

Tagged particle processes and their non-explosion criteria

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Abstract. We give a derivation of tagged particle processes from unlabeled interacting Brownian motions. We give a criteria of the non-explosion property of tagged particle processes. We prove the quasi-regularity of Dirichlet forms describing the environment seen from the tagged particle, which were used in previous papers to prove the invariance principle of tagged particles of interacting Brownian motions.

1. Introduction.

Interacting Brownian motions (IBMs) in infinite dimensions are diffusions $\mathbf{X}_t = (X_t^i)_{i \in \mathbf{Z}}$ consisting of infinitely many particles moving in \mathbf{R}^d with the effect of the external force coming from a self potential $\Phi: \mathbf{R}^d \rightarrow \mathbf{R} \cup \{\infty\}$ and that of the mutual interaction coming from an interacting potential $\Psi: \mathbf{R}^d \times \mathbf{R}^d \rightarrow \mathbf{R} \cup \{\infty\}$ such that $\Psi(x, y) = \Psi(y, x)$.

Intuitively, IBMs are described by the infinitely dimensional stochastic differential equation (SDE) of the form

$$dX_t^i = dB_t^i - \frac{1}{2} \nabla \Phi(X_t^i) dt - \frac{1}{2} \sum_{j \in \mathbf{Z}, j \neq i} \nabla \Psi(X_t^i, X_t^j) dt \quad (i \in \mathbf{Z}). \quad (1.1)$$

The state space of the process $\mathbf{X}_t = (X_t^i)_{i \in \mathbf{Z}}$ is $(\mathbf{R}^d)^{\mathbf{Z}}$ by construction. Let \mathfrak{X} be the configuration valued process given by

$$\mathfrak{X}_t = \sum_{i \in \mathbf{Z}} \delta_{X_t^i}. \quad (1.2)$$

Here δ_a denotes the delta measure at a and a configuration is a Radon measure consisting of a sum of delta measures. We call \mathbf{X} the labeled dynamics and \mathfrak{X} the

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unlabeled dynamics.

The SDE (1.1) was initiated by Lang [10], [11]. He studied the case $\Phi = 0$, and $\Psi(x, y) = \Psi(x - y)$, Ψ is of $C_0^3(\mathbf{R}^d)$, superstable and regular in the sense of Ruelle [21]. With the last two assumptions, the corresponding unlabeled dynamics \mathfrak{X} has Gibbsian equilibrium states. See [22], [5] and [24] for other works concerning on the SDE (1.1).

In [14] the unlabeled diffusion was constructed by the Dirichlet form approach. This method gives a general and simple proof of construction, and allows us to apply singular interaction potentials such as Lennard-Jones 6-12 potential, hard core potential and so on. See [27], [1] [25], and [26] for other works concerning on the Dirichlet form approach to IBMs.

In this paper we are interested in the property of each labeled particle of the unlabeled particle system given by the Dirichlet form. Such labeled particles are called tagged particles. By construction the unlabeled IBMs \mathfrak{X} are conservative since they have invariant probability measures and their state spaces are equipped with the vague topology. However, each labeled particle may explode under the Euclidean metric on \mathbf{R}^d in general. The first purpose of the paper is to give a criteria for the non-explosion of the labeled particles (Theorem 2.5).

Let us next assume the total system is translation invariant in space. More precisely, we assume the stationary measure μ and the energy form \mathcal{E}^μ of the Dirichlet space are translation invariant. Then the process \mathfrak{X} starting from μ is translation invariant in space. The above assumption means, for Ruelle's class potentials [21], $\Phi = 0$ and $\Psi(x, y) = \Psi(x - y)$.

This type of infinite-dimensional diffusions has been studied by the motivation from the statistical physics. One of the archetypical problem in this field is to investigate the large time property (the diffusive scaling limit, say) of tagged particles in the stationary system. This problem was solved for the simple exclusion process, which is a lattice analog of the hard core Brownian balls, by Kipnis-Varadhan [9]. For this they establish the celebrated Kipnis-Varadhan invariance principle.

As for the tagged particle problem of IBMs, Guo [6], Guo-Papanicolau [7] initiate the problem. Later De Masi *et al.* [2] study the problem for IBMs by using the Kipnis-Varadhan invariance principle. In [15], we convert the Kipnis-Varadhan invariance principle to the Dirichlet form theory. As a result, we weaken the assumption on the L^2 -integrability of the mean forward velocity. This enables us to apply the invariance principle to hard core Brownian balls [15] and [16].

In [15] we consider Dirichlet forms describing the tagged particle process and the environment process. These two Dirichlet forms are necessary to apply the Kipnis-Varadhan theory to IBMs. Although we gave the out line of the proof of the quasi-regularity of these Dirichlet forms and the relation between these two

processes and the original unlabeled diffusion, the details were postponed. The second purpose of the paper is to give these details (Theorems 2.4, 2.6 and 2.7).

We establish the quasi-regularity of the Dirichlet forms of k -labeled dynamics (Lemma 2.3) and prove the identity between k -labeled dynamics and additive functionals of unlabeled dynamics (Theorem 2.4). The 0-labeled dynamics are simply the unlabeled dynamics; the k -labeled dynamics are the processes of the form $(X_t^1, \dots, X_t^k, \sum_{j \neq 1, \dots, k} \delta_{X_t^j})$. The quasi-regularity of the Dirichlet form of the 0-labeled dynamics has been already proved in [14]. Although Lemma 2.3 is a straightforward generalization of it, we give a proof here for reader's convenience. On the other hand, the proof of Theorem 2.4 is complicated because there is no simple transformation between Dirichlet spaces of the 0-labeled dynamics and the k -labeled dynamics. Theorem 2.4 plays an important role not only in the present paper but also in [19]. In [19], Theorem 2.4 is used to solve the infinite-dimensional SDE (1.1) describing IBMs.

The organization of the paper is as follows: In Section 2, we give a set up and main results. In Section 3, we introduce a transformation of Dirichlet spaces. In Section 4, we prove the identity between unlabeled dynamics and the labeled dynamics (Theorem 2.4). In Section 5, we prove the quasi-regularity of tagged particle processes and environment processes (Theorems 2.6 and 2.7). In Section 6, we study a non-explosion criteria and prove Theorem 2.5. In Section 7, we prove the quasi-regularity of Dirichlet forms describing the k -labeled and other unlabeled particles.

2. Set up and main results.

Let S be a connected closed set in \mathbf{R}^d such that $S = \overline{(S^{\text{int}})}$; that is, S coincides with the closure of the open kernel of S . Let \mathfrak{S} be the set of the configurations on S , that is,

$$\mathfrak{S} = \left\{ \mathfrak{s} = \sum_i \delta_{s_i}; \mathfrak{s}(K) < \infty \text{ for all compact sets } K \subset S \right\}. \tag{2.1}$$

We endow \mathfrak{S} with the vague topology. Then \mathfrak{S} becomes a Polish space because S is a Polish space (see [20]). Let μ be a probability measure on $(\mathfrak{S}, \mathcal{B}(S))$.

We say a non-negative permutation invariant function ρ^n on S^k is the n -correlation function of μ if

$$\int_{A_1^{k_1} \times \dots \times A_m^{k_m}} \rho^n(x_1, \dots, x_n) dx_1 \cdots dx_n = \int_{\mathfrak{S}} \prod_{i=1}^m \frac{\mathfrak{s}(A_i)!}{(\mathfrak{s}(A_i) - k_i)!} d\mu \tag{2.2}$$

for any sequence of disjoint bounded measurable subsets $A_1, \dots, A_m \subset S$ and a sequence of natural numbers k_1, \dots, k_m satisfying $k_1 + \dots + k_m = n$.

For a subset $A \subset S$ we define the map $\pi_A : \mathfrak{S} \rightarrow \mathfrak{S}$ by $\pi_A(\mathfrak{s}) = \mathfrak{s}(A \cap \cdot)$. We say a function $f : \mathfrak{S} \rightarrow \mathbf{R}$ is local if f is $\sigma[\pi_A]$ -measurable for some compact set $A \subset S$. We say f is smooth if \tilde{f} is smooth, where $\tilde{f}((s_i))$ is the permutation invariant function in (s_i) such that $f(\mathfrak{s}) = \tilde{f}((s_i))$ for $\mathfrak{s} = \sum_i \delta_{s_i}$.

Let \mathcal{D}_\circ be the set of all local, smooth functions on \mathfrak{S} . For $f, g \in \mathcal{D}_\circ$ we set $D[f, g] : \mathfrak{S} \rightarrow \mathbf{R}$ by

$$D[f, g](\mathfrak{s}) = \frac{1}{2} \sum_i (\nabla_{s_i} \tilde{f}, \nabla_{s_i} \tilde{g})_{\mathbf{R}^d}. \tag{2.3}$$

Here $\nabla_{s_i} = (\frac{\partial}{\partial s_{i1}}, \dots, \frac{\partial}{\partial s_{id}})$ and $s_i = (s_{i1}, \dots, s_{id}) \in S$ and $\mathfrak{s} = \sum_i \delta_{s_i}$. Moreover, $(\cdot, \cdot)_{\mathbf{R}^d}$ is the standard inner product of \mathbf{R}^d . For given f and g in \mathcal{D}_\circ , it is easy to see that the right hand side depends only on \mathfrak{s} . So $D[f, g]$ is well defined.

Let $L^2(\mu) = L^2(\mathfrak{S}, \mu)$. We consider the bilinear form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ defined by

$$\mathcal{E}^\mu(f, g) = \int_{\mathfrak{S}} D[f, g] d\mu, \tag{2.4}$$

$$\mathcal{D}^\mu = \{f \in \mathcal{D}_\circ \cap L^2(\mu); \mathcal{E}^\mu(f, f) < \infty\}. \tag{2.5}$$

We now assume

(M.1.0) $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is closable on $L^2(\mu)$.

(M.2) The n -correlation function ρ^n of μ is locally bounded for all n .

We collect some known results.

LEMMA 2.1 ([14]). *Assume (M.1.0) and (M.2). Let $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ be the closure of $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ on $L^2(\mu)$. Then we have the following.*

- (1) $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mu))$ is a quasi-regular Dirichlet space.
- (2) There exists a diffusion $P^\mu = (\{P_s^\mu\}_{s \in \mathfrak{S}}, \mathfrak{X})$ associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mu))$.
- (3) The diffusion P^μ is reversible with respect to μ .

PROOF. (1) follows from [14, Theorem 1]. In [14, Theorem 1] we assume $S = \mathbf{R}^d$; the generalization to the present case is easy. (2) follows from (1) and the general theory of Dirichlet forms [12]. (3) is clear because P^μ has an invariant probability measure μ and the Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is μ -symmetric. \square

Let Cap^μ denote the capacity associated with the Dirichlet space $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mu))$. We refer to [4, p. 64] for the definition of the capacity. We remark that the diffusion P^μ in Lemma 2.1 (2) is unique up to quasi everywhere (q.e.).

Namely, if $\hat{P}^\mu = (\{\hat{P}_\mathfrak{s}^\mu\}_{\mathfrak{s} \in \mathfrak{G}}, \mathfrak{X})$ is another diffusion associated with $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mu))$, then there exists a set $\hat{\mathfrak{G}}$ such that $\text{Cap}^\mu(\hat{\mathfrak{G}}^c) = 0$ and that $P_\mathfrak{s}^\mu = \hat{P}_\mathfrak{s}^\mu$ for all $\mathfrak{s} \in \hat{\mathfrak{G}}$.

We assume:

$$(M.3) \text{Cap}^\mu(\mathfrak{G}_{\text{single}}^c) = 0.$$

Here $\mathfrak{G}_{\text{single}} = \{\mathfrak{s} \in \mathfrak{G}; \mathfrak{s}(x) \leq 1 \text{ for all } x \in S, \mathfrak{s}(x) \neq 0 \text{ for some } x \in S\}$.

LEMMA 2.2. Assume (M.1.0), (M.2), and (M.3). Then there exists a subset $\hat{\mathfrak{G}}_{\text{single}}$ such that

$$\hat{\mathfrak{G}}_{\text{single}} \subset \mathfrak{G}_{\text{single}}, \tag{2.6}$$

$$\text{Cap}^\mu(\hat{\mathfrak{G}}_{\text{single}}^c) = 0, \tag{2.7}$$

$$P_\mathfrak{s}^\mu(\mathfrak{X}_t \in \hat{\mathfrak{G}}_{\text{single}} \text{ for all } t) = 1 \text{ for all } \mathfrak{s} \in \hat{\mathfrak{G}}_{\text{single}}. \tag{2.8}$$

PROOF. By (M.3) and the general theory of Dirichlet forms we have

$$P_\mathfrak{s}^\mu(\mathfrak{X}_t \in \mathfrak{G}_{\text{single}} \text{ for all } t) = 1 \text{ for q.e. } \mathfrak{s} \in \mathfrak{G}_{\text{single}}.$$

Hence by taking a suitable version of P^μ we get a subset $\hat{\mathfrak{G}}_{\text{single}}$ satisfying (2.6), (2.7), and (2.8). □

We now introduce Dirichlet forms describing k -labeled dynamics. For this we recall the definition of Palm measures. Let $x = (x_1, \dots, x_k) \in S^k$. We set

$$\mu_x = \mu \left(\cdot - \sum_{i=1}^k \delta_{x_i} \mid \mathfrak{s}(x_i) \geq 1 \text{ for } i = 1, \dots, k \right). \tag{2.9}$$

Let ν^k be the measure defined by

$$\nu^k = \mu_x \rho^k(x) dx. \tag{2.10}$$

Here $\rho^k : S^k \rightarrow \mathbf{R}^+$ is the k -correlation function of μ as before, and $dx = dx_1 \cdots dx_k$ is the Lebesgue measure on S^k . We set $\nu = \nu^1$ when $k = 1$.

Let $\mathcal{D}_\circ^k = C_0^\infty(S^k) \otimes \mathcal{D}_\circ$. For $f, g \in \mathcal{D}_\circ^k$ let $\nabla^k[f, g]$ be such that

$$\nabla^k[f, g](x, \mathfrak{s}) = \frac{1}{2} \sum_{i=1}^k (\nabla_{x_i} f(x, \mathfrak{s}), \nabla_{x_i} g(x, \mathfrak{s}))_{\mathbf{R}^d}, \tag{2.11}$$

where $\nabla_{x_i} = (\frac{\partial}{\partial x_{i1}}, \dots, \frac{\partial}{\partial x_{id}})$ and $x = (x_1, \dots, x_k) \in S^k$. We set \mathbf{D}^k by

$$\mathbf{D}^k[f, g](x, \mathfrak{s}) = \nabla^k[f, g](x, \mathfrak{s}) + \mathbf{D}[f(x, \cdot), g(x, \cdot)](\mathfrak{s}). \tag{2.12}$$

Let $L^2(\nu^k) = L^2(S^k \times \mathfrak{S}, \nu^k)$. We set $(\mathcal{E}^{\nu^k}, \mathcal{D}_\circ^{\nu^k})$ by replacing \mathbf{D} , μ and \mathcal{D}_\circ in (2.4) and (2.5) with \mathbf{D}^k , ν^k and \mathcal{D}_\circ^k , respectively. For $k \in \mathbf{N}$ we consider the assumption analogous to (M.1.0).

(M.1.k) $(\mathcal{E}^{\nu^k}, \mathcal{D}_\circ^{\nu^k})$ is closable on $L^2(\nu^k)$.

LEMMA 2.3. *Assume (M.1.0), (M.1.k), and (M.2). Let $(\mathcal{E}^{\nu^k}, \mathcal{D}^{\nu^k})$ be the closure of $(\mathcal{E}^{\nu^k}, \mathcal{D}_\circ^{\nu^k})$ on $L^2(\nu^k)$. Then $(\mathcal{E}^{\nu^k}, \mathcal{D}^{\nu^k})$ is a quasi-regular Dirichlet form on $L^2(\nu^k)$.*

By Lemma 2.3 there exists a diffusion $\mathbf{P}^{\nu^k} = (\{\mathbf{P}_{(x, \mathfrak{s})}^{\nu^k}\}_{(x, \mathfrak{s}) \in S^k \times \mathfrak{S}}, \mathfrak{X}^1)$ associated with the Dirichlet space $(\mathcal{E}^{\nu^k}, \mathcal{D}^{\nu^k}, L^2(\nu^k))$. Here we set $\mathfrak{X}^1 = (X, \mathfrak{X}) \in C([0, \infty); S^k \times \mathfrak{S})$. By construction $(x, \mathfrak{s}) = \mathfrak{X}_0^1 = (X_0, \mathfrak{X}_0)$ $\mathbf{P}_{(x, \mathfrak{s})}^{\nu^k}$ -a.s.

Let $\kappa : S^k \times \mathfrak{S} \rightarrow \mathfrak{S}$ be such that $\kappa(x, \mathfrak{s}) = \sum_{j=1}^k \delta_{x_j} + \mathfrak{s}$, where $x = (x_1, \dots, x_k)$. By the correspondence $\kappa((X, \mathfrak{X})) = \{\sum_{j=1}^k \delta_{X_t^j} + \mathfrak{X}_t\}$ we regard κ as the map from $C([0, \infty); S^k \times \mathfrak{S})$ to $C([0, \infty); \mathfrak{S})$. We also denote by κ the map $\kappa : S^\infty \cup \sum_{k=1}^\infty S^k \rightarrow \mathfrak{S}$ such that $\kappa((x_i)) = \sum_i \delta_{x_i}$, and regard κ as the map from $C([0, \infty); S^\infty \cup \sum_{k=1}^\infty S^k)$ to $C([0, \infty); \mathfrak{S})$. For simplicity we denote these maps by the same symbol κ .

Let $j : \mathfrak{S}_{\text{single}} \rightarrow S^\infty \cup \sum_{k=1}^\infty S^k$ be a measurable map such that $\kappa \circ j$ is the identity map. We call this map a label map. Indeed, this map means labeling all the particles. We remark that plural maps satisfy the condition as above. So we choose any j of such maps in the sequel.

Once we fix a label map j , we can naturally extend the label map j to the map from $C([0, \infty); \mathfrak{S}_{\text{single}})$ to $C([0, \infty); S^\infty \cup \sum_{k=1}^\infty S^k)$. Indeed, for a path $\mathfrak{X} = \{\mathfrak{X}_t\} \in C([0, \infty); \mathfrak{S}_{\text{single}})$, there exists a unique $\{(X_t^i)\} \in C([0, \infty); S^\infty \cup \sum_{k=1}^\infty S^k)$ such that $(X_0^i) = j(\mathfrak{X}_0)$ and that $\sum_i \delta_{X_t^i} = \mathfrak{X}_t$ for all $t \in [0, \infty)$. We write this map by the same symbol j .

THEOREM 2.4. *Assume (M.1.0), (M.1.k), (M.2), and (M.3). Assume*

$$\mathbf{P}_\mathfrak{s}^\mu \left(\sup_{0 \leq t \leq u} |X_t^i| < \infty \text{ for all } u, i \in \mathbf{N} \right) = 1 \quad \text{for q.e. } \mathfrak{s}. \tag{2.13}$$

Here we initially label the process \mathfrak{X} as $\mathfrak{X}_0 = \sum_{i=0}^\infty \delta_{X_0^i}$. Let κ and j be maps given before Theorem 2.4. Let $\tilde{\mathfrak{S}}_{\text{single}}$ be as in Theorem 2.2. Then there exists a set $\tilde{\mathfrak{S}}$

satisfying

$$\tilde{\mathfrak{S}} \subset \hat{\mathfrak{S}}_{\text{single}}, \tag{2.14}$$

$$\text{Cap}^\mu(\tilde{\mathfrak{S}}^c) = 0, \tag{2.15}$$

$$\mathbb{P}_s^\mu(\mathfrak{X}_t \in \tilde{\mathfrak{S}} \text{ for all } t) = 1 \text{ for all } s \in \tilde{\mathfrak{S}}, \tag{2.16}$$

and for all $k \in \mathbf{N}$

$$\mathbb{P}_{s^k}^{\nu^k} = \mathbb{P}_{\kappa(s^k)}^\mu \circ j^{-1} \text{ for all } s^k \in j(\tilde{\mathfrak{S}}), \tag{2.17}$$

$$\mathbb{P}_s^\mu = \mathbb{P}_{j(s)}^{\nu^k} \circ \kappa^{-1} \text{ for all } s \in \tilde{\mathfrak{S}}. \tag{2.18}$$

REMARK 2.1.

- (1) Since j is any measurable map satisfying $\kappa \circ j = \text{id.}$, we see by (2.18) that $\mathbb{P}_s^\mu = \mathbb{P}_{s^k}^{\nu^k} \circ \kappa^{-1}$ for all $s^k \in \kappa^{-1}(s)$.
- (2) Let Cap^{ν^k} denote the capacity associated with $(\mathcal{E}^{\nu^k}, \mathcal{D}^{\nu^k}, L^2(\nu^k))$. Then by (2.15) and Lemma 4.1, we deduce

$$\text{Cap}^{\nu^k}(\kappa^{-1}(\tilde{\mathfrak{S}})^c) = 0. \tag{2.19}$$

We recall that \mathbb{P}^μ is conservative as a diffusion on \mathfrak{S} equipped with the vague topology. However, each of the tagged particles may explode under the usual metric on \mathbf{R}^d . So (2.13) does not hold in general. Next we prepare a sufficient condition for (2.13).

THEOREM 2.5. Assume (M.1.0), (M.1.1), (M.2), and (M.3). Assume there exists $T > 0$ such that for each $R > 0$

$$\liminf_{r \rightarrow \infty} \left\{ \int_{S_{r+R}} \rho^1(x) dx \right\} \cdot \ell \left(\frac{r}{\sqrt{(r+R)T}} \right) = 0, \tag{2.20}$$

where $\ell(x) = (2\pi)^{-1/2} \int_x^\infty e^{-x^2/2} dx$. Then we obtain (2.13).

REMARK 2.2. (2.20) is satisfied if there exists a positive constant $c_{2.1}$ such that

$$\sup_{x \in S} \rho^1(x) e^{-c_{2.1}|x|} < \infty. \tag{2.21}$$

We next proceed to the environment process. So we assume $S = \mathbf{R}^d$. Let $\vartheta_a: \mathfrak{S} \rightarrow \mathfrak{S}$ denote the translation defined by $\vartheta_a(\sum_i \delta_{x_i}) = \sum_i \delta_{x_i - a}$. We assume:

(M.4) μ is translation invariant, that is, $\mu \circ \vartheta_a^{-1} = \mu$ for all $a \in \mathbf{R}^d$.

By (M.4) we can and do choose the version μ_x in such a way that $\mu_x = \mu_0 \circ \vartheta_x^{-1}$ for all $x \in \mathbf{R}^d$. Here μ_x is the conditional probability given by (2.9) with $x \in \mathbf{R}^d$.

Let $\nabla_i = (\frac{\partial}{\partial s_{i1}}, \dots, \frac{\partial}{\partial s_{id}})$. Let $D: \mathcal{D}_\circ \rightarrow (\mathcal{D}_\circ)^d$ such that

$$Df(\mathfrak{s}) = \left\{ \sum_i \nabla_i \tilde{f} \right\} \quad \text{for } \mathfrak{s} \in \mathfrak{S} \quad (\tilde{f} \text{ is same as (2.3)}). \tag{2.22}$$

Note that D is the generator of the group of the unitary operators on $L^2(\mu)$ generated by the translation $\{\vartheta_a\}$. Let $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ be the nabla on \mathbf{R}^d . Let $(D - \nabla): \mathcal{D}_\circ^1 \rightarrow (\mathcal{D}_\circ^1)^d$ be such that

$$(D - \nabla)f(x, \mathfrak{s}) = \{Df(x, \cdot)\}(\mathfrak{s}) - \{\nabla f(\cdot, \mathfrak{s})\}(x) \quad \text{for } (x, \mathfrak{s}) \in \mathfrak{S}^1. \tag{2.23}$$

We set

$$\begin{aligned} \mathbf{D}^Y[f, g] &= \frac{1}{2}(Df, Dg)_{\mathbf{R}^d} + 2\mathbf{D}[f, g] && \text{for } f, g \in \mathcal{D}_\circ, \\ \mathbf{D}^{XY}[f, g] &= \frac{1}{2}((D - \nabla)f, (D - \nabla)g)_{\mathbf{R}^d} + 2\mathbf{D}[f, g] && \text{for } f, g \in \mathcal{D}_\circ^1. \end{aligned} \tag{2.24}$$

Here for $f, g \in \mathcal{D}_\circ^1$ we set $\mathbf{D}[f, g](x, \mathfrak{s}) = \mathbf{D}[f(x, \cdot), g(x, \cdot)](\mathfrak{s})$. Let

$$\mathcal{E}^Y(f, g) = \int_{\mathfrak{S}} \mathbf{D}^Y[f, g] d\mu_0, \tag{2.25}$$

$$\mathcal{E}^{XY}(f, g) = \int_{\mathbf{R}^d \times \mathfrak{S}} \mathbf{D}^{XY}[f, g] dx d\mu_0. \tag{2.26}$$

Let $L^2(\mu_0) = L^2(\mathfrak{S}, \mu_0)$ and $L^2(dx \times \mu_0) = L^2(\mathbf{R}^d \times \mathfrak{S}, dx \times \mu_0)$. Let

$$\mathcal{D}_\circ^Y = \{g \in \mathcal{D}_\circ \cap L^2(\mu_0); \mathcal{E}^Y(g, g) < \infty\}, \tag{2.27}$$

$$\mathcal{D}_\circ^{XY} = \{h \in \mathcal{D}_\circ^1 \cap L^2(dx \times \mu_0); \mathcal{E}^{XY}(h, h) < \infty\}. \tag{2.28}$$

THEOREM 2.6. *Assume (M.1.0), (M.1.1), (M.2), (M.3), and (M.4). Then*

- (1) The form $(\mathcal{E}^Y, \mathcal{D}_\circ^Y)$ is closable on $L^2(\mu_0)$. There exists a diffusion P^Y associated with its closure $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(\mu_0)$. Moreover, $(\mathcal{E}^Y, \mathcal{D}^Y)$ is a quasi-regular Dirichlet form on $L^2(\mu_0)$.
- (2) The form $(\mathcal{E}^{XY}, \mathcal{D}_\circ^{XY})$ is closable on $L^2(dx \times \mu_0)$. There exists a diffusion P^{XY} associated with its closure $(\mathcal{E}^{XY}, \mathcal{D}^{XY})$ on $L^2(dx \times \mu_0)$. Moreover, $(\mathcal{E}^{XY}, \mathcal{D}^{XY})$ is a quasi-regular Dirichlet form on $L^2(dx \times \mu_0)$.

REMARK 2.3. Fattler and Grothaus [3] prove the quasi-regularity of $(\mathcal{E}^Y, \mathcal{D}^Y, L^2(\mu_0))$ and $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$ for grand canonical Gibbs measures μ with translation invariant interaction potentials which are differentiable outside the origin. Their method is different from ours.

By (2.8) we can write $\mathfrak{X} \in C([0, \infty); \mathfrak{S})$ as

$$\mathfrak{X}_t = \sum_i \delta_{X_t^i} \quad P_\mathfrak{s}^\mu\text{-a.s. for all } \mathfrak{s} \in \hat{\mathfrak{S}}_{\text{single}}, \tag{2.29}$$

where $X^i \in C(I_i; \mathbf{R}^d)$ and I_i is the maximal interval in $[0, \infty)$ of the form $[0, b)$ or (a, b) satisfying the representation (2.29). Write $\mathfrak{s}(x) = \mathfrak{s}(\{x\})$ and let

$$\mathfrak{S}_x = \{ \mathfrak{s} \in \tilde{\mathfrak{S}}; \mathfrak{s}(x) = 1 \}. \tag{2.30}$$

If $\mathfrak{X}_0 = \mathfrak{s} \in \mathfrak{S}_x$, then there exists an $i(x, \mathfrak{s})$ such that $X_0^{i(x, \mathfrak{s})} = x$ and such \mathbf{R}^d -valued path $X^{i(x, \mathfrak{s})} = \{X_t^{i(x, \mathfrak{s})}\}$ is unique. For each $\mathfrak{s} \in \mathfrak{S}_x$ we regard $(X^{i(x, \mathfrak{s})}, P_\mathfrak{s}^\mu)$ as the tagged particle starting at x . Let \mathfrak{Y}^x be the process defined by

$$\mathfrak{Y}_t^x := \sum_{i \neq i(x, \mathfrak{s})} \delta_{X_t^i - X_t^{i(x, \mathfrak{s})}} \quad \text{under } P_\mathfrak{s}^\mu \text{ for } \mathfrak{s} \in \mathfrak{S}_x. \tag{2.31}$$

The process \mathfrak{Y}^x describes the environment seen from the tagged particle $X^{i(x, \mathfrak{s})}$.

Let P^{XY} be the diffusion associated with $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$. The following clarifies the relations among the diffusions P^μ , P^{XY} and P^Y .

THEOREM 2.7. Assume (M.1.0), (M.1.1), (M.2), (M.3), and (M.4). Let $X^{i(x, \mathfrak{s})}$ and \mathfrak{Y}^x be as above. Let $\mathfrak{X}^1 = (X, \mathfrak{X}) \in C([0, \infty); \mathbf{R}^d \times \mathfrak{S})$. Then (a version of) P^{XY} satisfies for each $x \in \mathbf{R}^d$

$$P_\mathfrak{s}^\mu(X^{i(x, \mathfrak{s})} \in \cdot) = P_{(x, \vartheta_x(\mathfrak{s} - \delta_x))}^{XY}(X \in \cdot) \quad \text{for all } \mathfrak{s} \in \mathfrak{S}_x, \tag{2.32}$$

$$P_\mathfrak{s}^\mu(\mathfrak{Y}^x \in \cdot) = P_{(x, \vartheta_x(\mathfrak{s} - \delta_x))}^{XY}(\mathfrak{X} \in \cdot) = P_{\vartheta_x(\mathfrak{s} - \delta_x)}^Y \quad \text{for all } \mathfrak{s} \in \mathfrak{S}_x. \tag{2.33}$$

REMARK 2.4. The total system of interacting Brownian motions is a priori given by the diffusion P^μ . The diffusion P^Y associated with $(\mathcal{E}^Y, \mathcal{D}^Y, L^2(\mu_0))$ describes the motion of the environment seen from the tagged particle, and the diffusion P^{XY} associated with $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$ corresponds to the motion of the coupling of the tagged particle and the environment seen from the tagged particle. Theorems 2.6 and 2.7 were used for the proof of the diffusive scaling limit of tagged particles of such translation invariant interacting Brownian motions in [15].

EXAMPLE 2.1.

- (1) Let μ be a canonical Gibbs measure with upper semicontinuous potentials. Assume the interaction potentials are super stable and regular in the sense of Ruelle. We refer to the reader [21]. Then μ satisfies (M.1.k) for all k and (M.2). (M.3) is satisfied if $d \geq 2$ or the interaction potential has repulsive enough. See [8] for the necessary and sufficient condition for this when the number of particles are finite. Since the Dirichlet forms of the infinite particle systems are decreasing limits of the finite particle systems [14], Inukai's result gives a sharp sufficient condition of (M.3).
- (2) Let μ be the Dyson's model in infinite dimension. This is a translation invariant probability measure on the one dimensional configuration space. Its correlation functions are given by the determinant of the sine kernel and related to the random matrices called GUE (see [23], [13]). This measure satisfies (M.1)–(M.4). Here (M.1) is the assumption that means (M.1.k) hold for all $k = 0, 1, \dots$. We refer to [18] and [17] for the proof of (M.1) and (M.2), respectively.
- (3) Let μ be the Ginibre random point field. This is a translation invariant probability measure on the two dimensional configuration space. Its correlation functions are given by the determinant of the exponential kernel and related to the random matrices called Ginibre Ensemble (see [23]). This measure satisfies (M.1)–(M.4). (see [18], [17]).
- (4) In [18] we introduce the notion of quasi Gibbs measures. This class contains all above examples. Measures in this class satisfies (M.1).

3. Transfer of Dirichlet spaces.

This section is devoted to the preparation of the proof of Theorem 2.4. We begin by considering the relation μ and ν^k under the map $\kappa: S^k \times \mathfrak{S} \rightarrow \mathfrak{S}$ defined before Theorem 2.4. Since these measures μ and ν^k are not directly related by the map $\kappa: S^k \times \mathfrak{S} \rightarrow \mathfrak{S}$, we consider the finite volume cut off of these measures instead.

Let $S_r = \{x \in S; |x| < r\}$ and $\mathfrak{S}_{r,m} = \{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s}(S_r) = m\}$. We define the

measures ν_r^k , $\nu_r^{k,N}$, μ_r^k , and $\mu_r^{k,N}$ by

$$\nu_r^k = \int 1_{S_r}(x) d\nu^k, \quad \nu_r^{k,N} = \int 1_{S_r^k}(x) \sum_{m=1}^{N-1} 1_{\mathfrak{S}_{r,m}}(\mathfrak{s}) d\nu^k, \tag{3.1}$$

$$\mu_r^k = \nu_r^k \circ \kappa^{-1}, \quad \mu_r^{k,N} = \nu_r^{k,N} \circ \kappa^{-1}. \tag{3.2}$$

Let $m^{[k]} = m(m-1)\cdots(m-k+1)$. Then it is not difficult to see that

$$\mu_r^k = \sum_{m=k}^{\infty} m^{[k]} \mu(\cdot \cap \mathfrak{S}_{r,m}), \quad \mu_r^{k,N} = \sum_{m=k}^N m^{[k]} \mu(\cdot \cap \mathfrak{S}_{r,m}). \tag{3.3}$$

Let $\partial S_r = \{|x| = r\}$ and $\partial \mathfrak{S}_r = \{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s}(\partial S_r) \geq 1\}$. We remark $\mathfrak{S}_{r,m}$ are open sets and their boundaries $\partial \mathfrak{S}_{r,m}$ are contained in $\partial \mathfrak{S}_r$. We define $\mathcal{D}_o^{\mu_r^k}$ in a similar fashion to $\mathcal{D}_o^{\nu^k}$ by replacing ν^k by μ_r^k . Let

$$\begin{aligned} \mathcal{D}_{o,r,D}^{\mu} &= \{f \in \mathcal{D}_o^{\mu}; f(\mathfrak{s}) = 0 \text{ if } \mathfrak{s} \in \partial \mathfrak{S}_r\}, \\ \mathcal{D}_{o,r,D}^{\mu,N} &= \left\{f \in \mathcal{D}_{o,r,D}^{\mu}; f(\mathfrak{s}) = 0 \text{ if } \mathfrak{s} \notin \sum_{m=1}^N \mathfrak{S}_{r,m}\right\}, \\ \mathcal{D}_{o,r,D}^{\mu_r^k,N} &= \left\{f \in \mathcal{D}_{o,r,D}^{\mu_r^k}; f(\mathfrak{s}) = 0 \text{ if } \mathfrak{s} \in \partial \mathfrak{S}_r \text{ or } \mathfrak{s} \notin \sum_{m=1}^N \mathfrak{S}_{r,m}\right\}. \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathcal{D}_{o,r,D}^{\nu^k} &= \{h \in \mathcal{D}_o^{\nu^k}; h(x, \mathfrak{s}) = 0 \text{ if } x \notin S_r^k \text{ or } \mathfrak{s} \in \partial \mathfrak{S}_r\}, \\ \mathcal{D}_{o,r,D}^{\nu_r^k,N} &= \left\{h \in \mathcal{D}_{o,r,D}^{\nu^k}; h(x, \mathfrak{s}) = 0 \text{ if } \mathfrak{s} \notin \sum_{m=0}^{N-1} \mathfrak{S}_{r,m}\right\}. \end{aligned} \tag{3.5}$$

Let $(\mathcal{E}^{\mu}, \mathcal{D}_{r,D}^{\mu})$ denote the closure of $(\mathcal{E}^{\mu}, \mathcal{D}_{o,r,D}^{\mu})$ on $L^2(\mu)$. We define the closures $(\mathcal{E}^{\mu}, \mathcal{D}_{r,D}^{\mu,N})$, $(\mathcal{E}^{\mu_r^k}, \mathcal{D}_{r,D}^{\mu_r^k,N})$, $(\mathcal{E}^{\nu^k}, \mathcal{D}_{r,D}^{\nu^k})$, and $(\mathcal{E}^{\nu_r^k}, \mathcal{D}_{r,D}^{\nu_r^k,N})$ similarly.

For an $h \in \mathcal{D}_{o,r,D}^{\nu_r^k}$ we set $h_{\text{sym}} \in \mathcal{D}_{o,r,D}^{\nu_r^k}$ by

$$h_{\text{sym}}(x, \mathfrak{s}) = \frac{1}{m!} \sum_{\sigma \in S_m} h\left(x_{\sigma(1)}, \dots, x_{\sigma(k)}, \sum_{i=k+1}^m \delta_{x_{\sigma(i)}}\right) \text{ if } \mathfrak{s} \in \mathfrak{S}_r^{m-k}. \tag{3.6}$$

Here $x = (x_1, \dots, x_k) \in S_r^k$, $\mathfrak{s} = \sum_{j=k+1}^m \delta_{x_j} \in \mathfrak{S}_r^{m-k}$, and S_m is the set consisting of the permutations of $(1, \dots, m)$.

If $h = h_{\text{sym}} \in \mathcal{D}_{\circ,r,D}^{\nu^k}$, then one can regard h as $h \in \mathcal{D}_{\circ,r,D}^{\mu}$, and we denote it by h^0 . Indeed, h^0 is defined by $h^0(\sum_{j=1}^k \delta_{x_j} + \mathfrak{s}) := h(x, \mathfrak{s})$ on $\sum_{m=k}^{\infty} \mathfrak{S}_r^m$ and by $h^0 = 0$ if $\mathfrak{s} \notin \sum_{m=k}^{\infty} \mathfrak{S}_r^m$. We remark $h_{\text{sym}}^0 \circ \kappa = h_{\text{sym}}$ by construction.

Let h_1 and $h_2 \in \mathcal{D}_{\circ,r,D}^{\nu^k}$. Assume $h_{2,\text{sym}} = h_2$. Then we have

$$\int_{S^k \times \mathfrak{S}} h_1 h_2 d\nu^k = \int_{S^k \times \mathfrak{S}} h_{1,\text{sym}} h_2 d\nu^k = \int_{\mathfrak{S}} h_{1,\text{sym}}^0 h_2^0 d\mu_r^k, \quad (3.7)$$

$$\mathcal{E}^{\nu^k}(h_1, h_2) = \mathcal{E}^{\nu^k}(h_{1,\text{sym}}, h_2) = \mathcal{E}^{\mu_r^k}(h_{1,\text{sym}}^0, h_2^0). \quad (3.8)$$

Let us take $h_1 = h$ and $h_2 = h_{\text{sym}}$ in (3.8). Then we have

$$\mathcal{E}^{\nu^k}(h, h_{\text{sym}}) = \mathcal{E}^{\nu^k}(h_{\text{sym}}, h_{\text{sym}}) = \mathcal{E}^{\mu_r^k}(h_{\text{sym}}^0, h_{\text{sym}}^0). \quad (3.9)$$

Applying Schwarz's inequality to the first equality of (3.9) yields

$$\mathcal{E}^{\nu^k}(h, h) \geq \mathcal{E}^{\nu^k}(h_{\text{sym}}, h_{\text{sym}}).$$

Hence we can define h_{sym} not only for $h \in \mathcal{D}_{\circ,r,D}^{\nu^k}$ but also for $h \in \mathcal{D}_{r,D}^{\nu^k}$ as the limit of the $\{\mathcal{E}_1^{\nu^k}\}^{1/2}$ -norm. Moreover, by (3.7) and (3.9) we have

$$\{h_{\text{sym}}^0; h \in \mathcal{D}_{r,D}^{\nu^k}\} = \mathcal{D}_{r,D}^{\mu_r^k}. \quad (3.10)$$

Similarly as (3.10) we have

$$\{h_{\text{sym}}^0; h \in \mathcal{D}_{r,D}^{\nu^k,N}\} = \mathcal{D}_{r,D}^{\mu_r^k,N}. \quad (3.11)$$

Since $\mu(\cdot) \leq \mu_r^k(\cdot) \leq N\mu(\cdot)$ on $\sum_{m=1}^N \mathfrak{S}_{r,m}$ by (3.3), we obtain

$$\mathcal{D}_{r,D}^{\mu,N} = \mathcal{D}_{r,D}^{\mu_r^k,N}. \quad (3.12)$$

4. Identities among k -labeled diffusions.

In this section we assume (M.1.0), (M.1. k), (M.2) and (M.3). The purpose of this section is to prove the identity between the diffusions associated with the Dirichlet spaces $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}, L^2(\mu))$ and $(\mathcal{E}^{\nu^k}, \mathcal{D}^{\nu^k}, L^2(\nu^k))$ introduced in Section 2.

This identity is a key to the proof of Theorem 2.6.

LEMMA 4.1. *Let $\mathfrak{A} \subset S^k \times \mathfrak{S}$ be such that $\kappa^{-1}(\kappa(\mathfrak{A})) = \mathfrak{A}$. Then $\text{Cap}^\mu(\kappa(\mathfrak{A})) = 0$ implies $\text{Cap}^{\nu^k}(\mathfrak{A}) = 0$. Here we regard κ as $\kappa: S^k \times \mathfrak{S} \rightarrow \mathfrak{S}$.*

PROOF. Without loss of the generality we can and do assume $\mathfrak{A} \subset S_{r-1}^k \times \mathfrak{S}$ and $\mathfrak{A} \cap (S^k \times \partial\mathfrak{S}_r) = \emptyset$ for some $r \in \mathbf{N}$. Since the capacity of a set B is given by the infimum of the capacity of the open sets including B , we can assume without loss of generality that $\kappa(\mathfrak{A})$ is an open set. Then \mathfrak{A} becomes an open set. So by definition we have

$$\text{Cap}^\mu(\kappa(\mathfrak{A})) = \inf \{ \mathcal{E}_1^\mu(f, f); f \in \mathcal{D}^\mu, f \geq 1 \text{ } \mu\text{-a.e. on } \kappa(\mathfrak{A}) \}, \tag{4.1}$$

$$\text{Cap}^{\nu^k}(\mathfrak{A}) = \inf \{ \mathcal{E}_1^{\nu^k}(g, g); g \in \mathcal{D}^{\nu^k}, g \geq 1 \text{ } \nu^k\text{-a.e. on } \mathfrak{A} \}. \tag{4.2}$$

Here $\mathcal{E}_1^\mu(f, f) = \mathcal{E}^\mu(f, f) + (f, f)_{L^2(\mu)}$ as usual and we set $\mathcal{E}_1^{\nu^k}$ similarly.

Since $\mathfrak{A} \subset S_{r-1}^k \times \mathfrak{S}$ and $\mathfrak{A} \cap (S^k \times \partial\mathfrak{S}_r) = \emptyset$, we deduce that

$$\text{Cap}^\mu(\kappa(\mathfrak{A})) = \inf \{ \mathcal{E}_1^\mu(f, f); f \in \mathcal{D}_{\circ, r, D}^\mu, f \geq 1 \text{ } \mu\text{-a.e. on } \kappa(\mathfrak{A}) \}. \tag{4.3}$$

If $f \in \mathcal{D}_{\circ, r, D}^\mu$, then $f \circ \kappa \in \mathcal{D}^{\nu^k}$. Combining this with (4.1)–(4.3) and the assumption $\text{Cap}^\mu(\kappa(\mathfrak{A})) = 0$ completes the proof. \square

We consider parts of P^μ and P^{ν^k} . We refer to [4] for the definition of a part of Dirichlet space and related results.

Let $\partial S_r = \{|x| = r\}$ and $\partial\mathfrak{S}_r = \{\mathfrak{s} \in \mathfrak{S}; \mathfrak{s}(\partial S_r) \geq 1\}$ as before. Let

$$\sigma_r^0(\mathfrak{X}) = \inf \{ t > 0; \mathfrak{X}_t \in \partial\mathfrak{S}_r \}, \tag{4.4}$$

$$\sigma_{r, N}^0(\mathfrak{X}) = \inf \left\{ t > 0; \mathfrak{X}_t \in \partial\mathfrak{S}_r \text{ or } \mathfrak{X}_t \notin \sum_{m=1}^N \mathfrak{S}_{r, m} \right\}, \tag{4.5}$$

$$\sigma_r^1(\mathfrak{X}^1) = \inf \{ t > 0; X_t \notin S_r^k \text{ or } \mathfrak{X}_t \in \partial\mathfrak{S}_r \}, \tag{4.6}$$

$$\sigma_{r, N}^1(\mathfrak{X}^1) = \inf \left\{ t > 0; X_t \notin S_r^k \text{ or } \mathfrak{X}_t \in \partial\mathfrak{S}_r \text{ or } \mathfrak{X}_t \notin \sum_{m=0}^{N-k} \mathfrak{S}_{r, m} \right\}, \tag{4.7}$$

where $\mathfrak{X} \in C([0, \infty); \mathfrak{S})$ and $\mathfrak{X}^1 = (X, \mathfrak{X}) \in C([0, \infty); S^k \times \mathfrak{S})$.

Let $\mathfrak{X}^{\sigma_r^0} = \{\mathfrak{X}_{t \wedge \sigma_r^0}\}$ and define $\mathfrak{X}^{\sigma_{r, N}^0}$, $\mathfrak{X}^{1, \sigma_r^1}$, and $\mathfrak{X}^{1, \sigma_{r, N}^1}$ in a similar fashion. Let $P^{\mu, \sigma_r^0} = (\{P_{\mathfrak{s}}^\mu\}_{\mathfrak{s} \in \mathfrak{S}}, \mathfrak{X}^{\sigma_r^0})$. Set $P^{\mu, \sigma_{r, N}^0}$, P^{ν^k, σ_r^1} , and $P^{\nu^k, \sigma_{r, N}^1}$ simi-

larly. Then P^{μ, σ_r^0} , $P^{\mu, \sigma_{r,N}^0}$, P^{ν^k, σ_r^1} , and $P^{\nu^k, \sigma_{r,N}^1}$ are diffusions associated with the Dirichlet spaces $(\mathcal{E}^\mu, \mathcal{D}_{r,D}^\mu, L^2(\mu))$, $(\mathcal{E}^\mu, \mathcal{D}_{r,D}^{\mu,N}, L^2(\mu))$, $(\mathcal{E}^{\nu^k}, \mathcal{D}_{r,D}^{\nu^k}, L^2(\nu^k))$, and $(\mathcal{E}^{\nu^k}, \mathcal{D}_{r,D}^{\nu^k,N}, L^2(\nu^k))$, respectively. Let $P_{\mathfrak{s}}^\mu = P_{\mathfrak{s}}^\mu(\mathfrak{X}^{\sigma_r^0} \in \cdot)$. We set $P_{\mathfrak{s}}^{\mu, \sigma_{r,N}^0}$, $P_{(x,\mathfrak{s})}^{\nu^k, \sigma_r^1}$, and $P_{(x,\mathfrak{s})}^{\nu^k, \sigma_{r,N}^1}$ similarly. We note that these are the distributions of P^{μ, σ_r^0} , $P^{\mu, \sigma_{r,N}^0}$, P^{ν^k, σ_r^1} , and $P^{\nu^k, \sigma_{r,N}^1}$, respectively. Let $\text{Cap}^{\mu, \sigma_r^0}$ and $\text{Cap}^{\mu, \sigma_{r,N}^0}$ be the capacities of P^{μ, σ_r^0} and $P^{\mu, \sigma_{r,N}^0}$, respectively.

LEMMA 4.2. *Assume (M.1.0), (M.1.k), (M.2), and (M.3). Then there exists $\mathfrak{A}_r^k \subset S_r^k \times \mathfrak{S}$ such that*

$$\kappa^{-1}(\kappa(\mathfrak{A}_r^k)) = \mathfrak{A}_r^k, \quad \kappa(\mathfrak{A}_r^k) \subset \mathfrak{S}_{\text{single}}, \tag{4.8}$$

$$\text{Cap}^{\mu, \sigma_r^0}(\kappa(S_r^k \times \mathfrak{S}) \setminus \kappa(\mathfrak{A}_r^k)) = 0, \tag{4.9}$$

$$P_{\kappa(x,\mathfrak{s})}^{\mu, \sigma_r^0}(\mathfrak{X}_t \in \kappa(\mathfrak{A}_r^k) \text{ for all } t) = 1 \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_r^k, \tag{4.10}$$

$$P_{\kappa(x,\mathfrak{s})}^{\mu, \sigma_r^0} = P_{(x,\mathfrak{s})}^{\nu^k, \sigma_r^1} \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_r^k. \tag{4.11}$$

PROOF. If for each $N \in \mathbf{N}$ there exists a set $\mathfrak{A}_{r,N}^k \subset S_r^k \times \sum_{m=0}^{N-k} \mathfrak{S}_{r,m}$ such that

$$\kappa^{-1}(\kappa(\mathfrak{A}_{r,N}^k)) = \mathfrak{A}_{r,N}^k, \quad \kappa(\mathfrak{A}_{r,N}^k) \subset \mathfrak{S}_{\text{single}}, \tag{4.12}$$

$$\text{Cap}^{\mu, \sigma_{r,N}^0} \left(\kappa \left(S_r^k \times \sum_{m=0}^{N-k} \mathfrak{S}_{r,m} \right) \setminus \kappa(\mathfrak{A}_{r,N}^k) \right) = 0, \tag{4.13}$$

$$P_{\kappa(x,\mathfrak{s})}^{\mu, \sigma_{r,N}^0}(\mathfrak{X}_t \in \kappa(\mathfrak{A}_{r,N}^k) \text{ for all } t) = 1 \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_{r,N}^k, \tag{4.14}$$

$$P_{\kappa(x,\mathfrak{s})}^{\mu, \sigma_{r,N}^0} = P_{(x,\mathfrak{s})}^{\nu^k, \sigma_{r,N}^1} \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_{r,N}^k, \tag{4.15}$$

then $\mathfrak{A}_r^k := \liminf_{N \rightarrow \infty} \mathfrak{A}_{r,N}^k$ satisfies (4.8)–(4.11). Hence it only remains to prove such an $\mathfrak{A}_{r,N}^k$ exists for each N .

Recall that $P^{\nu^k, \sigma_{r,N}^1}$ is the diffusion associated with $(\mathcal{E}^{\nu^k}, \mathcal{D}_{r,D}^{\nu^k,N}, L^2(\nu^k))$. Let $T_{r,D,t}^{\nu^k,N}$ be the semigroup associated with $P^{\nu^k, \sigma_{r,N}^1}$. Then for f and $g \in \mathcal{D}_{r,D}^{\nu^k,N}$

$$\int_{S^k \times \mathfrak{S}} T_{r,D,t}^{\nu^k,N} f \cdot g d\nu^k - \int_{S^k \times \mathfrak{S}} f \cdot g d\nu^k + \int_0^t \mathcal{E}^{\nu^k} (T_{r,D,u}^{\nu^k,N} f, g) du = 0. \tag{4.16}$$

Now suppose $g_{\text{sym}} = g$. Then by (3.7) and (3.8) we have

$$\int_{\mathfrak{S}} \left(T_{r,D,t}^{\nu^k,N} f \right)_{\text{sym}}^0 \cdot g^0 d\mu_r^k - \int_{\mathfrak{S}} f_{\text{sym}}^0 \cdot g^0 d\mu_r^k + \int_0^t \mathcal{E}^{\mu_r^k} \left(\left(T_{r,D,u}^{\nu^k,N} f \right)_{\text{sym}}^0, g^0 \right) du = 0. \tag{4.17}$$

Let $T_{r,D,t}^{\mu_r^k,N}$ be the semigroup associated with $(\mathcal{E}^{\mu_r^k}, \mathcal{D}_{r,D}^{\mu_r^k,N})$ on $L^2(\mu_r^k)$. Then by (3.11) and (4.17) we have

$$T_{r,D,t}^{\mu_r^k,N} (f_{\text{sym}}^0) = \left(T_{r,D,t}^{\nu^k,N} f \right)_{\text{sym}}^0. \tag{4.18}$$

Let $\text{Cap}^{\mu_r^k, \sigma_{r,N}^0}$ and $\text{P}^{\mu_r^k, \sigma_{r,N}^0}$ be the capacity and the diffusion associated with the Dirichlet space $(\mathcal{E}^{\mu_r^k}, \mathcal{D}_{r,D}^{\mu_r^k,N}, L^2(\mu_r^k))$, respectively. Then by (4.18) together with (M.3), we deduce that there exists $\mathfrak{A}_{r,N}^k \subset S_r^k \times \sum_{m=0}^{N-k} \mathfrak{S}_{r,m}$ satisfying (4.12) and

$$\text{Cap}^{\mu_r^k, \sigma_{r,N}^0} \left(\kappa \left(S_r^k \times \sum_{m=0}^{N-k} \mathfrak{S}_{r,m} \right) \setminus \kappa(\mathfrak{A}_{r,N}^k) \right) = 0, \tag{4.19}$$

$$\text{P}_{\kappa(x,\mathfrak{s})}^{\mu_r^k, \sigma_{r,N}^0} (\mathfrak{X}_t \in \kappa(\mathfrak{A}_{r,N}^k) \text{ for all } t) = 1 \text{ for all } (x, \mathfrak{s}) \in \mathfrak{A}_{r,N}^k, \tag{4.20}$$

$$\text{P}_{\kappa(x,\mathfrak{s})}^{\mu_r^k, \sigma_{r,N}^0} = \text{P}_{(x,\mathfrak{s})}^{\nu^k, \sigma_{r,N}^1} \circ \kappa^{-1} \text{ for all } (x, \mathfrak{s}) \in \mathfrak{A}_{r,N}^k. \tag{4.21}$$

Recall that the diffusions $\text{P}^{\mu, \sigma_{r,N}^0}$ and $\text{P}^{\mu_r^k, \sigma_{r,N}^0}$ in (4.15) and (4.21) are associated with the Dirichlet spaces $(\mathcal{E}^{\mu}, \mathcal{D}_{r,D}^{\mu,N}, L^2(\mu))$ and $(\mathcal{E}^{\mu_r^k}, \mathcal{D}_{r,D}^{\mu_r^k,N}, L^2(\mu_r^k))$, respectively. Note that $\mathcal{D}_{r,D}^{\mu,N} = \mathcal{D}_{r,D}^{\mu_r^k,N}$ by (3.12). Moreover, these two Dirichlet spaces have the common state space $\sum_{m=1}^N \mathfrak{S}_{r,m}$. On each connected component $\{\mathfrak{S}_{r,m}\}$ of the state space, the measures μ and μ_r^k are constant multiplication of each other. Hence the associated diffusions are the same until they hit the boundary. Since these Dirichlet forms enjoy the Dirichlet boundary conditions, we see that eventually these two Dirichlet spaces define the same diffusion. This combined with (4.19)–(4.21) we obtain (4.13)–(4.15), respectively.

We therefore deduce that $\mathfrak{A}_{r,N}^k$ ($N \in \mathbf{N}$) satisfy (4.12)–(4.15), which completes the proof of Lemma 4.2. \square

Let $r(i) = r$ if i is odd, and $r(i) = r + 1$ if i is even. Let for $i \geq 2$

$$\bar{\sigma}_i^0(\mathfrak{X}) = \inf \{t > \sigma_{i-1}^0; \mathfrak{X}_t \in \partial\mathfrak{S}_{r(i)}\}, \tag{4.22}$$

$$\bar{\sigma}_i^1(\mathfrak{X}^1) = \inf \{t > \sigma_{i-1}^1; X_t \in \partial S_{r(i)}^k \text{ or } \mathfrak{X}_t \in \partial\mathfrak{S}_{r(i)}\}, \tag{4.23}$$

where we set $\bar{\sigma}_1^a = \sigma_r^a$ ($a = 0, 1$). For $\mathfrak{X} = \{\sum_i \delta_{X_t^i}\} \in C([0, \infty); \mathfrak{S})$ satisfying $\mathfrak{X}_t \in \mathfrak{S}_{\text{single}}$ for all t and $\mathfrak{X}_0 = \sum_i \delta_{x_i}$, we choose the first k -particles

$$\{\bar{X}_t\} = \{(X_t^1, \dots, X_t^k)\} \in C([0, \infty); S^k)$$

such that $\bar{X}_0 = x = (x_1, \dots, x_k)$ and that $\mathfrak{X}_t = \sum_{j=1}^k \delta_{X_t^j} + \sum_{j>k} \delta_{X_t^j}$.

Let $\Omega_i^0 = \{\omega; \bar{X}_{\bar{\sigma}_i^0}(\omega) \in S_r^k\}$ and $\Omega_i^1 = \{\omega; X_{\bar{\sigma}_i^1}(\omega) \in S_r^k\}$. Let

$$\Omega_\infty^a = \bigcap_{i=1}^\infty \Omega_i^a, \quad \bar{\sigma}_\infty^a = \lim_{i \rightarrow \infty} \bar{\sigma}_i^a \quad (a = 0, 1). \tag{4.24}$$

Since $X_{\bar{\sigma}_i^1} \in S_r^k$ on Ω_i^1 , $S_r^k \subset S_{r+1}^k$, and $S_{r+1}^k \cap \partial S_{r+1}^k = \emptyset$, we deduce that

$$X_t(\omega) \in S_{r+1}^k \quad \text{for all } 0 \leq t < \bar{\sigma}_\infty^1(\omega), \text{ for all } \omega \in \Omega_\infty^1. \tag{4.25}$$

LEMMA 4.3. Assume (M.1.0), (M.1.k), (M.2), and (M.3). Let \mathfrak{A}_r^k be as in Lemma 4.2. Then for all $(x, \mathfrak{s}) \in \mathfrak{A}_r^k$ the following holds.

- (1) $\bar{\sigma}_\infty^0 = \infty$ for $P_{\kappa(x, \mathfrak{s})}^\mu(\cdot; \Omega_\infty^0)$ -a.e. ω ,
- (2) $\bar{\sigma}_\infty^1 = \infty$ for $P_{(x, \mathfrak{s})}^{\nu^k}(\cdot; \Omega_\infty^1)$ -a.e. ω .

PROOF. Let $\partial\mathfrak{S}_r = \{\mathfrak{s}; \mathfrak{s}(\partial S_r) \geq 1\}$ as before. Then by the continuity of the sample paths, (4.22) and (4.24), we deduce

$$\mathfrak{X}_{\bar{\sigma}_\infty^0} = \lim_{i \rightarrow \infty} \mathfrak{X}_{\bar{\sigma}_i^0} \in \partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1} \quad \text{on } \{\bar{\sigma}_\infty^0 < \infty\}. \tag{4.26}$$

Suppose $P_{\kappa(x, \mathfrak{s})}^\mu(\bar{\sigma}_\infty^0 < \infty; \Omega_\infty^0) > 0$. Then by (4.26) we have

$$P_{\kappa(x, \mathfrak{s})}^\mu(\mathfrak{X}_{\bar{\sigma}_\infty^0} \in \partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1}; \Omega_\infty^0) > 0. \tag{4.27}$$

Hence $\int_{\mathfrak{S}} P_{\mathfrak{s}}^\mu(\sigma_{\partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1}} < \infty) \mu(d\mathfrak{s}) > 0$, where $\sigma_{\partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1}}$ is the first hitting time to the set $\partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1}$. By the general theory of Dirichlet forms (see [4, Theorem 4.2.1. (ii)]) it follows from this that

$$\text{Cap}^\mu(\partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1}) > 0. \tag{4.28}$$

On the other hand, since the n -correlation functions ρ^n of μ are locally bounded by (M.2), it is not difficult to see that $\text{Cap}^\mu(\partial\mathfrak{S}_r \cap \partial\mathfrak{S}_{r+1}) = 0$. This contradicts (4.28). Hence we obtain $\mathbb{P}_{\kappa(x,\mathfrak{s})}^\mu(\bar{\sigma}_\infty^0 < \infty; \Omega_\infty^0) = 0$, which implies (1). The proof of (2) is similar to that of (1). \square

Let

$$\tau_{r,x}^0(\mathfrak{X}) = \inf \{ t > 0; \bar{X}_t \in \partial S_r^k \}, \tag{4.29}$$

$$\tau_r^1(\mathfrak{X}^1) = \inf \{ t > 0; X_t \in \partial S_r^k \}. \tag{4.30}$$

REMARK 4.1. The stopping times $\bar{\sigma}_i^0$, $\bar{\sigma}_i^1$ and τ_r^1 are the hitting times to the subsets of the state spaces. So one can relate the stopped processes to the parts of Dirichlet forms. However, $\tau_{r,x}^0$ is not a hitting time to any subset in the state space \mathfrak{S} . So one can not relate the associated stopped process to a part of the Dirichlet form, which is the reason we prepare Lemma 4.2 before Lemma 4.4.

LEMMA 4.4. Assume (M.1.0), (M.1.k), (M.2), and (M.3). Let \mathfrak{A}_r^k be as in Lemma 4.2. Then

$$\mathbb{P}_{\kappa(x,\mathfrak{s})}^{\mu, \tau_{r,x}^0} = \mathbb{P}_{(x,\mathfrak{s})}^{\nu^k, \tau_{r,x}^1} \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_r^k \cap \mathfrak{A}_{r+1}^k. \tag{4.31}$$

Let $\tau_{\infty,x}^0 = \lim_{r \rightarrow \infty} \tau_{r,x}^0$ and $\tau_\infty^1 = \lim_{r \rightarrow \infty} \tau_r^1$. Then

$$\mathbb{P}_{\kappa(x,\mathfrak{s})}^{\mu, \tau_{\infty,x}^0} = \mathbb{P}_{(x,\mathfrak{s})}^{\nu^k, \tau_\infty^1} \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \liminf_{r \rightarrow \infty} \mathfrak{A}_r^k. \tag{4.32}$$

PROOF. Suppose $\omega \in \Omega_\infty^1$. Then by (4.25) and Lemma 4.3 we have $\mathfrak{X}_t^1 \in S_{r+1}^k \times \mathfrak{S}$ for all $0 \leq t < \infty$. In particular, $X_t \in S_{r+1}^k$ for all $0 \leq t < \infty$. Hence by using Lemma 4.2 with r and $r + 1$ combined with the strong Markov property repeatedly, we obtain for all $(x, \mathfrak{s}) \in \mathfrak{A}_r^k \cap \mathfrak{A}_{r+1}^k$

$$\mathbb{P}_{\kappa(x,\mathfrak{s})}^{\mu, \bar{\sigma}_i^0}(\cdot; \Omega_\infty^0) = \mathbb{P}_{(x,\mathfrak{s})}^{\nu^k, \bar{\sigma}_i^1}(\cdot; \Omega_\infty^1) \circ \kappa^{-1} \quad \text{for all } i. \tag{4.33}$$

Hence by Lemma 4.3 we have

$$\mathbb{P}_{\kappa(x,\mathfrak{s})}^\mu(\cdot; \Omega_\infty^0) = \mathbb{P}_{(x,\mathfrak{s})}^{\nu^k}(\cdot; \Omega_\infty^1) \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_r^k \cap \mathfrak{A}_{r+1}^k. \tag{4.34}$$

Next suppose $\omega \notin \Omega_\infty^1$. Then there exists an i such that $X_{\bar{\sigma}_i^1} \notin S_r^k$ and $X_{\bar{\sigma}_j^1} \in S_r^k$ for all $j < i$. Let Ω_{i*}^1 denote the collection of such ω :

$$\Omega_{i*}^1 = \{\omega; X_{\bar{\sigma}_i^1}(\omega) \notin S_r^k, X_{\bar{\sigma}_j^1}(\omega) \in S_r^k (\forall j < i)\}.$$

By Lemma 4.2 and the strong Markov property we have

$$P_{\kappa(x, \mathfrak{s})}^{\mu, \bar{\sigma}_i^0}(\cdot; \Omega_{i*}^0) = P_{(x, \mathfrak{s})}^{\nu^k, \bar{\sigma}_i^1}(\cdot; \Omega_{i*}^1) \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_r^k \cap \mathfrak{A}_{r+1}^k. \quad (4.35)$$

By construction $\tau_{r,x}^a \leq \bar{\sigma}_i^a$ ($a = 0, 1$). Hence (4.35) implies

$$P_{\kappa(x, \mathfrak{s})}^{\mu, \tau_{r,x}^0}(\cdot; \Omega_{i*}^0) = P_{(x, \mathfrak{s})}^{\nu^k, \tau_{r,x}^1}(\cdot; \Omega_{i*}^1) \circ \kappa^{-1} \quad \text{for all } (x, \mathfrak{s}) \in \mathfrak{A}_r^k \cap \mathfrak{A}_{r+1}^k. \quad (4.36)$$

We now see that $\Omega^a = \Omega_\infty^a + \sum_{i=1}^\infty \Omega_{i*}^a$ ($a = 0, 1$). Hence (4.31) follows from (4.34) and (4.36). (4.32) follows from (4.31) immediately. \square

PROOF OF THEOREM 2.4. Let $\tilde{\mathfrak{S}} = \bigcap_{k \in \mathbf{N}} \{\liminf_{r \rightarrow \infty} \mathfrak{A}_r^k\}$. Then by (4.9) we have (2.15). Moreover, by (2.13) we deduce that $\tau_{\infty,x}^0 = \infty$ for $P_\mathfrak{s}^\mu$ -a.s. for all $\mathfrak{s} \in \tilde{\mathfrak{S}}$ such that $\mathfrak{s}(x) = 1$. Hence by (4.32) of Lemma 4.4 we obtain (2.17) and (2.18). \square

5. Tagged particle processes.

In this section, we prove Theorems 2.6 and 2.7. So we take $S = \mathbf{R}^d$ and $k = 1$. We set $\nu = \nu^1$. Let ι be the transformation on $\mathbf{R}^d \times \mathfrak{S}$ defined by

$$\iota(x, \mathfrak{s}) = (x, \vartheta_x(\mathfrak{s})). \quad (5.1)$$

Then by (M.4) we deduce that

$$\nu \circ \iota^{-1} = dx \times \mu_0. \quad (5.2)$$

We regard ι as the transformation on $C([0, \infty); \mathbf{R}^d \times \mathfrak{S})$, denoted by the same symbol ι , by $\iota(\mathfrak{X}^1) = \{\iota(\mathfrak{X}_t^1)\}$.

LEMMA 5.1. Assume (M.1.0), (M.1.1), (M.2)–(M.4). Then we have the following.

- (1) The bilinear form $(\mathcal{E}^{XY}, \mathcal{D}_\circ^{XY})$ is closable on $L^2(dx \times \mu_0)$.
- (2) Let $(\mathcal{E}^{XY}, \mathcal{D}^{XY})$ be the closure of $(\mathcal{E}^{XY}, \mathcal{D}_\circ^{XY})$ on $L^2(dx \times \mu_0)$. Let

$$P_{(x, \mathfrak{s})}^{XY} = P_{\iota^{-1}(x, \mathfrak{s})}^\nu \circ \iota^{-1}. \quad (5.3)$$

Then $P^{XY} = (\{P_{(x,\mathfrak{s})}^{XY}\}_{(x,\mathfrak{s}) \in \mathbf{R}^d \times \mathfrak{S}}, \mathfrak{X}^1)$ is a diffusion associated with the Dirichlet space $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$.

(3) The Dirichlet space $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$ is quasi-regular.

PROOF. By (5.2) we have

$$(f \circ \iota, g \circ \iota)_{L^2(\nu)} = (f, g)_{L^2(dx \times \mu_0)}. \tag{5.4}$$

We next calculate the transformation of D^1 under the change of coordinate induced by ι . By a straightforward calculation we see that

$$D^1[f \circ \iota, g \circ \iota] = (D^{XY}[f, g]) \circ \iota \quad \text{for } f, g \in \mathcal{D}_0^1. \tag{5.5}$$

By (5.2) and (5.5) we obtain the isometry of the bilinear forms $(\mathcal{E}^\nu, \mathcal{D}_0^\nu)$ and $(\mathcal{E}^{XY}, \mathcal{D}_0^{XY})$ under the transformation induced by ι . Indeed, the map $\iota^* : \mathcal{D}_0^{XY} \rightarrow \mathcal{D}_0^\nu$ defined by $\iota^*(f) = f \circ \iota$ is bijective and

$$\mathcal{E}^\nu(f \circ \iota, g \circ \iota) = \mathcal{E}^{XY}(f, g). \tag{5.6}$$

By (5.4) and (5.6) the closability of $(\mathcal{E}^{XY}, \mathcal{D}_0^{XY})$ on $L^2(dx \times \mu_0)$ follows from that of $(\mathcal{E}^\nu, \mathcal{D}_0^\nu)$ on $L^2(\nu)$, which is given by (M.1.1). We have thus proved (1).

Since ι is the transformation on $\mathbf{R}^d \times \mathfrak{S}$, it is clear that P^{XY} is a diffusion with state space $\mathbf{R}^d \times \mathfrak{S}$. Recall that $P_{(x,\mathfrak{s})}^{XY} = P_{\iota^{-1}(x,\mathfrak{s})}^\nu \circ \iota^{-1}$ and that P^ν is the diffusion associated with $(\mathcal{E}^\nu, \mathcal{D}^\nu, L^2(\nu))$. By (5.4) and (5.6) the Dirichlet spaces $(\mathcal{E}^\nu, \mathcal{D}^\nu, L^2(\nu))$ and $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$ are isometric. Hence we conclude $\{P_{(x,\mathfrak{s})}^{XY}\}$ is associated with the Dirichlet space $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$.

By the theorem due to Albeverio-Ma-Röckner (see [12, Theorem 5.1]), the quasi-regularity of the Dirichlet space follows from the existence of the associated diffusion. Hence (3) follows from (2) immediately. \square

LEMMA 5.2. Let Cap^{XY} be the capacity associated with the Dirichlet space $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$. Let P^{XY} be the associated diffusion as in Lemma 5.1. Then there exists a subset $\Xi \subset \mathbf{R}^d \times \mathfrak{S}$ such that

$$P_{(x,\mathfrak{s})}^{XY}(\mathfrak{X} \in \cdot) = P_{(y,\mathfrak{s})}^{XY}(\mathfrak{X} \in \cdot) \quad \text{for all } (x, \mathfrak{s}), (y, \mathfrak{s}) \in \Xi, \tag{5.7}$$

$$\text{Cap}^{XY}(\Xi^c) = 0. \tag{5.8}$$

Here we set $\mathfrak{X}^1 = (X, \mathfrak{X}) \in C([0, \infty); S \times \mathfrak{S})$ as before.

PROOF. It is clear that for each $a \in \mathbf{R}^d$

$$\begin{aligned} (f(\cdot - a, *), g(\cdot - a, *))_{L^2(dx \times \mu_0)} &= (f, g)_{L^2(dx \times \mu_0)}, \\ \mathcal{E}^{XY}(f(\cdot - a, *), g(\cdot - a, *)) &= \mathcal{E}^{XY}(f, g). \end{aligned}$$

Hence we see that the equality in (5.7) holds for a.e. $(x, \mathfrak{s}), (y, \mathfrak{s}) \in \mathbf{R}^d \times \mathfrak{S}$.

We next strength the equality in (5.7) from *a.e.* to *all* on Ξ for some Ξ satisfying $\text{Cap}^{XY}(\Xi^c) = 0$.

For each Borel set \mathfrak{A} of the form $\mathfrak{A} = \{\mathfrak{X}_{t_1} \in A_1, \dots, \mathfrak{X}_{t_i} \in A_i\}$, where $A_j \in \mathcal{B}(\mathfrak{S})$ ($j = 1, \dots, i$), we see that $P_{(x, \mathfrak{s})}^{XY}(\mathfrak{X} \in \mathfrak{A})$ is quasi-continuous in (x, \mathfrak{s}) . Hence there exists a subset $\Xi \subset \mathbf{R}^d \times \mathfrak{S}$ such that $\text{Cap}^{XY}(\Xi^c) = 0$ and that $\Xi = \bigcup_{n=1}^\infty K_n$ for some increasing sequence of closed set and, moreover, the restriction of $P_{(x, \mathfrak{s})}^{XY}(\mathfrak{X} \in A)$ on K_n is continuous in (x, \mathfrak{s}) for all n . This means, with a help of the monotone class theorem, (5.7) holds for Ξ as above. \square

LEMMA 5.3. *Assume (M.1.0), (M.1.1), (M.2)–(M.4). Then we have the following.*

- (1) *The bilinear form $(\mathcal{E}^Y, \mathcal{D}_\circ^Y)$ is closable on $L^2(\mu_0)$.*
- (2) *Let Ξ be as in Lemma 5.2. Let $\{P_\mathfrak{s}^Y\}_{\mathfrak{s} \in \mathfrak{S}}$ be the family of probability measures on $C([0, \infty); \mathfrak{S})$ defined by*

$$\begin{aligned} P_\mathfrak{s}^Y &= P_{(x, \mathfrak{s})}^{XY}(\mathfrak{X} \in \cdot) && \text{if } (x, \mathfrak{s}) \in \Xi \text{ for some } x \in \mathbf{R}^d, \\ P_\mathfrak{s}^Y(\mathfrak{X}_t = \mathfrak{s} \text{ for all } t) &= 1 && \text{otherwise.} \end{aligned}$$

Then $P^Y = (\{P_\mathfrak{s}^Y\}_{\mathfrak{s} \in \mathfrak{S}}, \mathfrak{X})$ is a diffusion.

PROOF. Let $\varphi \in C_0^\infty(\mathbf{R}^d)$ and $f \in \mathcal{D}_\circ^Y$. Then

$$\|\varphi \otimes f\|_{L^2(dx \times \mu_0)} = \|\varphi\|_{L^2(dx)} \|f\|_{L^2(\mu_0)}, \tag{5.9}$$

$$\mathcal{E}^{XY}(\varphi \otimes f, \varphi \otimes f) = \|\varphi\|_{L^2(dx)}^2 \mathcal{E}^Y(f, f) + \frac{1}{2} \|\nabla \varphi\|_{L^2(dx)}^2 \|f\|_{L^2(\mu_0)}^2. \tag{5.10}$$

Indeed, (5.9) is a straightforward calculation. As for (5.10) we see

$$D^{XY}[\varphi \otimes f, \varphi \otimes f] = \varphi^2 \otimes D^Y[f, f] + \frac{|\nabla \varphi|^2 \otimes f^2}{2} - (\varphi \nabla \varphi, f Df)_{\mathbf{R}^d}. \tag{5.11}$$

Then integrating over $\mathbf{R}^d \times \mathfrak{S}$ by $dx \times \mu_0$ and noticing

$$\int_{\mathbf{R}^d \times \mathfrak{S}} (\varphi \nabla \varphi, f Df)_{\mathbf{R}^d} dx \times \mu_0 = \left(\int_{\mathbf{R}^d} \varphi \nabla \varphi dx, \int_{\mathfrak{S}} f Df d\mu_0 \right)_{\mathbf{R}^d} = 0,$$

we obtain (5.10).

By (5.9) and (5.10) the closability of $(\mathcal{E}^Y, \mathcal{D}_\circ^Y)$ on $L^2(\mu_0)$ follows from the one of $(\mathcal{E}^{XY}, \mathcal{D}_\circ^{XY})$ on $L^2(dx \times \mu_0)$, which has been already obtained in Lemma 5.1 (1). We thus prove (1).

We next prove (2). By (5.7) we see that for any $A \in \mathcal{B}(C([0, \infty); \mathfrak{S}))$

$$P_s^Y(\mathfrak{X} \in A) = P_{(x, \mathfrak{s})}^{XY}((X, \mathfrak{X}) \in C([0, \infty); \mathbf{R}^d) \times A) \quad \text{for all } (x, \mathfrak{s}) \in \Xi. \quad (5.12)$$

We remark P^{XY} is a diffusion on $\mathbf{R}^d \times \mathfrak{S}$ and Ξ^c is an exceptional set, that is, $P_{(x, \mathfrak{s})}^{XY}(\sigma_{\Xi^c} < \infty) = 0$ for q.e. (x, \mathfrak{s}) because of $\text{Cap}^{XY}(\Xi^c) = 0$. Hence we deduce from (5.12) that P^Y is a diffusion with state space \mathfrak{S} . □

LEMMA 5.4. *Let $(\mathcal{E}^Y, \mathcal{D}^Y)$ be the closure of $(\mathcal{E}^Y, \mathcal{D}_\circ^Y)$ on $L^2(\mu_0)$.*

- (1) *The diffusion P^Y in Lemma 5.3 is associated with $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(\mu_0)$.*
- (2) *The Dirichlet form $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(\mu_0)$ is quasi-regular.*

PROOF. Let E_s^Y denote the expectation with respect to P_s^Y . Let $\{T_t^Y\}$ be the semigroup defined by $T_t^Y f = E_s^Y[f(\mathfrak{X}_t)]$. Let $\{T_t^{XY}\}$ be the semigroup associated with the Dirichlet space $(\mathcal{E}^{XY}, \mathcal{D}^{XY}, L^2(dx \times \mu_0))$. Then we deduce that

$$1 \otimes (T_t^Y f) = T_t^{XY}(1 \otimes f). \quad (5.13)$$

Let $\rho(x) = c_{5.1}(1 + |x|^{2(d+4)})^{-1/2}$ such that $\int \rho^2 dx = 1$, where $c_{5.1}$ is the normalizing constant. Let $L^2(\rho) = L^2(\mathbf{R}^d \times \mathfrak{S}, \rho^2 dx \times \mu_0)$ and

$$\mathcal{E}_{\rho, \lambda}^{XY}(f, g) = \mathcal{E}^{XY}(f, \rho^2 g) + \lambda(f, g)_{L^2(\rho)}. \quad (5.14)$$

Then there exists λ_0 such that $(\mathcal{E}_{\rho, \lambda}^{XY}, \mathcal{D}_\circ^1)$ is positive and closable on $L^2(\rho)$ for all $\lambda > \lambda_0$ (see [15, Lemma 2.1] for proof). We fix such a λ and denote by $\{T_t^\lambda\}$ the semigroup associated with the closure $(\mathcal{E}_{\rho, \lambda}^{XY}, \mathcal{D}_\rho^{XY})$ of $(\mathcal{E}_{\rho, \lambda}^{XY}, \mathcal{D}_\circ^1)$ on $L^2(\rho)$. It is known that (see [15, p. 234])

$$T_t^{XY}(1 \otimes f) = e^{\lambda t} T_t^\lambda(1 \otimes f). \quad (5.15)$$

By a direct calculation we see that

$$\mathcal{E}_{\rho,\lambda}^{XY}(1 \otimes f, 1 \otimes f) = \mathcal{E}^Y(f, f) + \lambda(f, f)_{L^2(\mu_0)}. \tag{5.16}$$

Let $\tilde{\mathcal{D}}^Y$ be the domain of the Dirichlet space associated with $\{T_t^Y\}$ on $L^2(\mu_0)$. By (5.13) and (5.15) we obtain $f \in \tilde{\mathcal{D}}^Y$ if and only if $1 \otimes f \in \mathcal{D}_\rho^{XY}$. By (5.16) we see that $1 \otimes f \in \mathcal{D}_\rho^{XY}$ if and only if $f \in \mathcal{D}^Y$. Collecting these we obtain that $\tilde{\mathcal{D}}^Y = \mathcal{D}^Y$.

Let $\psi \in C_0^\infty(\mathbf{R}^d)$ be such that $\int \psi \rho^2 dx \neq 0$. Then we have for any $f, g \in \mathcal{D}^Y$

$$\begin{aligned} & \lim_{\alpha \rightarrow \infty} \alpha^2 \left(\int_0^\infty e^{-\alpha t} \{1 \otimes f - T_t^{XY}(1 \otimes f)\} dt, \psi \otimes g \right)_{L^2(\rho)} \\ &= \int_{\mathbf{R}^d} \psi \rho^2 dx \cdot \mathcal{E}^Y(f, g). \end{aligned} \tag{5.17}$$

By using (5.13) and (5.17) and then by dividing the both sides by $\int_{\mathbf{R}^d} \psi \rho^2 dx$, we obtain

$$\lim_{\alpha \rightarrow \infty} \alpha^2 \left(\int_0^\infty e^{-\alpha t} \{f - T_t^Y f\} dt, g \right)_{L^2(\mu_0)} = \mathcal{E}^Y(f, g). \tag{5.18}$$

This implies $\{T_t^Y\}$ is the semigroup associated with the Dirichlet form $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(\mu_0)$ (see Lemma 1.3.4 in [4]). So we conclude P^Y is associated with $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(\mu_0)$.

(2) is immediate from (1) similarly as Lemma 5.1. □

PROOF OF THEOREM 2.6. (1) follows from Lemmas 5.3 and 5.4. (2) follows from Lemma 5.1. □

PROOF OF THEOREM 2.7. (2.32) follows from Theorem 2.4 and Lemma 5.1 (2). (2.33) follows from Theorem 2.4, Lemmas 5.1 and 5.3 immediately. □

6. Non-explosion of tagged particles.

Throughout this section we set $\nu_r = \nu_r^1$ and $\mu_r = \mu_r^1$. In this section we prove Theorem 2.5. By (4.32) in Lemma 4.4 the non-explosion property of tagged particles follows from the conservativeness of the diffusion P^ν . Then we apply a result in [4] to prove this as follows.

LEMMA 6.1. *Assume (M.1.0), (M.1.1), (M.2), and (M.3). Assume (2.20). Then P^ν is conservative.*

PROOF. Applying Theorem 5.7.2 in [4] to the diffusion P^ν yields Lemma 6.1. □

We next prepare several notations used in the rest of this section.

Let $\mathfrak{X} \in C([0, \infty); \mathfrak{S}_{\text{single}})$. We write $\mathfrak{X} = \{\sum_i \delta_{X_i^x}\}$ and set $X^i \in C(I_i; \mathbf{R}^d)$. We take I_i to be the maximal interval. By construction we deduce that I_i is of the form $[0, b_i)$ or (a_i, b_i) . Let $\mathbf{I} = \{i; I_i = [0, b_i)\}$ and $\mathbf{J} = \{i; I_i = (a_i, b_i)\}$. Then $\mathfrak{X} = \sum_{i \in \mathbf{I}} \delta_{X_i^x} + \sum_{i \in \mathbf{J}} \delta_{X_i^x} =: \mathfrak{X}^{\mathbf{I}} + \mathfrak{X}^{\mathbf{J}}$.

We relabel $\mathfrak{X}^{\mathbf{I}}$ as $\mathfrak{X}^{\mathbf{I}} = \{\sum_x \delta_{X_t^x}\}$, where $x \in S$ is such that $X_0^x = x$. Let

$$\xi^x(\mathfrak{X}) = \inf \left\{ t > 0; \sup_{0 \leq s < t} |X_s^x| = \infty \right\}, \tag{6.1}$$

$$\xi_r(\mathfrak{X}) = \inf \left\{ t > 0; \min_{|x| < r} \xi^x(\mathfrak{X}) < t \right\} \quad (r \in \mathbf{N} \cup \{\infty\}), \tag{6.2}$$

$$\mathfrak{A}_r = \{\mathfrak{s} \in \mathfrak{S}; P_{\mathfrak{s}}^\mu(\xi_r < \infty) > 0\}. \tag{6.3}$$

For a path $\mathfrak{X}^1 = (X, \mathfrak{X})$ we define the stopping time η by

$$\eta(\mathfrak{X}^1) = \inf \left\{ t > 0; \sup_{0 \leq s < t} |X_s| = \infty \right\}. \tag{6.4}$$

LEMMA 6.2. *Suppose $\int_{S \times \mathfrak{S}} P_{(x, \mathfrak{s})}^\nu(\eta < \infty) d\nu_r = 0$. Then $\mu(\mathfrak{A}_r) = 0$.*

PROOF. Let μ_r be as in (3.2). For $m \geq 1$ let $c_{6.1}$ be a constant such that

$$\mu(\cdot \cap \mathfrak{S}_{r,m}) \leq c_{6.1} \mu_r(\cdot \cap \mathfrak{S}_{r,m}).$$

Then we see that

$$\begin{aligned} \int_{\mathfrak{S}_{r,m}} P_{\mathfrak{s}}^\mu(\xi_r < \infty) \mu(d\mathfrak{s}) &\leq \int_{\mathfrak{S}_{r,m}} \sum_{\mathfrak{s}(x) \geq 1, |x| < r} P_{\mathfrak{s}}^\mu(\xi^x < \infty) \mu(d\mathfrak{s}) \\ &\leq c_{6.1} \int_{\mathfrak{S}_{r,m}} \sum_{\mathfrak{s}(x) \geq 1, |x| < r} P_{\mathfrak{s}}^\mu(\xi^x < \infty) \mu_r(d\mathfrak{s}) \\ &= c_{6.1} \int_{S \times \mathfrak{S}_r^{m-1}} P_{\kappa(x, \mathfrak{s})}^\mu(\xi^x < \infty) \nu_r(dx d\mathfrak{s}) \\ &= c_{6.1} \int_{S \times \mathfrak{S}_r^{m-1}} P_{(x, \mathfrak{s})}^\nu(\eta < \infty) \nu_r(dx d\mathfrak{s}) \quad \text{by (4.32)}. \end{aligned} \tag{6.5}$$

Hence we have $\int_{\mathfrak{S}_{r,m}} P_{\mathfrak{s}}^{\mu}(\xi_r < \infty)\mu(d\mathfrak{s}) = 0$ for all $m \geq 1$ by assumption. This equality also holds for $m = 0$ because $P_{\mathfrak{s}}^{\mu}(\xi_r < \infty) = 0$ for $\mathfrak{s} \in \mathfrak{S}_r^0$. Hence by $\mathfrak{S} = \sum_{m=0}^{\infty} \mathfrak{S}_{r,m}$ we deduce

$$\int_{\mathfrak{S}} P_{\mathfrak{s}}^{\mu}(\xi_r < \infty)\mu(d\mathfrak{s}) = 0. \tag{6.6}$$

By (6.3) and (6.6) obtain $\mu(\mathfrak{A}_r) = 0$. □

LEMMA 6.3. *Suppose $\mu(\mathfrak{A}_r) = 0$. Then $\text{Cap}^{\mu}(\mathfrak{A}_r) = 0$.*

PROOF. It is known that $\text{Cap}^{\mu}(\mathfrak{A}_r) = \sup\{\text{Cap}^{\mu}(K); K \subset \mathfrak{A}_r, K \text{ is compact}\}$ (see [4, (2.1.6) in p.66]). So let K be a compact set such that $K \subset \mathfrak{A}_r$.

Let $\sigma_K = \inf\{t > 0; \mathfrak{X}_t \in K\}$ be the first hitting time to K . Since K is compact, we deduce $\mathfrak{X}_{\sigma_K} \in K$ if $\sigma_K < \infty$.

Suppose $\mathfrak{s} \notin \mathfrak{A}_r$. Then $P_{\mathfrak{s}}^{\mu}(\xi_r < \infty) = 0$ by (6.3). Hence for $\mathfrak{s} \notin \mathfrak{A}_r$

$$0 = P_{\mathfrak{s}}^{\mu}(\xi_r < \infty; \sigma_K < \xi_r < \infty) = \int_K P_{\mathfrak{s}}^{\mu}(\mathfrak{X}_{\sigma_K} \in d\mathfrak{s}'; \sigma_K < \infty)P_{\mathfrak{s}}^{\mu}(\xi_r < \infty).$$

This combined with (6.3) and $K \subset \mathfrak{A}_r$ yields

$$P_{\mathfrak{s}}^{\mu}(\mathfrak{X}_{\sigma_K} \in K; \sigma_K < \infty) = 0 \text{ for } \mathfrak{s} \notin \mathfrak{A}_r. \tag{6.7}$$

Since $P_{\mathfrak{s}}^{\mu}(\mathfrak{X}_{\sigma_K} \in K; \sigma_K < \infty) = P_{\mathfrak{s}}^{\mu}(\sigma_K < \infty)$, we deduce from (6.7) that

$$P_{\mathfrak{s}}^{\mu}(\sigma_K < \infty) = 0 \text{ for } \mathfrak{s} \notin \mathfrak{A}_r. \tag{6.8}$$

By (6.8) and $\mu(\mathfrak{A}_r) = 0$ we have $\int_{\mathfrak{S}} P_{\mathfrak{s}}^{\mu}(\sigma_K < \infty)d\mu = 0$. From this we deduce $\text{Cap}^{\mu}(K) = 0$. We therefore obtain $\text{Cap}^{\mu}(\mathfrak{A}_r) = 0$. □

PROOF OF THEOREM 2.5. By Lemma 6.1 we see that P^{ν} is conservative. Hence $\int_{S \times \mathfrak{S}} P_{(x,\mathfrak{s})}^{\nu}(\eta < \infty)d\nu_r = 0$. Then by Lemmas 6.2 and 6.3 we obtain $\text{Cap}^{\mu}(\mathfrak{A}_r) = 0$ for all $r \in \mathbf{N}$, which yields $\text{Cap}^{\mu}(\mathfrak{A}_{\infty}) = 0$. Here

$$\mathfrak{A}_{\infty} = \{\mathfrak{s}; P_{\mathfrak{s}}^{\mu}(\xi_{\infty} < \infty) > 0\}.$$

By $\text{Cap}^{\mu}(\mathfrak{A}_{\infty}) = 0$ together with (6.1) and (6.2) we deduce (2.13). □

7. Quasi-regularity: Proof of Lemma 2.3.

In this section we prove the quasi-regularity of k -labeled Dirichlet forms. So we begin by recalling the definition of quasi-regular by following [12].

Let E be a Polish space. A Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(E, m)$ is called quasi-regular if it satisfies the following:

- (Q.1) There exists an increasing sequence of compact sets $\{K_n\}$ such that $\bigcup_n \mathcal{D}(K_n)$ is dense in \mathcal{D} w.r.t. $\mathcal{E}_1^{1/2}$ -norm. Here $\mathcal{D}(K_n)$ is the set of the elements f of \mathcal{D} such that $f(x) = 0$ a.e. $x \in K_n^c$, and $\mathcal{E}_1^{1/2}(f) = \mathcal{E}(f, f)^{1/2} + \|f\|_{L^2(E, m)}$.
- (Q.2) There exists a $\mathcal{E}_1^{1/2}$ -dense subset of \mathcal{D} whose elements have \mathcal{E} -quasi continuous m -version.
- (Q.3) There exist a countable set $\{u_n\}_{n \in \mathbf{N}}$ having \mathcal{E} -quasi continuous m -version \tilde{u}_n , and an exceptional set \mathcal{N} such that $\{\tilde{u}_n\}_{n \in \mathbf{N}}$ separates the points of $E \setminus \mathcal{N}$.

Let $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ be the closure of $(\mathcal{E}^\mu, \mathcal{D}_\circ^\mu)$ as before. By (M.2), $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ satisfies the quasi-regularity as seen in Lemma 2.1. We remark that $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ enjoys more strict conditions than the quasi-regularity. Indeed, we quote:

LEMMA 7.1 ([14]). *Assume (M.2). Then we have the following.*

- (1) *There exists a compact subset $\{K_n\}_{n \in \mathbf{N}}$ such that $\bigcup_n \mathcal{D}_\circ(K_n)$ is $\{\mathcal{E}_1^\mu\}^{1/2}$ -dense in \mathcal{D}_\circ^μ . Here $\mathcal{D}_\circ(K_n) = \{f \in \mathcal{D}_\circ; f(\mathfrak{s}) = 0 \text{ for all } \mathfrak{s} \in K_n^c\}$.*
- (2) *There exists countable elements $\{u_n\}_{n \in \mathbf{N}}$ of \mathcal{D}_\circ^μ that separate the points of \mathfrak{S} .*
- (3) *If $(\mathcal{E}^\mu, \mathcal{D}_\circ^\mu)$ is closable $L^2(\mu)$, then the closure $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ is quasi-regular.*

REMARK 7.1.

- (1) We remark $\mathcal{D}_\circ(K_n) \subset \mathcal{D}_\circ^\mu$. Indeed, $\mathbf{D}[f, g]$ with $f, g \in \mathcal{D}_\circ(K_n)$ is bounded because $\mathbf{D}[f, g]$ is continuous and K_n is compact.
- (2) Suppose $(\mathcal{E}^\mu, \mathcal{D}_\circ^\mu)$ is closable $L^2(\mu)$. Then we see that (1) implies (Q.1). Since $\mathcal{D}_\circ(K_n) \subset C(\mathfrak{S})$, (1) is more strict than (Q.1). Moreover, (Q.2) is trivially satisfied in the above case because \mathcal{D}^μ is the closure of \mathcal{D}_\circ^μ and $\mathcal{D}_\circ^\mu \subset C(\mathfrak{S})$. We see that (2) implies (Q.3) because $\mathcal{D}_\circ^\mu \subset \mathcal{D}^\mu$. The condition (2) is also more strict than (Q.3) in the sense that all $\{u_n\}$ are continuous and $\{u_n\}$ separate all the points of \mathfrak{S} .

LEMMA 7.2. *There exists an increasing sequence of compact sets $\{K_{r,n}\}$ such that $\bigcup_{n=1}^\infty \mathcal{D}_\circ(K_{r,n})$ is dense in $\mathcal{D}_\circ^{\mu_r^k}$ with respect to the $\{\mathcal{E}_1^{\mu_r^k}\}^{1/2}$ -norm.*

PROOF. By (M.2) μ_r^k becomes a finite measure. So the associated Dirichlet

space is same as the Dirichlet space with the probability measure $(\mu_r^k(\mathfrak{S}))^{-1}\mu_r^k$. We note here the measures in the energy form $\mathcal{E}^{\mu_r^k}$ and in the L^2 -space are common.

Applying Lemma 7.1 to the measure $(\mu_r^k(\mathfrak{S}))^{-1}\mu_r^k$ yields Lemma 7.2. □

Recall that $\mathcal{D}_\circ^k = C_0^\infty(S^k) \otimes \mathcal{D}_\circ$ and $\mathcal{D}_\circ^{\nu^k} = \{f \in \mathcal{D}_\circ^k; \mathcal{E}_1^{\nu^k}(f, f) < \infty\}$.

LEMMA 7.3. $\bigcup_{r,n=1}^\infty C_0^\infty(S^k) \otimes \mathcal{D}_\circ(K_{r,n})$ is dense in $\mathcal{D}_\circ^{\nu^k}$ with respect to the $\{\mathcal{E}_1^{\nu^k}\}^{1/2}$ -norm.

PROOF. Let $\varphi \in C_0^\infty(S^k)$ and $f \in \mathcal{D}_\circ$ such that $\varphi \otimes f \in \mathcal{D}_\circ^{\nu^k}$. It is sufficient for Lemma 7.3 to show that for such an f and all $\varepsilon > 0$ there exists $f_{r,n}$ such that $f_{r,n} \in \mathcal{D}_\circ(K_{r,n})$ and that

$$\mathcal{E}_1^{\nu^k}(\varphi \otimes (f - f_{r,n}), \varphi \otimes (f - f_{r,n})) \leq \varepsilon. \tag{7.1}$$

Since $\varphi \in C_0^\infty(S^k)$, there exists an r such that $\varphi = 0$ on $(S_r^k)^c$. Hence there exists a constant $c_{7.1} = c_{7.1}(\varphi)$ such that

$$\nabla[\varphi, \varphi](x) \leq c_{7.1}1_{S_r^k}(x), \quad \varphi^2(x) \leq c_{7.1}1_{S_r^k}(x) \quad \text{for all } x \in S^k. \tag{7.2}$$

We write $\nabla[\varphi] = \nabla[\varphi, \varphi]$ and $\mathbf{D}[f] = \mathbf{D}[f, f]$. By a direct calculation we have

$$\begin{aligned} &\mathcal{E}_1^{\nu^k}(\varphi \otimes (f - f_{r,n}), \varphi \otimes (f - f_{r,n})) \\ &= \int_{S^k \times \mathfrak{S}} \{ \nabla[\varphi] \otimes |f - f_{r,n}|^2 + \varphi^2 \otimes \mathbf{D}[f - f_{r,n}] + \varphi^2 \otimes |f - f_{r,n}|^2 \} d\nu^k \\ &\leq 2c_{7.1} \int_{S^k \times \mathfrak{S}} 1 \otimes \{ |f - f_{r,n}|^2 + \mathbf{D}[f - f_{r,n}] \} d\nu_r^k \\ &= 2c_{7.1} \int_{\mathfrak{S}} \{ |f - f_{r,n}|^2 + \mathbf{D}[f - f_{r,n}] \} d\mu_r^k \quad \text{by (3.2)} \\ &= 2c_{7.1} \mathcal{E}_1^{\mu_r^k}(f - f_{r,n}, f - f_{r,n}). \end{aligned} \tag{7.3}$$

By Lemma 7.2 we can take n and $f_{r,n} \in \mathcal{D}_\circ(K_{r,n})$ in such a way that

$$\mathcal{E}_1^{\mu_r^k}(f - f_{r,n}, f - f_{r,n}) \leq \frac{\varepsilon}{2c_{7.1}}.$$

This combined with (7.3) yields (7.1). □

PROOF OF LEMMA 2.3. (Q.1) follows from Lemma 7.3. (Q.2) is clear since \mathcal{D}^{ν^k} is the closure of $\mathcal{D}_\circ^{\nu^k}$ and $\mathcal{D}_\circ^{\nu^k} \subset C(S^k \times \mathfrak{S})$.

For $r \in \mathbf{N}$ let $\{u_{r,n}\}$ be a countable subset of $\mathcal{D}_\circ^{\mu_r^k}$ that separates the points of \mathfrak{S} . We can obtain this by applying Lemma 7.1 (2) to μ_r^k . We used here that (M.2) for μ_r^k follows from that for μ .

Let $\{\varphi_m\}$ be a countable subset of $C_0^\infty(S^k)$ that separates the points S^k . Then by the same calculation as (7.3) we have

$$\mathcal{E}_1^{\nu^k}(\varphi_m \otimes u_{r,n}, \varphi_m \otimes u_{r,n}) \leq 2c_{7.1} \mathcal{E}_1^{\mu_r^k}(u_{r,n}, u_{r,n}).$$

Here $c_{7.1}$ is a constant satisfying (7.2) for φ_m . Hence $\varphi_m \otimes u_{r,n} \in \mathcal{D}_\circ^{\nu^k}$. Since $\{\varphi_m \otimes u_{r,n}\}$ separates the points of $S^k \times \mathfrak{S}$, we obtain (Q.3). \square

References

- [1] S. Albeverio, Y. G. Kondratiev and M. Röckner, Analysis and geometry on configuration spaces: the Gibbsian case, *J. Funct. Anal.*, **157** (1998), 242–291.
- [2] A. De Masi, P. A. Ferrari, S. Goldstein and W. D. Wick, An invariance principle for reversible Markov processes, Applications to random motions in random environments, *J. Statist. Phys.*, **55** (1989), 787–855.
- [3] T. Fattler and M. Grothaus, Tagged particle process in continuum with singular interactions, preprint, available at arXiv:0804.4868v3.
- [4] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes, Walter de Gruyter, 1994.
- [5] J. Fritz, Gradient dynamics of infinite point systems, *Ann. Probab.*, **15** (1987), 478–514.
- [6] M. Z. Guo, Limit theorems for interacting particle systems, Thesis, Dept. of Mathematics, New York University, 1984.
- [7] M. Z. Guo and G. C. Papanicolaou, Self-diffusion of interacting Brownian particles, in “Probabilistic Method in Mathematical Physics”, Proc. Taniguchi International Sympo. at Katata and Kyoto, 1985 (eds. K. Ito and N. Ikeda), Kinokuniya, 1987, pp. 113–152.
- [8] K. Inukai, Collision or non-collision problem for interacting Brownian particles, *Proc. Japan Acad. Ser. A Math. Sci.*, **82** (2006), 66–70.
- [9] C. Kipnis and S. R. S. Varadhan, Central limit theorems for additive functional of reversible Markov process and applications to simple exclusions, *Comm. Math. Phys.*, **104** (1986), 1–19.
- [10] R. Lang, Unendlich-dimensionale Wienerprozesse mit Wechselwirkung I, *Z. Wahrsch. Verw. Gebiete*, **38** (1977), 55–72.
- [11] R. Lang, Unendlich-dimensionale Wienerprozesse mit Wechselwirkung II, *Z. Wahrsch. Verw. Gebiete*, **39** (1978), 277–299.
- [12] Z.-M. Ma and M. Röckner, Introduction to the theory of (non-symmetric) Dirichlet forms, Springer-Verlag, Berlin, 1992.
- [13] M. Mehta, Radom matrices, Third Edition, Elsevier, 2004.
- [14] H. Osada, Dirichlet form approach to infinitely dimensional Wiener processes with singular interactions, *Comm. Math. Phys.*, **176** (1996), 117–131.
- [15] H. Osada, An invariance principle for Markov processes and Brownian particles with

- singular interaction, *Ann. Inst. H. Poincaré*, **34** (1998), 217–248.
- [16] H. Osada, Positivity of the self-diffusion matrix of interacting Brownian particles with hard core, *Probab. Theory Related Fields*, **112** (1998), 53–90.
- [17] H. Osada, Non-collision and collision properties of Dyson’s model in infinite dimensions and other stochastic dynamics whose equilibrium states are determinantal random point fields, in *Stochastic Analysis on Large Scale Interacting Systems*, (eds. T. Funaki and H. Osada), *Adv. Stud. Pure Math.*, **39**, Math. Soc. Japan, Tokyo, 2004, pp. 325–343.
- [18] H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials, preprint, available at arXiv:0902.3561v1.
- [19] H. Osada, Infinite-dimensional stochastic differential equations related to random matrices, preprint, available at arXiv:1004.0301v1.
- [20] S. Resnick, *Extreme values, regular variation, and point processes*, Springer, 2000.
- [21] D. Ruelle, Superstable interactions in classical statistical mechanics, *Comm. Math. Phys.*, **18** (1970), 127–159.
- [22] T. Shiga, A remark on infinite-dimensional Wiener processes with interactions, *Z. Wahrsch. Verw. Gebiete*, **47** (1979), 299–304.
- [23] A. Soshnikov, Determinantal random point fields, *Russian Math. Surveys*, **55** (2000), 923–975.
- [24] H. Tanemura, A system of infinitely many mutually reflecting Brownian balls in \mathbf{R}^d , *Probab. Theory Related Fields*, **104** (1996), 399–426.
- [25] H. Tanemura, Uniqueness of Dirichlet forms associated with systems of infinitely many Brownian balls in \mathbf{R}^d , *Probab. Theory Related Fields*, **109** (1997), 275–299.
- [26] H. J. Yoo, Dirichlet forms and diffusion processes for fermion random point fields, *J. Funct. Anal.*, **219** (2005), 143–160.
- [27] M. W. Yoshida, Construction of infinite-dimensional interacting diffusion processes through Dirichlet forms, *Probab. Theory Related Fields*, **106** (1996), 265–297.

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