Toy models for D. H. Lehmer's conjecture

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Abstract. In 1947, Lehmer conjectured that the Ramanujan τ-function $\tau(m)$ never vanishes for all positive integers m, where $\tau(m)$ are the Fourier coefficients of the cusp form Δ_{24} of weight 12. Lehmer verified the conjecture in 1947 for $m < 214928639999$. In 1973, Serre verified up to $m < 10^{15}$, and in 1999, Jordan and Kelly for $m < 22689242781695999$.

The theory of spherical t-design, and in particular those which are the shells of Euclidean lattices, is closely related to the theory of modular forms, as first shown by Venkov in 1984. In particular, Ramanujan's τ-function gives the coefficients of a weighted theta series of the E_8-lattice. It is shown, by Venkov, de la Harpe, and Pache, that $\tau(m) = 0$ is equivalent to the fact that the shell of norm $2m$ of the E_8-lattice is an 8-design. So, Lehmer's conjecture is reformulated in terms of spherical t-design.

Lehmer's conjecture is difficult to prove, and still remains open. In this paper, we consider toy models of Lehmer's conjecture. Namely, we show that the m-th Fourier coefficient of the weighted theta series of the $\mathbf{Z}^2$-lattice and the A_2-lattice does not vanish, when the shell of norm m of those lattices is not the empty set. In other words, the spherical 5 (resp. 7)-design does not exist among the shells in the $\mathbf{Z}^2$-lattice (resp. A_2-lattice).

1. Introduction.

The concept of a spherical t-design is due to Delsarte-Goethals-Seidel [**6**]. For a positive integer t, a finite nonempty set X in the unit sphere

$$S^{n-1} = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \right\}$$

is called a spherical t-design in S^{n-1} if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x),$$

for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree not exceeding t. Here, the righthand side means the surface integral on the sphere, and $|S^{n-1}|$ denotes the

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volume of the sphere S^{n-1}. The meaning of spherical t-designs is that the average value of the integral of any polynomial of degree up to t on the sphere is replaced by the average value of a finite set on the sphere. A finite subset with norm r is also called a spherical t-design if its normalized set with norm 1 is a spherical t-design.

Here, we denote by $\mathrm{Harm}_j(\boldsymbol{R}^n)$ the set of homogeneous harmonic polynomials of degree j on $\boldsymbol{R}^n$. It is well known that X is a spherical t-design if and only if the condition

$$\sum_{x \in X} P(x) = 0$$

holds for all $P \in \mathrm{Harm}_j(\boldsymbol{R}^n)$ with $1 \le j \le t$. If the set X is antipodal, that is, $-X = X$, and j is odd, then the above condition is fulfilled automatically. So we reformulate the condition of spherical t-design on the antipodal set as follows:

PROPOSITION 1.1.   *A nonempty finite antipodal subset X is a spherical $2s + 1$-design if the condition*

$$\sum_{x \in X} P(x) = 0$$

holds for all $P \in \mathrm{Harm}_{2j}(\boldsymbol{R}^n)$ with $2 \le 2j \le 2s$.

A lattice in $\boldsymbol{R}^n$ is a subset $\Lambda \subset \boldsymbol{R}^n$ with the property that there exists a basis $\{e_1, \ldots, e_n\}$ of $\boldsymbol{R}^n$ such that $\Lambda = \boldsymbol{Z}e_1 \oplus \cdots \oplus \boldsymbol{Z}e_n$, i.e., Λ consists of all integral linear combinations of the vectors $e_1, \ldots, e_n$. The dual lattice Λ is the lattice

$$\Lambda^\sharp := \{y \in \boldsymbol{R}^n \mid \langle y|x \rangle \in \boldsymbol{Z}, \ \forall x \in \Lambda\},$$

where $\langle x|y \rangle$ is the standard inner product. In this paper, we assume that the lattice Λ is integral, that is, $\langle x|y \rangle \in \boldsymbol{Z}$ for all $x, y \in \Lambda$. An integral lattice Λ is called even if $\langle x|x \rangle \in 2\boldsymbol{Z}$ for all $x \in \Lambda$, and it is odd otherwise. An integral lattice Λ is called unimodular if $\Lambda^\sharp = \Lambda$. For a lattice Λ and a positive real number m, the shell of norm m of Λ is defined by

$$\Lambda_m := \{x \in \Lambda \mid \langle x|x \rangle = m\}.$$

Let $\boldsymbol{H} := \{z \in \boldsymbol{C} \mid \Im z > 0\}$ be the upper half-plane.

DEFINITION 1.1.   Let Λ be the lattice of $\boldsymbol{R}^n$. Then for a polynomial P, the function

$$\Theta_{\Lambda,P}(z) := \sum_{x \in \Lambda} P(x) e^{i\pi z \langle x|x \rangle}$$

is called the theta series of Λ weighted by P.

REMARK 1.1 (See Hecke [**7**], Schoeneberg [**14**], [**15**]).

(i) When $P = 1$, we get the classical theta series

$$\Theta_{\Lambda}(z) = \Theta_{\Lambda,1}(z) = \sum_{m \geq 0} |\Lambda_m| q^m, \quad \text{where } q = e^{\pi i z}.$$

(ii) The weighted theta series can be written as

$$\Theta_{\Lambda,P}(z) = \sum_{x \in \Lambda} P(x) e^{i\pi z \langle x|x \rangle}$$

$$= \sum_{m \geq 0} a_m^{(P)} q^m, \quad \text{where } a_m^{(P)} := \sum_{x \in \Lambda_m} P(x).$$

These weighted theta series have been used efficiently for the study of spherical designs which are the shells of Euclidean lattices. (See [**19**], [**20**], [**4**], [**12**], [**5**]. See also [**2**].)

LEMMA 1.1 (cf. [**19**], [**20**], [**12**, Lemma 5]). *Let Λ be an integral lattice in $\boldsymbol{R}^n$. Then, for $m > 0$, the non-empty shell Λ_m is a spherical t-design if and only if*

$$a_m^{(P)} = 0 \text{ for every } P \in \mathrm{Harm}_{2j}(\boldsymbol{R}^n), \quad 1 \leq 2j \leq t,$$

where $a_m^{(P)}$ are the Fourier coefficients of the weighted theta series

$$\Theta_{\Lambda,P}(z) = \sum_{m \geq 0} a_m^{(P)} q^m.$$

The theta series of Λ weighted by P is a modular form for some subgroup of $SL_2(\boldsymbol{R})$. We recall the definition of the modular forms.

DEFINITION 1.2. Let $\Gamma \subset SL_2(\boldsymbol{R})$ be a Fuchsian group of the first kind and let χ be a character of Γ. A holomorphic function $f : \boldsymbol{H} \to \boldsymbol{C}$ is called a modular form of weight k for Γ with respect to χ, if the following conditions are satisfied:

(i) $f\left(\dfrac{az+b}{cz+d}\right) = \left(\dfrac{cz+d}{\chi(\sigma)}\right)^k f(z)$ for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$

(ii) $f(z)$ is holomorphic at every cusp of Γ.

If $f(z)$ has a period N, then $f(z)$ has a Fourier expansion at infinity, [10]:

$$f(z) = \sum_{m=0}^{\infty} a_m q_N^m, \quad q_N = e^{2\pi i z/N}.$$

We remark that for $m < 0$, $a_m = 0$, by condition (ii). A modular form with constant term $a_0 = 0$, is called a cusp form. We denote by $M_k(\Gamma, \chi)$ (resp. $S_k(\Gamma, \chi)$) the space of modular forms (resp. cusp forms) with respect to Γ with the character χ. When f is the normalized eigenform of Hecke operators, [10, p. 163], the Fourier coefficients satisfy the following relations:

LEMMA 1.2 (cf. [10, Proposition 32, 37, 40, Exercise 2, p. 164]).   Let $f(z) = \sum_{m \geq 1} a(m) q^m \in S_k(\Gamma, \chi)$. If $f(z)$ is the normalized eigenform of Hecke operators, then the Fourier coefficients of $f(z)$ satisfy the following equations:

$$a(mn) = a(m)a(n) \qquad\qquad (m, n \text{ coprime})$$

$$a(p^{\alpha+1}) = a(p)a(p^\alpha) - \chi(p)p^{k-1}a(p^{\alpha-1}) \quad (p \text{ a prime}).$$

We set $f(z) = \sum_{m \geq 1} a(m) q^m \in S_k(\Gamma, \chi)$. When $\dim S_k(\Gamma, \chi) = 1$ and $a(1) = 1$, then $f(z)$ is the normalized eigenform of Hecke operators, [10]. So, the coefficients of $f(z)$ have the relations as mentioned in Lemma 1.2. It is known that

$$|a(p)| < 2p^{(k-1)/2} \tag{1}$$

for all primes p. Note that this is the Ramanujan conjecture and its generalization, called the Ramanujan-Petersson conjecture for cusp forms which are eigenforms for the Hecke operator. These conjectures were proved by Deligne as a consequence of his proof of the Weil conjectures, [10, p. 164], [9]. Moreover the following equation holds [11]. For a prime p with $\chi(p) = 1$,

$$a(p^\alpha) = p^{(k-1)\alpha/2}\frac{\sin(\alpha+1)\theta_p}{\sin \theta_p}, \tag{2}$$

where $2\cos\theta_p = a(p)p^{-(k-1)/2}$.

It is well known that the theta series of $\Lambda \subset \mathbf{R}^n$ weighted by harmonic polynomial $P \in \mathrm{Harm}_j(\mathbf{R}^n)$ is a modular form of weight $n/2 + j$ for some subgroup $\Gamma \subset SL_2(\mathbf{R})$. In particular, when $\deg(P(x)) \geq 1$, the theta series of Λ weighted by P is a cusp form.

For example, we consider an even unimodular lattice Λ. Then the theta series of Λ weighted by the harmonic polynomial P, $\Theta_{\Lambda,P}(z)$, is a modular form with respect to $SL_2(\mathbf{Z})$.

EXAMPLE 1.1 (cf. [**19**], [**20**]). Let Λ be the E_8-lattice. This is an even unimodular lattice of $\mathbf{R}^8$, generated by the E_8 root system. The theta series is as follows:

$$\Theta_\Lambda(z) = E_4(z) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}$$

$$= 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots,$$

where $\sigma_3(m)$ is a divisor function $\sigma_3(m) = \sum_{0 < d \mid m} d^3$.

For $j = 2, 4$ and 6, the theta series of Λ weighted by P, $P \in \mathrm{Harm}_j(\mathbf{R}^8)$ is a weight $6, 8$ and 10 cusp form with respect to $SL_2(\mathbf{Z})$. However, it is well known that for $k = 6, 8$ and 10, $\dim S_k(SL_2(\mathbf{Z})) = 0$, that is, $\Theta_{\Lambda,P}(z) = 0$. Then by Lemma 1.1, all the shells of E_8-lattice are spherical 7-designs.

For $j = 8$, the theta series of Λ weighted by P is a cusp form of weight 12 with respect to $SL_2(\mathbf{Z})$. Such a cusp form is uniquely determined up to constant, i.e., it is Ramanujan's delta function:

$$\Delta_{24}(z) = q^2 \prod_{m \geq 1} (1 - q^{2m})^{24} = \sum_{m \geq 1} \tau(z)q^{2m}.$$

The following proposition is due to Venkov, de la Harpe and Pache, [**4**], [**5**], [**12**], [**19**].

PROPOSITION 1.2 (cf. [**12**]). *Let the notation be the same as above. Then the following are equivalent*:

(i) $\tau(m) = 0$.
(ii) $(\Lambda)_{2m}$ *is a spherical 8-design.*

It is a famous conjecture of Lehmer that $\tau(m) \neq 0$. So, Proposition 1.2 gives a reformulation of Lehmer's conjecture. Lehmer proved in [**11**] the following theorem.

THEOREM 1.1 (cf. [11]).   Let m_0 be the least value of m for which $\tau(m) = 0$. Then m_0 is a prime if it is finite.

These are many attempts to study Lehmer's conjecture ([11], [16]), but it is difficult to prove and it is still open.

In this paper, we take the two cases $\mathbf{Z}^2$-lattice and A_2-lattice instead of E_8-lattice. Then, we consider the analogue of Lehmer's conjecture corresponding to the theta series weighted by some harmonic polynomial P. In Section 3, we show that the m-th coefficient of the weighted theta series of $\mathbf{Z}^2$-lattice does not vanish when the shell of norm m of those lattices is not an empty set. Or equivalently, we show the following result.

THEOREM 1.2.   The shells in $\mathbf{Z}^2$-lattice are not spherical 5-designs.

Similarly, in Section 4, we show the following result.

THEOREM 1.3.   The shells in A_2-lattice are not spherical 7-designs.

2.   Preliminaries.

First of all, we list up the classical modular forms needed later. More details about these functions appear in [3]. We use $q = e^{\pi i z}$.

$$\theta_2(z) = \sum_{m \in \mathbf{Z}+1/2} q^{m^2} = 2q^{1/4}(1 + q^2 + \cdots) \qquad \text{of weight } 1/2,$$

$$\theta_3(z) = \sum_{m \in \mathbf{Z}} q^{m^2} = 1 + 2q + 2q^4 + \cdots \qquad \text{of weight } 1/2,$$

$$\theta_4(z) = \sum_{m \in \mathbf{Z}} (-q)^{m^2} = 1 - 2q + 2q^4 + \cdots \qquad \text{of weight } 1/2,$$

$$\eta(z) = q^{1/12} \prod_{m \geq 1}(1 - q^{2m}) = q^{1/12}(1 - q^2 - q^4 + q^{10} + \cdots) \quad \text{of weight } 1/2,$$

$$\Phi(z) = \theta_4(z)^4 - \theta_2(z)^4 = 1 - 24q + 24q^2 - 96q^3 + \cdots \qquad \text{of weight } 2,$$

$$\Delta_8(z) = \frac{1}{16}\theta_2(z)^4\theta_4(z)^4 = q - 8q^2 + 28q^3 + \cdots \qquad \text{of weight } 4,$$

$$\Delta_{12}(z) = \eta(z)^6\eta(3z)^6 = q^2 - 6q^4 + 9q^6 + 4q^8 + \cdots \qquad \text{of weight } 6.$$

2.1.   The $\mathbf{Z}^2$-lattice.
Let

$$\boldsymbol{Z}^n := \{x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n \mid x_i \in \boldsymbol{Z}, i = 1, \ldots, n\}$$

be the cubic lattice of rank n. It is an odd unimodular lattice. The theta series of the $\boldsymbol{Z}^n$-lattice is $\Theta_{\boldsymbol{Z}^n}(z) = \theta_3(z)^n$. For example, if we take $n = 2$, then

$$\Theta_{\boldsymbol{Z}^2}(z) = \theta_3(z)^2 = \sum_{m=0}^{\infty} r_2(m) q^m$$

$$= 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13} + \cdots,$$

where the coefficient $r_2(m)$ is the number of ways of writing m as a sum of 2 squares.

LEMMA 2.1 (cf. [**12**, Lemma 24]).   *We have*

$$\Theta_{\boldsymbol{Z}^n,P} = \begin{cases} \theta_3^n & \text{if } P = 1, \\ 0 & \text{if } P \in \mathrm{Harm}_2(\boldsymbol{R}^n), \\ c_1(P)\Delta_8\theta_3^n & \text{if } P \in \mathrm{Harm}_4(\boldsymbol{R}^n), \\ c_2(P)\Phi\Delta_8\theta_3^n & \text{if } P \in \mathrm{Harm}_6(\boldsymbol{R}^n), \end{cases}$$

where c_1 is a nonzero linear form if and only if $n \geq 2$, and c_2 is a nonzero linear form if and only if $n \geq 3$.

By Lemma 1.1, For $n \geq 2$, all the nonempty shells of $\boldsymbol{Z}^n$ are spherical 3-designs. We consider the case $n = 2$. For $P \in \mathrm{Harm}_4(\boldsymbol{R}^2)$, $\Theta_{\boldsymbol{Z}^2,P} = c_1(P)\Delta_8\theta_3^2 = \sum_{m>0} a(m) q^m$.

We set

$$G(2) := \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Then, the functions Δ_8 and θ_3^2 are the modular forms with respect to $G(2)$ with the character χ,

$$\begin{cases} \chi\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right) = 1 \\ \chi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = i, \end{cases}$$

(cf. [**12**, Theorem 12], [**3**, p. 187]). Hence $\Delta_8\theta_3^2 \in S_5(G(2),\chi)$.

Then it is easy to see that the following proposition holds.

PROPOSITION 2.1 (cf. [**12**]).   *Let the notation be the same as above. Then the following assertions are equivalent*:

(i) $(\mathbf{Z}^2)_m \neq \emptyset$ *and* $a(m) = 0$.
(ii) $(\mathbf{Z}^2)_m$ *is a spherical* 5-*design*.

In Section 3, we will prove that $a(m) \neq 0$ if $(\mathbf{Z}^2)_m \neq \emptyset$, hence also show the non existence of the spherical 5-designs on the shells in $\mathbf{Z}^2$-lattice.

2.2.   The A^2-lattice.
Let

$$A_n := \{x = (x_0,\ldots,x_n) \in \mathbf{Z}^{n+1} \mid x_0 + \cdots + x_n = 0\}$$

be the A_n-lattice of rank n. It is an even lattice. When $n = 2$, the theta series of A_2-lattice is

$$\Theta_{A_2}(z) = \theta_3(2z)\theta_3(6z) + \theta_2(2z)\theta_2(6z)$$
$$= 1 + 6q^2 + 6q^6 + 6q^8 + 12q^{14} + 6q^{18} + 6q^{24} + 12q^{26} + 6q^{32} + 12q^{38} + \cdots.$$

LEMMA 2.2.   *We have*

$$\Theta_{A_2,P} = \begin{cases} \Theta_{A_2} & \text{if } P = 1, \\ 0 & \text{if } P \in \mathrm{Harm}_2(\mathbf{R}^2), \\ 0 & \text{if } P \in \mathrm{Harm}_4(\mathbf{R}^2), \\ c_1(P)\Delta_{12}\Theta_{A_2} & \text{if } P \in \mathrm{Harm}_6(\mathbf{R}^2), \end{cases}$$

where c_1 is a nonzero linear form.

PROOF.   First, we define the Fricke group; for any prime p,

$$\Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)\,W_p, \quad \text{where } W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}.$$

Then, the theta series of the A_2-lattice weighted by P are modular forms with respect to the Fricke group $\Gamma_0^*(3)$ with the character χ_s,

$$\begin{cases} \chi_s(A) = \left(\dfrac{(-3)^s}{d}\right) & \text{if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3) \\ \chi_s(W_3) = i^s, \end{cases}$$

(cf. [**1**, Theorem 3.1]). In fact, $\oplus_{s\geq 0} M_s(\Gamma_0^*(3), \chi_s) = \boldsymbol{C}[\Theta_{A_2}, \Delta_{12}]$, [**13**].
We take $P \in \text{Harm}_6(\boldsymbol{R}^2)$,

$$P(x) = (x^6 - y^6) - 15(x^4 y^2 - x^2 y^4).$$

Then $\Theta_{A_2, P}(z) = 4q^2 + \cdots$. So, c_1 is not identically zero. $\qquad \square$

By Lemma 1.1, all the nonempty shells of the A_2-lattice are spherical 5-designs. For $P \in \text{Harm}_6(\boldsymbol{R}^3)$, $\Theta_{A_2, P} = c_1(P)\Delta_{12}\Theta_{A_2} =: \sum_{m>0} a(m)q^{2m}$.

Then it is easy to see that the following proposition holds.

PROPOSITION 2.2. *Let the notation be the same as above. Then the following assertions are equivalent*:

 (i) $(A_2)_m \neq \emptyset$ *and* $a(m) = 0$.
 (ii) $(A_2)_{2m}$ *is a spherical 7-design.*

In Section 4, we will prove that $a(m) \neq 0$ if $(A_2)_{2m} \neq \emptyset$, hence also show the non existence of the spherical 7-designs on the shells in A_2-lattice.

Finally, we collect the results needed later.

PROPOSITION 2.3 (cf. [**8**]). *Let* $\sigma_k(m)$ *be the divisor function* $\sigma_k(m) = \sum_{0<d|m} d^k$. *Then the following equality holds*:

$$\frac{1}{16}\theta_2^4(z) = \sum_{m=1}^{\infty} \sigma_1(2m-1)q^{2m-1}.$$

PROPOSITION 2.4 (cf. [**12**, p. 127]). *Let* χ *be a Dirichlet character* $\bmod M$ *and* χ_1 *be a primitive Dirichlet character* $\bmod N$. *If* $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_0(M), \chi)$ *and* $f_{\chi_1}(z) = \sum_{n=0}^{\infty} a_n \chi_1(n)q^n$, *then* $f_{\chi_1}(z) \in M_k(\Gamma_0(MN^2), \chi\chi_1^2)$. *If* f *is a cusp form, then so is* f_{χ_1}. *In particular, if* $f(z) \in M_k(\Gamma_0(M))$ *and* χ_1 *is a quadratic (i.e., takes values* ± 1*), then* $f_{\chi_1}(z) \in M_k(\Gamma_0(MN^2))$.

THEOREM 2.1 (cf. [**18**, Theorem 1]). *Let* $f(z)$ *and* $g(z)$ *be holomorphic modular forms of weight* k *with respect to some congruence subgroup* Γ *of* $SL_2(\boldsymbol{Z})$. *If* $f(z)$ *and* $g(z)$ *have integer coefficients and there exists a prime* l *such that*

$$\mathrm{Ord}_l(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbf{Z}) : \Gamma],$$

then $\mathrm{Ord}_l(f(z) - g(z)) = \infty$. (i.e., $f(z) \equiv g(z) \pmod{l}$.)

3. $\mathbf{Z}^2$-lattice.

We recall the results:

$$\Theta_{\mathbf{Z}^2}(z) = \theta_3(z)^2 = \sum_{m=0}^{\infty} r_2(m)q^m.$$

LEMMA 3.1. *Assume that p is a prime. If $p \equiv 1 \pmod{4}$, then $r_2(p) = 8$. If $p \equiv 3 \pmod{4}$, then*

$$r_2(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

PROOF. Denote the number of divisors of n with $d \equiv a \pmod{4}$ by $d_a(n)$. It is well known that $r_2(m) = 4(d_1(m) - d_3(m))$, [**3**]. Then the results follow from easy calculations. □

For $P \in \mathrm{Harm}_4(\mathbf{R}^2)$,

$$\Theta_{\mathbf{Z}^2,P}(z) = c(P)\theta_3(z)^2\Delta_8(z)$$
$$= c(P)(q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + 81q^9 + \cdots)$$
$$=: c(P) \sum_{m \geq 1} a(m)q^m.$$

Then $\Theta_{\mathbf{Z}^2,P}(z) \in S_5(G(2), \chi)$.

Here, we define the following function:

$$\theta_3(2z)^2\Delta_8(2z) = \sum_{m \geq 1} a(m)q^{2m}$$
$$= q^2 - 4q^4 + 16q^8 - 14q^{10} - 64q^{16} + 81q^{18} + \cdots.$$

This is a modular form with respect to $\Gamma_0(4)$ with the character $\chi_4 = (-1)^{(d-1)/2}$ (cf. [**10**, Proposition 30, p. 145]).

Because of $\dim S_5(\Gamma_0(4), \chi_4) = 1$, [**17**], and $a(1) = 1$, by Lemma 1.2, the coefficients of $\Theta_{\mathbf{Z}^2, P}(z)$ satisfied the following equations:

$$a(mn) = a(m)a(n) \qquad\qquad (m, n \text{ coprime}) \qquad (3)$$

$$a(p^{\alpha+1}) = a(p)a(p^\alpha) - \chi_4(p)p^4 a(p^{\alpha-1}). \quad (p \text{ a prime}) \qquad (4)$$

By the equation (1) and (2), we get the following equations:

$$|a(p)| < 2p^2 \qquad\qquad (5)$$

for a prime p with $\chi(p) = 1$, $\quad a(p^\alpha) = p^{2\alpha}\dfrac{\sin(\alpha+1)\theta_p}{\sin\theta_p}, \qquad (6)$

where $2\cos\theta_p = a(p)p^{-2}$.

The coefficients $a(m)$ have the following crucial property.

LEMMA 3.2.   *If m is odd, then $a(m) \equiv \sigma_1(m) \pmod 4$, where $\sigma_1(m)$ is the divisor function* $\sigma_1(m) := \sum_{0<d|m} d$.

PROOF.   Because of the relation $\theta_3(2z)^2 \equiv \theta_4(2z)^4 \equiv 1 \pmod 4$ and Proposition 2.3,

$$\theta_3(2z)^2\Delta_8(2z) = \frac{1}{16}\theta_3(2z)^2\theta_4(2z)^4\theta_2(2z)^4$$

$$\equiv \frac{1}{16}\theta_2(2z)^4 \pmod 4$$

$$\equiv \sum_{m=1}^{\infty}\sigma_1(2m-1)q^{2(2m-1)} \pmod 4. \qquad \square$$

PROOF OF THEOREM 1.2.   We will show that $a(m) \neq 0$ when $(\mathbf{Z}^2)_m \neq \emptyset$. Assume that m is a power of prime, if not we could apply (3). We will divide into the three cases.

(i) Case $m = 2^\alpha$:
We consider the equation (4).

$$a(2^{n+1}) = a(2)a(2^n).$$

Hence we have $a(2^\alpha) \neq 0$, for $a(2) = -4$.

(ii) Case $m = p^\alpha$, $p \equiv 3 \pmod 4$:

By Lemma 3.1, $a(p^n) = 0$ if n is odd. Then, the equation (4) can be written as follows:

$$a(p^{n+1}) = p^4 a(p^{n-1}).$$

Thus we get

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ p^{4(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

Hence we have $a(p^\alpha) \neq 0$ when $(\mathbf{Z}^2)_m \neq \emptyset$.

(iii) Case $m = p^\alpha$, $p \equiv 1 \pmod 4$: First of all, we show the following proposition.

PROPOSITION 3.1.   *Let α_0 be the least value of α for which $a(p^\alpha) = 0$. Then $\alpha_0 = 1$.*

PROOF.   Assuming the contrary, that is, $\alpha_0 > 1$, so that $a(p) \neq 0$. By (6),

$$a(p^{\alpha_0}) = 0 = p^{2\alpha_0} \frac{\sin(\alpha + 1)\theta_p}{\sin \theta_p}.$$

This shows that θ_p is a real number of the form $\theta_p = \pi k/(1 + \alpha_0)$, where k is an integer. Now the number

$$z = 2 \cos \theta_p = a(p) p^{-2}, \tag{7}$$

being twice the cosine of a rational multiple 2π, is an algebraic integer. On the other hand z is a root of the obviously quadratic equation

$$p^4 z^2 - a^2(p) = 0. \tag{8}$$

Hence z is a rational integer. By (5) and (7), we have $z^2 \leq 3$. Therefore $z^2 = 1, 2$ and 3. If z=2 (resp. 3), the quadratic equation (8) becomes $a^2(2) = 2p^4$ (resp. $a^2(3) = 3p^4$). These are impossible because the right hand sides are not squares. If $z^2 = 1$, the quadratic equation (8) becomes

$$a(p) = \pm p^2. \tag{9}$$

By Lemma 3.2, we have

$$a(p) \equiv \sigma_1(p) \pmod 4$$
$$\equiv p + 1 \pmod 4$$
$$\equiv 2 \pmod 4. \tag{10}$$

However, if $p^2 \equiv 1 \pmod 4$ then the equation (9) becomes $a(p) = 1$ or 3. This is a contradiction. $\qquad\square$

So, it is enough to consider the case when m is a prime. By (10), $a(p) \equiv 2$ (mod 4), so, we have $a(p) \neq 0$. This completes the proof of Theorem 1.2. $\qquad\square$

4. A_2-lattice.

We recall the results.

$$\Theta_{A_2}(z) = \theta_3(2z)\theta_3(6z) + \theta_2(2z)\theta_2(6z)$$

$$=: \sum_{m=0}^{\infty} N(m)q^{2m}.$$

LEMMA 4.1 (cf. [**3**, p. 112]).

$$N(3^\alpha) = 6, \qquad \text{for all } a \geq 0,$$
$$N(p^\alpha) = 6(\alpha + 1), \qquad \text{for } p \equiv 1 \pmod 3,$$
$$N(p^\alpha) = \begin{cases} 0 & \text{for } p \equiv 2 \pmod 3, \ \alpha \text{ is odd,} \\ 6 & \text{for } p \equiv 2 \pmod 3, \ \alpha \text{ is even.} \end{cases}$$

For $P \in \mathrm{Harm}_6(\boldsymbol{R}^2)$,

$$\Theta_{A_2,P}(z) = c(P)\Theta_{A_2}(z)\Delta_{12}(z) =: \sum_{m \geq 1} a(m)q^{2m},$$

$$\Delta_{12}(z) = (\eta(z)\eta(3z))^6 =: \sum_{m \geq 1} b(m)q^{2m},$$

$$E_6(z) = 1 - 504\sum_{m \geq 1} \sigma_5(m)q^{2m},$$

where $E_6(z)$ is the Eisenstein series of weight 6 with respect to the group $SL_2(\mathbf{Z})$ and, $\sigma_5(m)$ is the divisor function $\sigma_5(m) := \sum_{0 < d|m} d^5$. As we saw above, $\Theta_{A_2,P}(z) \in S_7(\Gamma_0^*(3), \chi_7)$, hence $\Theta_{A_2,P}(z) \in S_7(\Gamma_0(3), \chi_7)$. Because $\dim S_7(\Gamma_0(3), \chi_7) = 1$, [17], and $a(1) = 1$, by Lemma 1.2, the coefficients of $\Theta_{A_2,P}(z)$ satisfies the following equations:

$$a(mn) = a(m)a(n) \qquad\qquad (m, n \text{ coprime}) \qquad (11)$$

$$a(p^{\alpha+1}) = a(p)a(p^\alpha) - \chi_7(p)p^6 a(p^{\alpha-1}). \quad (p \text{ a prime}) \qquad (12)$$

By the equations (1) and (2), we get the following equations:

$$|a(p)| < 2p^3 \qquad\qquad\qquad (13)$$

for a prime p with $\chi(p) = 1$, $\quad a(p^\alpha) = p^{3\alpha} \dfrac{\sin(\alpha+1)\theta_p}{\sin \theta_p}, \qquad\qquad (14)$

where $2\cos \theta_p = a(p)p^{-3}$.

The following lemma is useful later.

LEMMA 4.2. *For $m \not\equiv 0 \pmod 3$, $a(m) \equiv \sigma_5(m) \pmod 3$.*

PROOF. By Lemma 4.1, we remark that $\Theta_{A_2}(z) \equiv 1 \pmod 3$. So, we have $a(m) \equiv b(m) \pmod 3$. Next we consider the following function:

$$\Delta_{12}(z) + \frac{E_6(z)}{504} = \frac{1}{504} + \sum_{m \geq 0} (b(m) - \sigma_5(m))q^{2m} =: \sum_{m \geq 0} c(m)q^{2m}.$$

This is a modular form with respect to $\Gamma_0(3)$. Now we apply Proposition 2.4 and construct a modular form with respect to $\Gamma_0(27)$ by

$$\sum_{m \geq 0} \left(\frac{m}{3}\right) c(m)q^{2m} = 39q^4 - 1053q^8 + 3120q^{10} - 16848q^{14} + \cdots.$$

One finds that the first $(6/12)[SL_2(\mathbf{Z}) : \Gamma_0(27)] + 1 = 19$ terms are a multiple of 3 which completes the proof by an immediate application of Theorem 2.1. $\qquad\square$

LEMMA 4.3. *For $m \equiv 1 \pmod 3$ and m is odd, $a(m) \equiv \sigma_1(m) \pmod 2$.*

PROOF. By Lemma 4.1, we remark that $\Theta_{A_2}(z) \equiv \theta_3^4(z) \equiv \theta_4^4(z) \equiv 1 \pmod 2$. So, we have $a(m) \equiv b(m) \pmod 2$ and we take the following function:

$$\sum_{m \geq 1} c(m)q^{2m} := \frac{1}{16}\theta_3^4(2z)\theta_4^4(2z)\theta_2^4(2z) \equiv \frac{1}{16}\theta_2^4(2z) \pmod{2}$$

$$\equiv \sum_{m \geq 1} \sigma_1(2m-1)q^{2(2m-1)} \pmod{2}.$$

This is a modular form with respect to $\Gamma_0(4)$, [**10**]. So, for $m \equiv 1 \pmod{2}$, $c(m) \equiv \sigma_1(m)$. Namely, it is enough to show that for $m \equiv 1 \pmod{2}$, $b(m) \equiv c(m)$.

Next we consider the following function:

$$\sum_{m \geq 1} d(m)q^{2m} := \Delta_{12}(z) - \frac{1}{16}\theta_3^4(2z)\theta_4^4(2z)\theta_2^4(2z)$$

$$\equiv \Delta_{12}(z) - \sum_{m \geq 1} \sigma_1(2m-1)q^{2(2m-1)} \pmod{2}.$$

This is a modular form with respect to $\Gamma_0(12)$. Now we apply Proposition 2.4 and construct a modular form with respect to $\Gamma_0(108)$ by

$$\sum_{m \geq 1} \left(\frac{m}{3}\right)d(m)q^{2m} = 6q^4 + 4q^8 - 48q^{14} - 168q^{16} - 36q^{20} + \cdots.$$

One finds that the first $(6/12)[SL_2(\boldsymbol{Z}) : \Gamma_0(108)] + 1 = 109$ terms are multiples of 2 which completes the proof by an immediate application of Theorem 2.1. $\square$

PROOF OF THEOREM 1.3. We will show that $a(m) \neq 0$ when $(A_2)_m \neq \emptyset$. Assume that m is a power of prime, if not we could apply (11). We will divide into the three cases.

(i) Case $m = 3^\alpha$:
We consider the equation (12).

$$a(3^{n+1}) = a(3)a(3^n).$$

Hence we have $a(3^\alpha) \neq 0$, for $a(3) = -27$.

(ii) Case $m = p^\alpha$, $p \equiv 2 \pmod{3}$:
By Lemma 4.1, $a(p^n) = 0$ if n is odd. Then, the equation (12) can be written as follows:

$$a(p^{n+1}) = p^6 a(p^{n-1}).$$

Thus we get

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ p^{6(n-1)} & \text{if } n \text{ is even.} \end{cases}$$

Hence we have $a(p^\alpha) \neq 0$ when $(A_2)_m \neq \emptyset$.

(iii) Case $m = p^\alpha$, $p \equiv 1 \pmod 3$: First of all, we show the following proposition.

PROPOSITION 4.1.   Let α_0 be the least value of α for which $a(p^\alpha) = 0$. Then $\alpha_0 = 1$.

PROOF.   Assuming the contrary, that is, $\alpha_0 > 1$, so that $a(p) \neq 0$. By (13),

$$a(p^{\alpha_0}) = 0 = p^{2\alpha_0} \frac{\sin(\alpha+1)\theta_p}{\sin \theta_p}.$$

This shows that θ_p is a real number of the form $\theta_p = \pi k/(1+\alpha_0)$, where k is an integer. Now the number

$$z = 2\cos\theta_p = a(p)p^{-3}, \tag{15}$$

being twice the cosine of a rational multiple 2π, is an algebraic integer. On the other hand z is a root of the obviously quadratic equation

$$p^6 z^2 - a^2(p) = 0. \tag{16}$$

Hence z is a rational integer. By (13) and (15), we have $z^2 \leq 3$. Therefore $z^2 = 1, 2$ and 3. If z=2 (resp. 3), the quadratic equation (16) becomes $a^2(2) = 2p^6$ (resp. $a^2(3) = 3p^6$). These are impossible because the right hand sides are not squares. If $z^2 = 1$, we have

$$a(p) = \pm p^3. \tag{17}$$

By Lemma 4.2 and 4.3,

$$a(p) \equiv \sigma_5(p) \pmod 3$$
$$\equiv p^5 + 1 \pmod 3$$
$$\equiv 2 \pmod 3, \tag{18}$$

$$a(p) \equiv \sigma_1(p) \pmod 2$$
$$\equiv p + 1 \pmod 2$$
$$\equiv 0 \pmod 2. \tag{19}$$

By (17), $a(p) \equiv 1$ or $2 \pmod 3$. In the first case $a(p) \equiv 1 \pmod 3$, this is a contradiction to (18). In the second case $a(p) \equiv 2 \pmod 3$, this is a contradiction to (19). So, the proof is completed. $\square$

So it is enough to consider the case when m is a prime. By (18), $a(p) \equiv 2 \pmod 3$, so, we have $a(p) \neq 0$. This completes the proof of Theorem 1.3. $\square$

5. Concluding Remarks.

(1) In the last part of the proof of Theorem 1.2 (resp. Theorem 1.3), after we obtain Proposition 3.1 (resp. Proposition 4.1), we can directly show that all the nonempty shells $(\mathbf{Z}^2)_p$ (resp. $(A_2)_{2p}$), hence all the nonempty shells $(\mathbf{Z}^2)_m$ (resp. $(A_2)_{2m}$) are not spherical 5-designs (resp. 7-designs). Here we note that the irrationalities of $\tan(\pi/8)$ and $\tan(\pi/12)$ and the explicit structure of the shells $(\mathbf{Z}^2)_p$ and $(A_2)_{2p}$ play important roles. This gives an alternative approach to the proof of Theorem 1.2 (resp. Theorem 1.3).

(2) It is interesting to note that no spherical 12-design among the shells of any Euclidean lattice (of any dimension) is known. It is an interesting open problem to prove or disprove whether these exists any 12-design which is a shell of a Euclidean lattice.

(3) Responding to the author's request, Junichi Shigezumi performed computer calculations to determine whether there are spherical t-designs for bigger t, in the 2 and 3 dimensional cases. His calculation shows that among the shells of integral lattices in dimension 2 (with relatively small discriminant and small norms), there are only 5-designs. That is, no 6-designs were found. (So far, all examples of such 5-designs are vertices of a regular 6-gon, although they are the shells of many different lattices). In the 3 dimensional case, all examples are only 3-designs. No 4-designs which are shells of a lattice were found. It is an interesting open problem whether this is true in general for dimensions 2 and 3.

(4) For the lattices whose dimension is greater than 2, it is difficult for the results of this paper to extend to such lattices. One of the reasons is that the number of the points of the shell of norm p (prime) is not constant and we cannot know the structure of the shell Λ_p well, for dimension $n \geq 3$. On the other hand, it is in fact possible to extend our results for some other 2-dimensional lattices. In the subsequent paper, we will deal with the case of the 2-dimensional lattices

associated to the algebraic integers of imaginary quadratic number fields whose class number is either 1 or 2.

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