$-function whose derivatives of order 2 with respect to x: F''_{x_j,x_k}, $1 \leq j,k \leq n$, and whose derivative of order 1 with respect to t: F'_t, grow sub-exponentially at infinity with respect to x, uniformly with respect to $t \in [a,b]$. Then, for every s,t with $a \leq s \leq t \leq b$,

$$\begin{aligned}\boldsymbol{E}[F(G_t,t)] = \boldsymbol{E}[F(G_s,s)] &+ \int_s^t \boldsymbol{E}\big[F'_t(G_u,u)\big]\mathrm{d}u \\ &+ \frac{1}{2}\sum_{j,k}\int_s^t \boldsymbol{E}\big[F''_{x_j,x_k}(G_u,u)\big]\mathrm{d}c_{j,k}(u). \qquad (1)\end{aligned}$$

PROOF. The proof is based on the following lemma:

LEMMA 2.1. *Denote by $\mu_t(\mathrm{d}x)$ the law of the Gaussian variable G_t. Then, for every s,t with $a \le s \le t \le b$, there is the following identity, in the sense of Schwartz distributions*:

$$\mu_t = \mu_s + \frac{1}{2}\sum_{j,k}\int_s^t (\mu_u)''_{x_j,x_k}\,\mathrm{d}c_{j,k}(u). \tag{2}$$

PROOF. Let $\gamma = (\gamma_1,\dots,\gamma_n) \in \boldsymbol{R}^n$, and denote by $\langle x,y\rangle$ the scalar product of $x,y \in \boldsymbol{R}^n$. Set, for $t \in [a,b]$,

$$\varphi_\gamma(t) = \int_{\boldsymbol{R}^n} \mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle}\mu_t(\mathrm{d}x).$$

We have, for $t \in [a,b]$,

$$\varphi_\gamma(t) = \boldsymbol{E}[\exp(\mathrm{i}\langle\gamma,G_t\rangle)] = \exp\Bigg(-\frac{1}{2}\sum_{j,k}\gamma_j\,\gamma_k\,c_{j,k}(t)\Bigg).$$

Then, for $a \le s \le t \le b$,

$$\varphi_\gamma(t) = \varphi_\gamma(s) - \frac{1}{2}\sum_{j,k}\int_s^t \varphi_\gamma(u)\,\gamma_j\,\gamma_k\mathrm{d}c_{j,k}(u).$$

Now,

$$\varphi_\gamma(u)\,\gamma_j\,\gamma_k = \int_{\boldsymbol{R}^n}\mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle}\gamma_j\,\gamma_k\,\mu_u(\mathrm{d}x) = -\int_{\boldsymbol{R}^n}\frac{\partial^2}{\partial x_j\partial x_k}\mathrm{e}^{\mathrm{i}\langle\gamma,x\rangle}\mu_u(\mathrm{d}x).$$

Finally, we obtain that the Fourier transforms, in the sense of distributions, of both sides in (2) are equal, hence the desired result follows thanks to the injectivity of the Fourier transform. □

We now prove Theorem 2.1. We still denote by $\mu_t(\mathrm{d}x)$ the law of G_t.

We first suppose that

$$F(x,t) = g(t)\,h(x)$$

with $h \in C^2(\boldsymbol{R}^n)$ and $g \in C^1([a,b])$. We also assume that h has compact support.

Then for $t \in [a, b]$,

$$\boldsymbol{E}[F(G_t, t)] = g(t) \int h(x)\mu_t(\mathrm{d}x).$$

Lemma 2.1 ensures that, for $a \le s \le t \le b$,

$$\begin{aligned}\boldsymbol{E}[F(G_t, t)] = \boldsymbol{E}[F(G_s, s)] &+ \int_s^t g'(u)\Big(\int h(x)\mu_u(\mathrm{d}x)\Big)\mathrm{d}u \\ &+ \frac{1}{2}\sum_{j,k}\int_s^t g(u)\Big(\int h''_{x_j,x_k}(x)\mu_u(\mathrm{d}x)\Big)\mathrm{d}c_{j,k}(u). \qquad (3)\end{aligned}$$

Clearly, equality (3) yields formula (1) in this case.

Finally, if F satisfies the hypotheses of Theorem 2.1, the result follows easily by approximation of F by linear combinations of functions of the previous type. □

REMARKS.

1. In formula (1), only the law of G_t for *each* $t \in [a, b]$, and consequently only the matrices $C(t)$, are involved. This explains why this formula is well adapted to the study of PCOC's.
2. Suppose that

   $$t \in [a, b] \longrightarrow C(t) \in \boldsymbol{S}_n$$

   is an absolutely continuous function on $[a, b]$ and that the derivative $C'(t)$ is, for almost every $t \in [a, b]$, a positive symmetric matrix. Then, there exists a measurable function

   $$t \in [a, b] \longrightarrow D(t) \in \boldsymbol{S}_n^+$$

   such that, for almost every $t \in [a, b]$,

   $$[D(t)]^2 = C'(t).$$

   In particular,

   $$t \in [a, b] \longrightarrow D(t) \in \boldsymbol{S}_n$$

is a square integrable function. We set

$$M_t = G_a + \int_a^t D(s)\mathrm{d}B_s$$

where B denotes a standard $\boldsymbol{R}^n$-valued Brownian motion starting from 0, independent of G_a. Then $(M_t, t \in [a,b])$ is a continuous Gaussian martingale, and, for any $t \in [a,b]$, the covariance matrix of M_t is $C(t)$. Consequently, to prove (1) in this case, we may replace $(G_t, t \in [a,b])$ by $(M_t, t \in [a,b])$, but then (1) is a direct consequence of the classical Itô formula.

2.2. Examples.

In this subsection, we present some examples of application of our Itô-like formula in the scalar case $n = 1$. In particular, when $(G_t, t \in [a,b])$ has the same one-marginals as a semi-martingale, we compare our formula (1) with the one obtained by application of the classical Itô formula.

2.2.1. Time changed Brownian motion.

We consider continuous functions u and v from an interval $[a,b]$ into $\boldsymbol{R}$ and we suppose that u has a finite variation and v is increasing and nonnegative. Let (B_t) be the standard linear Brownian motion starting from 0 and consider the process:

$$G_t = u(t)B_{v(t)}, \quad t \in [a,b].$$

Recall this is the general form of a Gaussian Markovian process (see, for instance, [**N**]).

We have $c(t) = \boldsymbol{E}[G_t^2] = u^2(t)\,v(t)$. Let F be a C^2-function on $\boldsymbol{R}$ with compact support. Our formula (1) yields:

$$\mathrm{d}_t\boldsymbol{E}[F(G_t)] = \frac{1}{2}\boldsymbol{E}[F''(G_t)]\,[2u(t)v(t)\mathrm{d}u(t) + u^2(t)\mathrm{d}v(t)],$$

whereas the application of the classical Itô formula gives:

$$\mathrm{d}_t\boldsymbol{E}[F(G_t)] = \boldsymbol{E}[F'(G_t)B_{v(t)}]\mathrm{d}u(t) + \frac{1}{2}\boldsymbol{E}[F''(G_t)]u^2(t)\mathrm{d}v(t).$$

Consequently, we obtain:

$$\boldsymbol{E}[F''(u(t)B_{v(t)})]u(t)v(t) = \boldsymbol{E}[F'(u(t)B_{v(t)})B_{v(t)}].$$

The above equality may be written down as:

$$\boldsymbol{E}[F''(\alpha\, B_1)]\alpha = \boldsymbol{E}[F'(\alpha\, B_1)B_1] \tag{4}$$

with $\alpha = u(t)\sqrt{v(t)}$. Obviously, α may be taken equal to 1, and (4) is a well-known characterization of the law of B_1. In fact, formula (4) with g instead of F', is the starting point of Stein's method (see [**S**]); see also [**T1**, Appendix A.6, Gaussian r.v.].

The following Examples 2.2.2 and 2.2.3 are particular cases of the previous Example 2.2.1.

2.2.2. Ornstein-Uhlenbeck process.

The most famous example of a Gaussian process is the Ornstein-Uhlenbeck process. For $\lambda \in \boldsymbol{R}$, let $(U_t^\lambda, t \geq 0)$ be the scalar Ornstein-Uhlenbeck process with parameter λ, starting from 0. Thus, U^λ is the solution to the SDE:

$$U_t^\lambda = B_t + \lambda \int_0^t U_s^\lambda \mathrm{d}s. \tag{5}$$

We have:

$$\forall t \geq 0 \qquad U_t^\lambda = \mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{-\lambda s} \mathrm{d}B_s.$$

Consequently, there exists a Brownian motion (β_u) such that:

$$\forall t \geq 0 \qquad U_t^\lambda = \mathrm{e}^{\lambda t} \beta_{v(t)}$$

with

$$v(t) = \int_0^t \mathrm{e}^{-2\lambda s} \mathrm{d}s.$$

Thus, $(U_t^\lambda, t \geq 0)$ is a particular case of Example 2.2.1.

On the other hand, denoting by $\mu^{(\lambda)}(t,x)$ the density of $U_t^{(\lambda)}$, the Fokker-Planck equation corresponding to the SDE (5) yields:

$$\frac{\partial \mu^{(\lambda)}}{\partial t}(t,x) = -\lambda \frac{\partial}{\partial x}\big(x\mu^{(\lambda)}(t,x)\big) + \frac{1}{2}\frac{\partial^2 \mu^{(\lambda)}}{\partial x^2}(t,x),$$

whereas our Lemma 2.1 leads to:

$$\frac{\partial\mu^{(\lambda)}}{\partial t}(t,x) = \frac{1}{2}\mathrm{e}^{2\lambda t}\frac{\partial^2\mu^{(\lambda)}}{\partial x^2}(t,x).$$

Consequently, we obtain:

$$\frac{\partial}{\partial x}(x\mu^{(\lambda)}(t,x)) = \frac{1-\mathrm{e}^{2\lambda t}}{2\lambda}\frac{\partial^2\mu^{(\lambda)}}{\partial x^2}(t,x),$$

which is easy to verify directly.

2.2.3. Brownian bridge.

Let $(b_t, 0 \le t \le 1)$ be the standard Brownian bridge satisfying $b_0 = b_1 = 0$. It may be obtained as solution to the SDE:

$$X_t = B_t - \int_0^t \frac{X_s}{1-s}\mathrm{d}s \tag{6}$$

and one has: $\boldsymbol{E}[b_t^2] = t(1-t)$. Let F be a C^2-function on $\boldsymbol{R}$ with compact support. Our formula (1) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}[F(b_t)] = \frac{1-2t}{2}\boldsymbol{E}[F''(b_t)],$$

whereas the application of the classical Itô formula gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E}[F(b_t)] = -\frac{1}{1-t}\boldsymbol{E}[F'(b_t)b_t] + \frac{1}{2}\boldsymbol{E}[F''(b_t)].$$

Consequently, we obtain:

$$t(1-t)\boldsymbol{E}[F''(b_t)] = \boldsymbol{E}[F'(b_t)b_t],$$

which is equivalent to (4) with $\alpha = \sqrt{t(1-t)}$.

On the other hand, there exists a Brownian motion (β_u) such that:

$$\forall t \in [0,1) \qquad b_t = (1-t)\beta_{t/(1-t)}.$$

Thus, $(b_t, 0 \le t \le 1)$ is again a particular case of Example 2.2.1.

Besides, denoting by $\mu(t,x)$ the density of b_t for $0 < t < 1$, the Fokker-Planck equation corresponding to the SDE (6) yields:

$$\frac{\partial \mu}{\partial t}(t,x) = \frac{1}{1-t}\frac{\partial}{\partial x}(x\mu(t,x)) + \frac{1}{2}\frac{\partial^2 \mu}{\partial x^2}(t,x),$$

whereas our Lemma 2.1 leads to:

$$\frac{\partial \mu}{\partial t}(t,x) = \frac{1-2t}{2}\frac{\partial^2 \mu}{\partial x^2}(t,x).$$

Consequently, we obtain:

$$\frac{\partial}{\partial x}(x\mu(t,x)) = -t(1-t)\frac{\partial^2 \mu}{\partial x^2}(t,x),$$

which is easy to verify directly.

We note that, more generally than (6), we might consider the ε-generalized Brownian bridges on the time interval $[0,T]$, which solve:

$$X_t = B_t - \varepsilon \int_0^t \frac{X_s}{T-s}\mathrm{d}s, \quad t < T.$$

These processes are also particular cases of Example 2.2.1. They have been considered in [**M**].

2.2.4. Fractional Brownian motion.

Let $(B_t^H, t \geq 0)$ be the fractional Brownian motion with Hurst index $H \in (0,1)$. It is a continuous centered Gaussian process such that, for any $t \geq 0$, $\boldsymbol{E}[(B_t^H)^2] = t^{2H}$. Let F be a C^2-function on $\boldsymbol{R}$ with compact support. Our formula (1) yields, for $0 \leq s \leq t$:

$$\boldsymbol{E}[F(B_t^H)] = \boldsymbol{E}[F(B_s^H)] + H\int_s^t \boldsymbol{E}\left[F''(B_u^H)\right]u^{2H-1}\mathrm{d}u. \tag{7}$$

Although the fractional Brownian motion is not Markovian (except for $H = 1/2$), and hence is not a particular case of Example 2.2.1, there exists a Brownian motion (β_u) such that, for each $t \geq 0$,

$$B_t^H \stackrel{\mathrm{d}}{=} \beta_{t^{2H}}.$$

Formula (7) then also follows from the classical Itô formula applied to the martingale $(\beta_{t^{2H}})$.

2.3. An application of Theorem 2.1.

As an application of Theorem 2.1, we now present a simple proof of the Gordon-Slepian Lemma, the statement of which we recall below (see, for instance, [**Ka**] and [**T2**, Proposition 1.3.2]).

PROPOSITION 2.1 (Gordon-Slepian Lemma). *Let* $X = (X_1, \dots, X_n)$ *and* $Y = (Y_1, \dots, Y_n)$ *be two centered Gaussian vectors in* $\boldsymbol{R}^n$, *and let* A *and* B *be two subsets of* $\{1, \dots, n\} \times \{1, \dots, n\}$. *We assume*:

$$\begin{aligned} \boldsymbol{E}[X_j X_k] &\leq \boldsymbol{E}[Y_j Y_k] \quad \text{if } (j,k) \in A \\ \boldsymbol{E}[X_j X_k] &\geq \boldsymbol{E}[Y_j Y_k] \quad \text{if } (j,k) \in B \\ \boldsymbol{E}[X_j X_k] &= \boldsymbol{E}[Y_j Y_k] \quad \text{if } (j,k) \notin A \cup B. \end{aligned}$$

Let $F : \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ *be a* C^2*-function whose derivatives of order* 2 *are sub-exponential at infinity. We assume*:

$$F''_{x_j,x_k} \geq 0 \quad \text{if } (j,k) \in A \qquad \text{and} \qquad F''_{x_j,x_k} \leq 0 \quad \text{if } (j,k) \in B.$$

Then

$$\boldsymbol{E}[F(X)] \leq \boldsymbol{E}[F(Y)].$$

PROOF. As in Kahane's proof ([**Ka**]), we set, for $t \in [0,1]$,

$$G_t = \sqrt{t}\, Y + \sqrt{1-t} X$$

where X and Y are assumed to be independent (a special case of the *smart path method* used, again and again, in [**T2**]). Then, by Theorem 2.1,

$$\boldsymbol{E}[F(G_1)] = \boldsymbol{E}[F(G_0)] + \frac{1}{2} \sum_{j,k} \int_0^1 \boldsymbol{E}\big[F''_{x_j,x_k}(G_u)\big] (\boldsymbol{E}[Y_j Y_k] - \boldsymbol{E}[X_j X_k]) \mathrm{d}u.$$

Now, by hypothesis, for every (j,k),

$$(\boldsymbol{E}[Y_j Y_k] - \boldsymbol{E}[X_j X_k]) F''_{x_j,x_k} \geq 0. \qquad \square$$

2.4. An alternative proof of Theorem 2.1.

We now give an alternative proof of Theorem 2.1, based on a Gaussian integration by parts formula and on the smart path method mentioned in the proof of Proposition 2.1.

We first recall the Gaussian integration by parts formula stated, for instance, as Formula (A.41) in [**T1**, Appendix A.6, Gaussian r.v.].

PROPOSITION 2.2. *Let $(g_1, \dots, g_n, g)$ be an $\boldsymbol{R}^{n+1}$-valued centered Gaussian variable, and let F be a C^1-function on $\boldsymbol{R}^n$ with compact support. Then*

$$\boldsymbol{E}[F(g_1, \dots, g_n)\, g] = \sum_{k=1}^{n} \boldsymbol{E}[g_k\, g]\, \boldsymbol{E}\big[F'_{x_k}(g_1, \dots, g_n)\big].$$

Now, to prove formula (1) in Theorem 2.1, we may assume that the $C^{2,1}$-function F has a compact support. For $a \le s \le t \le b$ and $0 \le v \le 1$, we define $\widehat{G}_{s,t}(v)$ by the equality in law:

$$\widehat{G}_{s,t}(v) \overset{\mathrm{d}}{=} \sqrt{v}\, G_t + \sqrt{1-v}\, G_s,$$

where G_t and G_s are assumed to be independent. We then obtain directly, by differentiating the function of v:

$$\boldsymbol{E}\big[F\big(\widehat{G}_{s,t}(v), vt + (1-v)s\big)\big]$$

for v between 0 and 1:

$$\begin{aligned}\boldsymbol{E}[F(G_t, t)] = \boldsymbol{E}[F(G_s, s)] &+ (t-s)\int_0^1 \boldsymbol{E}\big[F'_t\big(\widehat{G}_{s,t}(v), vt + (1-v)s\big)\big]\mathrm{d}v \\ &+ \frac{1}{2}\sum_{j=1}^{n}\int_0^1 \boldsymbol{E}\bigg[F'_{x_j}\big(\widehat{G}_{s,t}(v), vt + (1-v)s\big) \\ &\qquad\qquad \cdot \bigg(\frac{1}{\sqrt{v}} G_t^{(j)} - \frac{1}{\sqrt{1-v}} G_s^{(j)}\bigg)\bigg]\mathrm{d}v.\end{aligned}$$

Moreover, by Proposition 2.2,

$$\begin{aligned}&\boldsymbol{E}\bigg[F'_{x_j}\big(\widehat{G}_{s,t}(v), vt + (1-v)s\big)\bigg(\frac{1}{\sqrt{v}} G_t^{(j)} - \frac{1}{\sqrt{1-v}} G_s^{(j)}\bigg)\bigg] \\ &\quad = \sum_{k=1}^{n} \boldsymbol{E}\big[F''_{x_j,x_k}\big(\widehat{G}_{s,t}(v), vt + (1-v)s\big)\big]\big(c_{j,k}(t) - c_{j,k}(s)\big).\end{aligned}$$

We now consider a subdivision $w = (w_1, \dots, w_r)$ of the interval $[s,t]$ ($w_1 = s$, $w_r = t$). Then, by telescoping,

$$\boldsymbol{E}[F(G_t,t)] = \boldsymbol{E}[F(G_s,s)] + \int_s^t H^w(u)\mathrm{d}u + \frac{1}{2}\sum_{j,k}\int_s^t K^w_{j,k}(u)\mathrm{d}c_{j,k}(u) \qquad (8)$$

with

$$H^w(u) = \sum_{l=1}^{r-1}\left(\int_0^1 \boldsymbol{E}\big[F'_t\big(\widehat{G}_{w_{l+1},w_l}(v), v w_{l+1} + (1-v)w_l\big)\big]\mathrm{d}v\right)1_{[w_l,w_{l+1})}(u)$$

and

$$K^w_{j,k}(u) = \sum_{l=1}^{r-1}\left(\int_0^1 \boldsymbol{E}\big[F''_{x_j,x_k}\big(\widehat{G}_{w_{l+1},w_l}(v), v w_{l+1} + (1-v)w_l\big)\big]\mathrm{d}v\right)1_{[w_l,w_{l+1})}(u).$$

Clearly, if $u \in [w_l, w_{l+1})$, then for every $v \in [0,1]$, $\widehat{G}_{w_{l+1},w_l}(v)$ tends to G_u in law when the mesh of w tends to 0. Consequently, for any $u \in [s,t]$, $H^w(u)$ (resp. $K^w_{j,k}(u)$) tends to $\boldsymbol{E}[F'_t(G_u,u)]$ (resp. $\boldsymbol{E}[F''_{x_j,x_k}(G_u,u)]$) when the mesh of w tends to 0. Therefore, passing to the limit in (8) as the mesh of w tends to 0, we obtain formula (1).

2.5. A variant of Theorem 2.1.

In this subsection, we keep the framework, the hypotheses and the notation of Subsection 2.1. A real valued function $h(x,t)$, defined on $\boldsymbol{R}\times\boldsymbol{R}_+$, will be called a *space-time harmonic function* if h is a $C^{2,1}$-function satisfying, on $\boldsymbol{R}\times\boldsymbol{R}_+$ the equation:

$$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = 0.$$

It is well-known that any nonnegative space-time harmonic function may be represented as:

$$h(x,t) = \int_{\boldsymbol{R}} \exp\left(y\,x - \frac{t\,y^2}{2}\right)\mathrm{d}\nu(y)$$

where ν denotes a positive finite measure on $\boldsymbol{R}$. We refer to [**Y**, Theorem 1.3] for a probabilistic proof.

Theorem 2.2. *Let $h^{(1)}, \dots, h^{(n)}$ be n space-time harmonic functions, and define the function $H = (H_1, \dots, H_n)$, from $\boldsymbol{R}^n \times [a, b]$ into $\boldsymbol{R}^n$, by*

$$H_j(x, t) = h^{(j)}(x_j, c_{j,j}(t)).$$

Let $\Phi : \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ be a C^2-function such that the derivatives of order 2 with respect to x of the function F defined by

$$F(x, t) = \Phi[H(x, t)],$$

are sub-exponential at infinity with respect to x, uniformly with respect to $t \in [a, b]$. Then, for every s, t with $a \le s \le t \le b$,

$$\begin{aligned} \boldsymbol{E}[\Phi[H(G_t, t)]] = \boldsymbol{E}[\Phi[H(G_s, s)]] + \frac{1}{2} \sum_{j,k} \int_s^t \boldsymbol{E}\Big[& \Phi''_{x_j, x_k}[H(G_u, u)] \\ & \cdot \frac{\partial h^{(j)}}{\partial x}\big(G_u^{(j)}, c_{j,j}(u)\big) \frac{\partial h^{(k)}}{\partial x}\big(G_u^{(k)}, c_{k,k}(u)\big)\Big] \mathrm{d}c_{j,k}(u). \end{aligned}$$

Proof. If the covariance matrix C is a C^1-function, the function F is of class $C^{2,1}$ and Theorem 2.2 follows by a direct application of Theorem 2.1, after simplifications which are consequences of the harmonicity property. Actually, the general case may be treated by a slight adaptation of the proof of Theorem 2.1. □

We now state two easy corollaries.

Corollary 2.1. *Let $h^{(1)}, \dots, h^{(n)}$ be n space-time harmonic functions and $a_1, \dots, a_n \in \boldsymbol{R}$. Define the function k, from $\boldsymbol{R}^n \times [a, b]$ into $\boldsymbol{R}$, by*

$$k(x, t) = \sum_j a_j \, h^{(j)}(x_j, c_{j,j}(t)).$$

Let $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be a C^2-function such that the derivatives of order 2 with respect to x of the function F defined by

$$F(x, t) = \varphi[k(x, t)],$$

are sub-exponential at infinity with respect to x, uniformly with respect to $t \in [a, b]$. Then, for every s, t with $a \le s \le t \le b$,

$$\boldsymbol{E}[\varphi[k(G_t,t)]] = \boldsymbol{E}[\varphi[k(G_s,s)]] + \frac{1}{2}\sum_{j,k} a_j\, a_k \int_s^t \boldsymbol{E}\Big[\varphi''[k(G_u,u)] \cdot \frac{\partial h^{(j)}}{\partial x}\big(G_u^{(j)}, c_{j,j}(u)\big)\frac{\partial h^{(k)}}{\partial x}\big(G_u^{(k)}, c_{k,k}(u)\big)\Big]\mathrm{d}c_{j,k}(u).$$

Corollary 2.2. *Let $a_1,\dots,a_n \in \boldsymbol{R}$ and let $\varphi : \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be a C^2-function whose second derivative has polynomial growth at infinity. We set, for $1 \le j \le n$,*

$$Y_u^{(j)} = \exp\left(G_u^{(j)} - \frac{c_{j,j}(u)}{2}\right)$$

and

$$K_u = \sum_j a_j Y_u^{(j)}.$$

Then, for every s,t with $a \le s \le t \le b$,

$$\boldsymbol{E}[\varphi(K_t)] = \boldsymbol{E}[\varphi(K_s)] + \frac{1}{2}\sum_{j,k} a_j\, a_k \int_s^t \boldsymbol{E}\big[\varphi''(K_u)\, Y_u^{(j)}\, Y_u^{(k)}\big]\mathrm{d}c_{j,k}(u).$$

3. Application to PCOC's.

3.1. Notation.

We first introduce the notation which will be in force in Section 3 and in Section 4.

- We denote by Λ a measure space.
- We consider, for each $t \ge 0$, a real valued measurable centered Gaussian process

$$G_{\bullet,t} = (G_{\lambda,t}, \lambda \in \Lambda).$$

- For $\lambda, \mu \in \Lambda$ and $t \ge 0$, we set:

$$c_{\lambda,\mu}(t) = \boldsymbol{E}[G_{\lambda,t}\, G_{\mu,t}].$$

- For any signed finite measure σ on Λ, we set, for $t \ge 0$,

$$A_t^{(\sigma)} = \int_\Lambda \exp\left(G_{\lambda,t} - \frac{1}{2}c_{\lambda,\lambda}(t)\right)\sigma(\mathrm{d}\lambda).$$

We now introduce various conditions which will appear in the sequel.

(M_1) Λ is a separable metric space equipped with its Borel σ-field.
(M_2) Λ is a metric σ-compact space equipped with its Borel σ-field.
(C_1) For all $t \geq 0$, the function:

$$(\lambda, \mu) \in \Lambda \times \Lambda \longrightarrow c_{\lambda,\mu}(t) \in \boldsymbol{R}$$

is continuous.
(C_2) The function:

$$(\lambda, \mu, t) \in \Lambda \times \Lambda \times \boldsymbol{R}_+ \longrightarrow c_{\lambda,\mu}(t) \in \boldsymbol{R}$$

is continuous.
(I_1) For every $\lambda, \mu \in \Lambda$, the function

$$t \in \boldsymbol{R}_+ \longrightarrow c_{\lambda,\mu}(t) \in \boldsymbol{R}$$

is increasing.
(I_2) For every $n \geq 1$, for every $\lambda_1, \dots, \lambda_n \in \Lambda$, the matrix function

$$t \in \boldsymbol{R}_+ \longrightarrow (c_{\lambda_j,\lambda_k}(t))_{1 \leq j,k \leq n} \in \boldsymbol{S}_n$$

is increasing with respect to the order on $\boldsymbol{S}_n$ induced by the convex cone $\boldsymbol{S}_n^+$.

Obviously, (M_2) entails (M_1) and (C_2) entails (C_1), whereas conditions (I_1) and (I_2) are not comparable.

3.2. Integrals of log-normal processes.

The next theorem provides sufficient conditions for the process $(A_t^{(\sigma)}, t \geq 0)$ to be a PCOC.

THEOREM 3.1. *We assume (M_2) and (C_1).*
If either (I_1) is satisfied and σ is a positive finite measure, or (I_2) is satisfied and σ is a signed finite measure, then $(A_t^{(\sigma)}, t \geq 0)$ is a PCOC.

PROOF. We begin with a lemma for which we refer to [**HPRY**, Chapter 1].

LEMMA 3.1. *Let $(X_t, t \geq 0)$ be a real valued integrable process, i.e.:*

$$\forall t \geq 0 \qquad \boldsymbol{E}[|X_t|] < \infty.$$

This process is a PCOC if (and only if), for any $\psi \in \mathscr{C}$, the function

$$t \geq 0 \longrightarrow \boldsymbol{E}[\psi(X_t)]$$

is increasing.

Here, $\mathscr{C}$ denotes the set of all convex C^2-functions ψ such that ψ'' has a compact support.

Now, the proof of Theorem 3.1 proceeds in three steps.

1. We first assume that

$$\sigma = \sum_{j=1}^{n} a_j \delta_{\lambda_j}$$

where δ_λ denotes the Dirac measure at λ and $a_1, \dots, a_n \in \boldsymbol{R}$.
We have

$$A_t^{(\sigma)} = \sum_{j=1}^{n} a_j \exp\left(G_{\lambda_j,t} - \frac{1}{2} c_{\lambda_j,\lambda_j}(t)\right).$$

Since for any t, $\boldsymbol{E}[|A_t^{(\sigma)}|] \leq \int_\Lambda |\sigma(\mathrm{d}\lambda)| < \infty$, to prove that $(A_t^{(\sigma)}, t \geq 0)$ is a PCOC, it suffices to prove (Lemma 3.1) that, for any $\psi \in \mathscr{C}$, the function $t \geq 0 \longrightarrow \boldsymbol{E}[\psi(A_t^{(\sigma)})]$ is increasing.
We fix $0 \leq s \leq t$. We set, for $1 \leq j \leq n$ and $u \in [0,1]$,

$$G_u^{(j)} = \sqrt{u}\, G_{\lambda_j,t} + \sqrt{1-u}\, G_{\lambda_j,s}$$

where the Gaussian vectors $(G_{\lambda_1,t}, \dots, G_{\lambda_n,t})$ and $(G_{\lambda_1,s}, \dots, G_{\lambda_n,s})$ are supposed to be independent. This will be yet another instance of application of the smart path method. Then, by Corollary 2.2, we have:

$$\boldsymbol{E}[\psi(K_1)] = \boldsymbol{E}[\psi(K_0)] + \frac{1}{2} \sum_{j,k} a_j\, a_k \int_0^1 \boldsymbol{E}\left[\psi''(K_u)\, Y_u^{(j)}\, Y_u^{(k)}\right] \cdot (c_{\lambda_j,\lambda_k}(t) - c_{\lambda_j,\lambda_k}(s))\mathrm{d}u$$

where, for $1 \le j \le n$,

$$Y_u^{(j)} = \exp\left(G_u^{(j)} - \frac{1}{2}\boldsymbol{E}\big[G_u^{(j)}G_u^{(j)}\big]\right)$$

and

$$K_u = \sum_{j=1}^{n} a_j Y_u^{(j)}.$$

Since

$$K_1 = A_t^{(\sigma)}, \quad K_0 = A_s^{(\sigma)}, \quad \psi'' \ge 0 \quad \text{and} \quad Y_u^{(j)} \ge 0,$$

if either (I_1) is satisfied and σ is a positive measure, or (I_2) is satisfied, then, for $0 \le s \le t$,

$$\boldsymbol{E}\big[\psi\big(A_t^{(\sigma)}\big)\big] \ge \boldsymbol{E}\big[\psi\big(A_s^{(\sigma)}\big)\big],$$

which proves the result.

2. By hypothesis (M_2), there exists a sequence $(\Lambda_n)_{n\ge 0}$ of compact subsets of Λ with $\bigcup_{n\ge 0}\Lambda_n = \Lambda$. We now assume that the support of σ is contained in some compact set Λ_{n_0}. Then, there exists a sequence $(\sigma_n, n \ge 0)$, weakly converging to σ, such that, for each n, σ_n is as in step 1 a linear combination of Dirac measures supported by Λ_{n_0}. Besides, we may suppose

$$\forall n \qquad \int |\sigma_n(\mathrm{d}\lambda)| \le \int |\sigma(\mathrm{d}\lambda)|. \tag{9}$$

Moreover, if σ is a positive measure, all measures σ_n may be assumed to be positive.

Let $\psi \in \mathscr{C}$. By step 1, if either (I_1) is satisfied and σ is a positive measure, or (I_2) is satisfied, then for any $n \ge 0$ and $0 \le s \le t$,

$$\boldsymbol{E}\big[\psi\big(A_t^{(\sigma_n)}\big)\big] \ge \boldsymbol{E}\big[\psi\big(A_s^{(\sigma_n)}\big)\big]. \tag{10}$$

On the other hand,

$$\boldsymbol{E}\big[\big(A_t^{(\sigma)} - A_t^{(\sigma_n)}\big)^2\big] = \int\int_{\Lambda_{n_0}^2} \mathrm{e}^{c_{\lambda,\mu}(t)}\mathrm{d}(\sigma - \sigma_n)(\lambda)\mathrm{d}(\sigma - \sigma_n)(\mu).$$

Consequently, using (9) and (C_1), we obtain the convergence, in L^2, of the sequence $(A_t^{(\sigma_n)}, n \geq 0)$ to $A_t^{(\sigma)}$.
Since ψ is affine outside of a compact interval, then ψ is a Lipschitz continuous function, and the sequence $(\psi(A_t^{(\sigma_n)}), n \geq 0)$ converges in L^2 to $\psi(A_t^{(\sigma)})$. We may then pass to the limit in (10) and obtain, for $0 \leq s \leq t$,

$$\boldsymbol{E}\big[\psi\big(A_t^{(\sigma)}\big)\big] \geq \boldsymbol{E}\big[\psi\big(A_s^{(\sigma)}\big)\big].$$

Then the desired result follows from Lemma 3.1.

3. In the general case, we set, for any $n \geq 0$

$$\sigma_n = 1_{\Lambda_n} \sigma.$$

We have, for $t \geq 0$,

$$\lim_{n \to \infty} A_t^{(\sigma_n)} = A_t^{(\sigma)} \text{ a.s.} \quad \text{and} \quad \big|A_t^{(\sigma_n)}\big| \leq A_t^{(|\sigma|)},$$

which allows to apply step 2 and to pass to the limit. □

Remark. Suppose that, for all $\lambda, \mu \in \Lambda$, the function $c_{\lambda,\mu}$ is absolutely continuous on $\boldsymbol{R}_+$. Then, Condition (I_2) may be written as:

For every $n \geq 1$, for every $\lambda_1, \dots, \lambda_n \in \Lambda$, the matrix

$$\big(c'_{\lambda_j,\lambda_k}(t)\big)_{1 \leq j,k \leq n}$$

is a positive symmetric matrix for a.e. $t \geq 0$.

3.3. Examples.

3.3.1. Processes $(t\, G_\lambda)$.

We assume (M_2) and we consider a real valued measurable centered Gaussian process $(G_\lambda, \lambda \in \Lambda)$. For $\lambda, \mu \in \Lambda$, we set:

$$c(\lambda, \mu) = \boldsymbol{E}[G_\lambda\, G_\mu].$$

We assume the following hypothesis:

$(\widetilde{C})$ The function:

$$(\lambda, \mu) \in \Lambda \times \Lambda \longrightarrow c(\lambda, \mu) \in \boldsymbol{R}$$

is continuous.

We set, for $\lambda \in \Lambda$ and $t \geq 0$,

$$G^{(1)}_{\lambda,t} = t\, G_\lambda.$$

Then, $G^{(1)}$ satisfies (C_2) and (I_2).

3.3.2. Processes $(G_{\lambda\, t})$.

Here, we consider the particular case $\Lambda = \boldsymbol{R}_+$, and a measurable centered Gaussian process $(G_\lambda, \lambda \geq 0)$ satisfying the previous condition $(\widetilde{C})$. We set, for $\lambda \geq 0$ and $t \geq 0$,

$$G^{(2)}_{\lambda,t} = G_{\lambda\, t}.$$

Furthermore, we assume the following hypothesis:

$(\widetilde{I})$ For $\lambda, \mu \geq 0$, the function

$$t \geq 0 \longrightarrow c(t\,\lambda, t\,\mu)$$

is increasing.

Then the process $G^{(2)}$ satisfies (C_2) and (I_1). In particular, Theorem 3.1 implies that the process

$$\left(\frac{1}{t}\int_0^t \exp\left(G_\lambda - \frac{1}{2}c(\lambda,\lambda)\right)\mathrm{d}\lambda, t \geq 0\right)$$

is a PCOC.

An example of a process $(G_\lambda, \lambda \geq 0)$ satisfying the above properties $(\widetilde{C})$ and $(\widetilde{I})$, is the fractional Brownian motion B^H with Hurst index $H \in (0,1)$. Indeed, then:

$$c(\lambda,\mu) = \frac{1}{2}\big(|\lambda|^{2H} + |\mu|^{2H} - |\lambda - \mu|^{2H}\big) \geq 0$$

and

$$c(t\,\lambda, t\,\mu) = t^{2H}\, c(\lambda,\mu).$$

Actually, for each $t \geq 0$, there is the equality in law:

$$B^H_{t\bullet} \overset{(\mathrm{d})}{=} t^H\, B^H_{\bullet}.$$

Therefore, we may as well apply the previous paragraph 3.3.1. Consequently, for any *signed* finite measure σ on $\boldsymbol{R}_+$,

$$\int_{\boldsymbol{R}_+} \exp\left(B^H_{t\,\lambda} - \frac{(t\,\lambda)^{2H}}{2}\right) \mathrm{d}\sigma(\lambda), \quad t \geq 0$$

is a PCOC.

We now introduce another example. Let

$$a : (\lambda, s) \in \boldsymbol{R}_+ \times \boldsymbol{R}_+ \longrightarrow a(\lambda, s) \in \boldsymbol{R}_+$$

be a nonnegative measurable function such that:

i) For every $\lambda \geq 0$, $a(\lambda, \bullet) \in L^2(\boldsymbol{R}_+)$.
ii) The function

$$(\lambda, \mu) \in \boldsymbol{R}_+ \times \boldsymbol{R}_+ \longrightarrow \int_0^\infty a(\lambda, s)\, a(\mu, s) \mathrm{d}s$$

is continuous.
iii) For any $s \geq 0$, the function

$$\lambda \in \boldsymbol{R}_+ \longrightarrow a(\lambda, s)$$

is increasing.

Setting

$$G_\lambda = \int_0^\infty a(\lambda, s) \mathrm{d}B_s, \quad \lambda \geq 0$$

where (B_s) is a standard Brownian motion, we see that properties $(\widetilde{C})$ and $(\widetilde{I})$ are satisfied.

Finally, we consider, for $\varepsilon \in \boldsymbol{R}$, the process:

$$G_\lambda = B_{\lambda \wedge 1} - \varepsilon(\lambda \wedge 1) B_1$$

where B denotes the standard Brownian motion. Obviously, $(\widetilde{C})$ is satisfied. An easy computation shows that $(\widetilde{I})$ is satisfied if and only if

$$|1-\varepsilon| \geq \frac{1}{\sqrt{2}}.$$

Consider now the case $\varepsilon = 1$. Then, $(G_t = B_t - t\,B_1, 0 \leq t \leq 1)$ is a representation of the standard Brownian bridge $(b_t, 0 \leq t \leq 1)$. The following proposition shows that, in this case, the conclusion of Theorem 3.1 fails.

PROPOSITION 3.1. *Let $(b_u, 0 \leq u \leq 1)$ be the standard Brownian bridge. We set, for $t \in [0,1]$,*

$$A_t^{(1)} = \int_0^1 \exp\left(b_{ut} - \frac{ut(1-ut)}{2}\right) \mathrm{d}u \quad \textit{and} \quad A_t^{(2)} = \exp\left(b_{at} - \frac{at(1-at)}{2}\right)$$

with $a \in (1/2, 1]$. Then, neither $(A_t^{(1)}, 0 \leq t \leq 1)$ nor $(A_t^{(2)}, 0 \leq t \leq 1)$ is a PCOC.

PROOF. It is not difficult to see that the left derivative at $t = 1$ of $\boldsymbol{E}[(A_t^{(j)})^2]$ is < 0 for $j = 1, 2$. Hence, for the convex function $\psi(x) = x^2$, the function

$$t \in [0,1] \longrightarrow \boldsymbol{E}[\psi(A_t^{(j)})]$$

is not increasing. □

Note that, for $t \in [0,1]$,

$$A_t^{(1)} = \frac{1}{t} \int_0^t \exp\left(b_s - \frac{s(1-s)}{2}\right) \mathrm{d}s.$$

Thus, replacing in the guiding Example 1.2, the Brownian motion (B_s) by the Brownian bridge (b_s), destroys the PCOC property.

3.3.3. Brownian sheet.

Let, for $\lambda, t \geq 0$,

$$G_{\lambda,t} = W_{\lambda,t}$$

where W denotes the standard Brownian sheet. We have:

$$c_{\lambda,\mu}(t) = t(\lambda \wedge \mu).$$

Then G satisfies hypotheses (C_2), (I_1) and (I_2). In fact, for any $t \geq 0$,

$$G_{\bullet,t} \overset{\text{(d)}}{=} \sqrt{t} B_\bullet$$

where B denotes the standard Brownian motion. We may then consider this example as a particular case of Example 3.3.1. (replacing t by $\sqrt{t}$). On the other hand, for any $\lambda \geq 0$,

$$\left(\exp \left(W_{\lambda,t} - \frac{t\lambda}{2} \right), t \geq 0 \right)$$

is a $(\mathscr{W}_t)$-martingale, with

$$\mathscr{W}_t = \sigma \big\{ W_{\lambda,s} \ ; \ \lambda \geq 0, \ 0 \leq s \leq t \big\}.$$

Therefore, Theorem 3.1 is obvious in this case since $(A_t^{(\sigma)})$ is a $(\mathscr{W}_t)$-martingale.

3.3.4. Stochastic integrals.

We assume (M_2). Let

$$h : (\lambda, s) \in \Lambda \times \boldsymbol{R}_+ \longrightarrow h(\lambda, s) \in \boldsymbol{R}$$

be a measurable function such that:

i) For every $\lambda \in \Lambda$, $h(\lambda, \bullet) \in L^2_{loc}(\boldsymbol{R}_+)$.
ii) For any $t \geq 0$, the function

$$(\lambda, \mu) \in \Lambda \times \Lambda \longrightarrow \int_0^t h(\lambda, s) h(\mu, s) \mathrm{d}s$$

is continuous.

We note that, for

$$G_{\lambda,t} = \int_0^t h(\lambda, s) \mathrm{d}B_s \ ; \ \lambda \in \Lambda, t \geq 0$$

where (B_s) is a standard Brownian motion, then:

$$c_{\lambda,\mu}(t) = \int_0^t h(\lambda, s) h(\mu, s) \mathrm{d}s.$$

Therefore, G satisfies hypotheses (C_1) and (I_2).

3.3.5. On a theorem of Kahane.

We assume (M_2). Let

$$X = (X_\lambda, \lambda \in \Lambda), \quad Y = (Y_\lambda, \lambda \in \Lambda)$$

be two real valued measurable centered Gaussian processes. We set, for $\lambda, \mu \in \Lambda$,

$$c_X(\lambda, \mu) = \boldsymbol{E}[X_\lambda X_\mu], \quad c_Y(\lambda, \mu) = \boldsymbol{E}[Y_\lambda Y_\mu]$$

and we assume that c_X and c_Y are continuous functions on $\Lambda \times \Lambda$. The following proposition is stated in [**Ka**] with the additional assumption that the convex function ψ below is increasing.

PROPOSITION 3.2. *We assume*:

$$\forall \lambda, \mu \in \Lambda \qquad c_X(\lambda, \mu) \le c_Y(\lambda, \mu).$$

Then, for any positive finite measure σ on Λ and for any convex function ψ on $\boldsymbol{R}$,

$$\boldsymbol{E}\left[\psi\left\{\int_\Lambda \exp\left(X_\lambda - \frac{c_X(\lambda,\lambda)}{2}\right)\sigma(\mathrm{d}\lambda)\right\}\right] \le \boldsymbol{E}\left[\psi\left\{\int_\Lambda \exp\left(Y_\lambda - \frac{c_Y(\lambda,\lambda)}{2}\right)\sigma(\mathrm{d}\lambda)\right\}\right].$$

PROOF. We shall use again the smart path method. We set, for $\lambda \in \Lambda$ and $t \in [0, 1]$,

$$G_{\lambda,t} = \sqrt{t}\, Y_\lambda + \sqrt{1-t}\, X_\lambda$$

where the processes X and Y are assumed to be independent. Then properties (C_1) and (I_1) are satisfied for $t \in [0, 1]$. Therefore, by Theorem 3.1, the process

$$\int_\Lambda \exp\left(G_{\lambda,t} - \frac{t\, c_Y(\lambda,\lambda) + (1-t)\, c_X(\lambda,\lambda)}{2}\right)\sigma(\mathrm{d}\lambda), \quad 0 \le t \le 1$$

is a PCOC, which leads to the desired result. □

4. PCOC's and Gaussian sheets.

In this section, our aim is to associate, to certain process $(A_t^{(\sigma)}, t \geq 0)$ as defined in Subsection 3.1, a martingale having the same one-dimensional marginals as this process. This will produce another proof that they are PCOC's.

Our main tool is the construction of Gaussian sheets.

4.1. Gaussian sheets.

Theorem 4.1. *Under* (M_1), (C_2) *and* (I_2), *there exists a measurable centered Gaussian process*:

$$(\Gamma_{\lambda,t}\ ;\ \lambda \in \Lambda, t \geq 0),$$

such that

$$\forall (\lambda, s), (\mu, t) \in \Lambda \times \boldsymbol{R}_+ \qquad \boldsymbol{E}[\Gamma_{\lambda,s}\, \Gamma_{\mu,t}] = c_{\lambda,\mu}(s \wedge t). \tag{11}$$

Proof. We first prove that

$$[(\lambda, s), (\mu, t)] \longrightarrow c_{\lambda,\mu}(s \wedge t)$$

is a covariance on $\Lambda \times \boldsymbol{R}_+$.

Let $\lambda_1, \dots, \lambda_n \in \Lambda$ and $t_1, \dots, t_n \in \boldsymbol{R}_+$. We denote by u a bijection from $\{1, 2, \dots, n\}$ onto $\{1, 2, \dots, n\}$ such that, setting $s_r = t_{u(r)}$, we have:

$$s_1 \leq s_2 \leq \cdots \leq s_n.$$

We also set $s_0 = 0$ and we denote by v the inverse of the bijection u. Condition (I_2) ensures the existence of matrices

$$D^{(\lambda_1,\dots,\lambda_n)}(r) \in \boldsymbol{S}_n^+, \quad 1 \leq r \leq n$$

such that,

$$\left[D^{(\lambda_1,\dots,\lambda_n)}(r)\right]^2 = \frac{1}{s_r - s_{r-1}} \left(c_{\lambda_j,\lambda_k}(s_r) - c_{\lambda_j,\lambda_k}(s_{r-1})\right)_{1 \leq j,k \leq n}$$

if $s_{r-1} < s_r$, and $D^{(\lambda_1,\dots,\lambda_n)}(r) = 0$ if $s_{r-1} = s_r$. Let

$$B_t = \left(B_t^1, \dots, B_t^n\right), \quad t \geq 0$$

be a standard $\boldsymbol{R}^n$-valued Brownian motion, independent of $G_{\bullet,0}$. We set, for $1 \le j \le n$,

$$Z_j = G_{\lambda_j,0} + \sum_{r=1}^{v(j)} \sum_{l=1}^{n} d_{j,l}^{(\lambda_1,\ldots,\lambda_n)}(r)\big(B^l_{s_r} - B^l_{s_{r-1}}\big)$$

where

$$d_{j,k}^{(\lambda_1,\ldots,\lambda_n)}(r), \quad 1 \le j,k \le n$$

denote the entries of the matrix $D^{(\lambda_1,\ldots,\lambda_n)}(r)$. Then,

$$\boldsymbol{E}[Z_j\, Z_k] = c_{\lambda_j,\lambda_k}(0) + \sum_{r=1}^{v(j)\wedge v(k)} (c_{\lambda_j,\lambda_k}(s_r) - c_{\lambda_j,\lambda_k}(s_{r-1})) = c_{\lambda_j,\lambda_k}(s_{v(j)\wedge v(k)}).$$

Since

$$s_{v(j)\wedge v(k)} = s_{v(j)} \wedge s_{v(k)} = t_j \wedge t_k,$$

we obtain

$$\boldsymbol{E}[Z_j\, Z_k] = c_{\lambda_j,\lambda_k}(t_j \wedge t_k),$$

which ensures the covariance property.

From the preceding, there exists a centered Gaussian process:

$$(\Gamma_{\lambda,t}\ ;\ \lambda \in \Lambda, t \ge 0),$$

such that

$$\boldsymbol{E}[\Gamma_{\lambda,s}\, \Gamma_{\mu,t}] = c_{\lambda,\mu}(s \wedge t).$$

Moreover, hypotheses (M_1) and (C_2) easily entail that the Gaussian space generated by this process Γ is separable. Therefore, by [**N**, Corollaire 3.8, p. 44] (see also, for instance, [**J**, Chapter VIII]), the process admits a measurable version. □

In some particular cases, we can give more explicit constructions, without assuming hypotheses (M_1), (C_2).

PROPOSITION 4.1. *Assume that there exists an increasing function φ on $\boldsymbol{R}_+$ such that, for every $\lambda, \mu \in \Lambda$ and $t \geq 0$,*

$$c_{\lambda,\mu}(t) = \varphi(t) c_{\lambda,\mu}(1),$$

i.e.: we may consider $(G_{\lambda,t}, \lambda \in \Lambda)$ as given by $(\sqrt{\varphi(t)} G_{\lambda,1}, \lambda \in \Lambda)$.

Let $(G_\bullet^{(n)}, n \geq 0)$ be a sequence of independent copies of $G_{\bullet,1}$, and let $(e_n, n \geq 0)$ be a Hilbert basis of $L^2(\boldsymbol{R}_+)$. We set

$$\Gamma_{\lambda,t} = \sum_{n=0}^{\infty} \left(\int_0^{\varphi(t)} e_n(s) \mathrm{d}s \right) G_\lambda^{(n)}.$$

Then,

$$(\Gamma_{\lambda,t} \ ; \ \lambda \in \Lambda, t \geq 0),$$

is a measurable centered Gaussian process such that (11) *is satisfied.*

PROOF. Since the function φ is increasing, the result follows from Parseval's identity. □

PROPOSITION 4.2. *Let $g : \Lambda \times \boldsymbol{R}_+ \longrightarrow \boldsymbol{R}$ be a measurable function such that, for every $\lambda \in \Lambda$, $g(\lambda, \bullet) \in L^2(\boldsymbol{R}_+)$. We suppose that*

$$G_{\lambda,t} = t \int_0^{\infty} g(\lambda, s) \mathrm{d}B_s$$

where (B_s) is a standard Brownian motion. We denote by $(W_{s,t} \ ; \ s, t \geq 0)$ the Brownian sheet and we set:

$$\Gamma_{\lambda,t} = \int_0^{\infty} g(\lambda, u) \mathrm{d}_u W_{u,t^2}.$$

Then,

$$(\Gamma_{\lambda,t} \ ; \ \lambda \in \Lambda, t \geq 0),$$

is a measurable centered Gaussian process such that (11) *is satisfied.*

PROOF. The result follows from the equality:

$$c_{\lambda,\mu}(t \wedge s) = (t \wedge s)^2 \int_0^{\infty} g(\lambda, u) g(\mu, u) \mathrm{d}u. \qquad \square$$

PROPOSITION 4.3. *Let $h : \Lambda \times \boldsymbol{R}_+ \longrightarrow \boldsymbol{R}$ be a measurable function such that, for every $\lambda \in \Lambda$, $h(\lambda, \bullet) \in L^2_{loc}(\boldsymbol{R}_+)$. We suppose that*

$$G_{\lambda,t} = \int_0^t h(\lambda, s) \mathrm{d}B_s$$

where (B_s) is a standard Brownian motion. We set:

$$\Gamma_{\lambda,t} = G_{\lambda,t}.$$

Then,

$$(\Gamma_{\lambda,t}\ ;\ \lambda \in \Lambda, t \geq 0),$$

is a measurable centered Gaussian process such that (11) *is satisfied.*

The proof is straightforward.

The following proposition states the properties of Γ which are essential in the sequel.

PROPOSITION 4.4. *Let $(\Gamma_{\lambda,t}\ ;\ \lambda \in \Lambda, t \geq 0)$ be a measurable centered Gaussian process such that* (11) *holds. We set, for $t \geq 0$,*

$$\mathscr{G}_t = \sigma\{\Gamma_{\lambda,s}\ ;\ \lambda \in \Lambda, 0 \leq s \leq t\}.$$

Then,

1) *For $0 \leq s \leq t$, the process $(\Gamma_{\lambda,t} - \Gamma_{\lambda,s}, \lambda \in \Lambda)$ is independent of the σ-field $\mathscr{G}_s$.*
2) *For any $t \geq 0$,*

$$\Gamma_{\bullet,t} \overset{\mathrm{(d)}}{=} G_{\bullet,t}.$$

The proof is straightforward.

4.2. Application to PCOC's.

PROPOSITION 4.5. *Assume there exists a measurable centered Gaussian process:*

$$(\Gamma_{\lambda,t}\ ;\ \lambda \in \Lambda, t \geq 0),$$

such that (11) *is satisfied. We set, for* $t \geq 0$,

$$\mathscr{G}_t = \sigma\{\Gamma_{\lambda,s}\ ;\ \lambda \in \Lambda, 0 \leq s \leq t\}.$$

Let σ *be a signed finite measure on* Λ. *We set, for* $t \geq 0$,

$$M_t^{(\sigma)} = \int_\Lambda \exp\left(\Gamma_{\lambda,t} - \frac{1}{2}c_{\lambda,\lambda}(t)\right)\sigma(\mathrm{d}\lambda).$$

Then $(M_t^{(\sigma)}, t \geq 0)$ *is a* $(\mathscr{G}_t)$*-martingale and, for each* $t \geq 0$,

$$M_t^{(\sigma)} \overset{\mathrm{d}}{=} A_t^{(\sigma)}.$$

In particular, $(A_t^{(\sigma)}, t \geq 0)$ *is a PCOC.*

Proof. This is a direct consequence of Proposition 4.4, using the following consequence of (11):

$$\forall 0 \leq s \leq t,\ \forall \lambda \in \Lambda, \qquad \boldsymbol{E}[(\Gamma_{\lambda,t} - \Gamma_{\lambda,s})^2] = c_{\lambda,\lambda}(t) - c_{\lambda,\lambda}(s). \qquad \square$$

Theorem 4.1, Proposition 4.1, Proposition 4.2 and Proposition 4.3 give conditions entailing the hypothesis of the above proposition. In particular, Theorem 4.1 and Proposition 4.5 yield another proof (with slightly different hypotheses) of Theorem 3.1 under Condition (I_2).

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