J. Math. Soc. Japan
Vol. 60, No. 4 (2008) pp. 1045–1064
doi: 10.2969/jmsj/06041045

On vanishing of certain Ext modules

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(Received Dec. 22, 2006)
(Revised Oct. 31, 2007)

Abstract. Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\dim R = 1$. In this paper, we shall prove that the module $\mathrm{Ext}_R^1(R/Q, R)$ does not vanish for every parameter ideal Q in R, if the embedding dimension $\mathrm{v}(R)$ of R is at most 4 and the ideal $\mathfrak{m}^2$ kills the $0^{\underline{th}}$ local cohomology module $H_{\mathfrak{m}}^0(R)$. The assertion is no longer true unless $\mathrm{v}(R) \leq 4$. Counterexamples are given. We shall also discuss the relation between our counterexamples and a problem on modules of finite G-dimension.

1. Introduction.

Throughout this paper let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim R$. The purpose of this research is to study the following problem concerning the vanishing of Ext modules. The motivation for the research comes from a conjecture posed by [**5**] on the modules of finite G-dimension.

QUESTION 1.1. Let M be an R-module of finite length. Then does it always hold true that $\mathrm{Ext}_R^d(M, R) \neq (0)$?

In [**5**] Takahashi studied a characterization of Gorenstein local rings in terms of G-dimension and posed the following conjecture: if a given Noetherian local ring R admits a non-zero R-module of finite length and of finite G-dimension, then the ring R would be Cohen-Macaulay. We can readily see that the conjecture holds true, if Question 1.1 has an affirmative answer. This is the reason why we are interested in Question 1.1. Later we shall closely discuss the relation between Question 1.1 and the conjecture.

In the present paper we shall restrict our attention on the following very special case of Question 1.1.

2000 *Mathematics Subject Classification.* Primary 13D07; Secondary 13D05.
Key Words and Phrases. vanishing of Ext, parameter ideal, G-dimension.
The first author was supported by the Grant-in-Aid for Scientific Research (C) (No. 19540054), Japan Society for the Promotion of Science.

Question 1.2. Assume that $d = 1$ and let Q be a parameter ideal in R. Then does it always hold true that $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$?

To the surprise of the authors, even in this case the answer is negative in general, while the answer is affirmative in certain special cases even if the base ring R is not Cohen-Macaulay, as we shall show in Section 5. Here let us summarize our conclusion into the following two theorems.

Theorem 1.3. *Let $d > 0$ be an integer. Then there exists a Noetherian local ring R such that* $\dim R = d$ *and* $\mathrm{Ext}_R^d(R/Q, R) = (0)$ *for some parameter ideal Q in R.*

Theorem 1.4. *Let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and* $\dim R = 1$. *Assume that $\mathfrak{m}^2 H_{\mathfrak{m}}^0(R) = (0)$, where $H_{\mathfrak{m}}^0(R)$ denotes the $0^{\underline{th}}$ local cohomology module of R. Then*

$$\mathrm{Ext}_R^1(R/Q, R) \neq (0)$$

for every parameter ideal Q in R, if $\mathrm{v}(R) \leq 4$. *Here* $\mathrm{v}(R) = \ell_R(\mathfrak{m}/\mathfrak{m}^2)$ *stands for the embedding dimension of R.*

Theorem 1.4 is no longer true unless $\mathrm{v}(R) \leq 4$. In Section 5 we shall construct examples, which show that for a given integer $v \geq 5$, there exists a parameter ideal Q in a certain one-dimensional Noetherian local ring $(R, \mathfrak{m})$ with the embedding dimension $\mathrm{v}(R) = v$ and $\mathfrak{m}^2 H_{\mathfrak{m}}^0(R) = (0)$, such that $\mathrm{Ext}_R^1(R/Q, R) = (0)$. Hence Question 1.2 does not hold true in general, and by adding indeterminates to the rings which are one-dimensional counterexamples, we have the negative answer Theorem 1.3 to Question 1.1 for arbitrary dimension $d > 0$.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1.4 in Section 3. For the purpose we need some preliminary results and some notation as well, which we will summarize in Section 2. In Section 4 we will explore some examples affirmative to Question 1.2, which do *not* satisfy conditions stated in Theorem 1.4. In Section 5 we shall prove Theorem 1.3, constructing counterexamples to Question 1.2. In the final Section 6 we will discuss the relation between our counterexamples constructed in Section 5 and the problem on the modules of finite G-dimension. We shall guarantee that the conjecture posed by the third author [**5**] remains open, showing that our counterexamples given in Section 5 are not counterexamples for the conjecture of the third author.

2. Preliminaries.

In this section, we shall summarize some preliminary results which we need to prove Theorem 1.4.

Let us fix our notation. Unless otherwise specified, let R be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\dim R = 1$. We set $W = H^0_{\mathfrak{m}}(R)$ the $0^{\underline{th}}$ local cohomology module and $\mathfrak{a} = (0) : W$ the annihilator of the ideal W. Note that W is the unmixed component of R, that is, $W = \bigcap_{\mathfrak{p} \in \mathrm{Min}\, R} \mathfrak{q}(\mathfrak{p})$, where $(0) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}\, R} \mathfrak{q}(\mathfrak{p})$ the primary decomposition of (0) in R. Also, unless otherwise specified, we denote by $Q = (a)$ the parameter ideal in R, and set $I = (0) : Q$. The parameter ideal Q is said to be *standard* if $QW = (0)$, that is, Q is contained in $\mathfrak{a}$. We denote by $\mu_R(M)$ the minimal number of generators of a finitely generated R-module M, i.e., $\mu_R(M) = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$. We denote by $\mathrm{v}(R)$ the *embedding dimension* of R, i.e., the minimal number of generators of the maximal ideal $\mathfrak{m}$.

Let us begin with the following.

LEMMA 2.1. *For every parameter ideal Q in R, one has an isomorphism*

$$\mathrm{Ext}^1_R(R/Q, R) \cong ((0) : I)/Q$$

of R-modules, where $I = (0) : Q$.

PROOF. Let $Q = (a)$. We first consider the following free resolution of R/Q

$$\cdots \longrightarrow R^l \xrightarrow{[x_1 \cdots x_l]} R \xrightarrow{a} R \longrightarrow R/Q \longrightarrow 0,$$

where $l = \mu_R(I)$ and $I = (x_1, \dots, x_l)$. Taking the R-dual of this resolution, we have a complex

$$0 \longrightarrow R \xrightarrow{a} R \xrightarrow{{}^t[x_1 \cdots x_l]} R^l \longrightarrow \cdots .$$

By this complex, we have an isomorphism $\mathrm{Ext}^1_R(R/Q, R) \cong ((0) : I)/Q$. $\square$

Consequently, Question 1.2 is the same as the following.

Does it always hold that $(0) : I \neq Q$?

We notice here that once a is a non-zero divisor on R, then $\mathrm{Ext}^1_R(R/Q, R) \neq (0)$, because $(0) : I = R \neq Q$. Hence $\mathrm{Ext}^1_R(R/Q, R) \neq (0)$ for every parameter ideal $Q = (a)$ in R, if R is a Cohen-Macaulay local ring.

The following assertions are easy but we shall use them frequently in this paper.

Lemma 2.2. *Let Q be a parameter ideal in R and $I = (0) : Q$. Then we have the following.*

(1) $Q \cap W = QW$.
(2) $I \subseteq W$.
(3) $\mathfrak{a} \subseteq (0) : I$.
(4) $Q : \mathfrak{m} \subseteq QW : I$.

Furthermore, if the ideal Q is standard, then we have

(5) $I = W$.
(6) $\mathfrak{a} = (0) : I$.
(7) $Q : \mathfrak{m} \subseteq (0) : I$.

Proof. Let $Q = (a)$. Since R/W is a 1-dimensional Cohen-Macaulay local ring, the parameter a is not a zero-divisor on R/W. Hence we have $Q \cap W = aW = QW$. (2) follows from the fact $QI = (0)$. (2) implies (3). (4) follows from the fact $I \subseteq \mathfrak{m}$ and assertions (1), (2). Assume $QW = (0)$, that is, $W \subseteq I$. Then (5) follows from (2). (5) implies (6). Assertion (7) follows from (4). $\square$

Proposition 2.3. *Let Q be a parameter ideal in R. Then we have*

$$\mathrm{Ext}_R^1(R/Q, R) \neq (0),$$

if either of the following conditions holds.

(1) *The ideal Q is standard.*
(2) *The ideal Q^2 is standard and $I = (0) : Q$ is contained in Q.*

Proof. (1) Suppose $\mathrm{Ext}_R^1(R/Q, R) = (0)$. Then we have the equality $Q : \mathfrak{m} = Q$ by Lemma 2.2 (7), which is impossible.

(2) Let $Q = (a)$. We consider the following exact sequence.

$$0 \longrightarrow R/(Q + I) \xrightarrow{a} R/Q^2 \longrightarrow R/Q \longrightarrow 0.$$

Since $I \subseteq Q$, the above short exact sequence yields an exact sequence

$$\mathrm{Ext}_R^1(R/Q, R) \longrightarrow \mathrm{Ext}_R^1(R/Q^2, R) \longrightarrow \mathrm{Ext}_R^1(R/Q, R).$$

Since the parameter ideal Q^2 is standard, we have $\mathrm{Ext}_R^1(R/Q^2, R) \neq (0)$ by (1).

Hence we get that $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$. $\square$

It is well known that every parameter ideal Q in a Buchsbaum local ring R is standard. So, by Proposition 2.3 (1), we have the following.

Corollary 2.4. *Assume that the ring R is Buchsbaum. Then we have*

$$\mathrm{Ext}_R^1(R/Q, R) \neq (0)$$

for every parameter ideal Q in R.

Theorem 2.5. *Suppose that a parameter ideal $Q = (a)$ is standard. Then, for any $z \in W$, the element $a+z$ is a parameter of R, and $\mathrm{Ext}_R^1(R/(a+z), R) \neq (0)$.*

Proof. Let $z \in W$. For every $\mathfrak{p} \in \mathrm{Min}\, R$, since $z \in \mathfrak{p}$, we have $a + z \notin \mathfrak{p}$. Hence $a + z$ is a parameter for R. We put $b = a + z$. Then we have equalities

$$\begin{aligned}(0) &= (b)[(0) : b] \\ &= a[(0) : b] + z[(0) : b] \\ &= z[(0) : b] \quad \text{(since } (0) : b \subseteq W \text{ and } a \text{ is standard).}\end{aligned}$$

Therefore $z \in (0) : ((0) : b)$. Suppose that $\mathrm{Ext}_R^1(R/(b), R) = (0)$. Then, since $\mathrm{Ext}_R^1(R/(b), R) \cong [(0) : ((0) : b)]/(b)$ by Lemma 2.1, we have $z \in (0) : ((0) : b) = (b)$. Hence we can write $z = by = (a+z)y$ for some $y \in R$. Then $y \in \mathfrak{m}$ because $b \notin W$. Since $z(1-y) = ay$ and $1-y$ is a unit in R, we have $z \in Q \cap W = QW = (0)$. Hence $b = a$, which is a contradiction by Proposition 2.3 (1). Therefore $\mathrm{Ext}_R^1(R/(b), R) \neq (0)$. $\square$

Proposition 2.6. *One has $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$ for every parameter ideal Q in R, if either of the following conditions holds.*

(1) *The ideal $\mathfrak{a}$ is not contained in $\mathfrak{m}^2$.*
(2) $W^2 = (0)$.

Proof. Suppose $\mathrm{Ext}_R^1(R/Q, R) = (0)$ for some parameter ideal Q in R.

(1) Take $x \in \mathfrak{a} \backslash \mathfrak{m}^2$. Since $\mathfrak{a} \subseteq (0) : I = Q$, we can write $x = ay$ for some $y \in R$. Then y is a unit in R. Hence $\mathfrak{a} = Q$. This implies that Q is standard. By Proposition 2.3 (1), this is impossible.

(2) By assumption $W^2 = (0)$, $W \subseteq \mathfrak{a} \subseteq (0) : I = Q$. Hence $W \subseteq Q \cap W = QW$ and we have $W = (0)$ by Nakayama's lemma. Therefore R is Cohen-Macaulay, which is a contradiction. $\square$

Before closing this section, let us give the following result.

Theorem 2.7. *If* $\mathrm{v}(R) \leq 2$, *then* $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$ *for every parameter ideal* Q *in* R.

Proof. We may assume that $\mathrm{v}(R) = 2$. Furthermore, passing to the completion, we may assume that R is complete. Then there exists a two-dimensional regular local ring S with the maximal ideal $\mathfrak{n}$ such that $R \cong S/J$, where J is an ideal in S whose height is one. Since we may assume that R is not Cohen-Macaulay, we can write $J = fL$ for some non-zero element $f \in \mathfrak{n}$ and some $\mathfrak{n}$-primary ideal L in S. Since $W \cong (f)/J$ is the unmixed component of R, we have $W \cong S/L$. Therefore $LR \subseteq (0) : W = \mathfrak{a}$. Here, suppose that $\mathrm{Ext}_R^1(R/Q, R) = (0)$ for some parameter ideal Q in R. Let $Q = gR$, where $g \in \mathfrak{n}$. Then, since $LR \subseteq \mathfrak{a} \subseteq (0) : I = Q$, we have $L \subseteq (g) + J$ and hence $L = [L \cap (g)] + J$. Since $J \subseteq \mathfrak{n}L$, $L = L \cap (g)$ by Nakayama's lemma. Hence we have $L \subseteq (g)$. But this is impossible because L is $\mathfrak{n}$-primary and $\dim S = 2$. □

3. Proof of Theorem 1.4.

The purpose of this section is to give a proof of Theorem 1.4. Recall that R is a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\dim R = 1$. Let $k = R/\mathfrak{m}$ be the residue field of R. We set $W = H_{\mathfrak{m}}^0(R)$ and $\mathfrak{a} = (0) : W$. We denote by $\mathrm{v}(R)$ the embedding dimension of R. With these notation and assumption, we shall prove the following.

Theorem 3.1. *Let* Q *be a parameter ideal in* R *and* $I = (0) : Q$. *Suppose that* $\mathfrak{m}^2 W = (0)$. *Then*

$$\mathrm{Ext}_R^1(R/Q, R) \neq (0),$$

if one of the following holds.

(1) $\mathrm{v}(R) \leq 4$.
(2) $\mu_R(W) \leq 1$.
(3) $\mu_R(I) \leq 1$.
(4) $\mathrm{v}(R/I) \leq 2$.

Proof. Suppose $\mathrm{Ext}_R^1(R/Q, R) = (0)$. Then, since $\mathrm{Ext}_R^1(R/Q, R) \cong ((0) : I)/Q$ by Lemma 2.1, we have $(0) : I = Q$. Let $Q = (a)$. We first note that $\mathfrak{m}^2 \subseteq \mathfrak{a}$ and Q^2 is standard by the assumption $\mathfrak{m}^2 W = (0)$. By Proposition 2.6 (1), we have $\mathfrak{a} \subseteq \mathfrak{m}^2$ and hence $\mathfrak{a} = \mathfrak{m}^2$. Also, $a \notin \mathfrak{m}^2$ because Q is not standard by Proposition 2.3 (1). Since $\mathfrak{m}^2 = \mathfrak{a} \subset (0) : I = Q$, we have $\mathfrak{m}^2 = a\mathfrak{m}$. Note that $\mathfrak{m}IW \subseteq \mathfrak{m}^2 W = (0)$. Hence $\mathfrak{m}W \subseteq ((0) : I) \cap W = Q \cap W = QW$. Therefore

$\mathfrak{m}W = QW$. Furthermore one can check that I is not contained in $\mathfrak{m}^2$. Indeed, if $I \subseteq \mathfrak{m}^2$, then $I \subseteq \mathfrak{m}^2 = \mathfrak{a} \subset Q$, which is impossible by Proposition 2.3 (2).

Now let $l = \ell_R((I + \mathfrak{m}^2)/\mathfrak{m}^2) > 0$ and take $x_1, \ldots, x_l \in I$ such that $\{x_i \bmod \mathfrak{m}^2 \mid 1 \leq i \leq l\}$ is a k-basis of $(I + \mathfrak{m}^2)/\mathfrak{m}^2$. Then we have the following.

Claim 1.

(i) *$a, x_1, \ldots, x_l$ is a part of a minimal system of generators for $\mathfrak{m}$.*
(ii) *The equality $(0) : (x_1, \ldots, x_l) = (0) : I$ holds.*

Proof of Claim.
(i) Let $\alpha, \beta_i \in R$ and suppose that $\alpha a + \sum_{i=1}^{l} \beta_i x_i \in \mathfrak{m}^2$. Since $\mathfrak{m}^2 = a\mathfrak{m}$, we can write $\alpha a + \sum_{i=1}^{l} \beta_i x_i = a\gamma$ for some $\gamma \in \mathfrak{m}$. Hence $(\alpha - \gamma)a \in (x_1, \ldots, x_l) \subseteq I = (0) : a$. If α is a unit in R, then $\alpha - \gamma$ is also a unit in R and hence $a \in I = (0) : a$ and $a^2 = 0$, which is impossible because a is a parameter. Therefore $\alpha \in \mathfrak{m}$ and hence each $\beta_i \in \mathfrak{m}$.

(ii) $I \subseteq (x_1, \ldots, x_l) + \mathfrak{m}^2$, so that the equality $I = [(x_1, \ldots, x_l) + \mathfrak{m}^2] \cap I = (x_1, \ldots, x_l) + [\mathfrak{m}^2 \cap I]$ holds. Since $\mathfrak{m}^2 \cap I \subseteq Q \cap W = aW = \mathfrak{m}W \subseteq (0) : \mathfrak{m}$, we then have the equality $(0) : (x_1, \ldots, x_l) = (0) : I$. $\square$

Let $\mathfrak{m} = (a, x_1, \ldots, x_l, x_{l+1}, \ldots, x_n)$, where $n + 1 = \mathrm{v}(R)$. Since $\mathfrak{m}^2 = a\mathfrak{m}$, we can write each $x_i x_j = a\delta_{ij}$ for some $\delta_{ij} \in \mathfrak{m}$. Then we may assume that $\delta_{ij} \in (a, x_{l+1}, \ldots, x_n)$ because $ax_i = 0$ for all $1 \leq i \leq l$. Let V be the k-subspace of $\mathfrak{m}/\mathfrak{m}^2$ spanned by $\{\delta_{ij} \bmod \mathfrak{m}^2 \mid 1 \leq i \leq l,\, 1 \leq j \leq n\}$ and let $q = \dim_k V$. Then

Claim 2. $q \leq n - l$.

Proof of Claim. It is clear that $q + l \leq n + 1$. Suppose $q + l = n + 1$. Then $(a, x_{l+1}, \ldots, x_n) \subseteq (\delta_{ij} \mid 1 \leq i \leq l,\ 1 \leq j \leq n) + \mathfrak{m}^2$, so that $(a, x_{l+1}, \ldots, x_n) = (\delta_{ij} \mid 1 \leq i \leq l,\ 1 \leq j \leq n) + [\mathfrak{m}^2 \cap (a, x_{l+1}, \ldots, x_n)]$. Since $\mathfrak{m}^2 = a\mathfrak{m}$, we have $(a, x_{l+1}, \ldots, x_n) = (\delta_{ij} \mid 1 \leq i \leq l,\ 1 \leq j \leq n) + \mathfrak{m}(a, x_{l+1}, \ldots, x_n)$. Hence $(a, x_{l+1}, \ldots, x_n) = (\delta_{ij} \mid 1 \leq i \leq l,\ 1 \leq j \leq n)$ by Nakayama's lemma and hence the equality

$$\mathfrak{m} = (x_1, \ldots, x_l) + (\delta_{ij} \mid 1 \leq i \leq l,\ 1 \leq j \leq n)$$

holds. Let $\mathfrak{p} \in \operatorname{Min} R$. Then $a \notin \mathfrak{p}$. Since $ax_i = 0$ for all $1 \leq i \leq l$, we have $(x_1, \ldots, x_l) \subseteq \mathfrak{p}$. Hence $a\delta_{ij} = x_i x_j \in \mathfrak{p}$ for all $1 \leq i \leq l$ and $1 \leq j \leq n$. Therefore $(\delta_{ij} \mid 1 \leq i \leq l,\ 1 \leq j \leq n) \subseteq \mathfrak{p}$. Hence $\mathfrak{m} \subseteq \mathfrak{p}$, which is impossible. Thus we have the inequality $q + l \leq n$. $\square$

For any n-elements $a_1, \ldots, a_n \in R$, we consider the following condition:

$$c_i := \sum_{j=1}^{n} a_j \delta_{ij} \in \mathfrak{m}^2 \quad \text{for all} \quad 1 \le i \le l. \tag{3.1.1}$$

The elements $a_1, \dots, a_n \in R$ satisfying condition 3.1.1 have the following property.

Claim 3. *If the elements $a_1, \dots, a_n \in R$ satisfy condition* 3.1.1*, then $a_i \in \mathfrak{m}$ for all $1 \le i \le n$.*

Proof of Claim. For any $1 \le i \le l$,

$$ac_i = \sum_{j=1}^{n} a_j(a\delta_{ij}) = \sum_{j=1}^{n} a_j(x_i x_j) = x_i \sum_{j=1}^{n} a_j x_j.$$

Hence we have

$$a^2 c_i = a x_i \sum_{j=1}^{n} a_j x_j = 0,$$

because $ax_i = 0$ for all $1 \le i \le n$. Therefore each c_i belongs to W. Since $a_1, \dots, a_n$ satisfy condition 3.1.1, $c_i \in \mathfrak{m}^2 \cap W \subseteq Q \cap W = aW$. Hence $ac_i = x_i\big(\sum_{j=1}^{n} a_j x_j\big) = 0$ for all $1 \le i \le l$. Therefore we have

$$\sum_{j=1}^{n} a_j x_j \in (0) : (x_1, \dots, x_l) = (0) : I = Q.$$

We write $\sum_{j=1}^{n} a_j x_j = az$ for some $z \in R$. Then $a_j \in \mathfrak{m}$ for all $1 \le j \le n$, because the set $\{a, x_1, \dots, x_n\}$ is a minimal system of generators for $\mathfrak{m}$. □

Claim 4. *$ql \ge n$. Hence we have $l \ge 2$ and $n - l \ge 2$.*

Proof of Claim. We take $\xi_1, \dots, \xi_q \in (\delta_{ij} \mid 1 \le i \le l,\ 1 \le j \le n)$ such that $\{\xi_k \bmod \mathfrak{m}^2 \mid 1 \le k \le q\}$ is a k-basis of V and write

$$\delta_{ij} \equiv \sum_{k=1}^{q} c_j^{ik} \xi_k \bmod \mathfrak{m}^2,$$

where $c_j^{ik} \in R$. We now consider the following system of linear equations in variables $y_1, \dots, y_n$ over $k = R/\mathfrak{m}$:

$$\sum_{j=1}^{n} \overline{c_j^{ik}} y_j = 0 \quad (1 \le i \le l,\ 1 \le k \le q), \tag{3.1.2}$$

where $\overline{*}$ denotes the reduction of $* \bmod \mathfrak{m}$. Suppose $\overline{a_1}, \dots, \overline{a_n} \in k$ is a solution of 3.1.2. Then $\sum_{j=1}^{n} a_j c_j^{ik} \in \mathfrak{m}$ for all $1 \le i \le l,\ 1 \le k \le q$. Therefore we have

$$c_i = \sum_{j=1}^{n} a_j \delta_{ij} \equiv \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{q} c_j^{ik} \xi_k \right) = \sum_{k=1}^{q} \left(\sum_{j=1}^{n} a_j c_j^{ik} \right) \xi_k \equiv 0 \bmod \mathfrak{m}^2.$$

Hence the elements $a_1, \dots, a_n$ satisfy condition 3.1.1. Thus $\overline{a_j} = 0$ for all $1 \le j \le n$ by Claim 3. Therefore 3.1.2 has the only trivial solution, which shows that $ql \ge n$. This implies $l \ge 2$ and $n - l \ge 2$. Indeed, if $l = 1$, then $n \le q \le n - 1$ by Claim 2. This is impossible. Also, if $n - l \le 1$, then $n \le ql \le (n-l)l \le l \le n$. Hence $n = ql = (n-l)l = l$. Therefore $q = 1$ and $n = l$. Again, by Claim 2, this is impossible. $\square$

Now suppose $\mathrm{v}(R) \le 4$. Then $\mathrm{v}(R) - 1 = n = (n-l) + l \ge 4$, which is a contradiction. Suppose $\mu_R(W) \le 1$. Since $(I + \mathfrak{m}^2)/\mathfrak{m}^2 \subseteq (W + \mathfrak{m}^2)/\mathfrak{m}^2 = W/\mathfrak{m}W$, it follows $l \le 1$, which is a contradiction. If $\mu_R(I) \le 1$, then $l \le 1$. If $\mathrm{v}(R/I) = \ell_R(\mathfrak{m}/(I + \mathfrak{m}^2)) \le 2$, then $l = \ell_R((I + \mathfrak{m}^2)/\mathfrak{m}^2) \ge \mathrm{v}(R) - 2 = (n+1) - 2 = n - 1$. Hence $n - l \le 1$, which is a contradiction.

Consequently, we have $\mathrm{Ext}_R^1(R/Q, R) \ne (0)$. This is a proof of Theorem 3.1. $\square$

As a direct consequence, we have the following, which is Theorem 1.4.

COROLLARY 3.2. *If* $\mathfrak{m}^2 W = (0)$ *and* $\mathrm{v}(R) \le 4$*, we then have*

$\mathrm{Ext}_R^1(R/Q, R) \ne (0)$ *for every parameter ideal* Q *in* R.

REMARK 3.3. The assumption in Corollary 3.2 is the best possible. Indeed, in Section 5, we shall construct examples, showing that for a given integer $v \ge 5$, there exists a one-dimensional Noetherian local ring $(R, \mathfrak{m})$ with the embedding dimension $\mathrm{v}(R) = v$ and $\mathfrak{m}^2 W = (0)$, which contains a parameter ideal Q such that $\mathrm{Ext}_R^1(R/Q, R) = (0)$.

4. Affirmative examples.

In this section, we shall give some affirmative examples. First, we give the following example, which follows from Theorem 1.4.

Example 4.1. Let k be a field and let $S = k[[X, Y, Z]]$ be a formal power series ring. Set $U = (X, Y)$, $L = (X^2, XY - YZ, Y^2 - XZ, Z^2)$ and $J = U \cap L$. We put $R = S/J$. Then $\dim R = 1$ and we have $\operatorname{Ext}_R^1(R/Q, R) \neq (0)$ for every parameter ideal Q in R.

Proof. One can check that $J = (X^2, XY - YZ, Y^2 - XZ, XZ^2, YZ^2)$. Let x, y denote respectively the reductions of X, Y mod J. Then, since the unmixed component of R is $\mathfrak{p} := (x, y)$, we have $W = \mathfrak{p}$. It is easy to see that $\mathrm{v}(R) = 3$ and $\mathfrak{m}^2 W = (0)$. Hence $\operatorname{Ext}_R^1(R/Q, R) \neq (0)$ for every parameter ideal Q in R by Theorem 1.4. $\square$

To construct another class of affirmative examples, we need the following.

Proposition 4.2. *Let $(S, \mathfrak{n})$ be a regular local ring with $\dim S > 0$. Let U, L be ideals in S satisfying the following three conditions.*

(i) *The ring S/U is a one-dimensional Cohen-Macaulay ring.*
(ii) *The ideal L is an $\mathfrak{n}$-primary ideal.*
(iii) *The ideal $J := U \cap L$ is contained in $\mathfrak{n}^2$.*

We put $R = S/J$. Then $\dim R = 1$ and we have

$$\operatorname{Ext}_R^1(R/Q, R) \neq (0) \text{ for every parameter ideal } Q \text{ in } R,$$

if either of the following conditions holds.

(1) *The ideal L is not contained in $\mathfrak{n}^2$.*
(2) *The ideal $J = U \cap L$ is contained in $\mathfrak{n}L$.*

Proof. We may assume that the ideal U is not contained in L. Since $W \cong U/J$, we have $LR \subseteq (0) : W = \mathfrak{a}$. Suppose L is not contained in $\mathfrak{n}^2$. Then $\mathfrak{a}$ is not contained in $\mathfrak{m}^2$, since $J = U \cap L \subseteq \mathfrak{n}^2$. By Proposition 2.6 (1), we have that $\operatorname{Ext}_R^1(R/Q, R) \neq (0)$ for every parameter ideal Q in R. Suppose $J \subseteq \mathfrak{n}L$. Assume the contrary and choose a parameter ideal $Q = fR$ in R such that $(0) : I = Q$, where $f \in \mathfrak{n}$. Since $\mathfrak{a} \subseteq (0) : I = Q$, we get $L \subseteq (f) + J$. Hence $L = [(f) \cap L] + \mathfrak{n}L$. By Nakayama's lemma, we have $L = (f) \cap L$. Therefore $\dim S = 1$ and hence $U = (0)$. This is a contradiction. $\square$

Using this, we have the following simple affirmative example, which does not follow from Theorem 1.4.

Example 4.3. Let $n > 0$ and $m > l > 0$ be integers. Let k be a field and let $S = k[[X_1, X_2, \ldots, X_n, Z]]$ be a formal power series ring. Set $U = (X_1^l, \ldots, X_n^l)$,

$L = (X_1^m, \ldots, X_n^m, Z)$ and $J = U \cap L$. Then $J \subseteq \mathfrak{n}^2$ and L is not contained in $\mathfrak{n}^2$, where $\mathfrak{n} = (X_1, X_2, \ldots, X_n, Z)$. Hence, for every parameter ideal Q in $R := S/J$, we have $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$.

The following example satisfies neither of the assumptions of Theorem 1.4 and Proposition 4.2. But $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$ holds for every parameter ideal Q in R.

EXAMPLE 4.4. Let k be a field and let $S = k[[X, Y, Z]]$ be a formal power series ring. Set $U = (X, Y)$, $L = (X^2, Y^2, Z^2)$ and $J = U \cap L$. We put $R = S/J$. Then $\dim R = 1$ and we have $\mathrm{Ext}_R^1(R/Q, R) \neq (0)$ for every parameter ideal Q in R.

PROOF. One can check that $J = (X^2, Y^2, XZ^2, YZ^2)$. Let x, y, z denote respectively the reduction of X, Y, Z mod J. Since the unmixed component of R is $\mathfrak{p} := (x, y)$, we have $W = \mathfrak{p}$. Then it is easy to see that $\mathfrak{m}^2 W \neq (0)$ and $\mathfrak{m}^3 W = (0)$. Suppose that there exists a parameter ideal $Q = (a)$ in R such that $\mathrm{Ext}_R^1(R/Q, R) = (0)$. We may assume that $a = z^n + b$ where $n > 0$ and $b \in \mathfrak{p}$. Furthermore, since z^2 is standard, we may assume that $a = z + b$. Indeed, if $a = z^n + b$ for some $n \geq 2$, then $\mathrm{Ext}_R^1(R/(a), R) \neq (0)$ by Theorem 2.5, because z^n is standard and $b \in W$.

Since $xyW = (0)$, $xy \in (0) : W = \mathfrak{a} \subseteq (0) : I = Q$. We write $xy = ac$ for some $c \in R$. Then $c \in W$ because $ac \in W = \mathfrak{p}$ and $a \notin \mathfrak{p}$. Here we write

$$b = b_1 x + b_2 y + b_3 xy + b_4 yz + b_5 zx + b_6 xyz,$$
$$c = c_1 x + c_2 y + c_3 xy + c_4 yz + c_5 zx + c_6 xyz,$$

where $b_i, c_i \in k$ for all $1 \leq i \leq 6$. Then we have

$$\begin{aligned} xy &= ac \\ &= (z + b)c \\ &= (b_1 c_2 + b_2 c_1)xy + c_2 yz + c_1 zx + (c_3 + b_1 c_4 + b_2 c_5 + b_4 c_1 + b_5 c_2)xyz. \end{aligned}$$

Since $\{xy, yz, zx, z^2\}$ is a minimal system of generators for $\mathfrak{m}^2$, we then have $c_1 = c_2 = 0$ and $b_1 c_2 + b_2 c_1 = 1$. This is a contradiction. $\square$

5. Counterexamples.

In this section, we will consider constructing examples which give a negative answer to Question 1.2.

Let n, l be integers with $2 \leq l \leq n-2$. Let k be a field, $S = k[X_1, \dots, X_n, A]$ a polynomial ring. The ring S is a $\boldsymbol{Z}$-graded ring with $S_0 = k$ and $\deg X_i = \deg A = 1$ for $1 \leq i \leq n$. Set $V = \sum_{j=l+1}^{n} kX_j$. Note that $\dim_k S_1 = n+1 \geq 5$.

Lemma 5.1. *There exists an $l \times n$ matrix $\Delta = (\Delta_{i,j})$ over S which satisfies the following.*

(1) *The submatrix $(\Delta_{i,j})_{1 \leq i,j \leq l}$ is symmetric.*
(2) $V = \sum_{1 \leq i \leq l,\ 1 \leq j \leq n} k\Delta_{i,j}$.
(3) *If $c_1, \dots, c_n \in k$ satisfies $\Delta \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0$, then $c_1 = \cdots = c_n = 0$.*

Proof. If $l \leq n-l$, then set

$$\Delta = \begin{pmatrix} X_{l+1} & X_{l+2} & \cdots & X_{2l} & 0 & \cdots & 0 \\ X_{l+2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ X_{2l-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ X_{2l} & 0 & \cdots & 0 & X_{l+1} & \cdots & X_n \end{pmatrix}.$$

It is easy to see that this matrix satisfies all the conditions in the lemma.

Let us consider the case where $l > n-l$. Put $\alpha = n-l$. We can write $l = \alpha q + r$ for some $0 \leq r < \alpha$.

Claim. *One has $0 < q \leq l-2$.*

Proof of Claim. If $q = 0$, then $l = r < \alpha$, which is a contradiction. Hence $q > 0$. Assume $q > l-2$. Then $q \geq l-1$, and we have $l-r = \alpha q \geq \alpha(l-1) \geq 2(l-1)$ since $\alpha \geq 2$. Hence $2 \leq l \leq 2-r \leq 2$. Therefore we obtain $l = 2$ and $r = 0$. It follows that $2 = \alpha q$. Since $\alpha \geq 2$, we get $\alpha = 2 = l > \alpha$. This is a contradiction. □

We construct a matrix Δ as follows:

$$\begin{cases} \Delta_{i,j} = \Delta_{j,i} = X_{l+j-\alpha(i-1)} & \text{if } 1 \leq i \leq q \text{ and } \alpha(i-1) < j \leq \alpha i, \\ \Delta_{q+1,j} = \Delta_{j,q+1} = X_{j+r} & \text{if } \alpha q < j \leq l, \\ \Delta_{l,j} = X_j & \text{if } l < j, \\ \Delta_{i,j} = 0 & \text{otherwise.} \end{cases}$$

Then we can check that this matrix Δ satisfies the three conditions in the lemma. □

Let Δ be a matrix satisfying the conditions in Lemma 5.1. We define an ideal J of S as follows:

$$J = (AX_1, \dots, AX_l) + (X_{l+1}, \dots, X_n)^2 + (X_iX_j - A\Delta_{i,j})_{1 \le i \le l,\, 1 \le j \le n}.$$

Since the submatrix $(\Delta_{i,j})_{1 \le i,j \le l}$ is symmetric, we have

$$\begin{aligned} J = {} & (AX_1, \dots, AX_l) + (X_{l+1}, \dots, X_n)^2 \\ & + (X_iX_j - A\Delta_{i,j})_{1 \le i \le j \le l} + (X_iX_j - A\Delta_{i,j})_{1 \le i \le l,\, l+1 \le j \le n}. \end{aligned}$$

Set $N = S_+ = \bigoplus_{m>0} S_m \subseteq S$. Lemma 5.1 says that $\Delta_{i,j}$ is in V. Since V is contained in S_1, the ideal J is graded and contained in N^2. Put $R = S/J$ and $M = R_+ = N/J$. Let $a, x_i, \delta_{i,j}$ be the residue classes of $A, X_i, \Delta_{i,j}$ in R, respectively.

Proposition 5.2. *One has the following.*

(1) $\dim R = 1$ *and* $\operatorname{Min} R = \{\mathfrak{p}\}$, *where* $\mathfrak{p} = (x_1, \dots, x_n)$.
(2) $M^2 = aM$, $M^3 = (a^3)$ *and* $M^2W = (0)$.
(3) $W = \mathfrak{p}$ *and* $W_i = (0)$ *for all* $i \ge 3$.

Proof. (1) We make a claim.

Claim. *One has* $\sqrt{J} = (X_1, \dots, X_n)$.

Proof of Claim. Note that $(X_1, \dots, X_n)$ is a radical ideal of S. Hence it is enough to show that $V(J) = V(X_1, \dots, X_n)$. Let $P \in V(J)$. Since $(X_{l+1}, \dots, X_n)^2$ is contained in J, the elements $X_{l+1}, \dots, X_n$ belong to P. Hence V is contained in P, and we have all $\Delta_{i,j}$ are in P by Lemma 5.1(2). As $X_iX_j - A\Delta_{i,j}$ is in J, all X_iX_j are in P. In particular, X_i^2 is in P for $1 \le i \le l$. Thus $X_i \in P$ for $1 \le i \le l$.

Conversely, let $P \in V(X_1, \dots, X_n)$. Then V is contained in P, and Lemma 5.1(2) says that all $\Delta_{i,j}$ are in P. Hence all $X_iX_j - A\Delta_{i,j}$ are in P, and therefore J is contained in P. □

It follows from the above claim that $\dim R = \dim S/J = \dim S/\sqrt{J} = \dim k[A] = 1$ and that $\min V(J) = \min V(\sqrt{J}) = \min V(X_1, \dots, X_n) = \{(X_1, \dots, X_n)\}$, hence $\operatorname{Min} R = \{\mathfrak{p}\}$.

(3) We begin with making the following claim.

Claim 1. *One has* $a^2x_i = 0$ *for* $1 \le i \le n$.

Proof of Claim. The claim is obvious for $1 \leq i \leq l$, so let $l+1 \leq i \leq n$. Then X_i is in $V = \sum_{1 \leq \alpha \leq l, 1 \leq \beta \leq n} k\Delta_{\alpha,\beta}$, and we have $X_i = \sum_{\alpha,\beta} c_{\alpha,\beta}\Delta_{\alpha,\beta}$ for some $c_{\alpha,\beta} \in k$. Hence $x_i = \sum_{\alpha,\beta} c_{\alpha,\beta}\delta_{\alpha,\beta}$, and we get $ax_i = \sum_{\alpha,\beta} c_{\alpha,\beta}(a\delta_{\alpha,\beta}) = \sum_{\alpha,\beta} c_{\alpha,\beta}(x_\alpha x_\beta)$. Therefore we obtain $a^2 x_i = \sum_{\alpha,\beta} c_{\alpha,\beta}(ax_\alpha)x_\beta = 0$ since $ax_\alpha = 0$. $\square$

Note that $R/(a)$ is artinian. Hence a is a homogeneous parameter of R. The above claim shows that $a^2\mathfrak{p} = (0)$. Since (a) is M-primary, we have $M^s\mathfrak{p} = (0)$ for some $s > 0$. It follows that $\mathfrak{p}$ is contained in W. On the other hand, as the ideal W has finite length, it is nilpotent. Therefore W is contained in $\mathfrak{p}$, and thus $W = \mathfrak{p}$.

Here we make the following two claims:

Claim 2. *One has $ax_ix_j = 0$ for $1 \leq i,\ j \leq n$.*

Proof of Claim. It holds that $ax_m = 0$ if $1 \leq m \leq l$. Hence we may assume $l+1 \leq i,j \leq n$. Since $(x_{l+1}, \ldots, x_n)^2 = (0)$, we have $x_ix_j = 0$. $\square$

Claim 3. *One has $x_ix_jx_h = 0$ for $1 \leq i,j,h \leq n$.*

Proof of Claim. The claim holds if $l+1 \leq i,j,h \leq n$ since $(x_{l+1}, \ldots, x_n)^2 = (0)$. Let $1 \leq i \leq l$. Then $x_ix_j = a\delta_{i,j}$, and $x_ix_jx_h = a\delta_{i,j}x_h$. Note that $\delta_{i,j}$ is in $kx_{l+1} + \cdots + kx_n$. If $l+1 \leq h \leq n$, then $\delta_{i,j}x_h = 0$ as $(x_{l+1}, \ldots, x_n)^2 = (0)$, and we get $x_ix_jx_h = 0$. If $1 \leq h \leq l$, then $ax_h = 0$, and $x_ix_jx_h = 0$. $\square$

It follows from Claims 1, 2 and 3 that $W_i = (0)$ for all $i \geq 3$.

(2) Since $M = (x_1, \ldots, x_n, a) = \mathfrak{p} + (a)$, we have $M^2 = aM + \mathfrak{p}^2$. For integers α, β with $1 \leq \alpha \leq \beta \leq n$, we have

$$x_\alpha x_\beta = \begin{cases} a\delta_{\alpha,\beta} & (\alpha \leq l), \\ 0 & (l+1 \leq \alpha), \end{cases}$$

which is in aM. Thus $M^2 = aM$, and $M^3 = aM^2 = a^2M = a^2\mathfrak{p} + (a^3) = (a^3)$ by Claim 1. Added to it, we have $M^2W = M^2\mathfrak{p} = aM\mathfrak{p} = a(\mathfrak{p} + (a))\mathfrak{p} = a\mathfrak{p}^2 = (0)$ by Claim 2. $\square$

Lemma 5.3. *The elements $ax_{l+1}, \ldots, ax_n, a^2$ form a k-basis of R_2.*

Proof. First of all, we claim the following.

Claim. *One has $J + (AX_{l+1}, \ldots, AX_n, A^2) = N^2$.*

PROOF OF CLAIM. Put $L = J + (AX_{l+1}, \dots, AX_n, A^2)$. It is obvious that L is contained in N^2. Fix integers α, β with $1 \le \alpha \le \beta \le n$. If $l + 1 \le \alpha$, then $X_\alpha X_\beta$ is in J, hence in L. If $\alpha \le l$, then $X_\alpha X_\beta - A\Delta_{\alpha,\beta}$ is in J. Since $\Delta_{\alpha,\beta}$ is in V, the element $A\Delta_{\alpha,\beta}$ is in the ideal $A(X_{l+1}, \dots, X_n)$, which is contained in L. Hence $X_\alpha X_\beta \in L$. Thus the element $X_\alpha X_\beta$ is in L for $1 \le \alpha \le \beta \le n$. Added to it, we have $NA = (AX_1, \dots, AX_l) + (AX_{l+1}, \dots, AX_n, A^2) \subseteq L$. Consequently, the ideal $N^2 = (X_1, \dots, X_n)^2 + NA$ is contained in L. $\square$

The above claim implies that $M^2 = (ax_{l+1}, \dots, ax_n, a^2)$. Hence we have $R_2 = (M^2)_2 = k \cdot ax_{l+1} + \cdots + k \cdot ax_n + k \cdot a^2$. Therefore $\dim_k R_2 \le \alpha + 1$, where $\alpha = n - l$. Assume $\dim_k R_2 < \alpha + 1$. Then $\dim_k S_2 = \dim_k R_2 + \dim_k J_2 < (\alpha + 1) + \dim_k J_2$. We have

$$\begin{aligned} J = {} & \underbrace{(AX_1, \dots, AX_l)}_{l} + \underbrace{(X_{l+1}, \dots, X_n)^2}_{\frac{\alpha(\alpha+1)}{2}} \\ & + \underbrace{(X_iX_j - A\Delta_{i,j})_{1 \le i \le j \le l}}_{\frac{l(l+1)}{2}} + \underbrace{(X_iX_j - A\Delta_{i,j})_{1 \le i \le l,\, l+1 \le j \le n}}_{l\alpha}. \end{aligned}$$

Hence $\dim_k J_2 \le l + \alpha(\alpha+1)/2 + l(l+1)/2 + l\alpha = (n^2 + n + 2l)/2$, and therefore $\dim_k S_2 < (\alpha + 1) + (n^2 + n + 2l)/2 = (n+1)(n+2)/2 = \dim_k S_2$. This is a contradiction, and it must hold that $\dim_k R_2 = \alpha + 1$. It follows that $ax_{l+1}, \dots, ax_n, a^2$ form a k-basis of R_2. $\square$

Now we are in the position to state and prove the main result of this section.

THEOREM 5.4. *The element a is a homogeneous parameter of R satisfying $(0) : ((0) : a) = (a)$, namely,* $\mathrm{Ext}^1_R(R/(a), R) = (0)$.

PROOF. Set $I = (0) : a$. Let us prove the theorem step by step.

STEP 1. The ideal I is contained in W.

Indeed, since (a) is an M-primary ideal, there is an integer $r > 0$ such that M^r is contained in (a). Since $aI = (0)$, we have $M^r I = (0)$.

STEP 2. We have $I = I_1 + I_2$.

Indeed, according to Proposition 5.2(3), it holds that $W = W_0 + W_1 + W_2$. Note that $W_0 = W \cap R_0 = W \cap k = (0)$. Hence $W = W_1 + W_2$. Since I is contained in W, we have $I = I_1 + I_2$.

STEP 3. We have $I_2 = W_2$.

In fact, since I is contained in W, I_2 is contained in W_2. Proposition 5.2(3) shows that $MW_2 = (0)$, hence $aW_2 = (0)$. Thus W_2 is contained in $I = (0) : a$. Hence W_2 is contained in I_2.

Step 4. We have $I_2 \subseteq (x_1, \ldots, x_l)$.

In fact, since $I_2 = W_2$, it suffices to check that W_2 is contained in $(x_1, \ldots, x_l)$. Let $\phi \in W_2$. Note that $W_2 = \mathfrak{p}_2 = \mathfrak{p} \cap R_2 = (x_1, \ldots, x_n) \cap R_2$. Hence we can write $\phi = \sum_{i=1}^{n} x_i \xi_i$ for some $\xi_i \in R_1$. Let us show that $x_i \xi_i$ is in $(x_1, \ldots, x_l)$ for $1 \leq i \leq n$. This is trivial if $1 \leq i \leq l$, so let $l+1 \leq i \leq n$. Then $x_i x_j = 0$ for $l+1 \leq j \leq n$, and hence the element $x_i x_j$ is in $(x_1, \ldots, x_l)$ for $1 \leq j \leq n$. As we saw in the proof of Claim 1 in the proof of Proposition 5.2, we can write $ax_i = \sum_{\alpha,\beta} c_{\alpha,\beta}(x_\alpha x_\beta)$ for some $c_{\alpha,\beta} \in k$. Hence $ax_i \in (x_1, \ldots, x_l)$. Since ξ_i belongs to $R_1 = kx_1 + \cdots + kx_n + ka$, we obtain $x_i \xi_i \in (x_1, \ldots, x_l)$.

Step 5. We have $I = (x_1, \ldots, x_l)$.

Indeed, as $ax_i = 0$ for $1 \leq i \leq l$, the ideal $(x_1, \ldots, x_l)$ is contained in $I = (0) : a$. Suppose that $(x_1, \ldots, x_l)$ is strictly contained in I, and choose a homogeneous element $\phi \in I - (x_1, \ldots, x_l)$. Since $I = I_1 + I_2$, the element ϕ is in either I_1 or I_2. However I_2 is contained in $(x_1, \ldots, x_l)$, ϕ must be in I_1, hence in $W_1 = \mathfrak{p}_1 = kx_1 + \cdots + kx_n$. Therefore $\phi = \psi + \sum_{i=l+1}^{n} c_i x_i$ for some $\psi \in kx_1 + \cdots + kx_l$ and $c_i \in k$. We have $0 = a\phi = \sum_{i=l+1}^{n} c_i(ax_i)$ since $ax_j = 0$ for $1 \leq j \leq l$. Lemma 5.3 shows that $c_i = 0$ for $l+1 \leq i \leq n$, and $\phi = \psi \in (x_1, \ldots, x_l)$. This is a contradiction.

Now, we shall prove that $(0) : ((0) : a) = (a)$. It is trivial that $(0) : ((0) : a)$ contains (a). Suppose that $(0) : ((0) : a) = (0) : I$ strictly contains the ideal (a), and choose a homogeneous element $\phi \in ((0) : I) - (a)$. Then, since the ideal $M^2 = aM$ is contained in (a) and ϕ is not in (a), ϕ is not in M^2. Hence $\deg \phi \leq 1$. Assume that $\deg \phi = 0$. Then ϕ is in k and is nonzero. Since $\phi I = (0)$, we have $(0) = I = (x_1, \ldots, x_l)$, which is a contradiction. Thus $\deg \phi = 1$, equivalently, the element ϕ is in R_1. We can write $\phi = \sum_{j=1}^{n} c_j x_j + ca$ for some $c_j, c \in k$. It holds that $(0) = \phi I = \phi(x_1, \ldots, x_l)$, and $0 = \phi x_i = \sum_{j=1}^{n} c_j(x_i x_j) + c(ax_i) = \sum_{j=1}^{n} c_j(a\delta_{i,j}) = a \sum_{j=1}^{n} c_j \delta_{i,j}$ for $1 \leq i \leq l$. Hence $\sum_{j=1}^{n} c_j \delta_{i,j} \in ((0) : a) = I = (x_1, \ldots, x_l)$. Noting that $\delta_{i,j}$ is in $V = kx_{l+1} + \cdots + kx_n$, we see that $\sum_j c_j \delta_{i,j} = 0$ for $1 \leq i \leq l$, and thus

$$\Delta \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = 0.$$

By Lemma 5.1(3) we have $c_i = 0$ for $1 \leq i \leq n$, and $\phi = ca \in (a)$, which is a contradiction. This contradiction completes the proof of the theorem. $\square$

6. Modules of finite G-dimension.

In this section, we will consider a problem on modules of finite G-dimension. We start by recalling the definition of G-dimension.

Definition 6.1. Let R be a Noetherian ring.

(1) Let $(-)^*$ denote the R-dual functor $\mathrm{Hom}_R(-, R)$. A finitely generated R-module M is said to be *totally reflexive* if M is isomorphic to M^{**} and $\mathrm{Ext}_R^i(M \oplus M^*, R) = (0)$ for all $i > 0$.
(2) The *Gorenstein dimension* (*G-dimension* for short) of a nonzero R-module M, which is denoted by $\mathrm{Gdim}_R\, M$, is defined as the infimum of integers r such that there exists an exact sequence

$$0 \to X_r \to X_{r-1} \to \cdots \to X_0 \to M \to 0$$

of R-modules, where each X_i is totally reflexive. The G-dimension of the zero module is defined as $-\infty$.

It is known that G-dimension has the following properties. For the details, see [**1**] and [**3**].

Proposition 6.2. *Let R be a Noetherian ring and M a finitely generated R-module. Then the following statements hold.*

(1) *There is an inequality* $\mathrm{Gdim}_R\, M \leq \mathrm{pd}_R\, M$.
(2) *If* $\mathrm{Gdim}_R\, M < \infty$, *then* $\mathrm{Gdim}_R\, M = \mathrm{depth}\, R - \mathrm{depth}_R\, M$.
(3) *If* $\mathrm{Gdim}_R\, M < \infty$, *then* $\mathrm{Gdim}_R\, M = \sup\{i \mid \mathrm{Ext}_R^i(M, R) \neq (0)\}$.

The third author gave the following conjecture in [**5**].

Conjecture 6.3. Let R be a Noetherian local ring. Suppose that there exists an R-module M of finite length and finite G-dimension. Then R is Cohen-Macaulay.

It is well-known that the statement with "G-dimension" replaced by "projective dimension" holds; it follows from the Peskine-Szpiro intersection theorem (cf. [**4**, Proposition 6.2.4]).

Let R be a d-dimensional Noetherian local ring, and M an R-module of finite length and finite G-dimension. Then one has $\mathrm{Gdim}_R\, M = \mathrm{depth}\ R - \mathrm{depth}_R\, M = \mathrm{depth}\ R$ by Proposition 6.2(2), and hence $\mathrm{Ext}_R^i(M, R) = (0)$ for $i > \mathrm{depth}\ R$ by Proposition 6.2(3). Therefore, if the R-module M satisfies

$$\mathrm{Ext}_R^d(M, R) \neq (0),$$

then one must have $d \leq \operatorname{depth} R$, that is to say, R is Cohen-Macaulay. So, if Question 1.1 has an affirmative answer, then the above conjecture is true. However, as we have already seen in the previous section, Question 1.1 does not have an affirmative answer.

Now, we are interested in whether the example which we constructed in the previous section is a counterexample to the above conjecture or not. The main result of this section is the following proposition, which says that it is not a counterexample.

Proposition 6.4. *Let R be the ring and a the homogeneous parameter of R which are constructed in Section 5. Then R is a standard graded algebra over a field with $\dim R = 1$ and $\operatorname{depth} R = 0$ (hence R is not Cohen-Macaulay) and $\mathrm{Ext}_R^1(R/(a), R) = (0)$, but the R-module $R/(a)$ is not of finite G-dimension.*

For a graded ring R and a graded R-module M, we denote by $H_M(t)$ the Hilbert series of M. To prove the above proposition, we prepare the following result, which is the main theorem in [**2**].

Theorem 6.5 (Avramov-Buchweitz-Sally). *Let k be a field and R a positively graded k-algebra. Let M, N be finitely generated graded R-modules with $\mathrm{Ext}_R^i(M, N) = (0)$ for $i \gg 0$. Then*

$$\sum_i (-1)^i H_{\mathrm{Ext}_R^i(M,N)}(t) = \frac{H_M(t^{-1}) \cdot H_N(t)}{H_R(t^{-1})}.$$

The lemma below follows from this theorem.

Lemma 6.6. *Let R be a positively graded algebra over a field k. Let M be a graded totally reflexive R-module of finite length. Then*

$$\ell_R(M) = \ell_R(M^*),$$

where $(-)^* = \mathrm{Hom}_R(-, R)$.

Proof. Since $\mathrm{Ext}_R^i(M, R) = (0)$ for $i > 0$, Theorem 6.5 yields an equality

$$H_{M^*}(t) = \frac{H_M(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})}.$$

Note by definition that the dual module M^* is also totally reflexive. Replacing M with M^* in the above equality, we get

$$H_M(t) = \frac{H_{M^*}(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})}.$$

Thus we obtain the following two equalities:

$$\begin{cases} H_{M^*}(t) \cdot H_R(t^{-1}) = H_M(t^{-1}) \cdot H_R(t), \\ H_M(t) \cdot H_R(t^{-1}) = H_{M^*}(t^{-1}) \cdot H_R(t). \end{cases}$$

Therefore we obtain

$$H_M(t) \cdot H_M(t^{-1}) = H_{M^*}(t) \cdot H_{M^*}(t^{-1}). \tag{6.6.1}$$

Since M has finite length, we can write $H_M(t) = a_0 + a_1 t + \cdots + a_s t^s$ for some integers $a_0, \ldots, a_s$, and so $H_M(1) = a_0 + a_1 + \cdots + a_s = \ell_R(M)$. Similarly we have $H_{M^*}(1) = \ell_R(M^*)$. Substituting $t = 1$ in the equality (6.6.1) yields $\ell_R(M)^2 = \ell_R(M^*)^2$. It follows that $\ell_R(M) = \ell_R(M^*)$, as desired. □

Now we can achieve the purpose of this section.

Proof of Proposition 6.4. Suppose that the R-module $R/(a)$ has finite G-dimension. Then $\mathrm{Gdim}_R\ R/(a) = \operatorname{depth} R - \operatorname{depth}_R\ R/(a) = 0$ since $\operatorname{depth} R = 0$. Hence $R/(a)$ is a totally reflexive R-module.

It is easy to see that $R/(a)$ is isomorphic to $k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^2$, which has dimension $n+1$ as a k-vector space. Hence $\ell_R(R/(a)) = n+1$.

On the other hand, the module $(R/(a))^*$ is isomorphic to the ideal $(0) : a = I = (x_1, \ldots, x_l)$, and it holds that $I = I_1 + I_2$. We have $I_1 = k \cdot x_1 + \cdots + k \cdot x_l$, hence $\dim_k I_1 = l$. The k-vector space I_2 is contained in $\sum_{1 \le i \le l,\, 1 \le j \le n} k \cdot x_i x_j + \sum_{1 \le i \le l} k \cdot a x_i$. We have $x_i x_j = a\delta_{i,j} \in k \cdot a x_{l+1} + \cdots + k \cdot a x_n$ and $a x_i = 0$, so I_2 is contained in $k \cdot a x_{l+1} + \cdots + k \cdot a x_n$. Conversely, since $a^2 x_m = 0$ for $l+1 \le m \le n$, we get $a x_m \in I$. It follows that $I_2 = k \cdot a x_{l+1} + \cdots + k \cdot a x_n$. Lemma 5.3 guarantees that $a x_{l+1}, \ldots, a x_n$ are linearly independent over k. Therefore $\dim_k I_2 = n - l$. Consequently, we obtain equalities

$$\ell_R((R/(a))^*) = \ell_R(I) = \dim_k I_1 + \dim_k I_2 = l + (n-l) = n.$$

In particular, we get $\ell_R(R/(a)) \neq \ell_R((R/(a))^*)$. Theorem 6.5 gives a contradiction. Thus, the R-module $R/(a)$ does not have finite G-dimension, and the proof is completed. □

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