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Twisted homology and cohomology groups associated to the Wirtinger integral

Dedicated to Professor Kazuo Okamoto on his sixtieth birthday

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Abstract. The first half of this paper (Section 1) deals with the structure of the twisted homology group associated to the Wirtinger integral. A basis of the first homology group is given, and the vanishing of the homology groups of the other dimensions is proved. The second half (Section 2) deals with the structure of the twisted cohomology groups associated to the Wirtinger integral. The isomorphism between the twisted cohomology groups and the cohomology groups associated to a certain subcomplex of the de Rham complex is established, and a basis of the first cohomology group of this subcomplex (therefore, of the first twisted cohomology group) is given. The vanishing of the cohomology groups of the other dimensions is also proved.

Introduction.

In his paper [**16**], Wirtinger showed, 1902, that the composite function $F(\alpha,\beta,\gamma,\lambda(\tau))$ of the Gauss hypergeometric function $F(\alpha,\beta,\gamma,z)$ and the lambda function

$$z=\lambda(\tau)=\frac{\theta_1(0,\tau)^4}{\theta_3(0,\tau)^4}$$

has the following integral representation

$$\begin{aligned}F(\alpha,\beta,\gamma,\lambda(\tau))=&\frac{2\pi\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\theta_1(0,\tau)^{2-2\gamma}\theta_2(0,\tau)^{2\gamma-2\alpha-2\beta}\theta_3(0,\tau)^{2\alpha+2\beta}\\ &\times\int_0^{\frac{1}{2}}\theta(u,\tau)^{2\alpha-1}\theta_1(u,\tau)^{2\gamma-2\alpha-1}\theta_2(u,\tau)^{2\beta-2\gamma+1}\theta_3(u,\tau)^{-2\beta+1}du.\end{aligned}$$

(In this paper we follow Chandrasekharan's notation for theta functions. See "Notation for theta functions" below.) The right-hand side, regarded as a function of τ, is single-valued and holomorphic on the upper-half plane H, and we proposed in our paper [**15**] to call it *Wirtinger integral*. This integral representation was forgotten for a long while in the study on hypergeometric functions, whereas we have recently given in [**15**] a new derivation of the connection formulas for the Gauss hypergeometric function by

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exploiting the Wirtinger integral and Jacobi's imaginary transformations for theta functions. Our result suggests a possibility to reconstruct the theory of the Gauss hypergeometric function on the basis of the Wirtinger integral and theta functions, and to generalize the theory of the Gauss hypergeometric function from the viewpoint of the Wirtinger integral. We can also give an intrinsic definition of the Wirtinger integral with the aide of the twisted de Rham theory. Such a treatment was given first by Aomoto [**1**] in the case of hypergeometric functions in several variables arising from hyperplane configurations of a complex projective space. Using the notation introduced in Section 1, we have a natural non-degenerate bilinear form of the twisted homology and cohomology groups $H_1(M, \check{\mathscr{L}}) \times H^1(M, \mathscr{L}) \to \boldsymbol{C}$ by which the image of any pair $(\sigma, \varphi) \in H_1(M, \check{\mathscr{L}}) \times H^1(M, \mathscr{L})$, where we can regard φ as a holomorphic 1-form defined on M, is expressed as $\int_\sigma T(u)\varphi \in \boldsymbol{C}$. Every image contains the parameter τ, and is regarded as a single-valued holomorphic function defined on H, which we also call Wirtinger integral. Therefore the computations of the twisted homology and cohomology groups are basic tasks for studying the properties of the Wirtinger integral further. The computation of the twisted (co)homology groups consists of the proof of the vanishing of the groups of various dimensions, and of the construction of a basis of a non-vanishing group. Thanks to the duality of the homology and cohomology, the vanishings of the homology and cohomology groups of a same dimension are concluded simultaneously if the vanishing of either the homology group or the cohomology group is proved.

Aomoto [**1**] studied the vanishing of cohomology groups of a complex projective space minus a divisor with coefficients in a local system of arbitrary rank. He also studied in [**2**] the vanishing of cohomology groups of a compact Kähler manifold minus a certain cycle of real codimension two with coefficients in a local system of rank one with the aide of the Morse theory. The vanishing of cohomology groups is discussed also in Aomoto [**3**] and [**4**]. Kita and Noumi [**13**] (see also Kita [**12**]) generalized the results in Aomoto [**1**] to the case of a complex projective space minus a divisor of more general class than Aomoto's with coefficients in a local system of arbitrary rank. A basis of a non-vanishing homology group on a complex projective space minus hyperplanes with local system coefficients was given by Kita [**10**] and [**11**] (see also Aomoto [**2**] and Orlik-Terao [**14**], Chapter 6). Deligne [**7**] (especially Corollaire 6.11) established the isomorphism between the cohomology groups with local system coefficients and the cohomology groups associated to the logarithmic complex of an affine variety minus a divisor. Aomoto [**1**], [**3**], [**4**] obtained some results about the structure of a non-vanishing cohomology group of a complex projective space minus a divisor with local system coefficients. Kita and Noumi [**13**] (see also Kita [**12**]) established the isomorphism between the cohomology groups with coefficients in a local system of arbitrary rank and the cohomology groups associated to the vector-valued logarithmic complex of a complex projective space minus a divisor, and Kita [**12**] constructed explicitly a basis of a non-vanishing cohomology group of a complex projective space minus hyperplanes associated to the logarithmic complex in the case where the rank of the local system is one. One can find a detailed account about the results mentioned above also from Aomoto-Kita's book [**5**].

The purpose of this paper is to develop an analogous theory of homology and cohomology of the complex torus $\boldsymbol{C}/\varGamma$ minus the four points $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ (where $\varGamma$

denotes the lattice generated by 1 and τ) with coefficients in the local system (and its dual) defined by the power product of theta functions $T(u)$ associated to the integrand of the Wirtinger integral. In Section 1 we give a computation (including the construction of a basis) of the twisted homology groups $H_\bullet(M, \check{\mathscr{L}})$ of $M = \boldsymbol{C}/\Gamma - \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$ with coefficients in the local system $\check{\mathscr{L}}$ defined by $T(u)$ by exploiting the Mayer-Vietoris exact sequence (Theorem 1). In Section 2, to investigate the structure of the twisted cohomology groups $H^\bullet(M, \mathscr{L})$, we first show the vanishing of the groups $H^0(M, \mathscr{L})$ and $H^2(M, \mathscr{L})$. Next, we establish the isomorphism between the group $H^1(M, \mathscr{L})$ and the first de Rham cohomology group $H^1_{\mathrm{DR}}(\Omega^\bullet(*D), \nabla)$ of the de Rham complex $\Omega^\bullet(*D)$ of differential forms meromorphic on $\boldsymbol{C}/\Gamma$ and holomorphic on M. Although this isomorphism has been already established algebro-geometrically by Deligne [**7**], we give here a complex-analytical proof by Mittag-Leffler's theorem (Lemma 2.2). Our proof leads us to introduce a subcomplex of $\Omega^\bullet(*D)$ consisting of differential forms φ with $\mathrm{ord}_p(\varphi) \geq -\mathrm{ord}_p(D)$ for $p \in \boldsymbol{C}/\Gamma$, where D denotes the effective divisor $D = 2[0] + [\frac{1}{2}] + [\frac{\tau}{2}] + [\frac{1+\tau}{2}]$ of degree 5 (for the notation, see [**8**]). Then we establish the isomorphism between the cohomology group $H^1_{\mathrm{DR}}(\Omega^\bullet(*D), \nabla)$ (and therefore $H^1(M, \mathscr{L})$) and the first cohomology group of the preceding subcomplex of $\Omega^\bullet(*D)$, and give a basis of the latter cohomology group, which makes clear the structure of the group $H^1(M, \mathscr{L})$ (Theorem 2). Our result tells us that a 1-form having a pole (or poles) of degree more than one is indispensable to giving a basis of the group $H^1(M, \mathscr{L})$. This situation is different from the situation in the case of the cohomology of a complex projective space minus hyperplanes considered by Kita and Noumi ([**13**], [**12**]). Finally, we refer to the recent work of Ito [**18**] briefly. He studied there the twisted homology and cohomology of $\boldsymbol{C}/\Gamma$ minus movable four points with coefficients in a local system arising from the configuration space of (essentially) two points of $\boldsymbol{C}/\Gamma$. His local system is of different kind from ours. Consequently, he obtained from the pairing of the homology and cohomology the integral representations which are functions defined on the configuration space.

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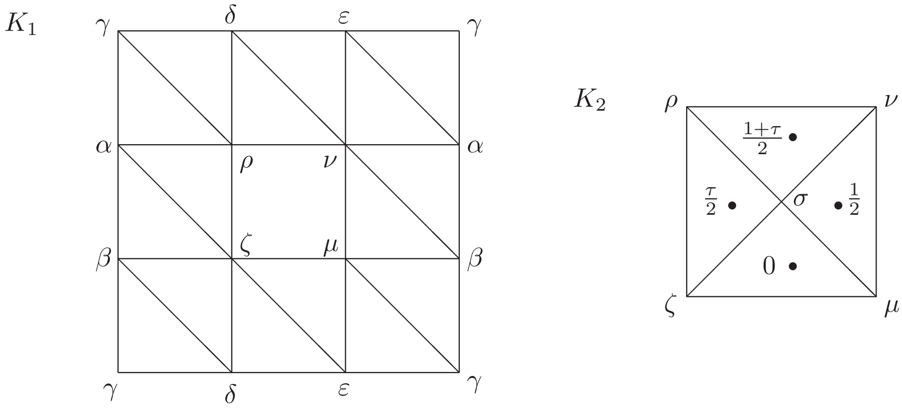
NOTATION FOR THETA FUNCTIONS. In this paper we adopt Chandrasekharan's notation for theta functions ([**6**]): $\theta(u,\tau)$, $\theta_1(u,\tau)$, $\theta_2(u,\tau)$ and $\theta_3(u,\tau)$. The relations between his theta functions and the theta functions $\theta_{ij}(u,\tau)$ $(i,j = 0,1)$ in Mumford's notation ([**19**]) are as follows: $\theta(u,\tau) = -\theta_{11}(u,\tau)$, $\theta_1(u,\tau) = \theta_{10}(u,\tau)$, $\theta_2(u,\tau) = \theta_{01}(u,\tau)$, $\theta_3(u,\tau) = \theta_{00}(u,\tau)$. The zeros of $\theta(u,\tau)$, $\theta_1(u,\tau)$, $\theta_2(u,\tau)$, $\theta_3(u,\tau)$ are congruent to $u = 0$, $u = \frac{1}{2}$, $u = \frac{\tau}{2}$, $u = \frac{1+\tau}{2}$ modulo Γ, respectively.

1. Twisted homology groups.

Let $\tau \in \boldsymbol{C}$ be such that $\mathrm{Im}(\tau) > 0$. We set $\Gamma = \boldsymbol{Z} + \boldsymbol{Z}\tau$, $D = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$, and $M = \boldsymbol{C}/\Gamma - D$, where $\boldsymbol{Z}$ denotes the additive group of integers, and $\boldsymbol{C}$ the additive group of complex numbers. Let p, q, r, s be complex numbers satisfying $p + q + r + s = 0$.

Throughout this paper we assume that p, q, r, s are not integers, and that the sum and the difference of any two of them are not integers either. If we set $T(u) = \theta(u)^p\theta_1(u)^q\theta_2(u)^r\theta_3(u)^s$, where $\theta_i(u)$ means $\theta_i(u,\tau)$, then we have $T(u+1) = e^{-(p+q)\pi i}T(u)$ and $T(u+\tau) = e^{(p+r)\pi i}T(u)$. We set $\omega = d(\log T(u))$. We define a connection ∇ by $\nabla\varphi = d\varphi + \omega\wedge\varphi$. Then we have $\nabla\nabla = 0$ and $\nabla(1) = \omega$. Let $\mathscr{L}$ and $\check{\mathscr{L}}$ be the local systems on M defined by $T(u)^{-1}$ and $T(u)$, respectively: $\mathscr{L} = \boldsymbol{C}T(u)^{-1}$ and $\check{\mathscr{L}} = \boldsymbol{C}T(u)$. They are dual to each other. In this section we compute the twisted homology groups $H_\bullet(M, \check{\mathscr{L}})$. Let us consider the following two-dimensional complex K_1 and one-dimensional complex K_2:



In the figure for K_1, let the 1-chains $\langle\gamma,\delta\rangle + \langle\delta,\varepsilon\rangle + \langle\varepsilon,\gamma\rangle$ and $\langle\gamma,\beta\rangle + \langle\beta,\alpha\rangle + \langle\alpha,\gamma\rangle$ in K_1 be homologous to the 1-chains defined by the periods 1 and τ of the torus $\boldsymbol{C}/\Gamma$, respectively, and let no 2-chain be contained inside the square $\zeta\mu\nu\rho\zeta$ of K_1. In the figure for K_2, we added the four points $0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ to the complex K_2 to indicate the configuration of chains of K_2. These points are not 0-chains of K_2. We set $K = K_1 \cup K_2$ and $K' = K_1 \cap K_2$. Since the complex K is homotopically equivalent to the real surface M, the group $H_\bullet(M, \check{\mathscr{L}})$ is isomorphic to $H_\bullet(K, \check{\mathscr{L}})$. In order to compute $H_\bullet(K, \check{\mathscr{L}})$, we need the homology groups of the three subcomplexes K', K_1, K_2. Since $p+q+r+s = 0$, the complex K' is homotopically equivalent to the circle S^1, from which we have immediately

Lemma 1.1. *$H_2(K', \check{\mathscr{L}}) = 0$, $H_1(K', \check{\mathscr{L}}) \cong \boldsymbol{C}$, $H_0(K', \check{\mathscr{L}}) \cong \boldsymbol{C}$.*

For the complex K_1, we have

Lemma 1.2. *$H_2(K_1, \check{\mathscr{L}}) = 0$, $H_1(K_1, \check{\mathscr{L}}) \cong \boldsymbol{C}$, $H_0(K_1, \check{\mathscr{L}}) = 0$.*

Proof. Since there is no non-zero element $c \in C_2(K_1, \check{\mathscr{L}})$ such that $\partial c = 0$, we have $Z_2(K_1, \check{\mathscr{L}}) = 0$, that is, $H_2(K_1, \check{\mathscr{L}}) = 0$. We see that two 1-chains $\langle\zeta,\mu\rangle + \langle\mu,\nu\rangle + \langle\nu,\rho\rangle + \langle\rho,\zeta\rangle$ and $\langle\gamma,\delta\rangle + \langle\delta,\varepsilon\rangle + \langle\varepsilon,\gamma\rangle + \langle\gamma,\beta\rangle + \langle\beta,\alpha\rangle + \langle\alpha,\gamma\rangle + \langle\gamma,\varepsilon\rangle + \langle\varepsilon,\delta\rangle + \langle\delta,\gamma\rangle + \langle\gamma,\alpha\rangle + \langle\alpha,\beta\rangle + \langle\beta,\gamma\rangle$ belong to $Z_1(K_1, \check{\mathscr{L}})$ but not to $B_1(K_1, \check{\mathscr{L}})$. Since they are homologous to each other, we have $H_1(K_1, \check{\mathscr{L}}) \cong \boldsymbol{C}(\langle\zeta,\mu\rangle + \langle\mu,\nu\rangle + \langle\nu,\rho\rangle + \langle\rho,\zeta\rangle)$. Finally, since $\partial(\langle\gamma,\delta\rangle + \langle\delta,\varepsilon\rangle + \langle\varepsilon,\gamma\rangle) = (e^{-(p+q)\pi i} - 1)\langle\gamma\rangle$, we have $\langle\gamma\rangle \in B_0(K_1, \check{\mathscr{L}})$. Similarly, we have $\langle\delta\rangle, \langle\varepsilon\rangle, \langle\alpha\rangle, \langle\beta\rangle \in B_0(K_1, \check{\mathscr{L}})$. Moreover, since $\langle\zeta\rangle = \langle\beta\rangle + \partial\langle\beta,\zeta\rangle$, we have $\langle\zeta\rangle \in B_0(K_1, \check{\mathscr{L}})$. Similarly, we have $\langle\mu\rangle, \langle\nu\rangle, \langle\rho\rangle \in B_0(K_1, \check{\mathscr{L}})$. Therefore we see that $Z_0(K_1, \check{\mathscr{L}}) = B_0(K_1, \check{\mathscr{L}})$, that is, $H_0(K_1, \check{\mathscr{L}}) = 0$. $\square$

The result for the complex K_2 is as follows:

LEMMA 1.3. *$H_2(K_2, \check{\mathscr{L}}) = 0$, $H_1(K_2, \check{\mathscr{L}}) \cong \boldsymbol{C}^3$, $H_0(K_2, \check{\mathscr{L}}) = 0$.*

PROOF. Since $\partial(\langle\zeta,\mu\rangle + \langle\mu,\sigma\rangle + \langle\sigma,\zeta\rangle) = (e^{2\pi i p} - 1)\langle\zeta\rangle$, we have $\langle\zeta\rangle \in B_0(K_2, \check{\mathscr{L}})$. Similarly, we have $\langle\mu\rangle, \langle\nu\rangle, \langle\rho\rangle, \langle\sigma\rangle \in B_0(K_2, \check{\mathscr{L}})$. So we have $H_0(K_2, \check{\mathscr{L}}) = 0$. We define four 1-chains $c_p, c_q, c_r, c_s \in C_1(K_2, \check{\mathscr{L}})$ by $c_p = \langle\sigma,\zeta\rangle + \langle\zeta,\mu\rangle + \langle\mu,\sigma\rangle$, $c_q = \langle\sigma,\mu\rangle + \langle\mu,\nu\rangle + \langle\nu,\sigma\rangle$, $c_r = \langle\sigma,\rho\rangle + \langle\rho,\zeta\rangle + \langle\zeta,\sigma\rangle$, $c_s = \langle\sigma,\nu\rangle + \langle\nu,\rho\rangle + \langle\rho,\sigma\rangle$, where we assume that the restrictions of the branches of $T(u)$ to the four chains define the same germ at the common initial point σ of those chains. Then we have $\partial c_p = (e^{2\pi i p} - 1)\langle\sigma\rangle$, $\partial c_q = (e^{2\pi i q} - 1)\langle\sigma\rangle$, $\partial c_r = (e^{2\pi i r} - 1)\langle\sigma\rangle$, $\partial c_s = (e^{2\pi i s} - 1)\langle\sigma\rangle$. For $k, l \in \{p,q,r,s\}$ $(k \neq l)$, we set $c_{kl} = (1/(e^{2\pi i k} - 1))c_k - (1/(e^{2\pi i l} - 1))c_l$. Then we see that $c_{kl} \in Z_1(K_2, \check{\mathscr{L}})$, $c_{kl} = -c_{lk}$, $c_{pq} + c_{qs} = c_{ps}$, $c_{rp} + c_{pq} = c_{rq}$, $c_{pq} + c_{qs} + c_{sr} + c_{rp} = 0$. Let us consider the chain $c = \langle\zeta,\mu\rangle + \langle\mu,\nu\rangle + \langle\nu,\rho\rangle + \langle\rho,\zeta\rangle$. Since $p+q+r+s=0$, we have $c \in Z_1(K_2, \check{\mathscr{L}})$. Here we assume that the restrictions of the branches of $T(u)$ to the chains c and c_p define the same germ at the common point ζ of those chains. Then we have $c = c_p + e^{2\pi i p}c_q + e^{2\pi i(p+q+s)}c_r + e^{2\pi i(p+q)}c_s$, from which it follows by simple calculation that $c = (e^{2\pi i p} - 1)c_{pq} + (e^{2\pi i(p+q)} - 1)c_{qs} + (e^{-2\pi i r} - 1)c_{sr}$. Since c_{pq}, c_{qs}, c_{sr} are linearly independent, we have $Z_1(K_2, \check{\mathscr{L}}) = \boldsymbol{C}c_{pq} \oplus \boldsymbol{C}c_{qs} \oplus \boldsymbol{C}c_{sr}$. Therefore we have $H_1(K_2, \check{\mathscr{L}}) \cong Z_1(K_2, \check{\mathscr{L}}) \cong \boldsymbol{C}^3$. $\square$

Let us now apply the Mayer-Vietoris exact sequence to the complexes K, K_1, K_2, K' (for the Mayer-Vietoris exact sequence, see [**9**]):

$$\begin{aligned} 0 &\to H_2(K, \check{\mathscr{L}}) \to H_1(K', \check{\mathscr{L}}) \to H_1(K_1, \check{\mathscr{L}}) \oplus H_1(K_2, \check{\mathscr{L}}) \to H_1(K, \check{\mathscr{L}}) \\ &\to H_0(K', \check{\mathscr{L}}) \to H_0(K_1, \check{\mathscr{L}}) \oplus H_0(K_2, \check{\mathscr{L}}) \to H_0(K, \check{\mathscr{L}}) \to 0. \end{aligned} \tag{1.1}$$

Since $H_0(K_1, \check{\mathscr{L}}) \oplus H_0(K_2, \check{\mathscr{L}}) = 0$ by Lemmas 1.2 and 1.3, we have $H_0(K, \check{\mathscr{L}}) = 0$. Furthermore, since the map $H_1(K', \check{\mathscr{L}}) \to H_1(K_1, \check{\mathscr{L}})$ is an isomorphism and the map $H_1(K', \check{\mathscr{L}}) \to H_1(K_2, \check{\mathscr{L}})$ is injective, the map $H_1(K', \check{\mathscr{L}}) \to H_1(K_1, \check{\mathscr{L}}) \oplus H_1(K_2, \check{\mathscr{L}})$ is also injective, from which it follows that $H_2(K, \check{\mathscr{L}}) = 0$. Therefore the exact sequence (1.1) is turned to

$$0 \to H_1(K', \check{\mathscr{L}}) \to H_1(K_1, \check{\mathscr{L}}) \oplus H_1(K_2, \check{\mathscr{L}}) \to H_1(K, \check{\mathscr{L}}) \to H_0(K', \check{\mathscr{L}}) \to 0,$$

from which we have the exact sequence

$$0 \to \big(H_1(K_1, \check{\mathscr{L}}) \oplus H_1(K_2, \check{\mathscr{L}})\big)/H_1(K', \check{\mathscr{L}}) \to H_1(K, \check{\mathscr{L}}) \xrightarrow{\varDelta} H_0(K', \check{\mathscr{L}}) \to 0. \tag{1.2}$$

By abuse of notation we may think $\big(H_1(K_1, \check{\mathscr{L}}) \oplus H_1(K_2, \check{\mathscr{L}})\big)/H_1(K', \check{\mathscr{L}}) \cong \boldsymbol{C}c_{pq} \oplus \boldsymbol{C}c_{qs} \oplus \boldsymbol{C}c_{sr}$. Without loss of generality we may think that $H_0(K', \check{\mathscr{L}}) = \boldsymbol{C}\langle\zeta\rangle$. Let us construct an element $c_0 \in H_1(K, \check{\mathscr{L}})$ such that $\varDelta(c_0) = \langle\zeta\rangle$. If we regard c_0 as an element in $Z_1(K, \check{\mathscr{L}})$, then we can write $c_0 = c_1 + c_2$ for some $c_i \in C_1(K_i, \check{\mathscr{L}})$ $(i = 1, 2)$. So it is sufficient to construct $c_i \in C_1(K_i, \check{\mathscr{L}})$ $(i = 1, 2)$ such that $\varDelta(c_0) = \partial(c_1) = -\partial(c_2) = \langle\zeta\rangle$.

We see that such elements c_1 and c_2 must satisfy the conditions $c_i \notin Z_1(K_i, \check{\mathscr{L}})$ $(i = 1, 2)$. We define c_1 and c_2 by $c_1 = (1/(e^{-\pi i(p+q)} - 1))(\langle\zeta,\mu\rangle + \langle\mu,\beta\rangle + \langle\beta,\zeta\rangle)$ and $c_2 = (1/(1 - e^{2\pi ip}))(\langle\zeta,\mu\rangle + \langle\mu,\sigma\rangle + \langle\sigma,\zeta\rangle)$, where we assume that the restrictions of the branches of $T(u)$ to the chains c_1 and c_2 define the same germ at the common initial point ζ of those chains. We have $\partial(c_1) = \langle\zeta\rangle$ and $\partial(c_2) = -\langle\zeta\rangle$. Then the sum $c_0 = c_1 + c_2$ is the desired solution of the equation $\varDelta(c_0) = \langle\zeta\rangle$. If we set $c_2' = (1/(1 - e^{2\pi ir}))(\langle\zeta,\sigma\rangle + \langle\sigma,\rho\rangle + \langle\rho,\zeta\rangle)$, the sum $c' = c_1 + c_2'$ is also a cycle satisfying the same equation. Since $c' = c_0 + c_{pr}$, we have $c' - c_0 \in \ker\varDelta$. Furthermore, if we set $c_1' = (1/(e^{\pi i(p+r)} - 1))(\langle\zeta,\rho\rangle + \langle\rho,\delta\rangle + \langle\delta,\zeta\rangle)$, the sum $c'' = c_1' + c_2$ is also a cycle satisfying the same equation. Since $c'' - c_0 \sim \langle\zeta,\mu\rangle + \langle\mu,\nu\rangle + \langle\nu,\rho\rangle + \langle\rho,\zeta\rangle \in Z_1(K', \check{\mathscr{L}})$, regarding c_0 and c'' as their homology classes, we have $c'' - c_0 \in \ker\varDelta$. Therefore we see that the set of the elements of $H_1(K, \check{\mathscr{L}})$ mapped by $\varDelta$ to $\langle\zeta\rangle$ coincides with $c_0 + \ker\varDelta$. We note that $\boldsymbol{C}c_0 \cap \ker\varDelta = 0$. If we define the map $\iota : H_0(K', \check{\mathscr{L}}) \to H_1(K, \check{\mathscr{L}})$ by $\iota(\langle\zeta\rangle) = c_0$, then we see that the exact sequence (1.2) is split. Namely we have $H_1(K, \check{\mathscr{L}}) \cong [(H_1(K_1, \check{\mathscr{L}}) \oplus H_1(K_2, \check{\mathscr{L}}))/H_1(K', \check{\mathscr{L}})] \oplus \iota(H_0(K', \check{\mathscr{L}}))$. Here we note that the map ι is an isomorphism. Since the complex K and the surface M are homotopically equivalent, we arrive at the following

THEOREM 1. *We have* $H_2(M, \check{\mathscr{L}}) = H_0(M, \check{\mathscr{L}}) = 0$, $H_1(M, \check{\mathscr{L}}) \cong \boldsymbol{C}c_{pq} \oplus \boldsymbol{C}c_{qs} \oplus \boldsymbol{C}c_{sr} \oplus \boldsymbol{C}c_0$, *where* c_{pq}, c_{qs}, c_{sr}, c_0 *are regarded as cycles on* M *by abuse of notation.*

REMARK 1. The homology groups of M with integral coefficients are given by $H_2(M, \boldsymbol{Z}) = 0$, $H_1(M, \boldsymbol{Z}) \cong \boldsymbol{Z}^5$, $H_0(M, \boldsymbol{Z}) \cong \boldsymbol{Z}$. We see that the Euler number of the homology with integral coefficients is equal to that of the homology with the local system coefficients.

REMARK 2. Theorem 1 follows also from the following fact (see Eilenberg [**17**], Section 28): Since M has the property $\pi_i(M) = 0$ $(i \neq 1)$, the homology groups $H_i(M, \check{\mathscr{L}})$ are isomorphic respectively to the homology groups $H_i(\pi_1(M, a_0), \check{\mathscr{L}}_{a_0})$ of the fundamental group $\pi_1(M, a_0)$, where $\check{\mathscr{L}}_{a_0}$ denotes the fibre of $\check{\mathscr{L}}$ on $a_0 \in M$. The latter groups are computable in a purely algebraic manner.

2. Twisted cohomology groups.

Let Ω^k $(k = 0, 1)$ be the sheaf of holomorphic k-forms on M. We have the exact sequence $0 \to \boldsymbol{C} \to \Omega^0 \xrightarrow{d} \Omega^1 \to 0$. Since the local system $\mathscr{L} = \boldsymbol{C}T(u)^{-1}$ is locally constant and without torsion, the tensor functor $\otimes_{\boldsymbol{C}}\mathscr{L}$ is exact. Namely we have the exact sequence $0 \to \mathscr{L} \to \Omega^0 \otimes_{\boldsymbol{C}} \mathscr{L} \xrightarrow{d} \Omega^1 \otimes_{\boldsymbol{C}} \mathscr{L} \to 0$. If we define the isomorphism between Ω^k and $\Omega^k \otimes_{\boldsymbol{C}} \mathscr{L}$ by $\Omega^k \ni \varphi \mapsto T(u)\varphi \in \Omega^k \otimes_{\boldsymbol{C}} \mathscr{L}$, then we have $d(T(u)\varphi) = T(u)\nabla\varphi$, which means that the following diagram is commutative:

$$\begin{array}{ccc}
\Omega^k & \xrightarrow{\nabla} & \Omega^{k+1} \\
\downarrow & & \downarrow \\
\Omega^k \otimes_{\boldsymbol{C}}\mathscr{L} & \xrightarrow[d]{} & \Omega^{k+1} \otimes_{\boldsymbol{C}}\mathscr{L},
\end{array}$$

where the vertical arrows represent isomorphisms. Combining this commutative diagram and the preceding exact sequence for $\mathscr{L}$, we have the exact sequence $0 \to \mathscr{L} \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \to 0$, from which it follows by the standard procedure that the following exact sequence holds:

$$0 \to H^0(M, \mathscr{L}) \to H^0(M, \Omega^0) \xrightarrow{\nabla} H^0(M, \Omega^1) \to H^1(M, \mathscr{L}) \to H^1(M, \Omega^0) \xrightarrow{\nabla} H^1(M, \Omega^1) \to 0.$$

Then we have

LEMMA 2.1. *$H^0(M, \mathscr{L}) = 0$, $H^1(M, \mathscr{L}) \cong H^0(M, \Omega^1)/\nabla(H^0(M, \Omega^0))$.*

PROOF. By definition we have $H^0(M, \mathscr{L}) = \{f \in \Gamma(M, \Omega^0) \mid \nabla f = 0\}$, where $\Gamma(M, \Omega^0)$ denotes the vector space of single-valued holomorphic functions on M. Since the function f satisfying the equation $\nabla f = 0$ is of the form $f(u) = c\theta(u)^{-p}\theta_1(u)^{-q}\theta_2(u)^{-r}\theta_3(u)^{-s}$ for some constant c, which is in general multivalued, we have $H^0(M, \mathscr{L}) = 0$. It is well-known that $H^1(U, \Omega^0) = 0$ for an arbitrary open Riemann surface U (e.g. [**8**]). Then we have the short exact sequence $0 \to H^0(M, \Omega^0) \xrightarrow{\nabla} H^0(M, \Omega^1) \to H^1(M, \mathscr{L}) \to 0$, and therefore $H^1(M, \mathscr{L}) \cong H^0(M, \Omega^1)/\nabla(H^0(M, \Omega^0))$. □

Let $\Omega^k(*D)$ $(k = 0, 1)$ be the sheaf of meromorphic k-forms on $\boldsymbol{C}/\Gamma$ which are holomorphic on $M = \boldsymbol{C}/\Gamma - D$ (where $D = \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$). The restriction of $\Omega^k(*D)$ to M is a subsheaf of Ω^k. We have a subcomplex $0 \to \mathscr{L} \to \Omega^0(*D) \xrightarrow{\nabla} \Omega^1(*D) \to 0$ of the complex $0 \to \mathscr{L} \to \Omega^0 \xrightarrow{\nabla} \Omega^1 \to 0$. The natural map of sheaf complexes, $\iota : (\Omega^\bullet(*D), \nabla) \to (\Omega^\bullet, \nabla)$, induces the natural homomorphism of the de Rham cohomologies: $\iota_* : H^1_{\mathrm{DR}}(\Omega^\bullet(*D), \nabla) \to H^1_{\mathrm{DR}}(\Omega^\bullet, \nabla)$, where $H^1_{\mathrm{DR}}(\Omega^\bullet(*D), \nabla) = H^0(M, \Omega^1(*D))/\nabla(H^0(M, \Omega^0(*D)))$ and $H^1_{\mathrm{DR}}(\Omega^\bullet, \nabla) = H^0(M, \Omega^1)/\nabla(H^0(M, \Omega^0))$. In fact we have

LEMMA 2.2. *ι_* is the isomorphism.*

It is well-known that this lemma is proved algebro-geometrically by the Grothendieck-Deligne comparison theorem ([**7**], II, Section 6). Nevertheless we give here a complex-analytical proof by exploiting Mittag-Leffler's theorem because our proof tells us what subcomplex of the de Rham complex $(\Omega^\bullet(*D), \nabla)$ with poles in D is suitable to take for establishing the isomorphism between the group $H^1_{\mathrm{DR}}(\Omega^\bullet(*D), \nabla)$ and the first cohomology group of such a subcomplex whose structure is easier to study. This will be explained in detail after the proof.

PROOF OF LEMMA 2.2. Let φ be an element in $H^0(M, \Omega^1)$. We set $\varphi = f(u)du$. Then the function $f(u)$ is single-valued and holomorphic on M, and may have isolated essential singularities at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ (and therefore $f(u)$ is expanded there in Laurent series). Let $P_0(u)$, $P_1(u)$, $P_2(u)$, $P_3(u)$ be the principal parts of the Laurent expansions of $f(u)$ at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, respectively. For a while, let us restrict ourselves to the case of the neighbourhood at $u = 0$. Let us find a function $Q_0(u)$ single-valued around $u = 0$

satisfying the equation $P_0(u)du = \nabla Q_0$, that is, $P_0(u) = (dQ_0/du) + Q_0(d/du)(\log T(u))$. Here we may assume that $P_0(u) = \sum_{n\geq 1} a_{-n}u^{-n}$. By the quadrature we have the general solution of this equation: $Q_0 = T(u)^{-1}[\int T(u)P_0(u)du + C]$ for some constant C. Since Q_0 is single-valued, the condition $C = 0$ is necessary. Let us investigate the behaviour of the solution $Q_0(u)$ with $C = 0$ around $u = 0$. Since p is not an integer (Section 1), the multivaluedness of $T(u)$ around $u = 0$ comes from the factor u^p. Namely we can write $T(u) = u^p \times$ (single-valued holomorphic function) around $u = 0$. Moreover, since we can write $T(u)P_0(u) = \sum_{n=-\infty}^{n=+\infty} c_n u^{p+n}$ around $u = 0$, we have

$$\int T(u)P_0(u)du = \sum_{n=-\infty}^{n=+\infty} \frac{c_n}{p+n+1} u^{p+n+1},$$

which is of the form $u^p \times$ (single-valued analytic function which may have an isolated singularity at $u = 0$) around $u = 0$. Consequently, the function $Q_0(u) = T(u)^{-1} \int T(u)P_0(u)du$ is a single-valued analytic function around $u = 0$ which may have an isolated singularity at $u = 0$, and therefore can be expanded in Laurent series at $u = 0$. We set $Q_0(u) = \sum_{n=-\infty}^{n=+\infty} b_n u^n$, the Laurent expansion at $u = 0$. Moreover we set $Q_{0-}(u) = \sum_{n\leq 0} b_n u^n$ and $Q_{0+}(u) = \sum_{n\geq 1} b_n u^n$. Substituting $Q_0 = Q_{0-} + Q_{0+}$ into the original equation above, we have $P_0 = Q'_{0-} + Q'_{0+} + Q_{0-}\cdot(\log T(u))' + Q_{0+}\cdot(\log T(u))'$. Since $(\log T(u))'$ has a pole of order one at $u = 0$ and Q_{0+} has a zero of order one at $u = 0$, the product $Q_{0+}\cdot(\log T(u))'$ is holomorphic at $u = 0$, and so is Q'_{0+}. Consequently, we see that in the right-hand side of the preceding relation the sum $Q'_{0-} + Q_{0-}\cdot(\log T(u))'$ contributes to the principal part P_0. Therefore, setting $\nabla Q_{0-} = g(u)du$, we see that the principal part of the Laurent expansion of $g(u)$ at $u = 0$ is equal to P_0. By the similar argument, we obtain functions $Q_{1-}(u)$, $Q_{2-}(u)$, $Q_{3-}(u)$ from the principal parts $P_1(u)$, $P_2(u)$, $P_3(u)$, respectively. We give the Laurent expansions for $Q_{k-}(u)$ $(k = 0,1,2,3)$ as follows: $Q_{0-}(u) = \sum_{n\leq 0} b_n^{(0)} u^n$, $Q_{1-}(u) = \sum_{n\leq 0} b_n^{(1)}(u - \frac{1}{2})^n$, $Q_{2-}(u) = \sum_{n\leq 0} b_n^{(2)}(u - \frac{\tau}{2})^n$, $Q_{3-}(u) = \sum_{n\leq 0} b_n^{(3)}(u - \frac{1+\tau}{2})^n$. We set $Q_{0*} = Q_{0-} - b_0^{(0)} - (b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)})u^{-1}$, $Q_{1*} = Q_{1-} - b_0^{(1)}$, $Q_{2*} = Q_{2-} - b_0^{(2)}$, $Q_{3*} = Q_{3-} - b_0^{(3)}$. We see that the residue of Q_{0*} at $u = 0$ is $-b_{-1}^{(1)} - b_{-1}^{(2)} - b_{-1}^{(3)}$, that of Q_{1*} at $u = \frac{1}{2}$ is $b_{-1}^{(1)}$, that of Q_{2*} at $u = \frac{\tau}{2}$ is $b_{-1}^{(2)}$, and that of Q_{3*} at $u = \frac{1+\tau}{2}$ is $b_{-1}^{(3)}$. By Mittag-Leffler's theorem (e.g. see [**8**]), there exists a global function $Q_* \in H^0(M,\Omega^0)$ whose principal parts of the Laurent expansions at $u = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ coincide with Q_{0*}, Q_{1*}, Q_{2*} and Q_{3*}, respectively. We note that the 1-form $P_0(u)du - \nabla b_0^{(0)} - \nabla((b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)})/u) - \nabla Q_{0*}$ is holomorphic at $u = 0$, and that the forms $P_k(u)du - \nabla b_0^{(k)} - \nabla Q_{k*}$ $(k = 1,2,3)$ are holomorphic at $u = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$, respectively. Then there exist a constant ξ and an Abelian 1-form η of third kind with poles in $\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$ such that the principal part of the Laurent expansion at $u = 0$ of the 1-form $\xi\mathscr{P}(u)du + \eta$, where $\mathscr{P}(u)$ denotes the Weierstrass P-function with periods 1 and τ, coincides with that of $\nabla(1/u)$, and that the 1-form $f(u)du - (b_0^{(0)} + b_0^{(1)} + b_0^{(2)} + b_0^{(3)})\nabla(1) - (b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)})(\xi\mathscr{P}(u)du + \eta) - \nabla Q_*$, say ζ, is holomorphic on the whole torus $\boldsymbol{C}/\Gamma$. Here we note that $\nabla(1)$ $(= \omega)$ is an Abelian 1-form of third kind, and $\xi\mathscr{P}(u)du$ is an Abelian 1-form of second kind. Setting $\psi = (b_0^{(0)} + b_0^{(1)} + b_0^{(2)} + b_0^{(3)})\nabla(1) + (b_{-1}^{(0)} + b_{-1}^{(1)} + b_{-1}^{(2)} + b_{-1}^{(3)})(\xi\mathscr{P}(u)du + \eta) + \zeta$, we see that $\psi \in H^0(M,\Omega^1(*D))$ and $\varphi = \psi + \nabla Q_*$. From this result we can show the surjectivity of the map ι_* as follows. Let us take $[\varphi] \in H^1_{\mathrm{DR}}(\Omega^\bullet,\nabla)$ arbitrarily, where $\varphi \in H^0(M,\Omega^1)$. If we form $[\psi] \in$

$H^1_{\mathrm{DR}}(\Omega^\bullet(*D),\nabla)$ from the element $\psi\in H^0(M,\Omega^1(*D))$ whose existence is guaranteed above, then we have $\iota_*[\psi]=[\varphi]$, which proves the surjectivity of ι_*. The proof of the injectivity of ι_* is as follows. For $[\psi]\in H^1_{\mathrm{DR}}(\Omega^\bullet(*D),\nabla)$, we set $\iota_*[\psi]=0$. This equation is translated into the assertion that there exists a single-valued function $g\in H^0(M,\Omega^0)$ such that $\psi=\nabla g$. If we set $\psi=f(u)du$, we see that $f(u)$ is holomorphic on M and has poles at $u=0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}$ if $f(u)$ is not holomorphic there. The equation is rewritten as $f(u)=(dg/du)+g(u)(d/du)(\log T(u))$, from which we have the solution $g(u)=T(u)^{-1}\int T(u)f(u)du$. By the same argument as when we constructed Q_0 from P_0 and investigated the behaviour of Q_0 at $u=0$, we see that $g(u)$ is single-valued and holomorphic on M, and has poles at $u=0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}$ if $g(u)$ is not holomorphic there. Therefore we conclude that $g(u)\in H^0(M,\Omega^0(*D))$, and $[\psi]=0$ as the equality in $H^1_{\mathrm{DR}}(\Omega^\bullet(*D),\nabla)$, which proves the injectivity of ι_*. □

Inspired by the proof of Lemma 2.2, we give the following formulation. Let D be the effective divisor on $\boldsymbol{C}/\Gamma$ given by $D=2[0]+[\frac{1}{2}]+[\frac{\tau}{2}]+[\frac{1+\tau}{2}]$. Let Ω_D be the sheaf of meromorphic 1-forms on $\boldsymbol{C}/\Gamma$ which are multiples of the divisor $-D$. Then Ω_D is a subsheaf of $\Omega^1(*D)$. Let $\mathscr{O}_D$ be the sheaf of meromorphic functions on $\boldsymbol{C}/\Gamma$ which are multiples of the divisor $-D$. We introduce two complexes:

$$0\to H^0(M,\Omega^0(*D))\xrightarrow{\nabla}H^0(M,\Omega^1(*D))\to 0,\quad 0\to\boldsymbol{C}\xrightarrow{\nabla}H^0(\boldsymbol{C}/\Gamma,\Omega_D)\to 0,$$

where the latter is a subcomplex of the former: $\boldsymbol{C}\subset H^0(M,\Omega^0(*D))$ and $H^0(\boldsymbol{C}/\Gamma,\Omega_D)\subset H^0(M,\Omega^1(*D))$, and $H^0(\boldsymbol{C}/\Gamma,\Omega_D)=\{\varphi:\text{holomorphic 1-form on } M\mid \mathrm{ord}_p(\varphi)\geq-\mathrm{ord}_p(D)\text{ for } p\in\boldsymbol{C}/\Gamma\}$. Let us observe the structure of the vector space $H^0(\boldsymbol{C}/\Gamma,\Omega_D)$. First we have

Lemma 2.3. $\dim H^0(\boldsymbol{C}/\Gamma,\Omega_D)=5$.

Proof. The Riemann-Roch formula for a compact Riemann surface X is given by $\dim H^0(X,\mathscr{O}_{-D})-\dim H^0(X,\Omega_D)=1-g-\deg D$. In our case, since $X=\boldsymbol{C}/\Gamma$, $g=1$, $\deg D=5$, $H^0(X,\mathscr{O}_{-D})=0$, we have $\dim H^0(X,\Omega_D)=5$. □

Let $\mathscr{P}(u)$ be the Weierstrass P-function with periods 1 and τ. For $i,j\in\{1,2,3\}$ $(i\neq j)$, we define 1-forms ω_i, ω_{ij} by

$$\omega_1=\frac{1}{2}d\log\left(\mathscr{P}(u)-\mathscr{P}\left(\frac{1}{2}\right)\right)=d\log\theta_1(u)-d\log\theta(u)=-\pi\theta_1^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du,$$

$$\omega_2=\frac{1}{2}d\log\left(\mathscr{P}(u)-\mathscr{P}\left(\frac{\tau}{2}\right)\right)=d\log\theta_2(u)-d\log\theta(u)=-\pi\theta_2^2\frac{\theta_1(u)\theta_3(u)}{\theta(u)\theta_2(u)}du,$$

$$\omega_3=\frac{1}{2}d\log\left(\mathscr{P}(u)-\mathscr{P}\left(\frac{1+\tau}{2}\right)\right)=d\log\theta_3(u)-d\log\theta(u)=-\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du,$$

$$\omega_{12}=d\log\theta_2(u)-d\log\theta_1(u)=\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du,$$

$$\omega_{13}=d\log\theta_3(u)-d\log\theta_1(u)=\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}du,$$

$$\omega_{23} = d\log\theta_3(u) - d\log\theta_2(u) = -\pi\theta_1^2\,\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)}\,du.$$

Moreover we set $\omega_{ij} = -\omega_{ji}$. Then we have $\omega_1 + \omega_{12} = \omega_2$, $\omega_1 + \omega_{13} = \omega_3$, $\omega_2 + \omega_{23} = \omega_3$, $\omega_{12} + \omega_{23} = \omega_{13}$. Therefore we see that the maximal number of linearly independent 1-forms among ones defined above is three. The 1-form ω_1 has poles of order one at $u = \frac{1}{2}, 0$ with residues $+1, -1$, respectively, ω_2 has poles of order one at $u = \frac{\tau}{2}, 0$ with residues $+1, -1$, respectively, ω_3 has poles of order one at $u = \frac{1+\tau}{2}, 0$ with residues $+1, -1$, respectively, ω_{12} has poles of order one at $u = \frac{\tau}{2}, \frac{1}{2}$ with residues $+1, -1$, respectively, ω_{13} has poles of order one at $u = \frac{1+\tau}{2}, \frac{\tau}{2}$ with residues $+1, -1$, respectively, and ω_{23} has poles of order one at $u = \frac{1+\tau}{2}, \frac{\tau}{2}$ with residues $+1, -1$, respectively. We see that all ω_i, ω_{ij} lie in $H^0(\boldsymbol{C}/\Gamma, \Omega_D)$. Besides, we note that the two forms du and $\mathscr{P}(u)du$ also lie in $H^0(\boldsymbol{C}/\Gamma, \Omega_D)$. Therefore we have

Lemma 2.4. *The five* 1*-forms*: du, $\mathscr{P}(u)du$ *and three linearly independent* 1*-forms among* ω_i, ω_{ij}, *form a basis of* $H^0(\boldsymbol{C}/\Gamma, \Omega_D)$.

The inclusion map between the two complexes defined above induces a natural map $I : H^0(\boldsymbol{C}/\Gamma, \Omega_D)/\nabla(\boldsymbol{C}) \longrightarrow H^1_{\mathrm{DR}}(\Omega^\bullet(*D), \nabla) = H^0(M, \Omega^1(*D))/\nabla H^0(M, \Omega^0(*D))$. We wish to prove that I is the isomorphism.

Lemma 2.5. *I is injective.*

Proof. It follows immediately from the fact $\nabla H^0(M, \Omega^0(*D)) \cap H^0(\boldsymbol{C}/\Gamma, \Omega_D) = \nabla(\boldsymbol{C})$. □

The surjectivity of I follows immediately from the following

Lemma 2.6. *For an arbitrary* $\varphi \in H^0(M, \Omega^1(*D))$, *there exist* $\psi \in H^0(\boldsymbol{C}/\Gamma, \Omega_D)$ *and* $f \in H^0(M, \Omega^0(*D))$ *such that* $\varphi = \psi + \nabla f$.

Proof. The lemma holds if it is proved for the following two cases: (i) φ has only one pole of order 2 at $u = \frac{1}{2}$ or $u = \frac{\tau}{2}$ or $u = \frac{1+\tau}{2}$; (ii) φ has only one pole of order more than 2 at $u = 0$ or $u = \frac{1}{2}$ or $u = \frac{\tau}{2}$ or $u = \frac{1+\tau}{2}$.

(i) Without loss of generality, we may concentrate our attention on the case where $u = \frac{1}{2}$. The other cases are treated similarly. Let us compute

$$\nabla\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) = \frac{d}{du}\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right)du + \frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\,\omega. \tag{2.1}$$

Here we have

$$\begin{aligned}
&\frac{d}{du}\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right) \\
&\quad = \frac{\theta_1(u)\theta_3(u)\{\theta(u)\theta_2'(u) - \theta'(u)\theta_2(u)\} + \theta(u)\theta_2(u)\{\theta_1(u)\theta_3'(u) - \theta_1'(u)\theta_3(u)\}}{\theta(u)^2\theta_1(u)^2}.
\end{aligned}$$

Applying the formulas $\{\theta'(u)\theta_2(u)-\theta(u)\theta_2'(u)\}\theta_1\theta_3=\theta_1(u)\theta_3(u)\theta_2\theta'$, $\{\theta_3'(u)\theta_1(u)-\theta_3(u)\theta_1'(u)\}\theta_1\theta_3=\theta_2(u)\theta(u)\theta_2\theta'$ and $\theta'=\pi\theta_1\theta_2\theta_3$ to the right-hand side of the preceding equality, we have

$$\frac{d}{du}\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right)=-\pi\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}+\pi\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2}. \tag{2.2}$$

The relation $\omega=-p\omega_3-q\omega_{13}-r\omega_{23}$ implies

$$\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\omega=\left\{p\pi\theta_3^2\frac{\theta_2(u)^2}{\theta(u)^2}-q\pi\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2}+r\pi\theta_1^2\right\}du. \tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), we have

$$\nabla\left(\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}\right)=\left\{-\pi\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}+(1-q)\pi\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2}+p\pi\theta_3^2\frac{\theta_2(u)^2}{\theta(u)^2}+r\pi\theta_1^2\right\}du. \tag{2.4}$$

Here we note that

$$\begin{aligned}
&\mathscr{P}\left(u+\frac{1}{2}\right)-\mathscr{P}\left(\frac{1+\tau}{2}\right)=\pi^2\theta_1^2\theta_2^2\frac{\theta_2(u)^2}{\theta_1(u)^2},\\
&\mathscr{P}(u)-\mathscr{P}\left(\frac{1+\tau}{2}\right)=\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2},\\
&\mathscr{P}(u)-\mathscr{P}\left(\frac{\tau}{2}\right)=\pi^2\theta_1^2\theta_3^2\frac{\theta_2(u)^2}{\theta(u)^2}.
\end{aligned}$$

Then the equality (2.4) means that, for $\varphi=\mathscr{P}(u+\frac{1}{2})du$, the lemma holds if we take $\psi=\mathscr{P}(u)du+$ (holomorphic 1-form).

(ii) Without loss of generality, we may assume that φ has only one pole of order ν (≥ 3) at $u=0$. Moreover, we may assume that such a 1-form φ is written by $\varphi=\mathscr{P}(u)^k\mathscr{P}'(u)^l du$ $(2k+3l=\nu\geq 3,\ k\geq 0,\ l\geq 0)$. In this case we prove the lemma by induction on ν. Let us first prove it for

$$\varphi=\mathscr{P}'(u)du=-2\pi^3\theta_1^2\theta_2^2\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du.$$

We have

$$\begin{aligned}
\nabla\left(\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\right)&=\pi^2\theta_1^2\theta_2^2 d\left(\frac{\theta_3(u)^2}{\theta(u)^2}\right)+\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}(-p\omega_3-q\omega_{13}-r\omega_{23})\\
&=\left\{-2\pi(\theta')^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}+\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\cdot p\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\right.
\end{aligned}$$

$$+\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\cdot(-q)\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}+\pi^2\theta_1^2\theta_2^2\frac{\theta_3(u)^2}{\theta(u)^2}\cdot r\pi\theta_1^2\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)}\Bigg\}du$$
$$=\left\{(p-2)\pi^3\theta_1^2\theta_2^2\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}-q\pi^3\theta_1^2\theta_2^4\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}+r\pi^3\theta_1^4\theta_2^2\frac{\theta_3(u)\theta_1(u)}{\theta(u)\theta_2(u)}\right\}du,$$

which proves the lemma for $\varphi=\mathscr{P}'(u)du$. Next we proceed to the general case. Since $\mathscr{P}(u)$ satisfies the differential equation $\mathscr{P}'(u)^2=4\mathscr{P}(u)^3-g_2\mathscr{P}(u)-g_3$ (g_2, g_3 are constants), we may assume without loss of generality that the general 1-form φ is of the form

$$\varphi=\left(\frac{\theta_3(u)}{\theta(u)}\right)^{2N}\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}\right)^M du\quad(N\geq 1,\ M=0\text{ or }1).$$

We have already proved the lemma in the case where $\nu=2N+3M\leq 3$. So we assume that $\nu\geq 4$. Let us compute

$$\nabla\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right)=d\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right)+\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\omega. \tag{2.5}$$

It holds:

$$\begin{aligned}
&\frac{d}{du}\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right)\\
&=\frac{\theta_1'(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}+\frac{\theta_1(u)\theta_2'(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}+(2N-3)\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-4}\theta_3'(u)}{\theta(u)^{2N-1}}\\
&\quad-(2N-1)\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}\theta'(u)}{\theta(u)^{2N}}\\
&=\frac{\theta_2(u)\theta_3(u)^{2N-3}\{\theta(u)\theta_1'(u)-\theta'(u)\theta_1(u)\}}{\theta(u)^{2N}}+\frac{\theta_1(u)\theta_3(u)^{2N-3}\{\theta(u)\theta_2'(u)-\theta'(u)\theta_2(u)\}}{\theta(u)^{2N}}\\
&\quad+(2N-3)\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-4}\{\theta(u)\theta_3'(u)-\theta'(u)\theta_3(u)\}}{\theta(u)^{2N}}.
\end{aligned}$$

Applying the formula $\{\theta'(u)\theta_1(u)-\theta(u)\theta_1'(u)\}\theta_2\theta_3=\theta_2(u)\theta_3(u)\theta_1\theta'$ and several similar ones to the right-hand side of the preceding equality, we have

$$\begin{aligned}
\frac{d}{du}\left(\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\right)&=-\pi\theta_1^2\frac{\theta_2(u)^2\theta_3(u)^{2N-2}}{\theta(u)^{2N}}-\pi\theta_2^2\frac{\theta_1(u)^2\theta_3(u)^{2N-2}}{\theta(u)^{2N}}\\
&\quad-(2N-3)\pi\theta_3^2\frac{\theta_1(u)^2\theta_2(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N}}.
\end{aligned}\tag{2.6}$$

Moreover it holds:

$$\begin{aligned}\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}\omega &= \frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-3}}{\theta(u)^{2N-1}}(-p\omega_3 - q\omega_{13} - r\omega_{23}) \\ &= \left\{p\pi\theta_3^2\frac{\theta_1(u)^2\theta_2(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N}} - q\pi\theta_2^2\frac{\theta_2(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N-2}}\right. \\ &\quad \left. + r\pi\theta_1^2\frac{\theta_1(u)^2\theta_3(u)^{2N-4}}{\theta(u)^{2N-2}}\right\}du.\end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5), we see that the lemma holds for

$$\varphi = \left(\frac{\theta_3(u)}{\theta(u)}\right)^{2N} du,$$

that is, for $M = 0$. Finally, it holds:

$$\begin{aligned}\nabla\left(\frac{\theta_3(u)^{2N}}{\theta(u)^{2N}}\right) &= d\left(\frac{\theta_3(u)^{2N}}{\theta(u)^{2N}}\right) + \frac{\theta_3(u)^{2N}}{\theta(u)^{2N}}(-p\omega_3 - q\omega_{13} - r\omega_{23}) \\ &= \left\{-2N\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N+1}} + p\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N+1}}\right. \\ &\quad \left. - q\pi\theta_2^2\frac{\theta_2(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N-1}\theta_1(u)} + r\pi\theta_1^2\frac{\theta_1(u)\theta_3(u)^{2N-1}}{\theta(u)^{2N-1}\theta_2(u)}\right\}du,\end{aligned}$$

from which we see that the lemma holds for

$$\varphi = \left(\frac{\theta_3(u)}{\theta(u)}\right)^{2N-2}\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du,$$

that is, for $M = 1$ too. Therefore Lemma 2.6 is proved completely. □

Combining everything above, we arrive at

THEOREM 2. *We have $H^0(M, \mathscr{L}) = H^2(M, \mathscr{L}) = 0$ and $H^1(M, \mathscr{L}) \cong H^0(\boldsymbol{C}/\Gamma, \Omega_D)/\nabla(\boldsymbol{C}) = \boldsymbol{C}[du] \oplus \boldsymbol{C}[\mathscr{P}(u)du] \oplus \boldsymbol{C}[\omega^{(1)}] \oplus \boldsymbol{C}[\omega^{(2)}]$. Here $[\varphi]$ denotes the image of an element φ in $H^0(\boldsymbol{C}/\Gamma, \Omega_D)$ by the natural map $H^0(\boldsymbol{C}/\Gamma, \Omega_D) \to H^0(\boldsymbol{C}/\Gamma, \Omega_D)/\nabla(\boldsymbol{C})$, and $\omega^{(1)}$ and $\omega^{(2)}$ denote vectors in the subspace generated by all ω_i and ω_{ij} in $H^0(\boldsymbol{C}/\Gamma, \Omega_D)$ such that the images $[\omega^{(1)}]$ and $[\omega^{(2)}]$ are linearly independent in $H^0(\boldsymbol{C}/\Gamma, \Omega_D)/\nabla(\boldsymbol{C})$.*

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