J. Math. Soc. Japan
Vol. 62, No. 1 (2010) pp. 95–133
doi: 10.2969/jmsj/06210095

On the Cauchy problem for hyperbolic operators of second order whose coefficients depend only on the time variable

By Seiichiro Wakabayashi

(Received Nov. 4, 2008)

Abstract. In this paper we deal with hyperbolic operators of second order whose coefficients depend only on the time variable and give necessary conditions and sufficient conditions for the Cauchy problem to be C^∞ well-posed. In particular, we give a necessary and sufficient condition (a complete characterization) for C^∞ well-posedness when the space dimension is equal to 2 and the coefficients are real analytic functions of the time variable.

1. Introduction.

The Cauchy problem for hyperbolic operators of second order has been investigated by many authors (see, *e.g.*, [**8**], [**5**], [**1**] and [**2**]). However, a complete characterization of C^∞ well-posedness of the Cauchy problem has not been obtained even if the coefficients of the operators depend only on the time variable. When the space dimension is equal to 1 and the coefficients are real analytic, Nishitani obtained a necessary and sufficient condition (a complete characterization) for C^∞ well-posedness of the Cauchy problem in [**8**].

In [**1**] Colombini, Ishida and Orrú studied the Cauchy problem for hyperbolic operators of second order whose coefficients depend only on the time variable, and they gave sufficient conditions for C^∞ well-posedness. However, their conditions are not always necessary ones (see Examples 7.1 and 7.2 below). In this paper we shall deal with the same problem and give a necessary and sufficient condition for C^∞ well-posedness when the space dimension is equal to 2 and the coefficients are real analytic functions of the time variable. Moreover, we shall also give a necessary and sufficient condition for C^∞ well-posedness without the restriction on the space dimension when the coefficients are semi-algebraic functions of the time variable (see Definition 1.6 below for the definition of semi-algebraic functions).

Let $P(t,x,\tau,\xi) \equiv \tau^2 + \sum_{j=0}^{1} \sum_{|\alpha|\le 2-j} a_{j,\alpha}(t,x)\tau^j\xi^\alpha$ be a polynomial of τ and $\xi = (\xi_1, \cdots, \xi_n)$ of degree 2 whose coefficients $a_{j,\alpha}(t,x)$ are C^∞ functions of

---

2000 *Mathematics Subject Classification.* Primary 35L15; Secondary 35L10.

Key Words and Phrases. Cauchy problem, hyperbolic, C^∞ well-posed, second order.

This research was supported by Grant-in-Aid for Scientific Research (No. 20540155), Japan Society for the Promotion of Science.

$(t, x) \equiv (t, x_1, \cdots, x_n) \in [0, \infty) \times \boldsymbol{R}^n$. Here $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\boldsymbol{Z}_+)^n$ is a multi-index, $|\alpha| = \sum_{j=1}^n \alpha_j$ and $\xi^\alpha = \xi_1^{\alpha_1}, \cdots, \xi_n^{\alpha_n}$, where $\boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$ $(= \{0, 1, 2, 3, \cdots\})$. We consider the Cauchy problem

$$\begin{cases} P(t, x, D_t, D_x)u(t, x) = f(t, x) & \text{in } [0, \infty) \times \boldsymbol{R}^n, \\ D_t^j u(t, x)|_{t=0} = u_j(x) & \text{in } \boldsymbol{R}^n \ (j = 0, 1) \end{cases} \tag{CP}$$

in the C^∞ category, where $D_t = -i\partial/\partial t$ $(= -i\partial_t)$, $D_x = (D_1, \cdots, D_n) = -i(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $f(t, x) \in C^\infty([0, \infty) \times \boldsymbol{R}^n)$ and $u_j(x) \in C^\infty(\boldsymbol{R}^n)$ $(j = 0, 1)$.

DEFINITION 1.1. We say that the Cauchy problem (CP) is C^∞ well-posed if the following conditions (E) and (U) are satisfied:

(E) For any $f \in C^\infty([0, \infty) \times \boldsymbol{R}^n)$ and $u_j \in C^\infty(\boldsymbol{R}^n)$ $(j = 0, 1)$ there is $u \in C^\infty([0, \infty) \times \boldsymbol{R}^n)$ satisfying (CP).

(U) If $s > 0$, $u \in C^\infty([0, \infty) \times \boldsymbol{R}^n)$, $u(0, x) = D_t u(t, x)|_{t=0} = 0$ and $\operatorname{supp} P(t, x, D_t, D_x)u(t, x) \subset [s, \infty) \times \boldsymbol{R}^n$, then $\operatorname{supp} u \subset [s, \infty) \times \boldsymbol{R}^n$.

We assume throughout the paper that $a_{j,\alpha}(t, x) \equiv a_{j,\alpha}(t)$ for $j \in \boldsymbol{Z}_+$ and $\alpha \in (\boldsymbol{Z}_+)^n$ with $j + |\alpha| = 2$, that is, the coefficients of the principal part of $P(t, x, D_t, D_x)$ do not depend on x. Moreover, in studying the Cauchy problem (CP), we may assume that $a_{1,\alpha}(t) \equiv 0$ if $|\alpha| = 1$. Indeed, if $\lambda(t, \xi) \equiv \sum_{|\alpha|=1} a_{1,\alpha}(t)\xi^\alpha/2$ does not vanish identically, we make a change of variables from x to y:

$$y_j = x_j - \frac{1}{2}\int_0^t a_{1,e_j}(r)\,dr \quad (1 \le j \le n),$$

where e_j denotes the vector in $(\boldsymbol{Z}_+)^n$ whose k-th component is equal to $\delta_{j,k}$ $(k = 1, 2, \cdots, n)$. Therefore, we can assume without loss of generality that

$$P(t, x, \tau, \xi) = \tau^2 - a(t, \xi) + b_0(t, x)\tau + b(t, x, \xi) + c(t, x),$$

where

$$a(t, \xi) = \sum_{j,k=1}^n a_{j,k}(t)\xi_j\xi_k, \quad b(t, x, \xi) = \sum_{j=1}^n b_j(t, x)\xi_j$$

and $a_{j,k}(t) = a_{k,j}(t)$. Taking account of the Lax-Mizohata theorem we assume that

(H) $a(t,\xi) \ge 0$ for $(t,\xi) \in [0,\infty) \times \boldsymbol{R}^n$

(see [**7**]). Define

$$V = \{\xi \in \boldsymbol{R}^n;\ a(t,\xi) \equiv 0 \text{ in } t \in [0,\infty)\}.$$

Then V is a subspace of $\boldsymbol{R}^n$ since $a(t,\xi) \ge 0$. It follows from Theorem 4.1 of [**4**] that $b(t,x,\xi) \equiv 0$ in $(t,x) \in [0,\infty) \times \boldsymbol{R}^n$ for $\xi \in V$ if the Cauchy problem (CP) is C^∞ well-posed (see, also, [**12**]). So we can also assume without loss of generality that

(F) $V = \{0\}$, *i.e.*, $a(t,\xi) \not\equiv 0$ in t for any $\xi \in \boldsymbol{R}^n \setminus \{0\}$.

Moreover, we assume that $a(t,\xi)$ satisfies the following condition (A):

(A) For any $t_0 \ge 0$ there are a neighborhood U of t_0 in $[0,\infty)$, $N \in \boldsymbol{N}$, Lebesgue measurable conic subsets Γ_j $(1 \le j \le N)$ of $\boldsymbol{R}^n$, $e_j(t,\xi) \in C^1(U; L^\infty(\Gamma_j))$, $C > 0$, $m_j \in \boldsymbol{Z}_+$, $a_k^j(\xi) \in L^\infty(\Gamma_j)$ $(1 \le k \le m_j)$ such that $\mu(\boldsymbol{R}^n \setminus (\bigcup_{j=1}^N \Gamma_j)) = 0$, the $e_j(t,\xi)$ are positively homogeneous of degree 2 in ξ, the $a_k^j(\xi)$ are positively homogeneous of degree 0, $e_j(t,\xi) \ge 0$, the $a_k^j(\xi)$ are real-valued and

$$\begin{aligned}
&\partial_t e_j(t,\xi) \le C e_j(t,\xi), \\
&a(t,\xi) = e_j(t,\xi) q_j(t,\xi), \\
&q_j(t,\xi) = (t-t_0)^{m_j} + a_1^j(\xi)(t-t_0)^{m_j-1} + \cdots + a_{m_j}^j(\xi)
\end{aligned}$$

for $(t,\xi) \in U \times \Gamma_j$, where μ denotes the Lebesgue measure on $\boldsymbol{R}^n$.

In the condition (A) we may assume that $\delta > 0$ satisfies $\delta \le t_0/2$ if $t_0 > 0$, and $U = [(t_0-\delta)_+, t_0+\delta]$, where $a_+ = \max\{a, 0\}$ for $a \in \boldsymbol{R}$. The condition (F) implies that $e_j(t,\xi) \not\equiv 0$ in t for any $\xi \in \boldsymbol{R}^n \setminus \{0\}$. We remark that the condition (A) is satisfied with $\inf\{e_j(t,\xi);\ t \in U \text{ and } \xi \in \Gamma_j\} > 0$ under the assumptions (H) and (F) if the $a_{j,k}(t)$ are real analytic in $[0,\infty)$ (see Lemma 2.1 below). In order to obtain a sufficient conditon on C^∞ well-posedness we impose the following two conditions (B) and (L):

(B) The coefficients do not depend on x, *i.e.*,

$$b_0(t,x) \equiv b_0(t), \quad b(t,x,\xi) \equiv b(t,\xi), \quad c(t,x) \equiv c(t).$$

(L) For any $t_0 \geq 0$ there is $C > 0$ such that for each j with $1 \leq j \leq N$

$$\min_{\tau \in \mathscr{R}_j(\xi)} |t-\tau| \cdot |b(t,\xi)| \leq C\sqrt{a(t,\xi)} \quad \text{for } (t,\xi) \in U \times \Gamma_j,$$

where

$$\mathscr{R}_j(\xi) = \{(\operatorname{Re}\lambda)_+;\ \lambda \in \boldsymbol{C},\ q_j(\lambda,\xi) = 0 \text{ and } \operatorname{Re}\lambda \in [t_0 - 2\delta, t_0 + 2\delta]\}$$

for $\xi \in \Gamma_j$, $\min_{\tau \in \mathscr{R}_j(\xi)} |t-\tau| = 1$ if $\mathscr{R}_j(\xi) = \emptyset$, and N, U, the Γ_j and the $q_j(t,\xi)$ are as in the condition (A) and depend on t_0.

Put $p(t,\tau,\xi) = \tau^2 - a(t,\xi)$ and define

$$\Gamma(p(t,\cdot,\cdot),\vartheta) = \{(\tau,\xi) \in \boldsymbol{R}^{n+1};\ \tau > \sqrt{a(t,\xi)}\},$$

where $\vartheta = (1,0,\cdots,0) \in \boldsymbol{R}^{n+1}$. We define

$$\begin{aligned} K^{\pm}_{(t_0,x^0)} = \{&(t(s),x(s)) \in [0,\infty) \times \boldsymbol{R}^n;\ \pm s \geq 0 \text{ and } \{(t(s),x(s))\} \text{ is} \\ &\text{a Lipschitz continuous curve in } [0,\infty) \times \boldsymbol{R}^n \text{ satisfying} \\ &(d/ds)(t(s),x(s)) \in \Gamma(p(t,\cdot,\cdot),\vartheta)^* \ (a.e.\ s) \text{ and} \\ &(t(0),x(0)) = (t_0,x^0)\}, \end{aligned}$$

where $(t_0,x^0) \in [0,\infty) \times \boldsymbol{R}^n$ and $\Gamma^* = \{(t,x) \in \boldsymbol{R}^{n+1};\ t\tau + x \cdot \xi \geq 0$ for any $(\tau,\xi) \in \Gamma\}$. $K^{\pm}_{(t_0,x^0)}$ are called generalized (half) flows for p. Concerning sufficiency of C^∞ well-posedness, we have the following

THEOREM 1.2. *Assume that the conditions* (B) *and* (L) *are satisfied* (*in addition to the assumptions* (H), (F) *and* (A)). *Then the Cauchy problem* (*CP*) *is* C^∞ *well-posed. Moreover, if* $(t_0,x^0) \in (0,\infty) \times \boldsymbol{R}^n$ *and* $u \in C^\infty([0,\infty) \times \boldsymbol{R}^n)$ *satisfies* (*CP*), $u_j(x) = 0$ *near* $\{x \in \boldsymbol{R}^n;\ (0,x) \in K^-_{(t_0,x^0)}\}$ $(j = 0,1)$ *and* $f = 0$ *near* $K^-_{(t_0,x^0)}$ (*in* $[0,\infty) \times \boldsymbol{R}^n$), *then* $(t_0,x^0) \notin \operatorname{supp} u$.

REMARK.
(i) If $u \in C^\infty([0,\infty) \times \boldsymbol{R}^n)$ satisfies the Cauchy problem (CP), then

$$\begin{aligned} \operatorname{supp} u \subset \{&(t,x) \in [0,\infty) \times \boldsymbol{R}^n;\ (t,x) \in K^+_{(s,y)} \text{ for} \\ &\text{some } (s,y) \in \left(\bigcup_{j=0}^{1} \{0\} \times \operatorname{supp} u_j\right) \cup \operatorname{supp} f\}. \end{aligned}$$

(ii) It follows from the proof given in Section 3 that one can replace $\mathscr{R}_j(\xi)$ $(1 \le j \le N)$ in the condition (L) by $\mathscr{R}'_j(\xi)$ satisfying $\mathscr{R}_j(\xi) \subset \mathscr{R}'_j(\xi)$ and $\sup_{\xi \in \Gamma_j} \#\mathscr{R}'_j(\xi) < \infty$, where $\#A$ denotes the number of the elements of a set A.

Let $(t_0, x^0) \in [0, \infty) \times \boldsymbol{R}^n$ and $\xi^0 \in S^{n-1}$ satisfy $a(t_0, \xi^0) = 0$. In the study on necessity of C^∞ well-posedness we impose the following conditions $(\mathrm{A})'_{(t_0,\xi^0)}$ which corresponds to the condition (A):

$(\mathrm{A})'_{(t_0,\xi^0)}$ There are a neighborhood U of t_0 in $[0, \infty)$, a conic neighborhood Γ of ξ^0, $e(t, \xi) \in C^\infty(U \times \Gamma)$, $m \in \boldsymbol{N}$, $a_k(\xi) \in C^\infty(\Gamma)$ $(1 \le k \le m)$ such that $e(t, \xi)$ is positively homogeneous of degree 2 in ξ, $e(t, \xi) > 0$, the $a_k(\xi)$ are positively homogeneous of degree 0 and real-valued, $a_k(\xi^0) = 0$ and

$$\begin{aligned} &a(t,\xi) = e(t,\xi)q(t,\xi), \\ &q(t,\xi) = (t - t_0)^m + a_1(\xi)(t - t_0)^{m-1} + \cdots + a_m(\xi) \end{aligned}$$

for $(t, \xi) \in U \times \Gamma$.

Note that the condition $(\mathrm{A})'_{(t_0,\xi^0)}$ is satisfied under the assumptions (H) and (F) if the $a_{j,k}(t)$ are real analytic in $[0, \infty)$ (see Lemma 2.1 below). Let $\theta_0 > 0$ and $\Xi_j(\theta)$ $(1 \le j \le n)$ be real-valued continuous functions defined in $[0, \theta_0]$ such that $\Xi_j(\theta) \in C^\infty((0, \theta_0])$, $\Xi(0) = \xi^0$ and the $\Xi_j(\theta)$ can be expanded into formal Puiseux series of θ, *i.e.*, $\Xi(\theta) \equiv (\Xi_1(\theta), \cdots, \Xi_n(\theta)) \sim \xi^0 + \sum_{k=1}^\infty \Xi^k \theta^{k/L}$, where $L \in \boldsymbol{N}$ and $\Xi^k \in \boldsymbol{R}^n$ $(k \in \boldsymbol{N})$. It is easy to see that the roots of the equation $q(t + t_0, \Xi(\theta)) = 0$ in t can be expanded into formal Puiseux series of θ (see, *e.g.*, [**10**] for general results). We denote by $\tau_j(\theta; \Xi)$ $(1 \le j \le m)$ the real parts of the roots of the equation $q(t + t_0, \Xi(\theta)) = 0$ in t which can be expanded into formal Puiseux series. When $t_0 > 0$, m is even and we can rearrange $\{\tau_j(\theta; \Xi)\}$ so that $\tau_j(\theta; \Xi) \equiv \tau_{m/2+j}(\theta; \Xi)$ $(1 \le j \le m/2)$. We may assume that $t_0 + \tau_j(\theta; \Xi) > 0$ for $\theta \in [0, \theta_0]$ when $t_0 > 0$, modifying θ_0 if necessary. Put $\mathrm{Ord}_{\theta \downarrow 0} f = \nu$ if $f(\theta) \in C([0, \theta_0])$ and there are $c \in \boldsymbol{C} \setminus \{0\}$ and $\nu \in \boldsymbol{R}$ satisfying $f(\theta) = c\theta^\nu(1 + o(1))$ as $\theta \downarrow 0$. If $f(\theta) = O(\theta^N)$ $(\theta \downarrow 0)$ for any $N \in \boldsymbol{Z}_+$, then we define $\mathrm{Ord}_{\theta \downarrow 0} f = \infty$.

THEOREM 1.3. *Assume that the condition* $(\mathrm{A})'_{(t_0,\xi^0)}$ *is satisfied* (*in addition to the assumptions* (H) *and* (F)). *Moreover, we assume that the following condition* $(\mathrm{C})_{(t_0,x^0,\xi^0)}$ *is satisfied*:

$(\mathrm{C})_{(t_0,x^0,\xi^0)}$ *There are* $\theta_0 > 0$ *and real-valued continuous functions* $T(\theta)$ *and* $\Xi_j(\theta)$ $(1 \le j \le n)$ *defined in* $[0, \theta_0]$ *such that* $T(\theta), \Xi_j(\theta) \in C^\infty((0, \theta_0])$, $t_0 + T(\theta) > 0$ *for* $\theta \in (0, \theta_0]$, $T(\theta)$ *and* $\Xi(\theta) \equiv (\Xi_1(\theta), \cdots, \Xi_n(\theta))$ *can be expanded into formal Puiseux series of* θ, $T(0) = 0$, $\Xi(0) = \xi^0$ *and*

$$\begin{aligned}&\mathrm{Ord}_{\theta\downarrow 0}\Big\{\min_{1\le j\le m}|t_0+T(\theta)-(t_0+\tau_j(\theta;\Xi))_+|\cdot|b(t_0+T(\theta),x^0,\Xi(\theta))|\Big\}\\&\quad<\mathrm{Ord}_{\theta\downarrow 0}\sqrt{a(t_0+T(\theta),\Xi(\theta))}.\end{aligned}$$

Then the Cauchy problem (CP) is not C^∞ well-posed.

Remark.

(i) We have

$$\min_{1\le j\le m}|t_0+T(\theta)-(t_0+\tau_j(\theta;\Xi))_+|=\min_{\tau\in\mathscr{R}(\Xi(\theta))}|t_0+T(\theta)-\tau|,$$

where

$$\mathscr{R}(\xi)=\{(\mathrm{Re}\,\lambda)_+;\ \lambda\in\boldsymbol{C}\text{ and }q(\lambda,\xi)=0\}. \tag{1.1}$$

(ii) The condition $(\mathrm{C})_{(t_0,x^0,\xi^0)}$ can be restated in terms of Newton polygons (see Lemma 2.2 below).

Under the condition $(\mathrm{A})'_{(t_0,\xi^0)}$ we define the condition $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ as follows:

$(\mathrm{L})_{(t_0,x^0,\xi^0)}$ There are a neighborhood U of t_0 in $[0,\infty)$, a conic neighborhood Γ of ξ^0 and $C>0$ such that

$$\min_{\tau\in\mathscr{R}(\xi)}|t-\tau|\cdot|b(t,x^0,\xi)|\le C\sqrt{a(t,\xi)}\quad\text{for }(t,\xi)\in U\times\Gamma, \tag{1.2}$$

where $\mathscr{R}(\xi)$ is the set defined by (1.1).

Assume that the $a_{j,k}(t)$ are real analytic functions of t on $[0,\infty)$. Then, for any $T>0$ there is $\delta_T>0$ such that the $a_{j,k}(t)$ are analytic in $\Omega_T\equiv\{t\in\boldsymbol{C};\ \mathrm{Re}\,t\in(-\delta_T,T+\delta_T)$ and $|\mathrm{Im}\,t|<\delta_T\}$. So the $a_{j,k}(t)$ are analytic in $\Omega_\infty\equiv\bigcup_{j=1}^\infty\Omega_j$. Moreover, there are a neighborhood $U_{(t_0,x^0)}$ of t_0 in $[0,\infty)$ and a conic neighborhood $\Gamma_{(t_0,x^0)}$ of ξ^0 such that the condition $(\mathrm{A})'_{(t_0,\xi^0)}$ is satisfied with $U=U_{(t_0,x^0)}$ and $\Gamma=\Gamma_{(t_0,x^0)}$ (see Lemma 2.1 below), and there are a neighborhood U_{t_0} of t_0 in $[0,\infty)$, $\xi^1,\cdots,\xi^N\in S^{n-1}$ and conic neighborhoods Γ_j of ξ^j $(1\le j\le N)$ such that $\bigcup_{j=1}^N\Gamma_j=\boldsymbol{R}^n\setminus\{0\}$ and the conddition (A) with $U=U_{t_0}$ is satisfied. It is obvious that the $\mathscr{R}_j(\xi)$ in the condition (L) and $\mathscr{R}(\xi)$ in the condition $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ can be replaced by

$$\mathscr{R}(\xi)(=\mathscr{R}_j(\xi))=\{(\mathrm{Re}\,\lambda)_+;\ \lambda\in\Omega_\infty\text{ and }a(\lambda,\xi)=0\}\quad(\ \xi\in\boldsymbol{R}^n\setminus\{0\}).$$

THEOREM 1.4. *Assume (in addition to the assumptions* (H) *and* (F)) *that* $n = 2$, *and that the* $a_{j,k}(t)$ *and* $b_j(t,x)$ $(j=1,2)$ *are real analytic functions of* $t \in [0,\infty)$. *Then the condition* $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ *is valid if the Cauchy problem* (*CP*) *is* C^∞ *well-posed.*

From Theorems 1.2 and 1.4 we have the following main result.

THEOREM 1.5. *Assume that* $n = 2$ *and the condition* (B) *is satisfied (in addition to the assumptions* (H) *and* (F)), *and that the* $a_{j,k}(t)$ *and* $b_j(t)$ $(j = 1, 2)$ *are real analytic functions of* t *in* $[0,\infty)$. *Then the condition* (L) *is a necessary and sufficient condition for the Cauchy problem* (*CP*) *to be* C^∞ *well-posed.*

REMARK. The condition (L) can be restated in terms of Newton polygons, which is similar to the condition given in [**8**] (see Lemma 2.2 below).

DEFINITION 1.6.

(i) Let f: $\boldsymbol{R}^N \ni X = (X_1, \cdots, X_N) \mapsto f(X) \in \boldsymbol{R}$. We say that $f(X)$ is a semi-algebraic function if the graph $\{(X,y) \in \boldsymbol{R}^{N+1};\ y = f(X)\}$ of f is a semi-algebraic set. For the definition of semi-algebraic sets we refer to [**3**], for example.

(ii) Let $X^0 \in \boldsymbol{R}^N$, U be a neighborhood of X^0, and let f: $U \to \boldsymbol{R}$. We say that f is semi-algebraic at X^0 if there is $c > 0$ such that the set $\{(X,y) \in \boldsymbol{R}^{N+1};\ y = f(X) \text{ and } |X - X^0| < c\}$ is a semi-algebraic set. Moreover, we say that f is semi-algebraic in U if f is semi-algebraic at each $X \in U$. When f: $U \to \boldsymbol{C}$, we say that f is semi-algebraic in U if $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are semi-algebraic in U.

For basic properties of semi-algebraic functions we refer to [**11**]. In the next theorem we impose the following conditions $(\text{A-}a)_{(t_0,\xi^0)}$ and $(\text{A-}b)_{(t_0,x^0,\xi^0)}$:

$(\text{A-a})_{(t_0,\xi^0)}$ The condition $(\mathrm{A})'_{(t_0,\xi^0)}$ is satisfied, and the $a_k(\xi)$ are semi-algebraic in Γ, where Γ and the $a_k(\xi)$ are as in $(\mathrm{A})'_{(t_0,\xi^0)}$.

$(\text{A-b})_{(t_0,x^0,\xi^0)}$ There are $\beta(t,\xi)$ and $\tilde{b}(t,\xi)$ in $C^\infty(U \times \Gamma)$ such that $\beta(t,\xi) \neq 0$ for $(t,\xi) \in U \times \Gamma$, $\tilde{b}(t,\xi)$ is semi-algebraic in $U \times \Gamma$ and

$$b(t,x^0,\xi) = \beta(t,\xi)\tilde{b}(t,\xi) \quad \text{in } U \times \Gamma,$$

where U and Γ are as in $(\mathrm{A})'_{(t_0,\xi^0)}$.

THEOREM 1.7. *Assume that the condition* (A-a)(t0,ξ0) *and* (A-b)(t0,x0,ξ0) *are satisfied (in addition to the assumptions* (H) *and* (F)). *Then the condition* $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ *is satisfied if the Cauchy problem* (*CP*) *is* C^∞ *well-posed.*

Remark.

(i) If the $a_{j,k}(t)$ and $b_j(t,x^0)$ $(1 \le j \le n)$ are semi-algebraic at t_0, then the conditions (A-a)$_{(t_0,\xi^0)}$ and (A-b)$_{(t_0,x^0,\xi^0)}$ are satisfied (see Lemma 2.3 below and [**11**]).

(ii) Assume that the condition (B) is satisfied (in addition to the assumptions (H) and (F)), and that the $a_{j,k}(t)$ and $b_j(t)$ $(1 \le j \le n)$ are semi-algebraic at any $t_0 \in [0,\infty)$. Then it follows from Theorems 1.2 and 1.7 that the Cauchy problem (CP) is C^∞ well-posed if and only if the condition (L) is satisfied, since the $a_{j,k}(t)$ and $b_j(t)$ $(1 \le j \le n)$ are real analytic in $[0,\infty)$ (see the proof of Theorem 10 of [**11**]). (iii) From Theorem 1.7 one may conjecture that the condition (L) is a necessary and sufficient condition for the Cauchy problem (CP) to be C^∞ well-posed under the conditions (H), (F) and (B) if the coefficients of $P(t,D_t,D_x)$ are real analytic in $[0,\infty)$.

The remainder of this paper is organized as follows. In Section 2 we shall give preliminary lemmas. Theorem 1.2 (sufficiency of C^∞ well-posedness) will be proved in Section 3. Theorem 1.3 will be proved in Section 4. In Section 5 and Section 6 we shall prove Theorems 1.4 and 1.7, respectively. Some examples and remarks will be given in Section 7.

2. Preliminaries.

First let us consider the condition (A).

Lemma 2.1. *Assume that the conditions* (H) *and* (F) *are satisfied, and that for any* $(t,\xi) \in [0,\infty) \times S^{n-1}$ *there is* $l \in \boldsymbol{Z}_+$ *satisfying* $\partial_t^l a(t,\xi) \ne 0$. *Then the condition* (A) *is satisfied. In particular, the condition* (A) *is satisfied* (*under the conditions* (H) *and* (F)) *if the* $a_{j,k}(t)$ *are real analytic on* $[0,\infty)$.

Remark. In the condition (A) we can choose the Γ_j as open cones in $\boldsymbol{R}^n \setminus \{0\}$, and $e_j(t,\xi) \in C^\infty(U \times \Gamma_j)$ so that $e_j(t,\xi) > 0$ for $(t,\xi) \in U \times \Gamma_j$, if the hypotheses of the lemma are fulfilled.

Proof. Let $(t_0,\xi^0) \in [0,\infty) \times S^{n-1}$ satisfy $a(t_0,\xi^0) = 0$. We may assume that $a(t,\xi)$ belongs to $C^\infty((-1,\infty) \times \boldsymbol{R}^n)$ and is real-valued. From the Malgrange preparation theorem there are a neighborhood $U_{(t_0,\xi^0)}$ of t_0 in $[0,\infty)$, an open conic neighborhood $\Gamma_{(t_0,\xi^0)}$ of ξ^0, $m \in \boldsymbol{N}$, $e(t,\xi) \in C^\infty(U_{(t_0,\xi^0)} \times \Gamma_{(t_0,\xi^0)})$ and real-valued functions $a_k(\xi)$ in $C^\infty(\Gamma_{(t_0,\xi^0)})$ $(1 \le k \le m)$ such that $e(t,\xi)$ and the $a_k(\xi)$ are positively homogeneous of degree 2 and 0, respectively, $e(t,\xi) > 0$, $a_k(\xi^0) = 0$ and

$$a(t,\xi) = e(t,\xi)\{(t-t_0)^m + a_1(\xi)(t-t_0)^{m-1} + \cdots + a_m(\xi)\} \quad \text{in } U_{(t_0,\xi^0)} \times \Gamma_{(t_0,\xi^0)}.$$

Since S^{n-1} is compact, we can choose $\xi^1, \cdots, \xi^N \in S^{n-1}$ so that $\bigcup_{j=1}^N \Gamma_{(t_0,\xi^j)} = \boldsymbol{R}^n \setminus \{0\}$. This proves the lemma. □

Let $(t_0, x^0, \xi^0) \in [0,\infty) \times \boldsymbol{R}^n \times S^{n-1}$ satisfy $a(t_0, \xi^0) = 0$, and assume that the condition $(\mathrm{A})'_{(t_0,\xi^0)}$ is satisfied. Let $\Xi_j(\theta)$ $(1 \le j \le n)$ be real-valued continuous functions defined on $[0,\theta_0]$ such that $\Xi_j(\theta) \in C^\infty((0,\theta_0])$, $\Xi(\theta) \equiv (\Xi_1(\theta), \cdots, \Xi_n(\theta))$ can be expanded into formal Puiseux series of θ and $\Xi(0) = \xi^0$, where $\theta_0 > 0$. We denote by $\tau_j(\theta;\Xi)$ $(1 \le j \le m)$ the real parts of the roots of the equation $q(t + t_0, \Xi(\theta)) = 0$ in t which can be expanded into formal Puiseux series, where m and $q(t,\xi)$ are as in $(\mathrm{A})'_{(t_0,\xi^0)}$. Let $1 \le j \le m$. If there is $l \in \boldsymbol{Z}_+$ such that

$$\mathrm{Ord}_{\theta\downarrow 0}(\partial_t^l b)((t_0 + \tau_j(\theta;\Xi))_+, x^0, \Xi(\theta)) < \infty, \tag{2.1}$$

then we can write

$$\begin{aligned} &tb((t_0 + \tau_j(\theta;\Xi))_+ + t, x^0, \Xi(\theta)) \sim \sum_{k=0}^\infty t\beta_{j,k}(t)\theta^{\nu_j + k/L}, \\ &\beta_{j,0}(t) \not\equiv 0, \end{aligned}$$

where $L \in \boldsymbol{N}$. Indeed, we write $\{l_0, l_1, l_2 \cdots\} = \{l \in \boldsymbol{Z}_+;\ l \text{ satisfies (2.1)}\}$, where $0 \le l_0 < l_1 < l_2 < \cdots$. Then, putting $\nu_{j,k} = \mathrm{Ord}_{\theta\downarrow 0}(\partial_t^{l_k} b)((t_0 + \tau_j(\theta;\Xi))_+, x^0, \Xi(\theta))$, we have $\nu_j = \min\{\nu_{j,k};\ k = 0,1,2,\cdots\}$. Moreover, we have

$$\begin{aligned} &\sum_{k=0}^{N-1} \beta_{j,k}(t)\theta^{\nu_j + k/L} \\ &= \sum_{\mu + |\alpha| < M_j(N)} \frac{((t_0 + \tau_j(\theta;\Xi))_+ - t_0)^\mu (\Xi(\theta) - \xi^0)^\alpha}{\mu!\alpha!} (\partial_t^\mu \partial_\xi^\alpha b)(t_0 + t, x^0, \xi^0) \\ &\quad + O(\theta^{\nu_j + N/L}) \quad \text{as } \theta \downarrow 0, \end{aligned}$$

where $M_j(N)$ is a positive integer satisfying $\mathrm{Ord}_{\theta\downarrow 0}\{((t_0 + \tau_j(\theta;\Xi))_+ - t_0)^\mu \times (\Xi(\theta) - \xi^0)^\alpha\} \ge \nu_j + N/L$ for $\mu + |\alpha| \ge M_j(N)$. This implies that $\beta_{j,k}(t) \in C^\infty([-t_0,\infty))$. If $\mathrm{Ord}_{\theta\downarrow 0}(\partial_t^l b)((t_0 + \tau_j(\theta;\Xi))_+, x^0, \Xi(\theta)) = \infty$ for every $l \in \boldsymbol{Z}_+$, then we put $\nu_j = \infty$. If $\nu_j < \infty$, we define

$$\mu_{j,k} = 1 + \mathrm{Ord}_{t\downarrow 0}\beta_{j,k}(t) \quad (\ k = 0,1,2,\cdots).$$

We denote by $\Gamma_{1,j}(\Xi)$ the Newton polygon of $tb((t_0 + \tau_j(\theta;\Xi))_+ + t, x^0, \Xi(\theta))$, *i.e.*,

$$\Gamma_{1,j}(\Xi) = \mathrm{ch}\Big[\bigcup_{k\geq 0,\, \mu_{j,k}<\infty} (\{(\nu_j + k/L, \mu_{j,k})\} + (\overline{\boldsymbol{R}}_+)^2\Big],$$

where ch$[A]$ denotes the convex hull of A, $\overline{\boldsymbol{R}}_+ = [0,\infty)$ and $\Gamma_{1,j}(\Xi) = \emptyset$ if $\nu_j = \infty$. We put

$$2\Gamma_{1,j}(\Xi) = \{(2\nu, 2\mu) \in \boldsymbol{R}^2;\ (\nu,\mu) \in \Gamma_{1,j}(\Xi)\}.$$

It is easily seen that

$$\Gamma_{1,j}(\Xi) = \bigcap_{p\geq 0} \{(\nu,\mu) \in (\overline{\boldsymbol{R}}_+)^2;$$
$$\nu + p\mu \geq \min\{\nu_j + k/L + p\mu_{j,k};\ k \geq 0 \text{ and } \mu_{j,k} < \infty\}\}.$$

Denote by $\Gamma_{0,j}(\Xi)$ the Newton polygon of $a((t_0 + \tau_j(\theta;\Xi))_+ + t, \Xi(\theta))$.

Lemma 2.2. *The following two conditions* (i) *and* (ii) *are equivalent*:

(i) *If $T(\theta)$ is a real-valued continuous function defined in $[0,\theta_0]$, $T(\theta) \in C^\infty((0,\theta_0])$, $T(0) = 0$, $t_0 + T(\theta) > 0$ for $\theta \in (0,\theta_0]$ and $T(\theta)$ can be expanded into a formal Puiseux series, then*

$$\mathrm{Ord}_{\theta\downarrow 0}\Big\{\min_{1\leq j\leq m} |t_0 + T(\theta) - (t_0 + \tau_j(\theta;\Xi))_+| \cdot |b(t_0 + T(\theta), x^0, \Xi(\theta))|\Big\}$$
$$\geq \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a(t_0 + T(\theta), \Xi(\theta))}. \tag{2.2}$$

(ii) $2\Gamma_{1,j}(\Xi) \subset \Gamma_{0,j}(\Xi) \quad (1 \leq j \leq m)$.

Proof. Choose real-valued continuous functions $\lambda_k(\theta)$ defined in $[0,\theta_0]$ and subsets I_k of $\{1,2,\cdots,m\}$ $(1 \leq k \leq r)$ so that $\lambda_k(\theta) \in C^\infty((0,\theta_0])$ can be expanded into formal Puiseux series, $\bigcup_{k=1}^r I_k = \{1,2,\cdots,m\}$, $\mathrm{Ord}_{\theta\downarrow 0}((t_0 + \tau_j(\theta;\Xi))_+ - t_0 - \lambda_k(\theta)) = \infty$ for $1 \leq k \leq r$ and $j \in I_k$,

$$\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_r(\theta) \quad \text{for } \theta \in (0,\theta_0],$$
$$\kappa_k \equiv \mathrm{Ord}_{\theta\downarrow 0}(\lambda_{k+1}(\theta) - \lambda_k(\theta)) < \infty \quad (\ 1 \leq k \leq r-1)$$

and $\lambda_1(\theta) \equiv 0$ if $\mathrm{Ord}_{\theta\downarrow 0}\lambda_1(\theta) = \infty$, modifying θ_0 if necessary. Put

$$\begin{aligned} S_0 &= \{(t,\theta) \in [(t_0-1)_+, t_0+1] \times [0,\theta_0];\ t-t_0 < \lambda_1(\theta)\},\\ S_{2k-1} &= \{(t,\theta) \in [(t_0-1)_+, t_0+1] \times [0,\theta_0];\\ &\qquad \lambda_k(\theta) \le t-t_0 < (\lambda_k(\theta)+\lambda_{k+1}(\theta))/2\},\\ S_{2k} &= \{(t,\theta) \in [(t_0-1)_+, t_0+1] \times [0,\theta_0];\\ &\qquad (\lambda_k(\theta)+\lambda_{k+1}(\theta))/2 \le t-t_0 < \lambda_{k+1}(\theta)\},\\ S_{2r-1} &= \{(t,\theta) \in [(t_0-1)_+, t_0+1] \times [0,\theta_0];\ \lambda_r(\theta) \le t-t_0 \le 1\}, \end{aligned}$$

where $1 \le k \le r-1$. First assume that (i) is valid. Let $1 \le j \le m$ and $p \ge 0$. Putting

$$T_p(t,\theta) = (t_0 + \tau_j(\theta;\Xi))_+ - t_0 + \theta^p t \quad (\ 1/2 \le t \le 1),$$

we have

$$\mathrm{Ord}_{\theta\downarrow 0} a(t_0 + T_p(t,\theta), \Xi(\theta)) = \min\{\nu + p\mu;\ (\nu,\mu) \in \Gamma_{0,j}(\Xi)\}$$

for a generic $t \in [1/2, 1]$. Moreover, we have

$$\mathrm{Ord}_{\theta\downarrow 0} \min_{1\le k\le m} |t_0 + T_p(t,\theta) - (t_0 + \tau_k(\theta;\Xi))_+| = p$$

for a generic $t \in [1/2, 1]$. By assumption we have

$$\begin{aligned} &\mathrm{Ord}_{\theta\downarrow 0}\{\theta^p t b((t_0+\tau_j(\theta;\Xi))_+ + \theta^p t, x^0, \Xi(\theta))\}\\ &\quad \ge \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a((t_0+\tau_j(\theta;\Xi))_+ + \theta^p t, \Xi(\theta))} \quad \text{for a generic } t \in [1/2,1]. \end{aligned}$$

This gives

$$\min\{\nu + p\mu;\ (\nu,\mu) \in 2\Gamma_{1,j}(\Xi)\} \ge \min\{\nu + p\mu;\ (\nu,\mu) \in \Gamma_{0,j}(\Xi)\},$$

which implies that (ii) is valid. Next assume that (ii) is valid. Let $T(\theta)$ be a real-valued continuous function defined in $[0,\theta_0]$ such that $T(\theta) \in C^\infty((0,\theta_0])$, $T(0) = 0,\ t_0 + T(\theta) > 0$ for $\theta \in (0,\theta_0]$ and $T(\theta)$ can be expanded into a formal Puiseux series. First consider the case $T(\theta) \in S_0$ for $0 < \theta \ll 1$. Similarly, we can deal with the case $T(\theta) \in S_{2r-1}$. Write

$$T(\theta) = \lambda_1(\theta) - c\theta^p(1 + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $c > 0$ and $p \geq 0$. We note that $S_0 = \emptyset$ if $t_0 = 0$ and $\lambda_1(\theta) \equiv 0$. This implies that $t_0 + \tau_j(\theta; \Xi) > 0$ for $0 < \theta \ll 1$ and $1 \leq j \leq m$ and m is even. So we can write

$$a(t, \Xi(\theta)) = e(t, \Xi(\theta)) \prod_{j=1}^{m/2} ((t - t_0 - \tau_j(\theta; \Xi))^2 + \sigma_j(\theta; \Xi)),$$

rearranging $\{\tau_j(\theta; \Xi)\}$ if necessary, where the $\sigma_j(\theta; \Xi)$ are continuous functions defined in $[0, \theta_0]$, expanded into formal Puiseux series and satisfying $\sigma_j(\theta; \Xi) \in C^\infty((0, \theta_0])$ and $\sigma_j(\theta; \Xi) \geq 0$. It is obvious that

$$\mathrm{Ord}_{\theta \downarrow 0}(T(\theta) - \tau_j(\theta; \Xi)) = \mathrm{Ord}_{\theta \downarrow 0}(\lambda_1(\theta) - \tau_j(\theta; \Xi) - t\theta^p)$$

for $t > 0$. Therefore, we have

$$\begin{aligned} &\mathrm{Ord}_{\theta \downarrow 0} a(t_0 + T(\theta), \Xi(\theta)) = \mathrm{Ord}_{\theta \downarrow 0} a(t_0 + \lambda_1(\theta) - t\theta^p, \Xi(\theta)) \\ &= \min\{\nu + p\mu;\ (\nu, \mu) \in \Gamma_{0,j}(\Xi)\} \quad \text{for } t > 0 \text{ and } j \in I_1. \end{aligned} \tag{2.3}$$

We have also

$$\mathrm{Ord}_{\theta \downarrow 0} \min_{1 \leq j \leq m} |t_0 + T(\theta) - (t_0 + \tau_j(\theta; \Xi))_+| = p. \tag{2.4}$$

It follows from assumption, (2.3) and (2.4) that

$$\begin{aligned} &2\,\mathrm{Ord}_{\theta \downarrow 0} \Big\{ \min_{1 \leq j \leq m} |t_0 + T(\theta) - (t_0 + \tau_j(\theta; \Xi))_+| \cdot |b(t_0 + T(\theta), x^0, \Xi(\theta))| \Big\} \\ &\geq 2\,\mathrm{Ord}_{\theta \downarrow 0} \{\theta^p t b((t_0 + \tau_l(\theta; \Xi))_+ + \theta^p t, x^0, \Xi(\theta))\} \\ &= \min\{\nu + p\mu;\ (\nu, \mu) \in 2\Gamma_{1,l}(\Xi)\} \geq \mathrm{Ord}_{\theta \downarrow 0} a(t_0 + T(\theta), \Xi(\theta)) \end{aligned} \tag{2.5}$$

for a generic $t < 0$, where $l \in I_1$. Next consider the case $T(\theta) \in S_{2k-1}$ for $0 < \theta \ll 1$, where $1 \leq k \leq r - 1$. Similarly, we can deal with the case $T(\theta) \in S_{2k}$ $(1 \leq k \leq r - 1)$. If $\mathrm{Ord}_{\theta \downarrow 0}(T(\theta) - \lambda_k(\theta)) = \infty$, then (2.2) holds trivially. Now write

$$T(\theta) = \lambda_k(\theta) + c\theta^p(1 + o(1)) \quad \text{as } \theta \downarrow 0,$$

where $c > 0$ and $p \geq \kappa_k$. We note that $k = 1$, $j \in I_1$, $t_0 = 0$ and $\lambda_1(\theta) \equiv 0$ if $j \in I_k$ and $t_0 + \tau_j(\theta; \Xi) < 0$ for some $\theta \in (0, \theta_0]$. For $j \in I_k$ we have

$$\mathrm{Ord}_{\theta\downarrow 0}\{T(\theta)-\tau_j(\theta;\Xi)\} = \begin{cases} p \quad \text{if } t_0+\tau_j(\theta;\Xi)\ge 0 \quad \text{for } \theta\in(0,\theta_0], \\ \min\{p,\ \mathrm{Ord}_{\theta\downarrow 0}\tau_j(\theta;\Xi)\} \\ \quad \text{if } t_0+\tau_j(\theta;\Xi)<0 \text{ for some } \theta\in(0,\theta_0]. \end{cases}$$

This yields

$$\mathrm{Ord}_{\theta\downarrow 0}\{T(\theta)-\tau_j(\theta;\Xi)\} = \mathrm{Ord}_{\theta\downarrow 0}\{\lambda_k(\theta)-\tau_j(\theta;\Xi)+\theta^p t\} \quad \text{for } t\in(0,c].$$

Therefore, we have

$$\begin{aligned} &\mathrm{Ord}_{\theta\downarrow 0}a(t_0+T(\theta),\Xi(\theta)) = \mathrm{Ord}_{\theta\downarrow 0}a(t_0+\lambda_k(\theta)+\theta^p t,\Xi(\theta)) \\ &= \min\{\nu+p\mu;\ (\nu,\mu)\in\Gamma_{0,j}(\Xi)\} \quad \text{for } t\in(0,c] \end{aligned}$$

if $j\in I_k$. It is obvious that (2.4) is valid in this case. Finally we see that (2.5) is valid with $l\in I_k$ for a generic $t\in(0,c]$, which proves the lemma. □

LEMMA 2.3. *Let* $(t_0,x^0,\xi^0)\in[0,\infty)\times \boldsymbol{R}^n\times S^{n-1}$ *satisfy* $a(t_0,\xi^0)=0$. *Then the conditions* (A-*a*)$_{(t_0,\xi^0)}$ *and* (A-*b*)$_{(t_0,x^0,\xi^0)}$ *are valid if the conditions* (H) *and* (F) *are satisfied and the* $a_{j,k}(t)$ *and* $b_j(t,x^0)$ $(1\le j\le n)$ *are semi-algebraic at* t_0.

PROOF. Assume (in addition to the conditions (H) and (F)) that the $a_{j,k}(t)$ and $b_j(t,x^0)$ $(1\le j\le n)$ are semi-algebraic at t_0. From Theorem 10 of [**11**] and its proof we can see that the $a_{j,k}(t)$ are real analytic at t_0 and that there are irreducible polynomials $P_{j,k}(z,t)$ $(\not\equiv 0)$ satisfying $P_{j,k}(a_{j,k}(t),t)=0$ in a neighborhood of t_0. Choose $\delta>0$ so that the $a_{j,k}(t)$ are continued analytically to $U_\delta\equiv\{t+is\in\boldsymbol{C};\ t,s\in\boldsymbol{R},\ |t-t_0|<\delta \text{ and } |s|<\delta\}$, and write

$$\begin{aligned} P_{j,k}(z,\omega) &= \alpha_0^{j,k}(\omega)z^{m(j,k)}+\alpha_1^{j,k}(\omega)z^{m(j,k)-1}+\cdots+\alpha_{m(j,k)}^{j,k}(\omega) \\ &= \alpha_0^{j,k}(\omega)\prod_{l=1}^{m(j,k)}(z-\lambda_l^{j,k}(\omega)), \end{aligned}$$

where the $\alpha_l^{j,k}(\omega)$ are polynomials of ω and $\alpha_0^{j,k}(\omega)\not\equiv 0$. We note that $a_{j,k}(\omega)\in\{\lambda_1^{j,k}(\omega),\cdots,\lambda_{m(j,k)}^{j,k}(\omega)\}$ if $\omega\in U_\delta$ and $\alpha_0^{j,k}(\omega)\ne 0$. Let us first prove that the $a_{j,k}(t+is)$ are semi-algebraic at $(t,s)=(t_0,0)$. Put

$$\begin{aligned} a_{j,k}^1(\omega) &= \mathrm{Re}\, a_{j,k}(\omega)\ (=(a_{j,k}(\omega)+\overline{a_{j,k}(\omega)})/2), \\ a_{j,k}^2(\omega) &= \mathrm{Im}\, a_{j,k}(\omega)\ (=(a_{j,k}(\omega)-\overline{a_{j,k}(\omega)})/(2i)) \end{aligned}$$

for $\omega \in U_\delta$, and $\widetilde{U}_\delta = \{(t,s) \in \boldsymbol{R}^2;\ t+is \in U_\delta\}$. For a polynomial $p(z,\omega) = \sum p_{\mu,\nu} z^\mu \omega^\nu$ we define $\bar{p}(z,\omega) = \sum \bar{p}_{\mu,\nu} z^\mu \omega^\nu$ $(= \overline{p(\bar{z},\bar{\omega})})$. Then it is obvious that $\overline{P}_{j,k}(\overline{a_{j,k}(t+is)}, t-is) = 0$ for $(t,s) \in \widetilde{U}_\delta$ and that

$$\overline{P}_{j,k}(z,\bar{\omega}) = \overline{\alpha_0^{j,k}(\omega)} \prod_{l=1}^{m(j,k)} (z - \overline{\lambda_l^{j,k}(\omega)}).$$

Put

$$\begin{aligned}\widetilde{P}^1_{j,k}(z,t,s) &= |\alpha_0^{j,k}(t+is)|^{2m(j,k)} \\ &\times \prod_{\mu,\nu=1}^{m(j,k)} \{z - (\lambda_\mu^{j,k}(t+is) + \overline{\lambda_\nu^{j,k}(t+is)})/2\}.\end{aligned}$$

$\widetilde{P}^1_{j,k}(z,t,s)$ is a polynomial of z, the $\alpha_l^{j,k}(t+is)$ and the $\overline{\alpha_l^{j,k}(t+is)}$ and, therefore, a polynomial of (z,t,s). Indeed, put

$$p(z;\alpha_0,\cdots,\alpha_m) = \alpha_0 z^m + \cdots + \alpha_m = \alpha_0 \prod_{j=1}^m (z - \lambda_j(\alpha_1/\alpha_0,\cdots,\alpha_m/\alpha_0)).$$

Then

$$\begin{aligned}&Q(z;\alpha_0,\cdots,\alpha_m) \\ &\equiv (\alpha_0\bar{\alpha}_0)^m \prod_{\mu,\nu=1}^m (2z - \lambda_\mu(\alpha_1/\alpha_0,\cdots,\alpha_m/\alpha_0) - \lambda_\nu(\bar{\alpha}_1/\bar{\alpha}_0,\cdots,\bar{\alpha}_m/\bar{\alpha}_0)) \\ &= \bar{\alpha}_0^m \prod_{\nu=1}^m p(2z - \lambda_\nu(\bar{\alpha}_1/\bar{\alpha}_0,\cdots,\bar{\alpha}_m/\bar{\alpha}_0);\alpha_0,\cdots,\alpha_m)\end{aligned}$$

is a polynomial of z, α_0, $\cdots$, α_m, $\bar{\alpha}_0$, $\bar{\alpha}_1/\bar{\alpha}_0$, $\cdots$, $\bar{\alpha}_m/\bar{\alpha}_0$. Similarly, $Q(z;\alpha_0,\cdots,\alpha_m)$ is a polynomial of z, $\bar{\alpha}_0$, $\cdots$, $\bar{\alpha}_m$, α_0, α_1/α_0, $\cdots$, α_m/α_0. This implies that $Q(z;\alpha_0,\cdots,\alpha_m)$ is a polynomial of z, α_0, $\cdots$, α_m, $\bar{\alpha}_0$, $\cdots$, $\bar{\alpha}_m$. So there is an irreducible polynomial $P^1_{j,k}(z,t,s)$ $(\not\equiv 0)$ satisfying $P^1_{j,k}(a^1_{j,k}(t+is),t,s) \equiv 0$. Similarly, there is an irreducible polynomial $P^2_{j,k}(z,t,s)$ $(\not\equiv 0)$ satisfying $P^2_{j,k}(a^2_{j,k}(t+is),t,s) \equiv 0$. Theorem 11 of [**11**] implies that the $a_{j,k}(t+is)$ are semi-algebraic at $(t,s) = (t_0,0)$. We define

$$a(\omega, \xi) = \sum_{j,k=1}^{n} a_{j,k}(\omega)\xi_j\xi_k \quad \text{for } \omega \in U_\delta \text{ and } \xi \in \boldsymbol{R}^n.$$

Then $a(t+is, \xi)$ is semi-algebraic at $(t, s, \xi) = (t_0, 0, \xi^0)$. By the proof of Lemma 2.1 and the Weierstrass preparation theorem there are an analytic function $e(\omega, \xi)$ defined in $U_\delta \times V_\delta$ $(\equiv U_\delta \times \{\xi \in \boldsymbol{R}^n;\ |\xi - \xi^0| < \delta\})$, $C_0 > 0$, $m \in \boldsymbol{N}$ and real analytic functions $a_j(\xi)$ $(1 \le j \le m)$ defined in V_δ such that the $a_j(\xi)$ are real-valued and

$$\begin{aligned}
&C_0^{-1} \le |e(\omega, \xi)| \le C_0, \\
&a(\omega, \xi) = e(\omega, \xi)((\omega - t_0)^m + a_1(\xi)(\omega - t_0)^{m-1} + \cdots + a_m(\xi))
\end{aligned}$$

in $U_\delta \times V_\delta$, with a modification of δ if necessary. Note that the $a_j(\xi)$ are uniquely determined. We define

$$\begin{aligned}
A = \{(\xi, a_1, a_2, \cdots, a_m) \in \boldsymbol{R}^n \times \boldsymbol{R}^m;\ &\xi \in V_\delta \text{ and for any } \omega \in U_\delta \\
&\text{there is } c \in \boldsymbol{C} \text{ satisfying } C_0^{-1} \le |c| \le C_0 \text{ and} \\
&a(\omega, \xi) = c((\omega - t_0)^m + a_1(\omega - t_0)^{m-1} + \cdots + a_m)\}.
\end{aligned}$$

The Tarski-Seidenberg theorem implies that A is a semi-algebraic set in $\boldsymbol{R}^{n+m}$. Choose $\delta' > 0$ so that $\delta' \le \delta$ and

$$(\omega - t_0)^m + a_1(\xi)(\omega - t_0)^{m-1} + \cdots + a_m(\xi) \ne 0 \quad \text{if } \xi \in V_{\delta'} \text{ and } \omega \in \boldsymbol{C} \setminus U_\delta.$$

Since $a(\omega, \xi)/((\omega - t_0)^m + a_1(\omega - t_0)^{m-1} + \cdots + a_m)$ can be regarded as an analytic function of ω in U_δ for $(\xi, a_1, \cdots, a_m) \in A$ with $\xi \in V_{\delta'}$, we have $a_j = a_j(\xi)$ $(1 \le j \le m)$, *i.e.*,

$$A \cap (V_{\delta'} \times \boldsymbol{R}^m) = \{(\xi, a_1(\xi), \cdots, a_m(\xi)) \in \boldsymbol{R}^n \times \boldsymbol{R}^m;\ \xi \in V_{\delta'}\}.$$

So the $a_j(\xi)$ are semi-algebraic at ξ^0, which proves the lemma. □

3. Proof of Theorem 1.2.

In this section we assume that the hypotheses of Theorem 1.2 are fulfilled, and we shall prove Theorem 1.2. Put

$$P_\varepsilon(t,\tau,\xi) = P(t,\tau,\xi) - \varepsilon|\xi|^2$$
$$(= \tau^2 - a(t,\xi) - \varepsilon|\xi|^2 + b_0(t)\tau + b(t,\xi) + c(t))$$

for $\varepsilon \in [0,1]$. We note that $P_\varepsilon(t,D_t,D_x)$ is strictly hyperbolic if $\varepsilon > 0$. Consider the Cauchy problem

$$\begin{cases} P_\varepsilon(t,D_t,D_x)u_\varepsilon(t,x) = f(t,x) \quad \text{in } [0,\infty)\times \boldsymbol{R}^n, \\ D_t^j u_\varepsilon(t,x)|_{t=0} = u_j(x) \quad \text{in } \boldsymbol{R}^n \ (j=0,1), \end{cases} \tag{CP}_\varepsilon$$

where $f \in C^\infty([0,\infty); H^\infty(\boldsymbol{R}^n_x))$ and $u_j \in H^\infty(\boldsymbol{R}^n)$ $(j=0,1)$. Here $H^s(\boldsymbol{R}^n)$ denotes the Sobolev space over $\boldsymbol{R}^n$ of order s and $H^\infty(\boldsymbol{R}^n) = \bigcap_{s\in\boldsymbol{R}} H^s(\boldsymbol{R}^n)$. By partial Fourier transformation in x, the Cauchy problem $(\mathrm{CP})_\varepsilon$ is reduced to the Cauchy problem for an ordinary differential operator with parameters ξ:

$$\begin{cases} P_\varepsilon(t,D_t,\xi)v_\varepsilon(t,\xi) = \hat{f}(t,\xi) \quad \text{for } (t,\xi)\in [0,\infty)\times \boldsymbol{R}^n, \\ D_t^j v_\varepsilon(t,\xi)|_{t=0} = \hat{u}_j(x) \quad \text{for } \xi \in \boldsymbol{R}^n \ (j=0,1), \end{cases} \tag{3.1}$$

where $\hat{f}(t,\xi)$ and $\hat{u}_j(\xi)$ $(j=0,1)$ denote the partial Fourier transforms of $f(t,x)$ and $u_j(x)$ with respect to x, respectively, for example, $\hat{f}(t,\xi) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} f(t,x)\,dx$. We note that the Cauchy problem (3.1) has a unique solution $v_\varepsilon(t,\xi) \in C^\infty([0,\infty); C^\infty(\boldsymbol{R}^n_\xi))$. Let $t_0 = 0$, and let U, N, the Γ_j, the $\mathscr{R}_j(\xi)$ and so forth be as in the conditions (A) and (L). We may assume that $U = [0,\delta]$ with some $\delta > 0$. Let $1 \le j \le N$, and put

$$\Phi_j(t,\xi) = t + \sum_{\tau\in\mathscr{R}_j(\xi)} \log(\sqrt{(t-\tau)^2\langle\xi\rangle + 1} + (t-\tau)\langle\xi\rangle^{1/2})$$

for $(t,\xi) \in [0,\delta]\times\Gamma_j$ if the equation $q_j(t,\eta) = 0$ in t does not have simple real roots for any $\eta \in \Gamma_j$, and

$$\begin{aligned} \Phi_j(t,\xi) =& t + \sum_{\tau\in\mathscr{R}_j(\xi)} \log(\sqrt{(t-\tau)^2\langle\xi\rangle + 1} + (t-\tau)\langle\xi\rangle^{1/2}) \\ &+ \log(\sqrt{t^2\langle\xi\rangle^{4/3} + 1} + t\langle\xi\rangle^{2/3}) \end{aligned} \tag{3.2}$$

for $(t,\xi) \in [0,\delta]\times\Gamma_j$ if the equation $q_j(t,\eta) = 0$ in t has a simple real root for some $\eta \in \Gamma_j$. We also put

$$W_j(t,\xi) = \partial_t \Phi_j(t,\xi) \quad \text{for } (t,\xi) \in [0,\delta] \times \Gamma_j.$$

We note that the $\Phi_j(t,\xi)$ and the $W_j(t,\xi)$ are measurable, and that

$$\partial_t \log(\sqrt{\lambda(t,\xi)^2+1} + \lambda(t,\xi)) = \partial_t \lambda(t,\xi)/\sqrt{\lambda(t,\xi)^2+1}.$$

For simplicity we define $\Phi_j(t,\xi)$ by (3.2) even if the equation $q_j(t,\eta)=0$ in t does not have simple real roots for any $\eta \in \Gamma_j$. We define, for $(t,\xi) \in [0,\delta] \times \Gamma_j$, $\varepsilon \in [0,1]$ and $\gamma > 0$,

$$\begin{aligned} \mathscr{E}_{\varepsilon,j}(t,\xi;\gamma) &= E_{\varepsilon,j}(t,\xi) \exp[-\gamma \Phi_j(t,\xi)], \\ E_{\varepsilon,j}(t,\xi) &= |\partial_t v_\varepsilon(t,\xi)|^2 + (a(t,\xi) + \varepsilon|\xi|^2 + W_j(t,\xi)^2)|v_\varepsilon(t,\xi)|^2. \end{aligned} \tag{3.3}$$

Let $(t,\xi) \in [0,\delta] \times \Gamma_j$ and $\varepsilon \in [0,1]$. A simple calculation yields

$$\begin{aligned} \partial_t \mathscr{E}_{\varepsilon,j}(t,\xi;\gamma) &= \{\partial_t E_{\varepsilon,j}(t,\xi) - \gamma W_j(t,\xi) E_{\varepsilon,j}(t,\xi)\} \exp[-\gamma \Phi_j(t,\xi)], \\ \partial_t E_{\varepsilon,j}(t,\xi) &= 2\,\mathrm{Re}\{(-\hat{f}(t,\xi) + (b(t,\xi) + c(t) + W_j(t,\xi)^2) v_\varepsilon(t,\xi)) \partial_t \overline{v_\varepsilon(t,\xi)}\} \\ &\quad + 2\,\mathrm{Im}\, b_0(t) \cdot |\partial_t v_\varepsilon(t,\xi)|^2 + (\partial_t a(t,\xi) + 2\partial_t W_j(t,\xi) \cdot W_j(t,\xi))|v_\varepsilon(t,\xi)|^2. \end{aligned}$$

Noting that $\partial_t W_j(t,\xi) \le W_j(t,\xi)^2$, we have

$$\begin{aligned} &\partial_t \mathscr{E}_{\varepsilon,j}(t,\xi;\gamma) \\ &\quad \le \big[|\hat{f}(t,\xi)|^2/W_j(t,\xi) - \{\gamma - 3 - (|c(t)| + 2\,\mathrm{Im}\, b_0(t))/W_j(t,\xi)\} \\ &\qquad\qquad \times W_j(t,\xi)|\partial_t v_\varepsilon(t,\xi)|^2 \\ &\qquad - (I_j(t,\xi;\gamma) + \gamma\varepsilon|\xi|^2 W_j(t,\xi)^2)|v_\varepsilon(t,\xi)|^2/W_j(t,\xi)\big] \exp[-\gamma \Phi_j(t,\xi)], \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} I_j(t,\xi;\gamma) =& \gamma a(t,\xi) W_j(t,\xi)^2 + (\gamma - 3) W_j(t,\xi)^4 - |b(t,\xi)|^2 \\ & - |c(t)| W_j(t,\xi) - \partial_t a(t,\xi) \cdot W_j(t,\xi). \end{aligned} \tag{3.5}$$

Choose $\gamma > 0$ so that

$$\gamma - 3 - |c(t)| - 2\,\mathrm{Im}\, b_0(t) \ge 0 \quad (\ t \in [0,\delta]). \tag{3.6}$$

First consider the case $t\langle\xi\rangle^{2/3} \le 1$. Then we have, with some $C > 0$,

$$\langle\xi\rangle^{2/3}/\sqrt{2} \le W_j(t,\xi) \le C\langle\xi\rangle^{2/3}$$

for $(t,\xi) \in [0,\delta] \times \Gamma_j$. Therefore, we have $I_j(t,\xi;\gamma) \ge 0$ if

$$\gamma \ge 3 + 2\sqrt{2} \sup_{t\in[0,\delta]} \{(\partial_t a(t,\xi) + |b(t,\xi)|^2)/|\xi|^2 + |c(t)|\}. \tag{3.7}$$

We choose $\gamma > 0$ so that (3.7) is valid. For each $\xi \in \Gamma_j$ there are $r_j(\xi), r_{0,j}(\xi) \in \boldsymbol{Z}_+$, $\nu_{j,l}(\xi) \in \boldsymbol{N}$ $(1 \le l \le r_{0,j}(\xi))$, $\tau_{j,k}(\xi) \in \boldsymbol{R}$ and $\sigma_{j,k}(\xi) \ge 0$ $(1 \le k \le r_j(\xi))$ and $\tau_{0,j,l}(\xi) \le 0$ $(1 \le l \le r_{0,j}(\xi))$ such that the $\nu_{j,l}(\xi)$ are odd and

$$a(t,\xi) = e_j(t,\xi) \prod_{k=1}^{r_j(\xi)} \{(t-\tau_{j,k}(\xi))^2 + \sigma_{j,k}(\xi)\} \prod_{l=1}^{r_{0,j}(\xi)} (t-\tau_{0,j,l}(\xi))^{\nu_{j,l}(\xi)} \tag{3.8}$$

for $t \in [0,\delta]$. We note that

$$m_j = 2r_j(\xi) + \sum_{l=1}^{r_{0,j}(\xi)} \nu_{j,l}(\xi), \tag{3.9}$$

$$\mathscr{R}_j(\xi) = \{(\tau_{j,k}(\xi))_+;\ 1 \le k \le r_j(\xi)\} \cup \{(\tau_{0,j,l}(\xi))_+;\ 1 \le l \le r_{0,j}(\xi)\},$$

$$|t-\tau_{j,k}(\xi)| \le \sqrt{(t-\tau_{j,k}(\xi))^2 + \sigma_{j,k}(\xi)}, \tag{3.10}$$

$$|t-\tau_{0,j,l}(\xi)|^{-1} \le t^{-1}, \tag{3.11}$$

$$|t-\tau_{0,j,l}(\xi)|^{\nu_{j,l}(\xi)-1} \le C|t-\tau_{0,j,l}(\xi)|^{\nu_{j,l}(\xi)/2} \quad \text{if } \nu_{j,l}(\xi) > 1 \tag{3.12}$$

for $(t,\xi) \in (0,\delta] \times \Gamma_j$, where $C > 0$. Next consider the case where $t\langle\xi\rangle^{2/3} \ge 1$ and $\min_{\tau\in\mathscr{R}_j(\xi)} |t-\tau| \le \langle\xi\rangle^{-1/2}$. Then we have

$$W_j(t,\xi) \ge (\langle\xi\rangle^{1/2} + t^{-1})/\sqrt{2}. \tag{3.13}$$

It follows from the condition (A) and (3.8) – (3.13) that there are positive constants C and C' such that

$$\begin{aligned}
\partial_t a(t,\xi) &\le C(a(t,\xi) + \sqrt{a(t,\xi)}|\xi| + t^{-1}a(t,\xi)) \\
&\le C'(a(t,\xi)W_j(t,\xi) + |\xi|^2/(4W_j(t,\xi))) \\
&\le C'(a(t,\xi)W_j(t,\xi) + W_j(t,\xi)^3)
\end{aligned}$$

for $(t,\xi) \in [0,\delta] \times \Gamma_j$. This, together with (3.5) and (3.13), gives $I_j(t,\xi;\gamma) \geq 0$ if

$$\gamma \geq 3 + C' + \sup_{t\in[0,\delta]} \{4|b(t,\xi)|^2/|\xi|^2 + |c(t)|\}. \tag{3.14}$$

Let us consider the case where $t\langle\xi\rangle^{2/3} \geq 1$ and $\min_{\tau\in\mathscr{R}_j(\xi)} |t-\tau| \geq \langle\xi\rangle^{-1/2}$. Then by (3.8) and (3.9) we have, with some positive constants C_1 and C_2,

$$W_j(t,\xi) \geq (\sqrt{2}\min_{\tau\in\mathscr{R}_j(\xi)} |t-\tau|)^{-1} + (\sqrt{2}t)^{-1}, \tag{3.15}$$

$$\partial_t a(t,\xi) \leq C_1 a(t,\xi) + \sqrt{2} m_j W_j(t,\xi) a(t,\xi) \leq C_2 W_j(t,\xi) a(t,\xi) \tag{3.16}$$

for $(t,\xi) \in [0,\delta] \times \Gamma_j$, since $|t-\tau_{j,k}(\xi)|/\{(t-\tau_{j,k}(\xi))^2 + \sigma_{j,k}(\xi)\} \leq |t-\tau_{j,k}(\xi)|^{-1} \leq (t-(\tau_{j,k}(\xi))_+)^{-1}$ and $0 \leq (t-\tau_{0,j,l}(\xi))^{-1} \leq t^{-1}$. It follows from the condition (L), (3.5), (3.15) and (3.16) that $I_j(t,\xi;\gamma) \geq 0$ if

$$\gamma \geq \max\{C_2 + 2C^2, 3 + \sup_{t\in[0,\delta]} |c(t)|\}, \tag{3.17}$$

where C is the constant in the condition (L). Choose $\gamma > 0$ so that (3.6), (3.7), (3.14) and (3.17) are valid. Then we have $I_j(t,\xi;\gamma) \geq 0$. Moreover, by (3.4) we have

$$\partial_t \mathscr{E}_{\varepsilon,j}(t,\xi;\gamma) \leq |\hat{f}(t,\xi)|^2 \exp[-\gamma\Phi_j(t,\xi)]/W_j(t,\xi)$$

for $(t,\xi) \in [0,\delta] \times \Gamma_j$ and $\varepsilon \in [0,1]$. This gives

$$\mathscr{E}_{\varepsilon,j}(t,\xi;\gamma) \leq \mathscr{E}_{\varepsilon,j}(0,\xi;\gamma) + \int_0^t \exp[-\gamma\Phi_j(s,\xi)]|\hat{f}(s,\xi)|^2/W_j(s,\xi)\,ds \tag{3.18}$$

for $(t,\xi) \in [0,\delta] \times \Gamma_j$ and $\varepsilon \in [0,1]$. By definition there is $\widehat{C} > 0$ such that

$$-(m/2)\log\langle\xi\rangle - \widehat{C} \leq \Phi_j(t,\xi) \leq (m/2 + 2/3)\log\langle\xi\rangle + \widehat{C}$$

for $1 \leq j \leq N$ and $(t,\xi) \in [0,\delta] \times \Gamma_j$, where $m = \max_{1\leq j\leq N} m_j$. This gives

$$e^{-\gamma\widehat{C}}\langle\xi\rangle^{-\kappa} \leq \exp[-\gamma\Phi_j(t,\xi)] \leq e^{\gamma\widehat{C}}\langle\xi\rangle^{\kappa} \tag{3.19}$$

for $1 \leq j \leq N$ and $(t,\xi) \in [0,\delta] \times \Gamma_j$, where $\kappa = \gamma(m/2 + 2/3)$. Therefore, from

(3.3), (3.18) and (3.19) there is $C > 0$ such that

$$|v_\varepsilon(t,\xi)|^2 + |\partial_t v_\varepsilon(t,\xi)|^2 \\ \le C\Big\{\langle\xi\rangle^{2\kappa+2}(|\hat{u}_0(\xi)|^2 + |\hat{u}_1(\xi)|^2) + \int_0^t \langle\xi\rangle^{2\kappa}|\hat{f}(s,\xi)|^2\,ds\Big\} \tag{3.20}$$

for $1 \le j \le N$ and $(t,\xi) \in [0,\delta] \times \Gamma_j$. Put

$$A(t,\xi) = \begin{pmatrix} 0 & \langle\xi\rangle \\ \langle\xi\rangle^{-1}(a(t,\xi) + \varepsilon|\xi|^2 - b(t,\xi) - c(t)) & -b_0(t) \end{pmatrix}.$$

Then $v_\varepsilon(t,\xi)$ satisfies

$$D_t\begin{pmatrix} \langle\xi\rangle v_\varepsilon(t,\xi) \\ D_t v_\varepsilon(t,\xi) \end{pmatrix} = A(t,\xi)\begin{pmatrix} \langle\xi\rangle v_\varepsilon(t,\xi) \\ D_t v_\varepsilon(t,\xi) \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{f}(t,\xi) \end{pmatrix}$$

and, therefore,

$$D_t^{k+1}\begin{pmatrix} \langle\xi\rangle v_\varepsilon(t,\xi) \\ D_t v_\varepsilon(t,\xi) \end{pmatrix} = \sum_{\mu=0}^{k}\binom{k}{\mu} D_t^{k-\mu}A(t,\xi)\cdot D_t^{\mu}\begin{pmatrix} \langle\xi\rangle v_\varepsilon(t,\xi) \\ D_t v_\varepsilon(t,\xi) \end{pmatrix} \\ + \begin{pmatrix} 0 \\ D_t^k\hat{f}(t,\xi) \end{pmatrix} \quad (\ k = 0,1,2,\cdots). \tag{3.21}$$

Put

$$u_\varepsilon(t,x) = \mathscr{F}_\xi^{-1}[v_\varepsilon(t,\xi)](x) \quad \text{for } (t,x) \in [0,\delta]\times \boldsymbol{R}^n \text{ and } \varepsilon \in [0,1],$$
$$E_{k,l}[u](t) = \sum_{\mu=0}^{k} \|\langle D_x\rangle^{l+k-\mu} D_t^{\mu} u(t,x)\|_{L^2}^2,$$

where $k \in \boldsymbol{Z}_+$, $l \in \boldsymbol{R}$, $u(t,x) \in C^\infty([0,\delta]; H^\infty(\boldsymbol{R}_x^n))$, $\|u(t,x)\|_{L^2} = (\int_{\boldsymbol{R}^n} |u(t,x)|^2\, dx)^{1/2}$ and $\mathscr{F}_\xi^{-1}$ denotes the inverse Fourier transformation with respect to ξ. It follows from (3.20) and Plancherel's theorem that $u_\varepsilon(t,x)$ and $E_{k,l}[u_\varepsilon](t)$ ($k = 0,1$, $l \in \boldsymbol{R}$) are well-defined and that

$$E_{k,l}[u_\varepsilon](t) \le C\Big\{ \sum_{\nu=0}^{1} \|\langle D_x\rangle^{k+l+\kappa+1} u_\nu(x)\|_{L^2}^2 + \int_0^t \|\langle D_x\rangle^{k+l+\kappa} f(s,x)\|_{L^2}^2\,ds \Big\} \tag{3.22}$$

for $k = 0, 1$, $l \in \boldsymbol{R}$, $t \in [0, \delta]$ and $\varepsilon \in [0, 1]$. Moreover, $u_\varepsilon(t, x)$ satisfies $(\mathrm{CP})_\varepsilon$, with $[0, \infty) \times \boldsymbol{R}^n$ replaced by $[0, \delta] \times \boldsymbol{R}^n$, for $\varepsilon \in [0, 1]$.

LEMMA 3.1. *For $\varepsilon \in [0,1]$ $u_\varepsilon(t,x) \in C^\infty([0,\delta]; H^\infty(\boldsymbol{R}_x^n))$. Moreover, for any $k \in \boldsymbol{Z}_+$ and $l \in \boldsymbol{R}$ there is $C_{k,l} > 0$ such that*

$$\max_{0\le t\le\delta} E_{k,l}[u_\varepsilon](t) \le C_{k,l}\Big\{ \sum_{\nu=0}^{1} \|\langle D_x\rangle^{k+l+\kappa+1} u_\nu(x)\|_{L^2}^2 + \max_{0\le t\le\delta} \|\langle D_x\rangle^{k+l+\kappa} f(t,x)\|_{L^2}^2 + \max_{0\le t\le\delta} \sum_{\mu=0}^{k-2} \|\langle D_x\rangle^{k+l-\mu-2} D_t^\mu f(t,x)\|_{L^2}^2 \Big\} \tag{3.23}$$

for $\varepsilon \in [0,1]$, where $\sum_{\mu=0}^{k-2} \cdots = 0$ if $k < 2$.

PROOF. By (3.22) it is obvious that (3.23) is valid for $k = 0, 1$. Now suppose that $u_\varepsilon(t,x) \in C^{K-1}([0,\delta]; H^\infty(\boldsymbol{R}_x^n))$ and (3.23) is valid if $k \le K-1$, where $K \in \boldsymbol{N}$ and $K \ge 2$. Then it follows from (3.21) with $k = K-2$ that $\langle\xi\rangle^l v_\varepsilon(t,\xi) \in C^K([0,\delta]; L^2(\boldsymbol{R}_\xi^n))$, *i.e.*, $u_\varepsilon(t,x) \in C^K([0,\delta]; H^\infty(\boldsymbol{R}_x^n))$. Note that $E_{K,l}[u_\varepsilon](t) = E_{K-1,l+1}[u_\varepsilon](t) + \|\langle D_x\rangle^l D_t^K u_\varepsilon(t,x)\|_{L^2}^2$. (3.21), with $k = K-2$, yields

$$\begin{aligned} &\|\langle D_x\rangle^l D_t^K u_\varepsilon(t,x)\|_{L^2} \\ &\le C_K \sum_{\mu=0}^{K-2} \Big\{ \|\langle D_x\rangle^{l+2} D_t^\mu u_\varepsilon(t,x)\|_{L^2} + \|\langle D_x\rangle^l D_t^{\mu+1} u_\varepsilon(t,x)\|_{L^2} \Big\} \\ &\quad + \|\langle D_x\rangle^l D_t^{K-2} f(t,x)\|_{L^2} \end{aligned}$$

for $t \in [0,\delta]$ and $\varepsilon \in [0,1]$, where $C_K > 0$. Therefore, (3.23) is valid for $k = K$, since $\langle\xi\rangle^{l+2} \le \langle\xi\rangle^{l+1+K-1-\mu}$ and $\langle\xi\rangle^l \le \langle\xi\rangle^{l+1+K-1-(\mu+1)}$ for $0 \le \mu \le K-2$ and

$$\begin{aligned} &\|\langle D_x\rangle^l D_t^K u_\varepsilon(t,x)\|_{L^2}^2 \\ &\le C_K' \{ E_{K-1,l+1}[u_\varepsilon](t) + \|\langle D_x\rangle^l D_t^{K-2} f(t,x)\|_{L^2}^2 \}. \end{aligned}$$

□

Put $u(t,x) = u_0(t,x)$ for $[0,\delta] \times \boldsymbol{R}^n$. Since

$$\begin{cases} P_\varepsilon(t,D_t,D_x)(u_\varepsilon(t,x) - u(t,x)) = -\varepsilon\Delta_x u(t,x), \\ D_t^j(u_\varepsilon(t,x) - u(t,x))|_{t=0} = 0 \quad (j = 0,1) \end{cases}$$

and $(u_\varepsilon(t,x) - u(t,x)), \Delta_x u(t,x) \in C^\infty([0,\delta]; H^\infty(\boldsymbol{R}_x^n))$, it follows from uniqueness theorem for ordinary differential equations and Lemma 3.1 that

$$\begin{aligned} &\max_{0\le t\le\delta} E_{k,l}[u_\varepsilon - u](t) \\ &\le C_{k,l}\varepsilon^2 \Big\{ \max_{0\le t\le\delta} E_{0,k+l+\kappa+2}[u](t) + \max_{0\le t\le\delta} E_{k-2,l+2}[u](t) \Big\} \\ &\le C'_{k,l}\varepsilon^2 \Big\{ \sum_{\nu=0}^{1} \|\langle D_x\rangle^{k+l+2\kappa+3} u_\nu(x)\|_{L^2}^2 + \max_{0\le t\le\delta} \|\langle D_x\rangle^{k+l+2\kappa+2} f(t,x)\|_{L^2}^2 \\ &\qquad + \max_{0\le t\le\delta} \sum_{\mu=0}^{k-4} \|\langle D_x\rangle^{k+l-\mu-2} D_t^\mu f(t,x)\|_{L^2}^2 \Big\} \end{aligned}$$

for $k \in \boldsymbol{Z}_+$, $l \in \boldsymbol{R}$ and $\varepsilon \in [0,1]$, where $E_{\mu,l}[u](t) \equiv 0$ if $\mu < 0$. This implies that for any $k \in \boldsymbol{Z}_+$ and $l \in \boldsymbol{R}$

$$D_t^k D_x^\alpha u_\varepsilon(t,x) \to D_t^k D_x^\alpha u(t,x) \quad \text{uniformly on } [0,\delta] \times \boldsymbol{R}^n \text{ as } \varepsilon \downarrow 0.$$

Denote by $K^\pm_{\varepsilon,(t_1,x^1)}$ the generalized flows for $p_\varepsilon(t,\tau,\xi) \equiv \tau^2 - a(t,\xi) - \varepsilon|\xi|^2$. It is easy to see that

$$\Gamma(p_{\varepsilon_1}(t,\cdot,\cdot),\vartheta) \supset \Gamma(p_{\varepsilon_2}(t,\cdot,\cdot),\vartheta) \quad \text{for } t \ge 0 \text{ and } 0 \le \varepsilon_1 \le \varepsilon_2 \le 1$$

and that for any $t \ge 0$ and any open conic set Γ with $\overline{\Gamma} \subset \Gamma(p(t,\cdot,\cdot),\vartheta) \cup \{0\}$ there is $\varepsilon_0 \in (0,1]$ satisfying

$$\Gamma \subset \Gamma(p_\varepsilon(t,\cdot,\cdot),\vartheta) \quad \text{for } \varepsilon \in [0,\varepsilon_0]$$

So, for any $(t_1,x^1) \in (0,\infty) \times \boldsymbol{R}^n$ and any neighborhood V of $K^-_{(t_1,x^1)} \cap \{t \ge 0\}$ there is $\varepsilon_0 \in (0,1]$ satisfying

$$K^-_{\varepsilon,(t_1,x^1)} \cap \{t \ge 0\} \subset V \quad \text{for } \varepsilon \in [0,\varepsilon_0] \tag{3.24}$$

(see, *e.g.*, Section 3 of [**14**] and [**13**]). Since $P_\varepsilon(t, D_t, D_x)$ is strictly hyperbolic for $\varepsilon \in (0,1]$, we can show that $(t_1, x^1) \notin \operatorname{supp} w$ if $\varepsilon \in (0,1]$, $(t_1, x^1) \in (0,\infty) \times \boldsymbol{R}^n$, $w(t,x) \in C^\infty([0,\infty) \times \boldsymbol{R}^n)$, $\operatorname{supp} P_\varepsilon(t, D_t, D_x)w(t,x) \cap K^-_{\varepsilon,(t_1,x^1)} \cap \{t \geq 0\} = \emptyset$ and $\{0\} \times (\operatorname{supp} w(0,x) \cup \operatorname{supp} (D_t w)(0,x)) \cap K^-_{\varepsilon,(t_1,x^1)} = \emptyset$ (see, *e.g.*, [**6**]). Let $\varphi(t,x) \in C_0^\infty(\boldsymbol{R}^{n+1})$ satisfy $\varphi(t,x) = 1$ $(|(t,x)| \leq R)$ and $\varphi(t,x) = 0$ $(|(t,x)| \geq R+1)$, where $R \gg 1$. Assume that $\tilde{u}(t,x) \in C^\infty([0,\infty) \times \boldsymbol{R}^n)$ satisfies

$$\begin{cases} P(t, D_t, D_x)\tilde{u}(t,x) = f(t,x) & \text{in } [0,\delta] \times \boldsymbol{R}^n, \\ D_t^j \tilde{u}(t,x)|_{t=0} = u_j(x) & \text{in } \boldsymbol{R}^n \ (j = 0,1). \end{cases} \tag{3.25}$$

Put $g(t,x) = P(t, D_t, D_x)(\varphi(t,x)\tilde{u}(t,x))$ $(\in C^\infty([0,\delta] \times \boldsymbol{R}^n))$. Since $\varphi\tilde{u}, g \in C^\infty([0,\infty); H^\infty(\boldsymbol{R}^n))$ and $w_0(x) \equiv \varphi(0,x)u_0(x)$, $w_1(x) \equiv (D_t\varphi)(0,x)u_0(x) + \varphi(0,x)u_1(x) \in H^\infty(\boldsymbol{R}^n)$, it follows from uniqueness theorem for ordinary differential equations that the Cauchy problem

$$\begin{cases} P_\varepsilon(t, D_t, D_x)w_\varepsilon(t,x) = g(t,x) & \text{in } [0,\delta] \times \boldsymbol{R}^n, \\ D_t^j w_\varepsilon(t,x)|_{t=0} = w_j(x) & \text{in } \boldsymbol{R}^n \ (j = 0,1) \end{cases}$$

has a unique solution $w_\varepsilon(t,x) \in C^\infty([0,\infty); H^\infty(\boldsymbol{R}^n))$ for $\varepsilon \in [0,1]$, and that $w_0(t,x) = \varphi(t,x)\tilde{u}(t,x)$. Let $(t_1, x^1) \in [0,\delta] \times \boldsymbol{R}^n$, and assume that $\operatorname{supp} f \cap K^-_{(t_1,x^1)} \cap \{t \geq 0\} = \emptyset$ and $\{0\} \times (\operatorname{supp} u_0 \cup \operatorname{supp} u_1) \cap K^-_{(t_1,x^1)} = \emptyset$. Then, taking $R \gg 1$ we have $\operatorname{supp} g \cap K^-_{(t_1,x^1)} \cap \{t \geq 0\} = \emptyset$. Therefore, by (3.24) there is $\varepsilon_0 \in (0,1]$ such that $(t_1, x^1) \notin \operatorname{supp} w_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0]$. Since $w_\varepsilon \to \varphi\tilde{u}$ uniformly on $[0,\delta] \times \boldsymbol{R}^n$ as $\varepsilon \downarrow 0$, We have $(t_1, x^1) \notin \operatorname{supp} \tilde{u}$. In particular, this proves uniqueness of the Cauchy problem (3.25) in $C^\infty([0,\delta] \times \boldsymbol{R}^n)$. Next consider the Cauchy problem

$$\begin{cases} P(t, D_t, D_x)u(t,x) = f(t,x) & \text{in } [\delta,\infty) \times \boldsymbol{R}^n, \\ D_t^j u(t,x)|_{t=\delta} = u_j(x) & \text{in } \boldsymbol{R}^n \ (j = 0,1), \end{cases}$$

and repeat the arguments. Finally we can prove Theorem 1.2, using finite propagation property.

4. Proof of Theorem 1.3.

In this section we assume that the hypotheses of Theorem 1.3 are fulfilled, and we shall prove Theorem 1.3. Define $\mu_0, \mu_1, \delta \in \boldsymbol{Q}$ by

$$\begin{aligned}\mu_0 &= \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a(t_0+T(\theta),\Xi(\theta))},\\ \mu_1 &= \mathrm{Ord}_{\theta\downarrow 0}\Big\{\min_{1\le j\le m}|t_0+T(\theta)-(t_0+\tau_j(\theta;\Xi))_+|\cdot|b(t_0+T(\theta),x^0,\Xi(\theta)|\Big\},\\ \delta &= \mathrm{Ord}_{\theta\downarrow 0}\min_{1\le j\le m}|t_0+T(\theta)-(t_0+\tau_j(\theta;\Xi))_+|\ (\ >0).\end{aligned}$$

The condition $(\mathrm{C})_{(t_0,x^0,\xi^0)}$ implies that $\mu_1<\mu_0$. Write

$$b(t_0+T(\theta)+v\theta^\delta,x,\Xi(\theta))=\theta^{\mu-\delta}(\hat{c}(v,x)+o(1))\quad \text{as } \theta\downarrow 0, \tag{4.1}$$

where $\mu\in\boldsymbol{Q}$, $\hat{c}(v,x)\not\equiv 0$ in (v,x). Then $\hat{c}(v,x)$ is a polynomial of v and $\mu\le\mu_1$. If $\hat{c}(v,x^0)\equiv 0$ in v, we replace $x^0\in\boldsymbol{R}^n$ so that $\hat{c}(v,x^0)\not\equiv 0$ in v. Let $c_0>0$ be a constant satisfying

$$\min_{1\le j\le m}|t_0+T(\theta)-(t_0+\tau_j(\theta;\Xi))_+|\ge c_0\theta^\delta \quad \text{for } \theta\in[0,\theta_0]. \tag{4.2}$$

If $\hat{c}(0,x^0)=0$, we replace $T(\theta)$ and μ_1 by $T(\theta)+v_0\theta^\delta$ and μ, respectively, choosing $v_0\in(0,c_0/2]$ so that $\hat{c}(v_0,x^0)\neq 0$. In fact, noting that

$$|T(\theta)-\tau_j(\theta;\Xi)|/2\le|T(\theta)+v_0\theta^\delta-\tau_j(\theta;\Xi)|\le 3|T(\theta)-\tau_j(\theta;\Xi)|/2$$

for $\theta\in[0,\theta_0]$, we have

$$\begin{aligned}\mu-\delta &= \mathrm{Ord}_{\theta\downarrow 0}b(t_0+T(\theta)+v_0\theta^\delta,x^0,\Xi(\theta))\\ &= \min_{x\in\boldsymbol{R}^n,\,v\in\boldsymbol{R}}\mathrm{Ord}_{\theta\downarrow 0}b(t_0+T(\theta)+v\theta^\delta,x,\Xi(\theta)),\\ \mu_0 &= \mathrm{Ord}_{\theta\downarrow 0}\sqrt{a(t_0+T(\theta)+v_0\theta^\delta,\Xi(\theta))}.\end{aligned}$$

Therefore, we may assume that $\hat{c}\equiv\hat{c}(0,x^0)\neq 0$ and $\mu=\mu_1$ in (4.1) and

$$\begin{aligned}\mu_1-\delta &= \mathrm{Ord}_{\theta\downarrow 0}b(t_0+T(\theta),x^0,\Xi(\theta))\\ &= \min_{x\in\boldsymbol{R}^n,\,v\in\boldsymbol{R}}\mathrm{Ord}_{\theta\downarrow 0}b(t_0+T(\theta)+v\theta^\delta,x,\Xi(\theta)).\end{aligned}$$

Let κ and δ' be positive rational constants satisfying $\delta'\kappa<1$, and choose $\varepsilon=\pm 1$ so that $\varepsilon\hat{c}\notin(-\infty,0]$. We shall impose further conditions on κ and δ'. Note that

$$\begin{aligned}&\exp[-i\varepsilon\rho x\cdot\Xi(\rho^{-\kappa})]P(t,x,D_t,D_x)\{\exp[i\varepsilon\rho x\cdot\Xi(\rho^{-\kappa})]u(t,x)\}\\ &=P(t,x,D_t,\varepsilon\rho\Xi(\rho^{-\kappa})+D_x)u(t,x),\end{aligned}$$

where $\rho \gg 1$. We make an asymptotic change of variables:

$$t = t(s;\rho) \equiv t_0 + T(\rho^{-\kappa}) + \rho^{-\delta\kappa}s,$$
$$x = x(y;\rho) \equiv x^0 + \rho^{\delta'\kappa-1}y.$$

Put

$$P_\rho(s,y,\sigma,\eta) = P(t(s;\rho), x(y;\rho), \rho^{\delta\kappa}\sigma, \varepsilon\rho\Xi(\rho^{-\kappa}) + \rho^{1-\delta'\kappa}\eta).$$

Then we have

$$\begin{aligned} P_\rho(s,y,\sigma,\eta) &= \rho^{2\delta\kappa}\sigma^2 - \rho^2 a(t(s;\rho), \Xi(\rho^{-\kappa}) + \varepsilon\rho^{-\delta'\kappa}\eta) \\ &+ \rho^{\delta\kappa}b_0(t(s;\rho), x(y;\rho))\sigma + \varepsilon\rho b(t(s;\rho), x(y;\rho), \Xi(\rho^{-\kappa}) + \varepsilon\rho^{-\delta'\kappa}\eta) \\ &+ c(t(s;\rho), x(y;\rho)). \end{aligned}$$

A simple calculation yields

$$\begin{aligned} &\exp[-i\rho^\nu\varphi(s,y;\rho)]P_\rho(s,y,D_s,D_y)\{\exp[i\rho^\nu\varphi(s,y;\rho)]u(s,y)\} \\ &= [\rho^{2\delta\kappa+2\nu}\varphi_s^2 + \rho^{2\delta\kappa+\nu}(-i\varphi_{ss} + 2\varphi_s D_s) + \rho^{2\delta\kappa}D_s^2 \\ &\quad - \rho^2 a(t(s;\rho), \Xi(\rho^{-\kappa}) + \varepsilon\rho^{-\delta'\kappa+\nu}\nabla_y\varphi) \\ &\quad - \rho^{2-2\delta'\kappa}\sum_{j,k=1}^n a_{j,k}(t(s;\rho))(-i\rho^\nu\varphi_{jk} + 2\rho^\nu\varphi_j D_k + D_j D_k) \\ &\quad - \varepsilon\rho^{2-\delta'\kappa}\sum_{j=1}^n (\partial_{\xi_j}a)(t(s;\rho), \Xi(\rho^{-\kappa}))D_j \\ &\quad + \rho^{\delta\kappa+\nu}b_0(t(s;\rho), x(y;\rho))\varphi_s + \rho^{\delta\kappa}b_0(t(s;\rho), x(y;\rho))D_s \\ &\quad + \varepsilon\rho b(t(s;\rho), x(y;\rho), \Xi(\rho^{-\kappa})) + \rho^{1-\delta'\kappa+\nu}b(t(s;\rho), x(y;\rho), \nabla_y\varphi) \\ &\quad + \rho^{1-\delta'\kappa}b(t(s;\rho), x(y;\rho), D_y) + c(t(s;\rho), x(y;\rho))]u(s,y), \end{aligned} \tag{4.3}$$

where $\nu\,(\in \boldsymbol{Q}) > 0$, $D_j = D_{y_j}$, $\varphi_s = \partial_s\varphi(s,y;\rho)$, $\varphi_j = \partial_{y_j}\varphi(s,y;\rho)$ and so on. We choose $\kappa, \delta', \nu \in \boldsymbol{Q}$ as follows:

$$\begin{cases} \kappa = (\mu_0 + (1+X)\delta)^{-1}, \quad \delta' = \mu_0 + \delta, \\ \nu = (1 - \kappa(\mu_1 + \delta))/2, \end{cases} \tag{4.4}$$

where $X = \min\{1/2, (\mu_0 - \mu_1)/(3\delta)\}$. Then we have

$$\begin{cases} 0 < \delta'\kappa < 1, \quad \nu > 0, \quad 2\delta\kappa + 2\nu = 1 - \kappa(\mu_1 - \delta), \\ 2\delta\kappa + \nu \ge 2 - 2\mu_0\kappa, \quad 2\delta\kappa + 2\nu > 2 - 2\delta'\kappa + 2\nu. \end{cases} \tag{4.5}$$

It is easy to see that there are $r \in \boldsymbol{Z}_+$, continuous functions $\sigma_k(\theta;\Xi)$ defined in $[0,\theta_0]$ $(1 \le k \le r)$ such that $\sigma_k(\theta;\Xi) \in C^\infty((0,\theta_0])$, $\sigma_k(\theta;\Xi) \ge 0$, the $\sigma_k(\theta;\Xi)$ can be expanded into formal Puiseux series of θ and

$$\begin{aligned} &q(t,\Xi(\theta)) \\ &= \prod_{k=1}^{r}\{(t-t_0-\tau_k(\theta;\Xi))^2 + \sigma_k(\theta;\Xi)\} \prod_{l=1}^{m-2r}(t-t_0-\tau_{2r+l}(\theta;\Xi)) \ (\ge 0), \end{aligned}$$

where $\prod_{k=1}^{0}\cdots = 1$ and $\{\tau_j(\theta;\Xi)\}$ is rearranged so that $\tau_k(\theta;\Xi) \equiv \tau_{r+k}(\theta;\Xi)$ for $1 \le k \le r$ and $\mathrm{Ord}_{\theta\downarrow 0}\tau_{2r+l}(\theta;\Xi) = \infty$ or $\tau_{2r+l}(\theta;\Xi) \le 0$ for $1 \le l \le m-2r$ and $\theta \in [0,\theta_0]$. Note that we can take $r = m/2$ if $t_0 > 0$. By (4.2) we have

$$\begin{aligned} &|T(\rho^{-\kappa}) - \tau_j(\rho^{-\kappa};\Xi)|/2 \\ &\le |t(s;\rho) - \tau_j(\rho^{-\kappa};\Xi)| \le 3|T(\rho^{-\kappa}) - \tau_j(\rho^{-\kappa};\Xi)|/2 \quad \text{if } |s| \le c_0/2. \end{aligned}$$

This gives

$$\begin{aligned} &\mathrm{Ord}_{\rho\to\infty}(T(\rho^{-\kappa}) - \tau_j(\rho^{-\kappa};\Xi)) = \mathrm{Ord}_{\rho\to\infty}(t(s;\rho) - \tau_j(\rho^{-\kappa};\Xi)), \\ &\mathrm{Ord}_{\rho\to\infty} a(t(s;\rho),\Xi(\rho^{-\kappa})) = -2\mu_0\kappa \end{aligned} \tag{4.6}$$

if $|s| \le c_0/2$. Here $\alpha = \mathrm{Ord}_{\rho\to\infty} a(\rho)$ means that, with $c \ne 0$, $a(\rho) = c\rho^\alpha(1+o(1))$ as $\rho \to \infty$. Write

$$\begin{aligned} &a(t(s;\rho),\Xi(\rho^{-\kappa}) + \varepsilon\rho^{-\delta'\kappa+\nu}\eta) \\ &= a(t(s;\rho),\Xi(\rho^{-\kappa})) + \varepsilon\rho^{-\delta'\kappa+\nu}\sum_{j=1}^{n}(\partial_{\xi_j}a)(t(s;\rho),\Xi(\rho^{-\kappa}))\eta_j \\ &\quad + \rho^{-2\delta'\kappa+2\nu}\sum_{j,k=1}^{n} a_{j,k}(t(s;\rho))\eta_j\eta_k \end{aligned} \tag{4.7}$$

for $\eta \in \boldsymbol{C}^n$. Noting that $a(t,\xi) \ge 0$, we have, with $C, C' > 0$,

$$\begin{aligned} &|(\partial_{\xi_j}a)(t(s;\rho),\Xi(\rho^{-\kappa}))| \\ &\le C\sqrt{a(t(s;\rho),\Xi(\rho^{-\kappa}))} \le C'\rho^{-\mu_0\kappa} \quad \text{if } |s| \le c_0/2. \end{aligned} \tag{4.8}$$

Since $\delta'\kappa - 1 < 0$, (4.1) with $\mu = \mu_1$ and $\hat{c} \equiv \hat{c}(0, x^0) \neq 0$ yield

$$\varepsilon\rho b(t(s;\rho), x(y;\rho), \Xi(\rho^{-\kappa})) = \rho^{1-\kappa(\mu_1-\delta)}(\varepsilon\hat{c}(s,x^0) + o(1)) \quad \text{as } \rho \to \infty \tag{4.9}$$

if $|s| \leq c_0/2$ and $|y| \leq 1$. Moreover, there is $s_0 > 0$ such that $s_0 \leq c_0/2$ and

$$\{\varepsilon\hat{c}(s,x^0);\ |s| \leq s_0\} \cap (-\infty, 0] = \emptyset. \tag{4.10}$$

It is easy to see that

$$\begin{cases} 2\delta\kappa + 2\nu = 1 - \kappa(\mu_1 - \delta) > 2 - 2\delta'\kappa + 2\nu \geq 2 - \delta'\kappa + \nu - \mu_0\kappa \\ \quad = 2\delta\kappa + \nu \qquad\qquad\qquad\qquad\qquad \text{if } X = 1/2, \\ 2\delta\kappa + 2\nu = 1 - \kappa(\mu_1 - \delta) > 2\delta\kappa + \nu \geq 2 - \delta'\kappa + \nu - \mu_0\kappa \\ \quad \geq 2 - 2\delta'\kappa + 2\nu > 0 \qquad\qquad\qquad \text{if } X = (\mu_0 - \mu_1)/(3\delta), \end{cases} \tag{4.11}$$
$$2 - 2\delta'\kappa - \nu \leq 0, \quad 2 - \kappa(\delta + 2\delta') \leq 0.$$

Put

$$\gamma_0 = (2\delta\kappa + 2\nu) - (2 - 2\delta'\kappa + 2\nu)\,(= 2\kappa(1-X)\delta \geq \kappa\delta),$$
$$\varphi(s,y;\rho) = \sum_{k=0}^{l_0} \rho^{-k\gamma_0}\varphi_k(s,y;\rho), \quad l_0 = -[-\nu/\gamma_0] - 1,$$

where $[a]$ denotes the largest integer $\leq a$, *i.e.*, $-[-a]$ is equal to the smallest integer $\geq a$. We note that $l_0 = 0$ if $X = (\mu_0 - \mu_1)/(3\delta)$, *i.e.*, if $\mu_0 - \mu_1 \leq 3\delta/2$. Then, by (4.3) – (4.8) and (4.11) we have

$$\begin{aligned}
&\exp[-i\rho^\nu\varphi(s,y;\rho)]P_\rho(s,y,D_s,D_y)\{\exp[i\rho^\nu\varphi(s,y;\rho)]u(s,y)\} \\
&= \rho^{2\delta\kappa+2\nu}[\varphi_{0,s}(s,y;\rho)^2 + \varepsilon\rho^{\kappa(\mu_1-\delta)}b(t(s;\rho), x(y;\rho), \Xi(\rho^{-\kappa})) \\
&\quad + \sum_{k=1}^{l_0}\rho^{-k\gamma_0}\{2\varphi_{0,s}(s,y;\rho)\varphi_{k,s}(s,y;\rho) + \Phi_k^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{k-1})\} \\
&\quad + \rho^{-\nu}\{2\varphi_{0,s}(s,y;\rho)D_s - i\varphi_{0,ss}(s,y;\rho) - \rho^{2-2\delta'\kappa-\nu+2\mu_0\kappa}a(t(s;\rho),\Xi(\rho^{-\kappa})) \\
&\quad - \varepsilon\rho^{2-\kappa(\delta+2\delta')+\mu_0\kappa}\sum_{j=1}^{n}(\partial_{\xi_j}a)(t(s;\rho),\Xi(\rho^{-\kappa}))\varphi_{0,j}(s,y;\rho) \\
&\quad + \delta_{l_0,0}\rho^{\nu-\gamma_0}\Phi_1^\varepsilon(s,y;\rho;\varphi_0) \\
&\quad + \rho^{-1/L}\mathscr{L}^\varepsilon(s,y,D_s,D_y;\rho;\varphi_0,\cdots,\varphi_{l_0})\}]u(s,y),
\end{aligned}$$

where $(s,y,\rho^{-1})\in\Omega\equiv[-s_0,s_0]\times V_0\times(0,\rho_0^{-1}]$, $V_0=\{y\in\boldsymbol{R}^n;\ |y|\le 1\}$, $L\in\boldsymbol{N}$,

$$\Phi_1^\varepsilon(s,y;\rho;\varphi_0)=-\sum_{j,k=1}^n a_{j,k}(t(s;\rho))\varphi_{0,j}(s,y;\rho)\varphi_{0,k}(s,y;\rho),$$

the $\varphi_k(s,y;\rho)$ are bounded continuous functions of $(s,y,\rho^{-1})\in\Omega$ and their derivatives with respect to s and y are all bounded and continuous in $(s,y,\rho^{-1})\in\Omega$, $\varphi_{k,s}=\partial_s\varphi_k$, $\varphi_{k,ss}=\partial_s^2\varphi_k$ and so on. Here the $\Phi_k^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{k-1})$ are functions of $(s,y,\rho^{-1})\in\Omega$, which depend on $\varphi_0(s,y;\rho)$, $\cdots$, $\varphi_{k-1}(s,y;\rho)$ and their first order derivatives, and the $D_s^jD_y^\alpha\Phi_k^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{k-1})$ ($j\in\boldsymbol{Z}_+$ and $\alpha\in(\boldsymbol{Z}_+)^n$) are bounded and continuous in $(s,y,\rho^{-1})\in\Omega$. $\mathscr{L}^\varepsilon(s,y,D_s,D_y;\rho;\varphi_0,\cdots,\varphi_{l_0})$ is a differential operator of second order, whose coefficients are functions of $(s,y,\rho^{-1})\in\Omega$ and depend on $\varphi_0(s,y;\rho)$, $\cdots$, $\varphi_{l_0}(s,y;\rho)$ and their derivatives up to order 2. Moreover, the derivatives of the coefficients with respect to s and y are all bounded and continuous in $(s,y,\rho^{-1})\in\Omega$. It follows from (4.9) and (4.10) that

$$\varepsilon\rho^{\kappa(\mu_1-\delta)}b(t(s;\rho),x(y;\rho),\Xi(\rho^{-\kappa}))\notin(-\infty,0]\quad\text{for }(s,y,\rho^{-1})\in\Omega,$$

with a modification of ρ_0 if necessary. Define

$$\begin{aligned}\varphi_0(s,y;\rho)=&-i\int_{s_0}^s\sqrt{\varepsilon\rho^{\kappa(\mu_1-\delta)}b(t(\tau;\rho),x(y;\rho),\Xi(\rho^{-\kappa}))}\,d\tau\\&+i|y|^2\quad\text{for }(s,y,\rho^{-1})\in\Omega,\end{aligned}$$

where $\sqrt{z}$ for $z\notin(-\infty,0]$ is the branch satisfying $\mathrm{Re}\sqrt{z}>0$. Then there is $c_1>0$ such that

$$\begin{aligned}&\mathrm{Im}\,\varphi_0(s,y;\rho)\ge c_1(s_0-s)+|y|^2,\\&\varphi_{0,s}(s,y;\rho)=-i\sqrt{\varepsilon\rho^{\kappa(\mu_1-\delta)}b(t(s;\rho),x(y;\rho),\Xi(\rho^{-\kappa}))}\ne 0\end{aligned}$$

for $(s,y,\rho^{-1})\in\Omega$. So we can determine inductively $\varphi_k(s,y;\rho)$ $(1\le k\le l_0)$ so as to satisfy the equations

$$\begin{cases}2\varphi_{0,s}(s,y;\rho)\varphi_{k,s}(s,y;\rho)+\Phi_k^\varepsilon(s,y;\rho;\varphi_0,\cdots,\varphi_{k-1})=0,\\ \varphi_k(s_0,y;\rho)=0\end{cases}$$

$(k = 1, 2, \cdots, l_0)$. Next, putting

$$u(s, y) \sim \sum_{j=0}^{\infty} \rho^{-j/L} u_j(s, y; \rho),$$

we determine inductively $\{u_j(s, y; \rho)\}_{j=0,1,\cdots}$ so as to satisfy

$$\begin{cases} 2\varphi_{0,s}(s, y; \rho) D_s u_j(s, y; \rho) - i\varphi_{0,ss}(s, y; \rho) u_j(s, y; \rho) \\ \quad -\rho^{2-2\delta'\kappa-\nu+2\mu_0\kappa} a(t(s; \rho), \Xi(\rho^{-\kappa})) u_j(s, y; \rho) \\ \quad -\varepsilon\rho^{2-\kappa(\delta+2\delta')+\mu_0\kappa} \sum_{j=1}^{n} (\partial_{\xi_j} a)(t(s; \rho), \Xi(\rho^{-\kappa})) \varphi_{0,j}(s, y; \rho) u_j(s, y; \rho) \\ \quad +\delta_{l_0,0} \rho^{\nu-\gamma_0} \Phi_1^{\varepsilon}(s, y; \rho; \varphi_0) u_j(s, y; \rho) \\ \quad +\mathscr{L}^{\varepsilon}(s, y, D_s, D_y; \rho; \varphi_o, \cdots, \varphi_{l_0}) u_{j-1}(s, y; \rho) = 0, \\ u_j(s_0, y; \rho) = 0 \end{cases}$$

$(j = 0, 1, 2, \cdots)$, where $u_{-1}(s, y; \rho) \equiv 0$. Let $\chi(s, y)$ be a function in $C_0^\infty(\boldsymbol{R} \times \boldsymbol{R}^n)$ satisfying $\chi(s, y) = 1$ near $(s, y) = (s_0, 0)$ and $\operatorname{supp} \chi \subset (0, \infty) \times V_0$, and put

$$u^N(s, y; \rho) = \sum_{j=0}^{N-1} \rho^{-j/L} \exp[i\rho^\nu \varphi(s, y; \rho)] u_j(s, y; \rho) \chi(s, y).$$

Then, applying the same argument as in Ivrii-Petkov [**4**] we can prove Theorem 1.3.

5. Proof of Theorem 1.4.

Let $n = 2$, and let $(t_0, x^0, \xi^0) \in [0, \infty) \times \boldsymbol{R}^2 \times S^1$ satisfy $a(t_0, \xi^0) = 0$. In this section we assume that the hypotheses of Theorem 1.4 are fulfilled, and that the Cauchy problem (CP) is C^∞ well-posed. By assumption we take $e(t, \xi)$ and the $a_j(\xi)$ in the condition $(\mathrm{A})'_{(t_0,\xi^0)}$ to be real analytic. Let $\boldsymbol{e}$ be a vector in S^1 satisfying $\boldsymbol{e} \perp \xi^0$, and choose $\theta_0 > 0$ so that $\Gamma_0 \equiv \{\lambda(\xi^0 + \theta \boldsymbol{e}); \lambda > 0 \text{ and } |\theta| \le \theta_0\} \subset \Gamma$, where Γ is as in $(\mathrm{A})'_{(t_0,\xi^0)}$. Since $n = 2$, Γ_0 is a conic neighborhood of ξ^0. We put

$$\begin{aligned} &a^\pm(t, \theta) = a(t, \xi^0 \pm \theta \boldsymbol{e}), \quad e^\pm(t, \theta) = e(t, \xi^0 \pm \theta \boldsymbol{e}), \\ &q^\pm(t, \theta) = (t - t_0)^m + a_1(\xi^0 \pm \theta \boldsymbol{e})(t - t_0)^{m-1} + \cdots + a_m(\xi^0 \pm \theta \boldsymbol{e}), \\ &b^\pm(t, \theta) = b(t, x^0, \xi^0 \pm \theta \boldsymbol{e}). \end{aligned}$$

Since the $a_j(\xi^0+\theta\boldsymbol{e})$ are real analytic in θ, with a modification of θ_0 if necessary, there are $r_\pm \in \boldsymbol{Z}_+$ and real-valued continuous functions $\tau_k^\pm(\theta)$ and $\sigma_k^\pm(\theta)$ $(1 \le k \le r_\pm)$ and $\tau_{0,l}^\pm(\theta)$ $(1 \le l \le r_{0,\pm})$ defined in $[0,\theta_0]$ such that $2r_\pm = m$ if $t_0 > 0$, the $\tau_k^\pm(\theta)$, the $\sigma_k^\pm(\theta)$ and the $\tau_{0,l}^\pm(\theta)$ can be expanded into convergent Puiseux series in $[0,\theta_0]$,

$$\begin{aligned}
&\tau_1^\pm(\theta) \le \tau_2^\pm(\theta) \le \cdots \le \tau_{r_\pm}^\pm(\theta), \quad \tau_{0,1}^\pm(\theta) \le \cdots \le \tau_{0,r_{0,\pm}}^\pm(\theta) \le 0,\\
&\sigma_k^\pm(\theta) \ge 0 \quad (\, 1 \le k \le r_\pm),\\
&q^\pm(t,\theta) = \prod_{k=1}^{r_\pm}\{(t-t_0-\tau_k^\pm(\theta))^2+\sigma_k^\pm(\theta)\}\prod_{l=1}^{r_{0,\pm}}(t-t_0-\tau_{0,l}^\pm(\theta)) \qquad (5.1)
\end{aligned}$$

for $\theta \in [0,\theta_0]$, where $r_{0,\pm} = m-2r_\pm$. Note that

$$\tau_k^\pm(0) = \sigma_k^\pm(0) = \tau_{0,l}^\pm(0) = 0 \quad (\, 1 \le k \le r_\pm,\ 1 \le l \le r_{0,\pm}),$$

and that

$$\begin{aligned}
&\tau_k^\pm(\theta) \ne \tau_\mu^\pm(\theta) \quad \text{for } \theta \in (0,\theta_0] \quad \text{if } \tau_k^\pm(\theta) \not\equiv \tau_\mu^\pm(\theta) \text{ in } [0,\theta_0],\\
&\tau_{0,l}^\pm(\theta) \ne \tau_{0,\mu}^\pm(\theta) \quad \text{for } \theta \in (0,\theta_0] \quad \text{if } \tau_{0,l}^\pm(\theta) \not\equiv \tau_{0,\mu}^\pm(\theta) \text{ in } [0,\theta_0].
\end{aligned}$$

Let us consider only the "+" case, since the "−" case can be treated similarly. Let $r \in \boldsymbol{N}$ and $\lambda_j(\theta)$ $(1 \le j \le r)$ be continuous functions satisfying

$$\begin{aligned}
&0 \le \lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_r(\theta),\\
&\{\lambda_1(\theta),\cdots,\lambda_r(\theta)\} = \begin{cases} \{(t_0+\tau_1^+(\theta))_+ - t_0, \cdots, (t_0+\tau_{r_+}^+(\theta))_+ - t_0\} \\ \qquad\qquad \text{if } m = 2r_+, \\ \{0, (t_0+\tau_1^+(\theta))_+ - t_0, \cdots, (t_0+\tau_{r_+}^+(\theta))_+ - t_0\} \\ \qquad\qquad \text{if } m > 2r_+, \end{cases}
\end{aligned}$$

for $\theta \in [0,\theta_0]$. We may assume that $|\tau_k^+(\theta)| \le 1$ and $|\tau_{0,l}^+(\theta)| \le 1$ for $1 \le k \le r_+$, $1 \le l \le r_{0,+}$ and $\theta \in [0,\theta_0]$, modifying θ_0 if necessary. It follows from Theorem 1.3 with $\Xi(\theta) = \xi^0+\theta\boldsymbol{e}$ and Lemma 2.2 that

$$2\Gamma_{1,j}^+ \subset \Gamma_{0,j}^+ \quad (\, 1 \le j \le r), \qquad (5.2)$$

where $\Gamma_{0,j}^+$ and $\Gamma_{1,j}^+$ denote the Newton polygons of $a^+(t_0+\lambda_j(\theta)+t,\theta)$ and

$tb^+(t_0 + \lambda_j(\theta) + t, \theta)$, respectively. We put

$$\nu_k = \mathrm{Ord}_{\theta\downarrow 0}\sigma_k^+(\theta)\ (\ >0) \quad \text{for } 1 \le k \le r_+,$$
$$\kappa_{j,k} = \mathrm{Ord}_{\theta\downarrow 0}(\lambda_j(\theta) - \tau_k^+(\theta))\ (\ >0) \quad \text{for } 1 \le j \le r \text{ and } 1 \le k \le r_+,$$
$$\kappa_{0,j,l} = \mathrm{Ord}_{\theta\downarrow 0}(\lambda_j(\theta) - \tau_{0,l}^+(\theta))\ (\ >0) \quad \text{for } 1 \le j \le r \text{ and } 1 \le l \le r_{0,+}.$$

First consider the case where $\theta \in [0, \theta_0]$ and $(t_0 - 1)_+ \le t < t_0 + \lambda_1(\theta)$. Similarly we can deal with the case where $t_0 + \lambda_r(\theta) \le t \le t_0 + 1$. Note that there does not exist $(t, \theta) \in [0, \infty) \times [0, \theta_0]$ satisfying $(t_0 - 1)_+ \le t < t_0 + \lambda_1(\theta)$ if $t_0 + \lambda_1(\theta) \equiv 0$. So we may assume that $t_0 + \lambda_1(\theta) > 0$, $r_{0,+} = 0$ and $r_+ = m/2$. Write $t = t_0 + \lambda_1(\theta) - \tau$, and put

$$\Omega_1 = \{(\tau, \theta) \in \boldsymbol{R} \times [0, \theta_0];\ 0 < \tau \le \lambda_1(\theta) + t_0 - (t_0 - 1)_+\}.$$

By (5.1) we have

$$a^+(t_0 + \lambda_1(\theta) - \tau, \theta) \approx \tau^{2r_1} \prod_{k \in I_1} (\tau^2 + \theta^{\hat{\kappa}_{1,k}}) \quad \text{uniformly in } \Omega_1,$$

i.e., with $C > 0$,

$$C^{-1}\tau^{2r_1} \prod_{k \in I_1} (\tau^2 + \theta^{\hat{\kappa}_{1,k}}) \le a^+(t_0 + \lambda_1(\theta) - \tau, \theta)$$
$$\le C\tau^{2r_1} \prod_{k \in I_1} (\tau^2 + \theta^{\hat{\kappa}_{1,k}}) \quad \text{for } (\tau, \theta) \in \Omega_1,$$

where $\hat{\kappa}_{1,k} = \min\{2\kappa_{1,k}, \nu_k\}$ (>0), $r_1 = \#\{k \in \boldsymbol{N};\ k \le r_+ \text{ and } \hat{\kappa}_{1,k} = \infty\}$ and $I_1 = \{k \in \boldsymbol{N};\ k \le r_+ \text{ and } \hat{\kappa}_{1,k} < \infty\}$. Therefore, we have

$$a^+(t_0 + \lambda_1(\theta) - \tau, \theta) \approx \tau^{2r_1} \sum_{l=0}^{r_+ - r_1} \tau^{2(r_+ - r_1 - l)} \theta^{\nu_{1,l}} \quad \text{uniformly in } \Omega_1,$$

where $\nu_{1,0} = 0$ and $0 < \nu_{1,l} < \infty$ $(1 \le l \le r_+ - r_1)$. This gives

$$\Gamma_{0,1} = \mathrm{ch}\Big[\bigcup_{l=0}^{r_+ - r_1} (\{(\nu_{1,l}, 2(r_+ - l))\} + (\overline{\boldsymbol{R}}_+)^2)\Big]. \tag{5.3}$$

On the other hand, we have

$$\min_{\zeta\in\mathscr{R}(\xi^0+\theta\boldsymbol{e})}|t_0+\lambda_1(\theta)-\tau-\zeta|=\min_{1\le j\le r}|\lambda_1(\theta)-\tau-\lambda_j(\theta)|=\tau. \tag{5.4}$$

We can assume without loss of generality that $b^+(t,\theta)\not\equiv 0$. Then there is $l\in\boldsymbol{Z}_+$ satisfying $(\partial_t^l b^+)(t_0+\lambda_1(\theta),\theta)\not\equiv 0$ in θ. So we can write

$$b^+(t_0+\lambda_1(\theta)-\tau,\theta)=\sum_{k=0}^{\infty}\beta_{1,k}(\tau)\theta^{\hat\nu_1+k/L},\quad \beta_{1,0}(\tau)\not\equiv 0, \tag{5.5}$$

where $L\in\boldsymbol{N}$ and $\hat\nu_1\,(\in\boldsymbol{Q})\ \geq 0$. Note that the $\beta_{1,k}(\tau)$ are analytic in a neighborhood of $(-\infty,t_0+\lambda_1(\theta)]$. (5.2) gives

$$2\tilde\nu+2p\tilde\mu\ge\min\{\nu+p\mu;\ (\nu,\mu)\in\Gamma_{0,1}^+\}\quad\text{if } (\tilde\nu,\tilde\mu)\in\Gamma_{1,1}^+ \text{ and } p\ge 0.$$

Tending p to ∞, by (5.3) and (5.5) we have

$$\mathrm{Ord}_{\tau\downarrow 0}\beta_{1,k}(\tau)\ge r_1-1\quad\text{if } \beta_{1,k}(\tau)\not\equiv 0. \tag{5.6}$$

Put $\hat\kappa_1=\sum_{k\in I_1}\hat\kappa_{1,k}/2$. From (5.5) and (5.6) we can write

$$\tau b^+(t_0+\lambda_1(\theta)-\tau,\theta)=\sum_{0\le k<(\hat\kappa_1-\hat\nu_1)L}\tau\beta_{1,k}(\tau)\theta^{\hat\nu_1+k/L}+\tau^{r_1}\beta_1(\tau,\theta)\theta^{\hat\kappa_1}.$$

Here the $\beta_{1,k}(\tau)$ are analytic in τ and $\beta_1(\tau,\theta)$ is continuous. For $0\le k<(\hat\kappa_1-\hat\nu_1)L$ we can also write

$$\tau\beta_{1,k}(\tau)=\tau^{r_1}\sum_{0\le j<r_+-r_1}\beta_{1,k,j}\tau^j+\tilde\beta_{1,k}(\tau)\tau^{r_+},$$

where $\beta_{1,k,j}\in\boldsymbol{C}$ and $\tilde\beta_{1,k}(\tau)$ is analytic. If $\beta_{1,k,j}\neq 0$, then $(\hat\nu_1+k/L,r_1+j)\in\Gamma_{1,1}^+$ and, therefore, $(2(\hat\nu_1+k/L),2(r_1+j))\in\Gamma_{0,1}^+$. So we have, with $C>0$,

$$\tau^{r_1+j}\theta^{\hat\nu_1+k/L}\le C\sqrt{a^+(t_0+\lambda_1(\theta)-\tau,\theta)}\quad\text{for } (\tau,\theta)\in\Omega_1$$

if $0\le k<(\hat\kappa_1-\hat\nu_1)L,\ 0\le j<r_+-r_1$ and $\beta_{1,k,j}\neq 0$. This, together with (5.4), yields

$$\begin{aligned}&\min_{\zeta\in\mathscr{R}(\xi^0+\theta\boldsymbol{e})}|t_0+\lambda_1(\theta)-\tau-\zeta|\cdot|b^+(t_0+\lambda_1(\theta)-\tau,\theta)|\\&\le C\sqrt{a^+(t_0+\lambda_1(\theta)-\tau,\theta)}\quad\text{for } (\tau,\theta)\in\Omega_1.\end{aligned}$$

Next consider the case where $1 \le j \le r-1$, $\theta \in [0,\theta_0]$ and $\lambda_j(\theta) \le t - t_0 < (\lambda_j(\theta)+\lambda_{j+1}(\theta))/2$. Similarly we can deal with the case where $\theta \in [0,\theta_0]$ and $(\lambda_j(\theta)+\lambda_{j+1}(\theta))/2 \le t-t_0 < \lambda_{j+1}(\theta)$. Put

$$\widetilde{\Omega}_j = \{(\tau,\theta) \in \boldsymbol{R} \times [0,\theta_0];\ 0 \le \tau < c_j\},$$

where $p_j = \mathrm{Ord}_{\theta\downarrow 0}(\lambda_{j+1}(\theta)-\lambda_j(\theta))$ and

$$c_j = \lim_{\theta\to+0} 2\theta^{-p_j}(\lambda_{j+1}(\theta)-\lambda_j(\theta))/3.$$

Then we have

$$(\lambda_{j+1}(\theta)-\lambda_j(\theta))/2 \le c_j\theta^{p_j} \le 5(\lambda_{j+1}(\theta)-\lambda_j(\theta))/6$$

for $\theta \in [0,\theta_0]$, modifying θ_0 if necessary. (5.1) gives

$$\begin{aligned} a^+(t_0+\lambda_j(\theta)+\tau\theta^{p_j},\theta) \approx \prod_{k\in I_{1,j}} (\theta^{2p_j}\tau^2+\theta^{\hat{\kappa}_{j,k}}) \prod_{k\in I_{2,j}} \theta^{2p_j}\tau^2 \\ \times \prod_{k\in I_{3,j}} \theta^{\hat{\kappa}_{j,k}} \prod_{l\in I_{0,j}} \theta^{p_j}\tau \prod_{l\in I'_{0,j}} (\theta^{p_j}\tau+\theta^{\kappa_{0,j,l}}) \quad \text{uniformly in } \widetilde{\Omega}_j, \end{aligned}$$

where $\hat{\kappa}_{j,k} = \min\{2\kappa_{j,k},\nu_k\}$, $I_{1,j} = \{k \in \boldsymbol{N};\ k \le r_+,\ \kappa_{j,k}=\infty \text{ and } \hat{\kappa}_{j,k}<\infty\} \cup \{k \in \boldsymbol{N};\ k \le r_+ \text{ and } \tau_k^+(\theta) < \lambda_j(\theta) \text{ for } \theta \in (0,\theta_0]\}$, $I_{2,j} = \{k \in \boldsymbol{N};\ k \le r_+ \text{ and } \hat{\kappa}_{j,k} = \infty\}$, $I_{3,j} = \{k \in \boldsymbol{N};\ k \le r_+ \text{ and } k \notin I_{1,j}\cup I_{2,j}\}$, $I_{0,j} = \{l \in \boldsymbol{N};\ l \le r_{0,+} \text{ and } \kappa_{0,j,l} = \infty\}$, and $I'_{0,j} = \{l \in \boldsymbol{N};\ l \le r_{0,+} \text{ and } \kappa_{0,j,l} < \infty\}$. Therefore, we can write

$$\begin{aligned} &a^+(t_0+\lambda_j(\theta)+\tau\theta^{p_j},\theta) \\ &\approx \tau^{2r'_j}\theta^{2\delta_j} \sum_{l=0}^{r''_j} (\theta^{p_j}\tau)^{r''_j-l}\theta^{\nu'_{j,l}} \quad \text{uniformly in } \widetilde{\Omega}_j, \end{aligned} \tag{5.7}$$

where $r'_j = \#I_{2,j} + \#I_{0,j}/2$, $\delta_j = p_j r'_j + \sum_{k\in I_{3,j}} \hat{\kappa}_{j,k}/2$, $r''_j = 2(\#I_{1,j}) + \#I'_{0,j}$, $\nu'_{j,0} = 0$ and $0 < \nu'_{j,l} < \infty$ $(1 \le l \le r''_j)$. Here we have used the fact that

$$\theta^{2p_j}\tau^2 + \theta^{\hat{\kappa}_{j,k}} \approx (\theta^{p_j}\tau + \theta^{\hat{\kappa}_{j,k}/2})^2 \quad \text{uniformly in } \widetilde{\Omega}_j.$$

Let $\widetilde{\Gamma}_{0,j}$ be the Newton polygon of $a^+(t_0+\lambda_j(\theta)+\tau\theta^{p_j},\theta)$. Then we have

$$\begin{aligned}\widetilde{\Gamma}_{0,j} &= \mathrm{ch}\Big[\bigcup_{l=0}^{r''_j}(\{(2\delta_j + p_j(r''_j - l) + \nu'_{j,l}, 2r'_j + r''_j - l)\} + (\overline{\boldsymbol{R}}_+)^2)\Big] \\ &= \bigcap_{p\ge 0}\{(\tilde{\nu},\tilde{\mu}) \in \boldsymbol{R}^2;\ \tilde{\nu} + (p_j+p)\tilde{\mu} \ge \min\{\nu + (p_j+p)\mu;\ (\nu,\mu) \in \Gamma^+_{0,j}\}\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\min_{\zeta\in\mathscr{R}(\xi^0+\theta\boldsymbol{e})} |t_0 + \lambda_j(\theta) + \tau\theta^{p_j} - \zeta| \\ &= \min_{1\le l\le r} |\lambda_j(\theta) + \tau\theta^{p_j} - \lambda_l(\theta)| \approx \tau\theta^{p_j} \quad \text{uniformly in } \widetilde{\Omega}_j. \end{aligned} \tag{5.8}$$

We can also write

$$b^+(t_0 + \lambda_j(\theta) + \tau\theta^{p_j}, \theta) = \sum_{k=0}^{\infty} \tilde{\beta}_{j,k}(\tau)\theta^{\hat{\nu}'_j + k/L}, \quad \tilde{\beta}_{j,0}(\tau) \not\equiv 0, \tag{5.9}$$

where $L \in \boldsymbol{N}$, $\hat{\nu}'_j\,(\in \boldsymbol{Q}) \ge 0$ and the $\tilde{\beta}_{j,k}(\tau)$ are polynomials of τ. Similarly, it follows from (5.2) and (5.7) – (5.9) that

$$\begin{aligned}&2\widetilde{\Gamma}^+_{1,j} \subset \widetilde{\Gamma}^+_{0,j}, \\ &p_j + \hat{\nu}'_j \ge \delta_j + \min_{0\le l\le r''_j}\{p_j(r''_j - l) + \nu'_{j,l}\}/2, \\ &\mathrm{Ord}_{\tau\downarrow 0}\tilde{\beta}_{j,k}(\tau) \ge r'_j - 1, \\ &\min_{\zeta\in\mathscr{R}(\xi^0+\theta\boldsymbol{e})} |t_0 + \lambda_j(\theta) + \tau\theta^{p_j} - \zeta| \cdot |b^+(t_0 + \lambda_j(\theta) + \tau\theta^{p_j}, \theta)| \\ &\le C\sqrt{a^+(t_0 + \lambda_j(\theta) + \tau\theta^{p_j}, \theta)} \quad \text{for } (\tau,\theta) \in \widetilde{\Omega}_j,\end{aligned}$$

where $\widetilde{\Gamma}^+_{1,j}$ denotes the Newton polygon of $\tau\theta^{p_j}b^+(t_0 + \lambda_j(\theta) + \tau\theta^{p_j}, \theta)$. Therefore, by homogeneity the condition $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ with Γ replaced by Γ_0 is valid, which proves Theorem 1.4.

6. Proof of Theorem 1.7.

Let $(t_0, x^0, \xi^0) \in [0,\infty) \times \boldsymbol{R}^n \times S^{n-1}$ satisfy $a(t_0, \xi^0) = 0$. In this section we assume that the hypotheses of Theorem 1.7 are fulfilled. Moreover, we assume that the condition $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ is not satisfied. Choose $\delta > 0$ so that $(t,\xi) \in U \times \Gamma$ if $|\xi - \xi^0|^2 + |t - t_0|^2 \le \delta^2$ and $t \ge 0$. We put

$$
\begin{aligned}
A &= \{(t,\xi,y) \in \boldsymbol{R}^{n+2};\ |\xi-\xi^0|^2 + |t-t_0|^2 \le \delta^2,\ t \ge 0 \text{ and } y = q(t,\xi)\},\\
B &= \{(t,\xi,y) \in \boldsymbol{R}^{n+2};\ |\xi-\xi^0|^2 + |t-t_0|^2 \le \delta^2,\ t \ge 0 \text{ and } y = |\tilde{b}(t,\xi)|^2\},\\
C &= \{(t,\xi,y) \in \boldsymbol{R}^{n+2};\ |\xi-\xi^0|^2 + |t-t_0|^2 \le \delta^2,\ t \ge 0 \text{ and } y = \min_{\tau \in \mathscr{R}(\xi)} |t-\tau|^2\},
\end{aligned}
$$

where $\tilde{b}(t,\xi)$ is as in $(\text{A-}b)_{(t_0,x^0,\xi^0)}$. It is obvious that A and B are semi-algebraic sets. Since $a_+ = (a+|a|)/2$ for $a \in \boldsymbol{R}$ and, with $A_j \equiv \{(\xi,\lambda) \in \boldsymbol{R}^{n+1};\ \lambda = a_j(\xi)\}$ $(1 \le j \le m)$,

$$
\begin{aligned}
C = \{(t,\xi,y) \in \boldsymbol{R}^{n+2};\ & |\xi-\xi^0|^2 + |t-t_0|^2 \le \delta^2,\ t \ge 0, \text{ and there are}\\
& (\xi,\lambda_j) \in A_j\ (\ 1 \le j \le m) \text{ and } \tau_j, \sigma_j, \tilde{\tau}_j \in \boldsymbol{R}\ (\ 1 \le j \le m)\\
& \text{such that } \tilde{\tau}_j \ge 0,\\
& (t_0+\tau_j)^2 = \tilde{\tau}_j^2,\ s^m + \lambda_1 s^{m-1} + \cdots + \lambda_m = \prod_{j=1}^m (s - \tau_j - i\sigma_j)\\
& \text{for } s \in \boldsymbol{C},\ |t-(t_0+\tau_1+\tilde{\tau}_1)/2|^2 \le |t-(t_0+\tau_2+\tilde{\tau}_2)/2|^2\\
& \le \cdots \le |t-(t_0+\tau_m+\tilde{\tau}_m)/2|^2 \text{ and } y = |t-(t_0+\tau_1+\tilde{\tau}_1)/2|^2\},
\end{aligned}
$$

C is a semi-algebraic set. Put

$$
\begin{aligned}
\Lambda = \{(\rho,t,\xi,\lambda) \in \boldsymbol{R}^{n+3};\ & \text{there are } y,u,v,w \in \boldsymbol{R} \text{ satisfying}\\
& (t,\xi,y) \in A,\ (t,\xi,u) \in B,\ (t,\xi,v) \in C,\ \rho y = 1,\\
& w((|\xi-\xi^0|^2 + |t-t_0|^2)\rho uv + 1) = 1 \text{ and } \lambda = \rho uvw\}.
\end{aligned}
$$

Then Λ is a semi-algebraic set and

$$
\begin{aligned}
\Lambda = \{(\rho,t,\xi,\lambda) \in \boldsymbol{R}^{n+3};\ & |\xi-\xi^0|^2 + |t-t_0|^2 \le \delta^2,\ t \ge 0,\ \rho q(t,\xi) = 1\\
& \text{and } \lambda = \rho \min_{\tau \in \mathscr{R}(\xi)} |t-\tau|^2 \cdot |\tilde{b}(t,\xi)|^2\\
& \qquad \times ((|\xi-\xi^0|^2 + |t-t_0|^2)\rho \min_{\tau \in \mathscr{R}(\xi)} |t-\tau|^2 \cdot |\tilde{b}(t,\xi)|^2 + 1)^{-1}\}.
\end{aligned}
$$

For $\rho > 0$ we put

$$
K(\rho) = \{(t,\xi) \in \boldsymbol{R}^{n+1};\ |\xi-\xi^0|^2 + |t-t_0|^2 \le \delta^2,\ t \ge 0 \text{ and } \rho q(t,\xi) = 1\}.
$$

Then $K(\rho)$ is compact and there is $\rho_0 > 0$ such that $K(\rho) \ne \emptyset$ for $\rho \ge \rho_0$. Indeed, we can take

$$\rho_0^{-1} = \max\{q(t,\xi);\ |\xi-\xi^0|^2+|t-t_0|^2 \le \delta^2,\ \text{and } t \ge 0\},$$

since $a(t_0,\xi^0)=0$. This yields

$$\{\rho \in \boldsymbol{R};\ (\rho,t,\xi,\lambda) \in \Lambda \text{ for some } (t,\xi,\lambda) \in \boldsymbol{R}^{n+2}\} \supset \{\rho;\ \rho \ge \rho_0\}.$$

Therefore, we can define

$$f(\rho) = \sup\{\lambda;\ (\rho,t,\xi,\lambda) \in \Lambda \text{ for some } (t,\xi) \in \boldsymbol{R}^{n+1}\}$$

for $\rho \ge \rho_0$. Note that

$$\begin{aligned} f(\rho) = \max\Big\{ & \frac{\rho \min_{\tau\in\mathscr{R}(\xi)} |t-\tau|^2 \cdot |\tilde{b}(t,\xi)|^2}{((|\xi-\xi^0|^2+|t-t_0|^2)\rho \min_{\tau\in\mathscr{R}(\xi)} |t-\tau|^2 \cdot |\tilde{b}(t,\xi)|^2+1)}; \\ & (t,\xi) \in K(\rho)\Big\} \end{aligned} \tag{6.1}$$

since $K(\rho)$ is compact. It follows from Theorem A.2.8 of [**3**] that there are continuous functions $\widetilde{T}(\rho)$, $\widetilde{\Xi}(\rho)$ and $\lambda(\rho)$ such that $\widetilde{T}(\rho)$, $\widetilde{\Xi}(\rho)$ and $\lambda(\rho)$ can be expanded into convergent Puiseux series for $\rho \gg 1$ and

$$(\rho, t_0+\widetilde{T}(\rho), \widetilde{\Xi}(\rho), \lambda(\rho)) \in \Lambda, \quad f(\rho) = \lambda(\rho)\ (\ \ge 0) \tag{6.2}$$

(see, also, [**9**]). Since the condition $(\mathrm{L})_{(t_0,x^0,\xi^0)}$ does not hold, there is $\{(t_k,\xi^k)\} \in U \times \Gamma$ satisfying $(t_k,\xi^k) \to (t_0,\xi^0)$ and

$$\min_{\tau\in\mathscr{R}(\xi)} |t_k-\tau| \cdot |\tilde{b}(t_k,\xi^k)|/\sqrt{q(t_k,\xi^k)} \to \infty \tag{6.3}$$

as $k \to \infty$. Put $\delta_k = (|\xi^k-\xi^0|^2+|t-t_k|^2)^{1/2}$ and $\rho_k = q(t_k,\xi^k)^{-1}$. Then we have $\delta_k \to 0$ and $\rho_k \to \infty$ as $k \to \infty$. (6.2), together with (6.1) and (6.3), gives

$$\begin{aligned} \lambda(\rho_k) \ge & \rho_k \min_{\tau\in\mathscr{R}(\xi)} |t_k-\tau|^2 \cdot |\tilde{b}(t_k,\xi^k)|^2 \\ & \times (\delta_k^2 \rho_k \min_{\tau\in\mathscr{R}(\xi)} |t_k-\tau|^2 \cdot |\tilde{b}(t_k,\xi^k)|^2+1)^{-1} \\ & \to \infty \quad \text{as } k \to \infty, \end{aligned}$$

since $\delta_k \to 0$ and $\rho_k \min_{\tau\in\mathscr{R}(\xi)} |t_k-\tau|^2 \cdot |\tilde{b}(t_k,\xi^k)|^2 \to \infty$ as $k \to \infty$. So we have $\lambda(\rho) \to \infty$, which implies that

$$\min_{\tau\in\mathscr{R}(\xi)} |t_0+\widetilde{T}(\rho)-\tau|\cdot|\tilde{b}(t_0+\widetilde{T}(\rho),\widetilde{\Xi}(\rho))|/\sqrt{a(t_0+\widetilde{T}(\rho),\widetilde{\Xi}(\rho))}\to\infty,$$
$$(\widetilde{T}(\rho),\widetilde{\Xi}(\rho))\to(0,\xi^0)$$

as $\rho\to\infty$. If we put $T(\theta)=\widetilde{T}(\theta^{-1})$ and $\Xi(\theta)=\widetilde{\Xi}(\theta^{-1})$, then $T(\theta)$ and $\Xi(\theta)$ satisfy the condition $(\mathrm{C})_{(t_0,x^0,\xi^0)}$ except for $t_0+T(\theta)>0$. If $t_0+T(\theta)\equiv 0$, then, with $N\gg 1$ and $T(\theta)$ replaced by $T(\theta)+\theta^N$, $(\mathrm{C})_{(t_0,x^0,\xi^0)}$ is satisfied. By Theorem 1.3 the Cauchy problem (CP) is not C^∞ well-posed. This proves Theorem 1.7.

7. Some remarks and examples.

Colombini, Ishida and Orrú proved in [**1**] that the Cauchy problem (CP) is C^∞ well-posed if $b(t,x,\xi)\equiv b(t,\xi)$ and there are $k\in\boldsymbol{N}$ and $C>0$ such that $k\ge 2$ and

$$\sum_{j=0}^{k}|\partial_t^j a(t,\xi)|\neq 0\quad\text{for } (t,\xi)\in[0,\infty)\times S^{n-1},$$
$$|b(t,\xi)|\le Ca(t,\xi)^{1/2-1/k}\quad\text{for } (t,\xi)\in[0,\infty)\times S^{n-1}. \tag{7.1}$$

The following two examples show that (7.1) is not a necessary condition for C^∞ well-posedness.

EXAMPLE 7.1. Let $n=2$, and let $a(t,\xi)=t^2(t\xi_1-\xi_2)^2$ and $P(t,x,\tau,\xi)=\tau^2-a(t,\xi)+b_0(t)\tau+b(t,\xi)+c(t)$. Assume that $b_j(t)$ $(j=1,2)$ are real analytic. Let $t_0>0$ and $\xi^0=(\xi_1^0,\xi_2^0)\in S^1$ satisfy $a(t_0,\xi^0)=0$. Then we have $\xi_1^0\neq 0$, $t_0=\xi_2^0/\xi_1^0$, and $\mathscr{R}(\xi)=\{\xi_2/\xi_1\}$ in a conic neighborhood of $(t,\xi)=(t_0,\xi^0)$, where $\mathscr{R}(\xi)$ is the set defined by (1.1). It is obvious that (1.2) is valid in a neighborhood of (t_0,ξ^0). Let $t_0=0$ and $\xi^0\in S^1$. If $\xi_2^0\neq 0$, then (1.2) is also valid in a conic neighborhood of (t_0,ξ^0). Assume that $\xi_2^0=0$. Since $\mathscr{R}(\xi)=\{0,(\xi_2/\xi_1)_+\}$, (1.2) is valid in a conic neighborhood of $(0,\xi^0)$ if and only if

$$|b(t,\xi)|\le C\{t|\xi|+|t\xi_1-\xi_2|\}\quad\text{in a conic neighborhood of } (0,\xi^0).$$

Therefore, by Theorem 1.5 the Cauchy problem (CP) is C^∞ well-posed if and only if $b_1(0)=0$. On the other hand, in (7.1) we can take $k=4$ and (7.1) is valid if and only if there is $\beta(t)\in C^\infty([0,\infty))$ satisfying $b(t,\xi)=\beta(t)t(t\xi_1-\xi_2)$. This implies that (7.1) is not necessary for C^∞ well-posedness.

Assume that $P(t,x,\tau,\xi) \equiv P(t,\tau,\xi)$. Let $n' \in \boldsymbol{N}$ satisfy $n' < n$, and write $x' = (x_1, \cdots, x_{n'})$ and $x'' = (x_{n'+1}, \cdots, x_n)$ for $x = (x_1, \cdots, x_n)$. As $P(t, D_t, D_x)$ we take $\widetilde{P}(t, D_t, D_{x'}) = P(t, D_t, D_1, \cdots, D_{n'}, 0, \cdots 0)$. Then the Cauchy problem (CP) for $\widetilde{P}(t, D_t, D_{x'})$ is C^∞ well-posed if the Cauchy problem (CP) for $P(t, D_t, D_x)$ is C^∞ well-posed. This implies that a necessary condition for $\widetilde{P}(t, D_t, D_{x'})$ is also a necessary one for $P(t, D_t, D_x)$. In the next example we use this fact to obtain a necessary and sufficient condition.

EXAMPLE 7.2. Let $m_j \in \boldsymbol{Z}_+$ $(1 \le j \le n)$, and let $a(t,\xi) = t^{m_1}\xi_1^2 + t^{m_2}\xi_2^2 + \cdots + t^{m_n}\xi_n^2$ and $P(t,x,\tau,\xi) = \tau^2 - a(t,\xi) + b_0(t)\tau + b(t,\xi) + c(t)$. Let us prove that the Cauchy problem (CP) is C^∞ well-posed if and only if

$$b_j^{(k)}(0) = 0 \quad \text{for } 1 \le j \le n \text{ and } k < [(m_j - 1)/2] \tag{7.2}$$

If (7.2) holds, then we have, with some $C > 0$,

$$\min_{\tau \in \mathscr{R}(\xi) \cup \{0\}} |t - \tau| \cdot |b(t,\xi)| \le C\sqrt{a(t,\xi)} \quad \text{for } (t,\xi) \in [0,1] \times S^{n-1}.$$

Therefore, it follows from Theorem 1.2 and its remark that the Cauchy problem (CP) is C^∞ well-posed. Assume that the Cauchy problem (CP) is C^∞ well-posed. Then, for a fixed j with $1 \le j \le n$ the Cauchy problem (CP) with $n = 1$ and $P(t,\tau,\xi_1)$ replaced by $P(t,\tau,\xi_1 e_j)$ is also C^∞ well-posed, where $e_j \in \boldsymbol{R}^n$ and the k-th component of e_j is equal to $\delta_{j,k}$ $(1 \le k \le n)$. It follows from [**4**] or the proof of Theorem 1.3 that (7.2) is valid.

References

[1] F. Colombini, H. Ishida and N. Orrú, On the Cauchy problem for finitely degenerate hyperbolic equations of second order, Ark. Mat., **38** (2000), 223–230.

[2] F. Colombini and T. Nishitani, On finitely degenerate hyperbolic operators of second order, Osaka J. Math., **41** (2004), 933–947.

[3] L. Hörmander, The Analysis of Linear Partial Differential Operators II, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

[4] V. Ja. Ivrii and V. Petkov, Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Uspehi Mat. Nauk, **29** (1974), 3–70. (Russian; English translation in Russian Math. Surveys.)

[5] K. Kajitani and S. Wakabayashi, The Cauchy problem for a class of hyperbolic operators with double characteristics, Funkcial. Ekvac., **39** (1996), 235–307.

[6] K. Kajitani and S. Wakabayashi, Microlocal *a priori* estimates and the Cauchy problem I, Japan. J. Math. (N.S.), **19** (1993), 353–418.

[7] S. Mizohata, Some remarks on the Cauchy problem, J. Math. Kyoto Univ., **1** (1961), 109–127.

[8] T. Nishitani, A necessary and sufficient condition for the hyperbolicity of second order equations in two independent variables, J. Math. Kyoto Univ., **24** (1984), 91–104.

[9] S. L. Svensson, Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part, Ark. Mat., **8** (1969), 145–162.
[10] S. Wakabayashi, Asymptotic expansions of the roots of the equations of pseudo-polynomials with a small parameter, located in http://www.math.tsukuba.ac.jp/˜wkbysh/
[11] S. Wakabayashi, Remarks on semi-algebraic functions, located in http://www.math.tsukuba.ac.jp/˜wkbysh/
[12] S. Wakabayashi, An alternative proof of Ivrii-Petkov's necessary condition for C^∞ well-posedness of the Cauchy problem, located in http://www.math.tsukuba.ac.jp/˜wkbysh
[13] S. Wakabayashi, Generalized flows and their applications, Proc. NATO ASI on Advances in Microlocal Analysis, Series C, D. Reidel, 1986, pp. 363–384.
[14] S. Wakabayashi, Singularities of solutions of the Cauchy problem for hyperbolic systems in Gevrey classes, Japan. J. Math. (N.S.), **11** (1985), 157–201.

Seiichiro Wakabayashi
Institute of Mathematics
University of Tsukuba
Tsukuba
Ibaraki 305-8571 Japan
E-mail: wkbysh@math.tsukuba.ac.jp