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Decay rates of the derivatives of the solutions of the heat equations in the exterior domain of a ball

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Abstract. We consider the initial-boundary value problem

$$(P)\qquad \begin{cases} \dfrac{\partial}{\partial t}u = \Delta u - V(|x|)u & \text{in } \ \Omega_L \times (0,\infty), \\ \mu u + (1-\mu)\dfrac{\partial}{\partial \nu}u = 0 & \text{on } \ \partial\Omega_L \times (0,\infty), \\ u(\cdot,0) = \phi(\cdot) \in L^p(\Omega_L), & p \geq 1, \end{cases}$$

where $\Omega_L = \{x \in \boldsymbol{R}^N : |x| > L\}$, $N \geq 2$, $L > 0$, $0 \leq \mu \leq 1$, ν is the outer unit normal vector to $\partial\Omega_L$, and V is a nonnegative smooth function such that $V(r) = O(r^{-2})$ as $r \to \infty$. In this paper, we study the decay rates of the derivatives $\nabla_x^j u$ of the solution u to (P) as $t \to \infty$.

1. Introduction.

The linear heat equation, which has been studied for more than two centuries, is still one of the main topics in the theory of partial differential equations. The decay rate of the derivatives of a solution to the linear heat equation is one of worth challenging problems and will give some insight to the behavior of solutions to the semilinear heat equations.

We consider the initial-boundary value problem of the heat equation in the exterior domain of a ball,

$$\begin{cases} \dfrac{\partial}{\partial t}u = \Delta u - V(|x|)u & \text{in } \ \Omega_L \times (0,\infty), \\ \mu u + (1-\mu)\dfrac{\partial}{\partial \nu}u = 0 & \text{on } \ \partial\Omega_L \times (0,\infty), \\ u(\cdot,0) = \phi(\cdot) \in L^p(\Omega_L), \end{cases} \tag{1.1}$$

where $0 \leq \mu \leq 1$, $p \geq 1$, $\Omega_L = \{x \in \boldsymbol{R}^N : |x| > L\}$, $N \geq 2$, $L > 0$, and ν is the outer unit normal vector to $\partial\Omega_L$. Throughout this paper, we assume that $V = V(|x|)$ satisfies the following condition (V_ω^l) for some $\omega \geq 0$ and $l \in \boldsymbol{N}$:

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$$(\boldsymbol{V}^l_\omega)\qquad \begin{cases} \text{(0)} & V = V(|x|) \in C^l(\boldsymbol{R}^N), \quad V \ge 0 \text{ in } \boldsymbol{R}^N, \\ \text{(i)} & \displaystyle\lim_{r\to\infty} r^2 V(r) = \omega, \\ \text{(ii)} & \displaystyle\int_L^\infty r\left|V(r) - \frac{\omega}{r^2}\right| dr < \infty, \\ \text{(iii)} & \displaystyle\sup_{r\ge L}\left| r^{2+j}\left(\frac{d^j}{dr^j}V\right)(r)\right| < \infty, \qquad j = 1,\dots,l. \end{cases}$$

The purpose of this paper is to study the decay rates of the derivatives of the solution of (1.1) under the condition (V^l_ω), as $t \to \infty$.

To explain the precedent works and our results, we introduce some notations. For any set A and B, let $f = f(\lambda,\nu)$ and $g = g(\lambda,\mu)$ be maps from $A \times B$ to $(0,\infty)$. Then we say

$$f(\lambda,\mu) \preceq g(\lambda,\mu) \quad \text{for all } \lambda \in A$$

if, for any $\mu \in B$, there exists a positive constant C such that $f(\lambda,\mu) \le Cg(\lambda,\mu)$ for all $\lambda \in A$. Furthermore, we say

$$f(\lambda,\mu) \asymp g(\lambda,\mu) \quad \text{for all } \lambda \in A$$

if $f(\lambda,\mu) \preceq g(\lambda,\mu)$ and $g(\lambda,\mu) \preceq f(\lambda,\mu)$ for all $\lambda \in A$. We put

$$\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}, \quad \boldsymbol{N}_0^N = \big\{(n_1,\dots,n_N) \,:\, n_i \in \boldsymbol{N}_0, i = 1,\dots,N\big\}.$$

Furthermore, for any $j = (j_1,\dots,j_N) \in \boldsymbol{N}_0^N$, we write $|j| = \sum_{i=1}^N j_i$ and $\nabla_x^j = \partial^{|j|}/\partial x_1^{j_1}\cdots\partial x_N^{j_N}$.

Let Ω be an unbounded domain in $\boldsymbol{R}^N$. Then, under the suitable assumptions on Ω and V, for any $j \in \boldsymbol{N}_0^N$, the solution u of (1.1) in the domain Ω satisfies

$$\big\|(\nabla_x^j u)(\cdot,t)\big\|_{L^\infty(\Omega)} \preceq t^{-\frac{N}{2p}}\|\phi\|_{L^p(\Omega)} \tag{1.2}$$

for all sufficiently large t. (See Theorem 10.1 of Chapters 3 and 4 in [**5**].) On the other hand, for the case when $\Omega = \boldsymbol{R}^N$ (or $\Omega = \boldsymbol{R}^N_+$) and $V \equiv 0$, the explicit representation of the fundamental solution of the heat equation implies that, for any $j \in \boldsymbol{N}_0^N$,

$$\big\|(\nabla_x^j u)(\cdot,t)\big\|_{L^\infty(\boldsymbol{R}^N)} \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}}\|\phi\|_{L^p(\boldsymbol{R}^N)} \tag{1.3}$$

for all $t > 0$. Furthermore, for the case when Ω is a convex domain in $\boldsymbol{R}^N$ and $V \equiv 0$, Li and Yau [**6**] studied the behavior of the nonnegative solution of (1.1) with $\mu = 0$, and obtained the inequality

$$\frac{|\nabla_x u|^2}{u^2} - \frac{\partial_t u}{u} \preceq \frac{1}{t}, \quad (x,t) \in \Omega \times (0,\infty). \tag{1.4}$$

Then, by the standard arguments in the parabolic equations, we see that, for any $j \in \boldsymbol{N}_0^N$ with $|j| \le 1$, the inequality (1.3) holds for all $t > 0$. (We remark that the inequality (1.4) holds for all sufficiently small $t > 0$ without the convexity of the domain Ω (see [**10**]).) On the other hand, Grigor'yan and Saloff-Coste [**2**] studied the asymptotic behavior of the Green function $G_\mu^V = G_\mu^V(x,y,t)$ of (1.1) for the case when Ω is the exterior domain of a compact set, $\mu = 1$, and $V \equiv 0$. They proved that, for any fixed $x, y \in \Omega$,

$$G_1^V(x,y,t) \asymp t^{-\frac{N}{2}}$$

for all sufficiently large t (see also [**11**]) if $N \ge 3$. This together with the mean value theorem, the Dirichlet boundary condition, and (1.2) implies that

$$\big\|(\nabla_x G_1^V)(\cdot,\cdot,t)\big\|_{L^\infty(\Omega\times\Omega)} \asymp t^{-\frac{N}{2}}$$

for all sufficiently large t. So we see that the solution of (1.1) with $\mu = 1$ does not necessarily satisfy the inequality (1.3) even for the case $|j| = 1$. The first author of this paper studied the asymptotic behavior of the solution of the heat equation under the Neumann boundary condition in the exterior domain of a ball in [**3**]. His results imply that, for the case $\mu = 0$ and $V \equiv 0$ on Ω_L, the inequality (1.3) does not necessarily hold for the case $|j| = 2$. Recently, Shibata and Shimizu [**8**] studied the decay properties of the Stokes semigroup in the exterior domain of a compact set, under the Neumann boundary condition. Their results are applicable to the heat equation, and we see that the inequality (1.3) holds for the case when $N \ge 3$, Ω is the exterior domain of a compact set, $V \equiv 0$ on Ω, and $\mu = 0$. (For further informations on the behavior of the derivatives of the solutions of the heat equations, see [**1**], [**7**] and [**9**].)

Let $u_\mu^V = u_\mu^V(x,t:\phi)$ be a solution of the initial-boundary value problem (1.1) in the exterior domain Ω_L. For any $p \ge 1$ and $t > 0$, put

$$\big\|\nabla_x^j G_\mu^V(t)\big\|_{p\to\infty} = \sup\big\{\big\|(\nabla_x^j u_\mu^V)(\cdot,t:\phi)\big\|_{L^\infty(\Omega_L)} : \|\phi\|_{L^p(\Omega_L)} = 1\big\},$$

where $j \in \boldsymbol{N}_0^N$. In particular, for the case $p = 1$, we see that

$$\big\|\nabla_x^j G_\mu^V(t)\big\|_{1\to\infty} = \big\|(\nabla_x^j G_\mu^V)(\cdot,\cdot,t)\big\|_{L^\infty(\Omega_L\times\Omega_L)}, \quad t > 0.$$

Let $\Delta_{\boldsymbol{S}^{N-1}}$ be the Laplace-Beltrami operator on $\boldsymbol{S}^{N-1}$ and $\{\omega_k\}_{k=0}^\infty$ the eigenvalues of

$$-\Delta_{\boldsymbol{S}^{N-1}} Q = \omega Q \quad \text{on} \quad \boldsymbol{S}^{N-1}, \quad Q \in L^2(\boldsymbol{S}^{N-1}), \tag{1.5}$$

that is,

$$\omega_k = k(N+k-2), \qquad k \in \boldsymbol{N}_0. \tag{1.6}$$

Furthermore, let $\{Q_{k,i}\}_{i=1}^{l_k}$ and l_k be the orthonormal system and the dimension of the eigenspace corresponding to ω_k, respectively. Let $U_{\mu,L}^V(r)$ be a solution of the initial value problem for the ordinary differential equation,

$$(O_V)\qquad \begin{cases} \partial_r^2 U + \dfrac{N-1}{r}\partial_r U - V(r)U = 0 \quad \text{in} \ \ (L,\infty), \\ (\partial_r U)(L) = \mu, \quad U(L) = 1-\mu, \end{cases}$$

where $0 \le \mu \le 1$. Put

$$g(t:\omega) = (1+t)^{-\frac{\alpha(\omega)}{2}} \tag{1.7}$$

Here $\alpha = \alpha(\omega)$ is a nonnegative root of the equation $\alpha(\alpha + N - 2) = \omega$, that is,

$$\alpha(\omega) = \frac{-(N-2) + \sqrt{(N-2)^2 + 4\omega}}{2}. \tag{1.8}$$

Then, under the condition (V_ω^1), we see that

$$g(t:\omega) \asymp \left[U_{\mu,L}^V(t^{1/2})\right]^{-1}$$

for all sufficiently large t (see Proposition 3.1).

In this paper, we consider the initial-boundary value problem (1.1), and study the decay rate of $\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty}$ as $t \to \infty$, by using the asymptotic behavior of $U_{\mu,L}^V(r)$ as $r \to \infty$. Here, we give the main results of this paper for the case $N \ge 3$.

Theorem 1.1. *Let $N \ge 3$ and consider the initial-boundary value problem* (1.1) *under the condition (V_ω^l) with $\omega \ge 0$ and $l \in \boldsymbol{N}$. Let $p \ge 1$. Assume either*

$$\mu \ne \frac{2n'}{2n' + L} \quad \text{or} \quad V(r) \not\equiv \frac{\omega_{2n'}}{r^2} \quad \text{on} \quad [L,\infty) \tag{1.9}$$

for any $n' \in \boldsymbol{N}_0$ with $2n' \le l+1$. Then, for any $j \in \boldsymbol{N}_0^N$ with $|j| \le l+1$,

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}} \quad \text{if} \ \ |j| \le \alpha(\omega), \tag{1.10}$$

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{p\to\infty} \asymp t^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \quad \text{if} \ \ |j| > \alpha(\omega) \tag{1.11}$$

for all sufficiently large t.

If, for some $n' \in \boldsymbol{N}_0$, the equalities hold in (1.9), we have another decay property.

Theorem 1.2. *Let $N \ge 3$ and consider the initial-boundary value problem* (1.1)*. Assume that there exists a nonnegative integer n' such that*

$$n = 2n', \quad V(r) \equiv \frac{\omega_n}{r^2} \quad on \quad [L, \infty), \quad \mu = \frac{n}{n+L}. \tag{1.12}$$

Let $p \geq 1$. Then, for any $j \in \boldsymbol{N}_0^N$,

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{p \to \infty} \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} \qquad if \quad |j| \leq n, \tag{1.13}$$

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{p \to \infty} \asymp t^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}} \qquad if \quad |j| > n \tag{1.14}$$

for all sufficiently large t.

Here we remark that, under the condition (1.12), V satisfies the condition (V_ω^l) for all $l \in \boldsymbol{N}$ and $\alpha(\omega) = \alpha(\omega_n) = n$. Furthermore, as a corollary of Theorems 1.1 and 1.2, we have

Corollary 1.1. *Let $N \geq 3$ and $u_\mu^V = u_\mu^V(x, t : \phi)$ be a solution of the initial-boundary value problem* (1.1) *with $\phi \in L^p(\Omega_L)$, under the condition (V_ω^l) with $\omega \geq 0$ and $l \in \boldsymbol{N}$. Let $p \geq 1$ and $j \in \boldsymbol{N}_0^N$ with $|j| \leq l+1$. Then there exist positive constants C and T such that*

$$\left\|(\nabla_x^j u_\mu^V)(\cdot, t : \phi)\right\|_{L^\infty(\Omega_L)} \leq C t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $t \geq T$ and all $\phi \in L^p(\Omega_L)$ if and only if, either $\omega \geq \omega_{|j|}$ or

$$|j| = 1, \quad V(r) \equiv 0 \quad on \quad [L, \infty), \quad \mu = 0.$$

The latter case of Corollary 1.1 comes from Theorem 1.2 with $n = 0$. Indeed, in this case, there hold $\omega_0 = 0$, $\omega_1 = N - 1$ and $\alpha(\omega_0 + \omega_1) = 1$. By (1.14), the conclusion follows. The former case is an immediate consequence of Theorems 1.1 and 1.2.

For the decay rates of the derivatives of the solution for case $N = 2$, see Section 7.

Now, we give a rough sketch of the proof of the upper estimates in Theorems 1.1 and 1.2, which will be discussed in-depth in Section 5. Under the suitable assumptions on the initial data ϕ, the solution u_μ^V of (1.1) is written by

$$u_\mu^V(x, t) = \sum_{k=0}^{\infty} \sum_{i=1}^{l_k} v_\mu^{k,i}(x, t) Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad \Omega_L \times (0, \infty).$$

Here $v_\mu^{k,i}$ is a radial solution of (1.1) with $V(r)$ replaced by $V_k(r) \equiv V(r) + \omega_k/r^2$. Assume (V_ω^l) and let $j \in \boldsymbol{N}_0^N$ with $1 \leq |j| \leq l$. Then, roughly speaking, we see that there exists a function $\zeta_{k,i} = \zeta_{k,i}(t)$ such that

$$v_\mu^{k,i}(x, t) \asymp \zeta_{k,i}(t) U_{\mu,L}^{V_k}(|x|), \tag{1.15}$$

$$\left|\nabla_x^j v_\mu^{k,i}(x, t)\right| \asymp \zeta_{k,i}(t) \left|\nabla_x^j U_{\mu,L}^{V_k}(|x|)\right| \asymp \zeta_{k,i}(t) |x|^{\alpha(\omega + \omega_k) - |j|} \tag{1.16}$$

for all $x \in \Omega_L$ and all sufficiently large t with $|x| \le t^{1/2}$. By (1.16), we have

$$\begin{aligned}\big|(\nabla_x^j u_\mu^V)(x,t)\big| &\asymp \left|\sum_{k=0}^\infty \sum_{i=1}^{l_k} \zeta_{k,i}(t)\nabla_x^j \left[U_{\mu,L}^{V_k}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right)\right]\right| \\ &\preceq \sum_{k=0}^\infty \sum_{i=1}^{l_k} \zeta_{k,i}(t)|x|^{\alpha(\omega+\omega_k)-|j|} \end{aligned} \tag{1.17}$$

for all $x \in \Omega_L$ and all sufficiently large t with $|x| \le t^{1/2}$. Furthermore, by (1.7) and (1.15), we have

$$\zeta_{k,i}(t) \preceq \left[U_{\mu,L}^{V_k}(t^{1/2})\right]^{-1}\big\|v_\mu^{k,i}(\cdot,t)\big\|_{L^\infty(\Omega_L)} \asymp g(t:\omega+\omega_k)t^{-\frac{N}{2p}}\|\phi\|_{L^p(\Omega_L)}$$

for all sufficiently large t (see also (2.12)). By (1.7) and (1.17), for any sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}\big|(\nabla_x^j u_\mu^V)(x,t)\big| &\preceq t^{-\frac{N}{2p}} \sum_{k=0}^\infty \sum_{i=1}^{l_k} g(t:\omega+\omega_k)|x|^{\alpha(\omega+\omega_k)-|j|}\|\phi\|_{L^p(\Omega_L)} \\ &\preceq t^{-\frac{N}{2p}} \sum_{k=0}^\infty \sum_{i=1}^{l_k} \epsilon^{[\alpha(\omega+\omega_k)-|j|]_+} \max\left\{g(t:\omega+\omega_k), t^{-\frac{|j|}{2}}\right\}\|\phi\|_{L^p(\Omega_L)} \\ &\preceq t^{-\frac{N}{2p}} \max\left\{g(t:\omega), t^{-\frac{|j|}{2}}\right\}\|\phi\|_{L^p(\Omega_L)} \end{aligned} \tag{1.18}$$

for all $x \in \Omega_L$ and all sufficiently large t with $|x| \le \epsilon t^{1/2}$, where $[a]_+ = \max\{a, 0\}$. On the other hand, by using the standard arguments in the parabolic equations, we see

$$\big|(\nabla_x^j u_\mu^V)(x,t)\big| \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}}\|\phi\|_{L^p(\Omega_L)} \tag{1.19}$$

for all $(x,t) \in \Omega_L \times (0,\infty)$ with $|x| \ge \epsilon t^{1/2} > L+2$ (see Lemma 2.3). By (1.18) and (1.19), we have the upper estimates of $\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty}$ in Theorem 1.1. (For more details, see the proof of Proposition 5.1.)

Next, for the case (1.12), we see that $\nabla_x^j U_\mu^V \equiv 0$ in Ω_L for all $j \in \boldsymbol{N}_0^N$ with $|j| \ge n+1$ (see Proposition 3.3). Then, in a similar way to (1.17) and (1.18), we have

$$\begin{aligned}\big|(\nabla_x^j u_\mu^V)(x,t)\big| &\asymp \left|\sum_{k=1}^\infty \sum_{i=1}^{l_k} \zeta_{k,i}(t)\nabla_x^j \left[U_{\mu,L}^{V_k}(|x|)Q_{k,i}\left(\frac{x}{|x|}\right)\right]\right| \\ &\preceq t^{-\frac{N}{2p}} \max\left\{g(t:\omega+\omega_1), t^{-\frac{|j|}{2}}\right\}\|\phi\|_{L^p(\Omega_L)} \end{aligned} \tag{1.20}$$

for all $x \in \Omega_L$ and all sufficiently large t with $|x| \le \epsilon t^{1/2}$, where ϵ is a sufficiently small positive constant and $j \in \boldsymbol{N}_0^N$ with $|j| \ge n+1$. This together with (1.19) implies the upper estimates of $\|\nabla_x^j G_\mu^V(t)\|_{p\to\infty}$ in Theorem 1.2. (For more details, see the proof of Proposition 5.2.) Note that the lower estimates in (1.11) and (1.14) are essential parts of this paper and will be discussed in Section 6.

The rest of this paper is organized as follows: In Section 2, we give preliminary lemmas in order to study the decay rates of the derivatives of the solution (1.1) for the case $N \geq 3$. Section 3 is devoted to the study of the asymptotic behavior of $U_{\mu,L}^V$. In Section 4, by using the similar arguments to in [**3**] and [**4**], we study the large time behavior of derivatives of the solution v of the initial-boundary value problem (P_μ^k):

$$(P_\mu^k) \qquad \begin{cases} \partial_t v = \Delta v - \left(V(|x|) + \dfrac{\omega_k}{|x|^2}\right) v & \text{in } \ \Omega_L \times (0,\infty), \\ \mu v - (1-\mu)\partial_r v = 0 & \text{on } \ \partial\Omega_L \times (0,\infty), \\ v(\cdot,0) = \psi(\cdot) \in L^p(\Omega_L), \end{cases}$$

where $0 \leq \mu \leq 1$, $p \geq 1$, $k \in \boldsymbol{N}_0$, and ψ is a radial function in Ω_L. However, it seems difficult to obtain optimal decay rates of the derivatives of the solution v_μ^k for all $k = 0, 1, 2, \dots$ by using the arguments in [**3**] and [**4**] directly. So we construct a super-solution of (P_μ^k), and obtain estimates of the derivatives of v_μ^k. In Section 5, we give upper estimates of the derivatives of the solution u of (1.1). Lower estimates of the derivatives of the solution u_μ^V for some initial data ϕ are given in Section 6, and complete the proofs of Theorems 1.1 and 1.2 there. Furthermore, as corollaries of Theorems 1.1 and 1.2, we give two results on the decay rates of derivatives of u_μ^V. In Section 7, we study the decay rate of the solutions for the case $N = 2$.

2. Preliminaries.

In this section, we give preliminary lemmas in order to study the decay rates of the derivatives of the solution (1.1) for the case $N \geq 3$. For any $\mu \in [0,1]$, $R \geq L$, and $\omega \geq 0$, let $U_{\mu,R}^\omega$ be the solution of

$$(O_\omega) \qquad \begin{cases} \partial_r^2 U + \dfrac{N-1}{r}\partial_r U - \dfrac{\omega}{r^2} U = 0 & \text{in } \ (R,\infty), \\ (\partial_r U)(R) = \mu, \quad U(R) = 1-\mu. \end{cases}$$

Put

$$U_+^\omega(r) = \left(\frac{r}{L}\right)^{\alpha(\omega)}, \quad U_-^\omega(r) = \left(\frac{r}{L}\right)^{-\beta(\omega)}, \tag{2.1}$$

where $\beta(\omega) = N - 2 + \alpha(\omega)$. Then the functions $U_+^\omega(r)$ and $U_-^\omega(r)$ are solutions of the ordinary differential equation

$$\partial_r^2 U + \frac{N-1}{r}\partial_r U - \frac{\omega}{r^2} U = 0 \ \text{ in } \ (0,\infty), \tag{2.2}$$

and $U_+^\omega(r) \not\equiv U_-^\omega(r)$ on $(0,\infty)$. So, by the uniqueness of the solution of (O_ω), there exist constants c_1 and c_2 such that

$$U_{\mu,R}^\omega(r) = c_1 U_+^\omega(r) + c_2 U_-^\omega(r), \quad r \ge R.$$

Therefore, by $U_{\mu,R}^\omega(R) = 1-\mu$ and $\partial_r U_{\mu,R}^\omega(R) = \mu$, we obtain

$$U_{\mu,R}^\omega(r) = \frac{\alpha - \mu\alpha - R\mu}{\alpha+\beta}\left(\frac{r}{R}\right)^{-\beta} + \frac{R\mu - \beta\mu + \beta}{\alpha+\beta}\left(\frac{r}{R}\right)^{\alpha} \tag{2.3}$$

where $\alpha = \alpha(\omega)$ and $\beta = \beta(\omega)$. In what follows, we put

$$U_{\mu,R}^{\omega,k}(r) = U_{\mu,R}^{\omega+\omega_k}(r), \quad U_+^{\omega,k}(r) = U_+^{\omega+\omega_k}(r), \quad U_-^{\omega,k}(r) = U_-^{\omega+\omega_k}(r),$$

for simplicity. Then we have the following lemma on $U_{\mu,R}^\omega$.

Lemma 2.1. *Let $L \le R < S$ and a, $b \ge 0$. Assume $N \ge 3$. Then*

$$U_{\mu,R}^{a,k}(r) \asymp U_{\mu,R}^{b,k}(r) \tag{2.4}$$

for all $r \in [R,S]$, $\mu \in [0,1]$, and $k \in \boldsymbol{N}_0$,

$$U_{\mu,R}^{a,k}(r) \asymp \left[\frac{\mu}{k+1} + 1 - \mu\right]\left(\frac{r}{R}\right)^{\alpha(a+\omega_k)} \tag{2.5}$$

for all $r \ge S$, $\mu \in [0,1]$, and $k \in \boldsymbol{N}_0$, and

$$U_{0,R}^{a,k}(r) \asymp U_+^{a,k}(r) \tag{2.6}$$

for all $r \ge R$ and $k \in \boldsymbol{N}_0$. Furthermore

$$0 \le \frac{d}{dr} U_{\mu,R}^{a,k}(r) \preceq \frac{\mu + (k+1)(1-\mu)}{R}\left(\frac{r}{R}\right)^{\alpha(a+\omega_k)-1}, \tag{2.7}$$

$$0 < U_{\mu,R}^{a,k}(r) \preceq \left[\frac{\mu}{k+1} + 1 - \mu\right]\left(\frac{r}{R}\right)^{\alpha(a+\omega_k)}, \tag{2.8}$$

for all $r > R$, $0 \le \mu \le 1$, and $k \in \boldsymbol{N}_0$.

Proof. Let a, $b \ge 0$. Put $\alpha_k(a) = \alpha(a+\omega_k)$ and $\beta_k(a) = \beta(a+\omega_k)$. Then we have

$$\lim_{k\to\infty} k^{-1}\alpha_k(a) = \lim_{k\to\infty} k^{-1}\beta_k(a) = 1, \tag{2.9}$$

$$\lim_{k\to\infty} k|\alpha_k(a)-\alpha_k(b)| = 2|a-b|. \tag{2.10}$$

Then, by the Cauchy mean value theorem and (2.3), for any $r \in (R,S)$ and $k = 1,2,\dots$, there exists an $\tilde{r} \in (R,r)$ such that

$$\begin{aligned}\frac{U^{a,k}_{1,R}(r)}{U^{b,k}_{1,R}(r)} &= \frac{\alpha_k(b)+\beta_k(b)}{\alpha_k(a)+\beta_k(a)}\left(\frac{r}{R}\right)^{\alpha_k(a)-\alpha_k(b)} \frac{1-(r/R)^{-\alpha_k(a)-\beta_k(a)}}{1-(r/R)^{-\alpha_k(b)-\beta_k(b)}} \\ &= \left(\frac{r}{R}\right)^{\alpha_k(a)-\alpha_k(b)}\left(\frac{\tilde{r}}{R}\right)^{2(\alpha_k(b)+\alpha_k(a))} \geq \left(\frac{S}{R}\right)^{-|\alpha_k(a)-\alpha_k(b)|}.\end{aligned}$$

This together with (2.10) implies (2.4) for the case $\mu = 1$. Similarly, we have (2.4) for the case $\mu = 0$. Therefore, since

$$U^{\omega,k}_{\mu,R}(r) = (1-\mu)U^{\omega,k}_{0,R}(r) + \mu U^{\omega,k}_{1,R}(r), \quad r \geq R, \tag{2.11}$$

we have (2.4). On the other hand, by (2.3), we have (2.5)–(2.8), and the proof of Lemma 2.1 is complete. □

Next we recall the following two lemmas on the decay rate of the solutions of the initial-boundary value problem (1.1) under the condition (V^l_ω).

LEMMA 2.2. *Let u^V_μ be a solution of* (1.1) *under the condition (V^1_ω) with $\omega \geq 0$. Let $1 \leq p \leq q \leq \infty$ and $i = 1,2,\dots$. Then there exists a positive constant C, independent of V, such that*

$$\left\|u^V_\mu(\cdot,t)\right\|_{L^q(\Omega_L)} \leq Ct^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|\phi\|_{L^p(\Omega_L)} \tag{2.12}$$

for all $t > 0$.

PROOF. Let G^V_μ be the Green function of (1.1). By the comparison principle, we see that

$$0 < G^V_\mu(x,y,t) \leq G^0_0(x,y,t), \quad x,y \in \Omega_L,\ t > 0. \tag{2.13}$$

Furthermore, there exists a positive constant C such that

$$G^0_0(x,y,t) \leq Ct^{-\frac{N}{2}}\exp\left(-\frac{|x-y|^2}{Ct}\right), \quad x,y \in \Omega_L,\ t > 0$$

(see [**10**] and [**11**]). This together with (2.13) implies (2.12), and the proof of Lemma 2.2 is complete. □

LEMMA 2.3. *Let u^V_μ be a solution of* (1.1) *under the condition (V^l_ω) with $\omega \geq 0$ and $l \geq 1$. Then, for any $\epsilon \in (0,1)$ and $p \geq 1$, there exists a positive constant C such that*

$$\left|(\partial_t^i \nabla_x^j u_\mu^V)(x,t)\right| \le Ct^{-\frac{N}{2p}-\frac{|j|}{2}-i}\|\phi\|_{L^p(\Omega_L)}, \tag{2.14}$$

for all $(x,t) \in \Omega_L \times (0,\infty)$ *with* $|x| \ge \epsilon t^{1/2} > L+2$ *and all* $i \in \boldsymbol{N}_0$ *and* $j \in \boldsymbol{N}_0^N$ *with* $2i + |j| \le l+1$.

Proof. Let $(x_0,t_0) \in \Omega_L \times (0,\infty)$ with $|x_0| \ge \epsilon t_0^{1/2} > L+2$. Let $k = \epsilon t_0^{1/2}/2 > 1$ and put

$$\tilde{u}(x,t) = \frac{1}{k^2} u_\mu^V(x_0 + kx, t_0 + k^2 t)$$

for $(x,t) \in Q \equiv B(0,1) \times (-1,1)$. Then $\tilde{u}$ satisfies

$$\partial_t \tilde{u} = \Delta\tilde{u} - \tilde{V}(x)\tilde{u} \quad \text{in} \quad Q,$$

where $\tilde{V}(x) = k^2 V(|x_0 + kx|)$. Furthermore, by (V_ω^l)–(iii) and $|x_0| \ge 2k$, there exist constants C_1 and C_2 such that

$$\left\|\nabla_x^j \tilde{V}\right\|_{L^\infty(B(0,1))} \le C_1 \max_{x \in B(0,1)} \sum_{m=1}^{|j|} \frac{k^{m+2}}{|x_0 + kx|^{m+2}} \le C_2$$

for all $j \in \boldsymbol{N}_0^N$ with $|j| \le l$. Therefore, by Theorem 10.1 of Chapter 4 in [**5**] and Lemma 2.2, there exist constants C_3, C_4, and C_5 such that

$$\begin{aligned} k^{2i+|j|-2}\left|(\partial_t^i \nabla_x^j u_\mu^V)(x_0,t_0)\right| &= \left|(\partial_t^i \nabla_x^j \tilde{u})(0,0)\right| \le C_3\|\tilde{u}\|_{L^p(Q)} \\ &\le C_4 k^{-\frac{N}{p}-2} \sup_{t>0}\left\|u_\mu^V(\cdot,t)\right\|_{L^p(\Omega_L)} \le C_5 k^{-\frac{N}{p}-2}\|\phi\|_{L^p(\Omega_L)} \end{aligned}$$

for all $i \in \boldsymbol{N}_0$ and $j \in \boldsymbol{N}_0^N$ with $2i + |j| \le l+1$. This implies (2.14), and the proof of Lemma 2.3 is complete. □

3. Behavior of $U_{\mu,L}^V(r)$ as $r \to \infty$.

In this section, we study the behavior of the solution $U_{\mu,L}^V(r)$ of (O_V) under the assumption (V_ω^l). Put

$$V_k(r) = V(r) + \frac{\omega_k}{r^2}, \quad k \in \boldsymbol{N}_0.$$

In what follows, for $k \in \boldsymbol{N}_0$ and $\lambda \in \boldsymbol{R}$, we put

$$\alpha_k = \alpha(\omega + \omega_k), \quad \beta_k = N - 2 + \alpha_k, \quad h_\lambda(r) = V(r) - \frac{\lambda}{r^2}$$

for simplicity. We first prove the following lemma.

LEMMA 3.1. *Let $R \geq L$, $a \geq 0$, and $k \in \boldsymbol{N}_0$. For any $g \in C([R,\infty))$, put*

$$H_R^{a,k}[g](r) = U_-^{a,k}(r) \int_R^r s^{1-N}[U_-^{a,k}(s)]^{-2} \biggl(\int_R^s \tau^{N-1} U_-^{a,k}(\tau) g(\tau) d\tau \biggr) ds.$$

Then

(i) $H_R^{a,k}[g](r)$ *is a solution of the ordinary differential equation*

$$U'' + \frac{N-1}{r}U' - \frac{a+\omega_k}{r^2}U = g \quad in \quad (R,\infty),$$

with $U(R) = U'(R) = 0$. In particular,

$$U_{\mu,R}^{V_k}(r) = U_{\mu,R}^{a,l}(r) + H_R^{a,l}\bigl[h_{\omega_l+a-\omega_k} U_{\mu,R}^{V_k}\bigr](r)$$

for all $r \geq R$, $k \in \boldsymbol{N}_0$, and $l = 0,\dots,k$.

(ii) *If $g(r) \geq 0$ on $[R,R_1]$ with $R_1 > R$, then*

$$H_R^{a,k}[g](r) \geq 0, \quad H_R^{a,k}[g]'(r) \geq 0, \quad R \leq r \leq R_1. \tag{3.1}$$

(iii) *Assume that there exists a positive constant A such that*

$$|g(r)| \leq A|h_a(r)| U_{\mu,R}^{a,k}(r), \quad r \geq R. \tag{3.2}$$

Then there exist positive constants C_1 and C_2, independent of R and k, such that

$$\bigl|H_R^{a,k}[g]'(r)\bigr| \leq C_1 A r^{-1} U_{\mu,R}^{a,k}(r) \int_R^r \tau |h_a(\tau)| d\tau, \tag{3.3}$$

$$\bigl|H_R^{a,k}[g](r)\bigr| \leq C_2 A U_{\mu,R}^{a,k}(r) \int_R^r \tau |h_a(\tau)| d\tau \tag{3.4}$$

for all $r \geq R$.

PROOF. The statement (i) comes from the variation of constants for the second order ordinary differential equations for (O_V). Furthermore, we can switch the role of $U_-^{a,k}$ to $U_+^{a,k}$ in the definition of $H_R^{a,k}$ and we have

$$H_R^{a,k}[g](r) = U_+^{a,k}(r) \int_R^r s^{1-N}\bigl[U_+^{a,k}(s)\bigr]^{-2} \biggl(\int_R^s \tau^{N-1} U_+^{a,k}(\tau) g(\tau) d\tau \biggr) ds.$$

This implies the statement (ii). Next we prove (iii) for the case $\mu = 0$. Put

$$U_g(r) = r^{-\beta_k(a)} \int_R^r s^{2\beta_k(a)+1-N} I_g(s) ds, \quad I_g(s) = \int_R^s \tau^{1-\alpha_k(a)} g(\tau) d\tau,$$

where $\alpha_k(a) = \alpha(a+\omega_k)$ and $\beta_k(a) = \beta(a+\omega_k)$. Then $H_R^{a,k}[g](r) = U_g(r)$ for $r \ge R$. By (2.6) and (3.2), there exists a constant C_1 such that

$$\max_{R\le s\le r} |I_g(s)| \le C_1 A \int_R^r \tau^{1-\alpha_k(a)} \left(\frac{\tau}{R}\right)^{\alpha_k(a)} |h_a(\tau)| d\tau \\ \le C_1 A R^{-\alpha_k(a)} \int_R^r \tau |h_a(\tau)| d\tau. \tag{3.5}$$

By (3.5) and $\beta_k(a) > 0$, we have

$$\begin{aligned} \left|U_g'(r)\right| &\le \beta_k(a) r^{-\beta_k(a)-1} \int_R^r s^{2\beta_k(a)+1-N} |I_g(s)| ds + r^{\beta_k(a)+1-N} |I_g(r)| \\ &\le \left[1 + \frac{\beta_k(a)}{\alpha_k(a)+\beta_k(a)}\right] r^{\alpha_k(a)-1} \max_{R\le s\le r} |I_g(s)| \\ &\le 2C_1 A r^{-1} \big(\frac{r}{R}\big)^{\alpha_k(a)} \int_R^r \tau |h_a(\tau)| d\tau \end{aligned}$$

and

$$\begin{aligned} |U_g(r)| &\le r^{-\beta_k(a)} \max_{R\le s\le r} |I_g(s)| \int_R^r s^{2\beta_k(a)+1-N} ds \\ &\le \frac{C_1 A}{\alpha_k(a)+\beta_k(a)} \left(\frac{r}{R}\right)^{\alpha_k(a)} \int_R^r \tau |h_a(\tau)| d\tau \end{aligned}$$

for all $r \ge R$ and $k \in \boldsymbol{N}_0$. These inequalities together with (2.6) imply (3.3) and (3.4) for the case $\mu = 0$. Next we consider the case $\mu = 1$. Then

$$\left(\frac{r}{R}\right)^{-\alpha_k(a)} U_{1,R}^{a,k}(r) = \frac{R}{\alpha_k(a)+\beta_k(a)} - \frac{R}{\alpha_k(a)+\beta_k(a)} \left(\frac{r}{R}\right)^{-\alpha_k(a)-\beta_k(a)}$$

is a monotone increasing function, and we have

$$\begin{aligned} \max_{R\le s\le r} |I_g(s)| &\le A \left(\frac{r}{R}\right)^{-\alpha_k(a)} U_{1,R}^{a,k}(r) \int_R^r \tau^{1-\alpha_k(a)} \left(\frac{\tau}{R}\right)^{\alpha_k(a)} |h_a(\tau)| d\tau \\ &\le \frac{A R^{-\alpha_k(a)}}{\alpha_k(a)+\beta_k(a)} \left(\frac{r}{R}\right)^{-\alpha_k(a)} U_{1,R}^{a,k}(r) \int_R^s \tau |h_a(\tau)| d\tau. \end{aligned}$$

Therefore, by the same argument as in the proof of (3.3) and (3.4) for the case $\mu = 0$, we have (3.3) and (3.4) for the case $\mu = 1$. Finally, by (2.11), we have (3.3) and (3.4) for the case $0 < \mu < 1$, and the proof of Lemma 3.1 is complete. □

In view of Lemma 3.1, we have the following proposition on the behavior of $U_{\mu,L}^{V_k}(r)$

as $r \to \infty$, by using the function $U^{\omega,k}_{\mu,L}(r) = U^{\alpha(\omega+\omega_k)}_{\mu,L}(r)$.

PROPOSITION 3.1. *Assume (V^1_ω) with $\omega \geq 0$ and $N \geq 3$. Then*

$$0 \leq \big(\partial_r U^{V_k}_{\mu,L}\big)(r) \preceq (k+1)\left(\frac{r}{L}\right)^{\alpha_k - 1} \tag{3.6}$$

for all $r > L$, $0 \leq \mu \leq 1$, and $k \in \boldsymbol{N}_0$. Furthermore

$$U^{V_k}_{\mu,L}(r) \asymp U^{\omega,k}_{\mu,L}(r), \quad 0 \leq \mu \leq 1, \tag{3.7}$$

$$U^{V_k}_{0,L}(r) \asymp U^{\omega,k}_{+}(r) \tag{3.8}$$

for all $r \geq L$ and $k \in \boldsymbol{N}_0$. In particular,

$$U^{V_k}_{\mu,L}(r) \asymp \left[\frac{\mu}{k+1} + 1 - \mu\right] U^{\omega,k}_{+} \tag{3.9}$$

for all sufficiently large r, $0 \leq \mu \leq 1$, and $k \in \boldsymbol{N}_0$.

PROOF. By $U^{V_k}_{\mu,L}(L) = 1 - \mu$, $(U^{V_k}_{\mu,L})'(L) = \mu$, and the continuity of $U^{V_k}_{\mu,L}$, there exists a constant $r_0 > L$ such that $U^{V_k}_{\mu,L}(r) > 0$ for all $r \in (L, r_0)$. Assume that there exists a constant $r_1 > r_0$ such that

$$U^{V_k}_{\mu,L}(r) > 0, \quad r \in (L, r_1), \quad U^{V_k}_{\mu,L}(r_1) = 0.$$

By $h_0(r) = V(r) \geq 0$, we see $h_0(r) U^{V_k}_{\mu,L}(r) \geq 0$ on $[R, r_1]$. So, by Lemma 3.1–(i), (ii), we have

$$U^{V_k}_{\mu,L}(r) = U^{0,l}_{\mu,L}(r) + H^{0,l}_L\big[h_{\omega_l - \omega_k} U^{V_k}_{\mu,L}\big](r) \geq U^{0,k}_{\mu,L}(r) > 0$$

for all $r \in (L, r_1]$ and $l = 0, \ldots, k$. This contradicts $U^{V_k}_{\mu,L}(r_1) = 0$. So we see that $U^{V_k}_{\mu,L}(r) > 0$ on (L, ∞), and obtain

$$U^{V_k}_{\mu,L}(r) = U^{0,k}_{\mu,L}(r) + H^{0,k}_L\big[V U^{V_k}_{\mu,L}\big](r) \geq U^{0,k}_{\mu,L}(r) > 0, \quad r > L. \tag{3.10}$$

Furthermore, by Lemma 3.1–(ii), (2.7), and (3.10), we see that

$$\big(\partial_r U^{V_k}_{\mu,L}\big)(r) \geq 0, \quad r > L. \tag{3.11}$$

Let S be a constant to be chosen later such that $S > L$. Put $\omega_S = S^2 \max_{L \leq r \leq S} V(r)$. Then $h_{\omega_S}(r) \leq 0$ in (L, S). So, by (3.10), we have $h_{\omega_S}(r) U^{V_k}_{\mu,L}(r) \leq 0$ in (L, S). Hence, by Lemma 3.1–(i), (ii), we have

$$U^{V_k}_{\mu,L}(r) = U^{\omega_S,k}_{\mu,L}(r) + H^{\omega_S,k}_L\big[h_{\omega_S}U^{V_k}_{\mu,L}\big](r) \le U^{\omega_S,k}_{\mu,L}(r) \tag{3.12}$$

for all $r \in [L,S]$. By (2.4), (3.10), and (3.12), we have

$$U^{V_k}_{\mu,L}(r) \asymp U^{\omega,k}_{\mu,L}(r), \quad r \in [L,S],\ 0 \le \mu \le 1,\ k \in \boldsymbol{N}_0. \tag{3.13}$$

Furthermore, by (2.5), we have

$$U^{V_k}_{\mu,L}(S) \asymp U^{\omega,k}_{\mu,L}(S) \asymp \left[\frac{\mu}{k+1} + \mu - 1\right]\left(\frac{S}{L}\right)^{\alpha_k} \tag{3.14}$$

for all $k \in \boldsymbol{N}_0$ and $0 \le \mu \le 1$. On the other hand, by Lemma 3.1–(i), we have

$$U^{V_k}_{\mu,L}(r) = U^{\omega,k}_{\mu,L}(r) + H^{\omega,k}_L\big[h_\omega U^{V_k}_{\mu,L}\big](r), \quad r \ge L. \tag{3.15}$$

Then, by (V^l_ω)–(ii), Lemma 3.1–(iii), and (3.13), we have

$$0 \le \frac{d}{dr}U^{V_k}_{\mu,L}(r) \preceq [\mu + (k+1)(1-\mu)]\left(\frac{r}{L}\right)^{\alpha_k - 1} \tag{3.16}$$

for all $r \in [L,S]$ and $k \in \boldsymbol{N}_0$.

Let $\epsilon > 0$ be a sufficiently small constant to be chosen later. By (V^l_ω)–(ii), we may take a sufficiently large S so that

$$\int_S^\infty \tau|h_\omega(\tau)|d\tau < \epsilon. \tag{3.17}$$

By Lemma 3.1–(i), we have

$$U^{V_k}_{\mu,S}(r) = U^{\omega,k}_{\mu,S}(r) + H^{\omega,k}_S\big[h_\omega U^{V_k}_{\mu,S}\big](r), \quad r \ge S.$$

Put

$$U_1(r) = U^{\omega,k}_{\mu,S}(r), \quad U_{j+1}(r) = U^{\omega,k}_{\mu,S}(r) + H^{\omega,k}_S[h_\omega U_j](r), \quad j \in \boldsymbol{N}. \tag{3.18}$$

Then there exists $C_0 > 0$ independent of j such that

$$\big|U^{V_k}_{\mu,S} - U_{j+1}\big| \le C_0\epsilon\big|U^{V_k}_{\mu,S} - U_j\big|, \quad j \in \boldsymbol{N}.$$

By the standard arguments in the ordinary differential equations, we see that

$$U^{V_k}_{\mu,S}(r) = \lim_{j\to\infty} U_j(r), \quad r \ge S. \tag{3.19}$$

By (3.4), (3.17), and (3.18), there exists a positive constant C such that

$$\left|U_2(r) - U^{\omega,k}_{\mu,S}(r)\right| \le C U^{\omega,k}_{\mu,S}(r) \int_S^\infty \tau |h_\omega(\tau)| d\tau \le C\epsilon U^{\omega,k}_{\mu,S}(r) \tag{3.20}$$

for all $r \ge S$. Similarly, we have

$$\left|U_3(r) - U^{\omega,k}_{\mu,S}(r)\right| \le (C\epsilon + C^2\epsilon^2) U^{\omega,k}_{\mu,S}(r), \quad r \ge S.$$

Let ϵ be a sufficiently small positive constant such that $C\epsilon < 1/4$. By repeating this argument, we have

$$\left|U_j(r) - U^{\omega,k}_{\mu,S}(r)\right| \le U^{\omega,k}_{\mu,S}(r) \sum_{k=1}^{j-1} (C\epsilon)^k \le \frac{1}{2} U^{\omega,k}_{\mu,S}(r)$$

for all $r \ge S$ and $j = 2, 3, \dots$. By (3.19), we have

$$\left|U^{V_k}_{\mu,S}(r) - U^{\omega,k}_{\mu,S}(r)\right| \le \frac{1}{2} U^{\omega,k}_{\mu,S}(r), \quad r \ge S, \tag{3.21}$$

and obtain

$$U^{V_k}_{\mu,S}(r) \asymp U^{\omega,k}_{\mu,S}(r) \quad r \ge S,\ k \in \boldsymbol{N}_0.$$

Then, by (2.5), (2.6), (3.14), and (3.16), we have

$$\begin{aligned} U^{V_k}_{\mu,L}(r) &= U^{V_k}_{\mu,L}(S) U^{V_k}_{0,S}(r) + \left(\partial_r U^{V_k}_{\mu,L}\right)(S) U^{V_k}_{1,S}(r) \\ &\ge U^{V_k}_{\mu,L}(S) U^{V_k}_{0,S}(r) \asymp \left[\frac{\mu}{k+1} + (1-\mu)\right] \left(\frac{S}{L}\right)^{\alpha_k} \left(\frac{r}{S}\right)^{\alpha_k} \asymp U^{\omega,k}_{\mu,L}(r) \end{aligned} \tag{3.22}$$

for all $r \ge S$ and $k \in \boldsymbol{N}_0$. Furthermore, by (2.3) and (2.9), we have

$$U^{V_k}_{1,S}(r) \asymp U^{\omega,k}_{1,S}(r) \le \frac{S}{\alpha_k + \beta_k} \left(\frac{r}{S}\right)^{\alpha_k} \preceq \frac{1}{k+1} \left(\frac{r}{S}\right)^{\alpha_k}$$

and obtain

$$\begin{aligned} U^{V_k}_{\mu,L}(r) &= U^{V_k}_{\mu,L}(S) U^{V_k}_{0,S}(r) + \left(\partial_r U^{V_k}_{\mu,L}\right)(S) U^{V_k}_{1,S}(r) \\ &\preceq U^{V_k}_{\mu,L}(S) U^{V_k}_{0,S}(r) + \left[\frac{\mu}{k+1} + (1-\mu)\right] \left(\frac{S}{L}\right)^{\alpha_k - 1} \left(\frac{r}{S}\right)^{\alpha_k} \asymp U^{\omega,k}_{\mu,L}(r) \end{aligned} \tag{3.23}$$

for all $r \ge S$ and $k \in \boldsymbol{N}_0$. Therefore, by (2.5), (2.6), (3.13), (3.22) and (3.23), we have (3.7)–(3.9). Furthermore, by Lemma 3.1, (2.7), (2.8), (3.7), and (3.16), we have (3.6), and the proof of Proposition 3.1 is complete. □

Furthermore, by Proposition 3.1, we have the following proposition.

Proposition 3.2. *Assume* (V_ω^1) *with* $\omega \ge 0$ *and* $N \ge 3$. *For any* $g \in C([L,\infty))$, *put*

$$F_L^V[g](r) = U_{0,L}^V(r)\int_L^r s^{1-N}\big[U_{0,L}^V(s)\big]^{-2}\bigg(\int_L^s \tau^{N-1}U_{0,L}^V(\tau)g(\tau)\,d\tau\bigg)ds.$$

Then, for any $k \in \boldsymbol{N}_0$, $F_L^{V_k}[g](r)$ *is a solution of*

$$\begin{cases} U'' + \dfrac{N-1}{r}U' - V_k(r)U = g \quad in \ \ (L,\infty), \\ U(L) = U'(L) = 0. \end{cases} \tag{3.24}$$

If there exist constants $A > 0$ *such that*

$$|g(r)| \le AU_{0,L}^{V_k}(r), \quad r \ge L,$$

then there exists a positive constant C, *independent of* k, *such that*

$$\big|F_L^{V_k}[g](r)\big| \le CA(k+1)^{-1}r^2U_{0,L}^{V_k}(r), \tag{3.25}$$

$$\big|F_L^{V_k}[g]'(r)\big| \le CArU_{0,L}^{V_k}(r), \tag{3.26}$$

for all $r \ge L$.

Proof. As in Lemma 3.1, by the definition of $F_L^{V_k}$, we see that $F_L^{V_k}$ satisfies (3.24). Put

$$J(r) = \int_L^r s^{1-N}\big[U_{0,L}^{V_k}(s)\big]^{-2}\bigg(\int_L^s \tau^{N-1}\big(U_{0,L}^{V_k}(\tau)\big)^2\,d\tau\bigg)ds.$$

By (3.8), there exists a constant C_1 such that

$$\begin{aligned} |J(r)| &\le C_1\int_L^r s^{1-N}\big[U_+^{\omega,k}(s)\big]^{-2}\bigg(\int_L^s \tau^{N-1}\big[U_+^{\omega,k}(\tau)\big]^2\,d\tau\bigg)ds \\ &\le C_1\int_L^r s^{-2\alpha_k+1-N}\bigg(\int_L^s \tau^{2\alpha_k+N-1}\,d\tau\bigg)ds \le \frac{C_1}{2(2\alpha_k+N)}r^2, \end{aligned} \tag{3.27}$$

$$\begin{aligned} |J'(r)| &\le C_1r^{1-N}\big[U_+^{\omega,k}(r)\big]^{-2}\int_L^r \tau^{N-1}\big[U_+^{\omega,k}(\tau)\big]^2\,d\tau \\ &\le C_1r^{-2\alpha_k+1-N}\int_L^r \tau^{2\alpha_k+N-1}\,d\tau \le \frac{C_1}{2\alpha_k+N}r \end{aligned} \tag{3.28}$$

for all $r \geq L$. By (2.9) and (3.27), we have (3.25). Furthermore, by (3.6) and (3.28), we have

$$\left|F_L^{V_k}[g]'(r)\right| \preceq (k+1)\left(\frac{r}{L}\right)^{\alpha_k-1}|J(r)| + \left(\frac{r}{L}\right)^{\alpha_k}|J'(r)| \preceq r\left(\frac{r}{L}\right)^{\alpha_k}.$$

Therefore, by (3.8), we obtain (3.26), and the proof of Proposition 3.2 is complete. □

Next we consider the condition (1.12).

PROPOSITION 3.3. *Assume* (V_ω^l) *with* $\omega \geq 0$ *and* $l \in \boldsymbol{N}$. *Furthermore assume that there exists a multi-index* $J \in \boldsymbol{N}_0^N$ *with* $|J| = n+1 \leq l+2$ *such that*

$$\begin{aligned}&\left(\nabla_x^j U_{\mu,L}^V\right)(|x|) \not\equiv 0 \ \textit{in}\ \Omega_L, \quad \textit{for all}\ j \in \boldsymbol{N}_0^N\ \textit{with}\ |j| \leq n,\\ &\left(\nabla_x^J U_{\mu,L}^V\right)(|x|) \equiv 0 \ \textit{in}\ \Omega_L.\end{aligned} \tag{3.29}$$

Then there exists a nonnegative integer n' *such that* (1.12),

$$U_{\mu,L}^V(|x|) = \frac{1-\mu}{L^n}\left(x_1^2 + \cdots + x_N^2\right)^{n'} = \frac{1-\mu}{L^n}|x|^n, \quad x \in \Omega_L, \tag{3.30}$$

and

$$\left(\nabla_x^j U_{\mu,L}^V\right)(|x|) \equiv 0 \ \ \textit{in}\ \ \Omega_L \tag{3.31}$$

hold for all $j \in \boldsymbol{N}_0^N$ *with* $|j| \geq n+1$.

PROOF. Let $J = (J_1, \cdots, J_N) \in \boldsymbol{N}_0^N$ with $|J| = \sum_{i=1}^N J_i = n+1 \leq l+2$ such that $(\nabla_x^J U_\mu^V)(x) \equiv 0$ in Ω_L. Put

$$\boldsymbol{Z}(J) = \left\{j = (j_1, \cdots, j_N) \in \boldsymbol{N}_0^N : 0 \leq j_i \leq J_i,\ i = 1, \ldots, N\right\},$$

$$\varphi_0(r) = U_{\mu,L}^V(r), \quad \varphi_{k+1}(r) = \frac{1}{r}(\partial_r\varphi_k)(r), \quad k = 0, \ldots, n.$$

Then we have

$$\frac{\partial}{\partial x_i}\varphi_k(|x|) = (\partial_r\varphi_k)(|x|)\frac{x_i}{|x|} = \varphi_{k+1}(|x|)x_i, \quad i = 1, \ldots, N.$$

So there exist radial functions $\{f_j\}_{j\in\boldsymbol{Z}(J)\setminus\{J\}}$ such that

$$\begin{aligned}0 = \left(\nabla_x^J\varphi_0\right)(|x|) &= \varphi_{n+1}(|x|)x^J + \sum_{j\in\boldsymbol{Z}(J)\setminus\{J\}} f_j(|x|)x^j\\ &= |x|^{n+1}\varphi_{n+1}(|x|)y^J + \sum_{j\in\boldsymbol{Z}(J)\setminus\{J\}} |x|^{|j|}f_j(|x|)y^j\end{aligned}$$

for all $x \in \Omega_L$ and $y \in \boldsymbol{S}^{N-1}$ with $y = x/|x|$, and we have

$$r^{n+1}\varphi_{n+1}(r) = 0, \quad r \geq L. \tag{3.32}$$

This implies that $U^V_{\mu,L}(r)$ is a polynomial at most of $2n$ degree, that is, there exist a natural number $l(\leq 2n)$ and constants $\{a_i\}_{i=0}^{l}$ such that $a_l \neq 0$ and

$$U^V_{\mu,L}(r) = \sum_{i=0}^{l} a_i r^i, \quad r \geq L. \tag{3.33}$$

Then, by (O_V) and (3.33), we have

$$\sum_{i=2}^{l} i(i-1)a_i r^{i-2} + (N-1)\sum_{i=1}^{l} i a_i r^{i-2} - \omega \sum_{i=0}^{l} a_i r^{i-2} = h_\omega(r) U^V_{\mu,L}(r)$$

for all $r \geq L$. Let

$$\begin{aligned} &b_0 = -\omega a_0, \quad b_1 = (N-1)a_1 - \omega a_1, \\ &b_i = i(i-1)a_i + (N-1)i a_i - \omega a_i. \end{aligned} \tag{3.34}$$

Then, by (V^l_ω)–(i), we have

$$r^2 h_\omega(r) U^V_{\mu,L}(r) = \sum_{i=0}^{l} b_i r^i = \sum_{i=0}^{l} a_i h_\omega(r) r^{i+2} = o(1)\sum_{i=0}^{l} a_i r^i \tag{3.35}$$

for all sufficiently large r. So we have $b_l = 0$, and by $a_l \neq 0$, we obtain

$$\omega = l(l-1) + (N-1)l = \omega_l. \tag{3.36}$$

By (3.34) and (3.36), for any $i = 0, 1, \dots, l-1$, if $b_i \neq 0$, then $a_i \neq 0$. However, since (3.35) holds identically, we see that

$$a_0 = a_1 = \cdots = a_{l-1} = 0, \quad h_\omega \equiv 0 \text{ on } [L, \infty). \tag{3.37}$$

Therefore $U^V_{\mu,L}(r) = a_l r^l$, and by (3.32), we have

$$r^{n+1}\varphi_{n+1}(r) = a_l\biggl(\prod_{k=0}^{n}(l-2k)\biggr) r^{l-(n+1)} = 0, \quad r \geq L.$$

So there exists a nonnegative integer $n' \in \boldsymbol{N} \cup \{0\}$ such that $l = 2n'$, and we have

$$U^V_{\mu,L}(|x|) = a_l\bigl(x_1^2 + \cdots + x_N^2\bigr)^{n'}.$$

Then, by (3.29), we have $n = 2n' = l$ and (3.31). Furthermore, by (3.36), (3.37), and the boundary condition, we have (1.12). Finally, by $U^V_{\mu,L}(L) = 1 - \mu$, we have (3.30), and the proof of Proposition 3.3 is complete. □

4. Derivatives of the solutions of (P^k_μ).

In this section, we consider the radial solution v of the initial-boundary value problem

$$(P^k_\mu) \qquad \begin{cases} \partial_t v = \Delta v - V_k(|x|)v & \text{in } \Omega_L \times (0,\infty), \\ \mu v - (1-\mu)\partial_r v = 0 & \text{on } \partial\Omega_L \times (0,\infty), \\ v(\cdot,0) = \psi(\cdot) \in L^p(\Omega_L), \end{cases}$$

where $0 \le \mu \le 1$, $p \ge 1$, $k \in \boldsymbol{N}_0$, and ψ is a radial function in Ω_L. For any positive ϵ and T, put

$$D_\epsilon(T) = \{(x,t) \in \Omega_L \times (T,\infty) : |x| < \epsilon(1+t)^{1/2}\},$$
$$\Gamma_\epsilon(T) = \{(x,t) \in \Omega_L \times (T,\infty) : |x| = \epsilon(1+t)^{1/2}\}$$
$$\cup \{(x,T) : x \in \Omega_L,\ |x| \le \epsilon(1+T)^{1/2}\}.$$

In this section, we construct a super-solution of (P^k_μ) in $D_\epsilon(T)$ for some positive constants ϵ and T, and give some estimates on the derivatives of the solution v^k_μ of (P^k_μ) in $D_\epsilon(T)$. In what follows, under the assumption (V^l_ω), we put

$$U_k(r) = U^{V_k}_{0,L}(r), \quad g_k(t) = g(t : \omega + \omega_k)$$

for simplicity. We first construct a super-solution of (P^k_μ).

LEMMA 4.1. *Assume $N \ge 3$ and (V^l_ω) with $\omega \ge 0$ and $k \in \boldsymbol{N}_0$. Let $\gamma > 0$. Then there exist positive constants T, ϵ, and C, which are independent of k, and a function $W = W(x,t)$ in $\Omega_L \times (0,\infty)$ such that*

$$\partial_t W \ge \Delta W - V_k(|x|)W \qquad \text{in } D_\epsilon(T), \tag{4.1}$$

$$\mu W(x,t) + (1-\mu)\frac{\partial}{\partial\nu}W(x,t) \ge 0 \quad \text{on } \partial\Omega_L \times (T,\infty), \tag{4.2}$$

$$W(x,t) \ge C^{-\alpha_k}(1+t)^{-\gamma} \qquad \text{on } \Gamma_\epsilon(T), \tag{4.3}$$

and

$$0 < W(x,t) \le (1+t)^{-\gamma} g_k(t) U_k(|x|) \quad \text{in } D_\epsilon(T). \tag{4.4}$$

PROOF. Let A and ϵ be constants to be chosen later such that $A > 0$ and $0 < \epsilon < 1$. Let T_ϵ be a positive constant such that $\epsilon(1+T_\epsilon)^{1/2} = L + 1$. Put

$$W(x,t) = (1+t)^{-\gamma} g_k(t)\big[U_k(|x|) - A(1+k)(1+t)^{-1} F_L^{V_k}[U_k](|x|)\big]$$

for all $(x,t) \in \Omega_L \times (T_\epsilon, \infty)$. Then, by (1.7) and (2.9), there exists a constant $C_1 = C_1(\gamma)$ such that

$$\begin{aligned}\partial_t W &\ge \big[-\gamma(1+t)^{-\gamma-1} g_k(t) + (1+t)^{-\gamma} g_k'(t)\big] U_k(|x|) \\ &\ge -C_1(1+k)(1+t)^{-\gamma-1} g_k(t) U_k(|x|) \end{aligned} \tag{4.5}$$

and by (3.24), we have

$$\Delta W - V_k(|x|)W = -A(1+k)(1+t)^{-\gamma-1} g_k(t) U_k(|x|) \tag{4.6}$$

in $\Omega_L \times (T_\epsilon, \infty)$. Let $A = C_1$. Then, by (4.5) and (4.6), we have

$$\partial_t W \ge \Delta W - V_k(|x|)W \quad \text{in} \quad \Omega_L \times (T_\epsilon, \infty). \tag{4.7}$$

On the other hand, by Proposition 3.2, there exists a positive constant C_2, independent of ϵ, such that

$$\begin{aligned} 0 &\le A(1+k)(1+t)^{-1} F_L^{V_k}[U_k](|x|) \\ &\le C_2 A (1+t)^{-1} |x|^2 U_k(|x|) \le C_2 A \epsilon U_k(|x|) \end{aligned}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Let $0 < \epsilon \le \min\{1, 1/2C_2A\}$. Then we have

$$\frac{1}{2} g_k(t) U_k(|x|) \le (1+t)^\gamma W(x,t) \le g_k(t) U_k(|x|) \tag{4.8}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Then, by the definition of W, we have

$$\mu W + (1-\mu)\frac{\partial}{\partial \nu} W = \mu W \ge 0 \quad \text{on} \quad \partial\Omega_L \times (0,\infty). \tag{4.9}$$

By Proposition 3.1 and (1.7), we see that

$$\begin{aligned} U_k\big(\epsilon(1+t)^{1/2}\big) &\asymp U_{\mu,L}^{\omega,k}\big(\epsilon(1+t)^{1/2}\big) \\ &\succeq (k+1)^{-1}\left(\frac{\epsilon(1+t)^{1/2}}{L}\right)^{\alpha_k} \asymp (k+1)^{-1}\left(\frac{\epsilon}{L}\right)^{\alpha_k} [g_k(t)]^{-1} \end{aligned} \tag{4.10}$$

for all $t \ge T_\epsilon$ and $k \in \boldsymbol{N}_0$. By (4.8) and (4.10), there exists a positive constant C_3 such that

$$(1+t)^\gamma W(x,t) \ge \frac{1}{2} g_k(t) U_k(|x|) = \frac{1}{2} g_k(t) U_k\big(\epsilon(1+t)^{1/2}\big)$$
$$\ge C_3^{-1}(k+1)^{-1}\left(\frac{\epsilon}{L}\right)^{\alpha_k} \tag{4.11}$$

for all $(x,t) \in \Gamma_\epsilon(T_\epsilon)$ with $t > T_\epsilon$. Furthermore, by (3.6), (4.8), and $\epsilon(1+T_\epsilon)^{1/2} = L+1$, there exists a positive constant C_4 such that

$$W(x,T_\epsilon) \ge \frac{1}{2}(1+T_\epsilon)^{-\gamma-\frac{\alpha_k}{2}} U_k(L) = \frac{1}{2}(1+T_\epsilon)^{-\gamma}\left(\frac{\epsilon}{L+1}\right)^{\alpha_k}$$
$$\ge C_4^{-\alpha_k}(1+T_\epsilon)^{-\gamma} \tag{4.12}$$

for all $(x,T_\epsilon) \in \Gamma_\epsilon(T_\epsilon)$ and $k \in \boldsymbol{N}_0$. By (4.7), (4.9), (4.11), and (4.12), we have (4.1)–(4.4), and the proof of Lemma 4.1 is complete. □

Next we give the following lemmas on the estimates of derivatives of v_μ^k. First, we estimate v and its time derivatives.

LEMMA 4.2. *Assume that ψ is a radial function in Ω_L such that $\|\psi\|_{L^p(\Omega_L)} = 1$ with $p \ge 1$. Let $N \ge 3$ and v be a solution of (P_μ^k) with $v(\cdot,0) = \psi(\cdot)$ under the condition (V_ω^l) with $\omega \ge 0$. Put*

$$w(x,t) = F_L^{V_k}[(\partial_t v)(\cdot,t)](|x|).$$

Then there exist positive constants T, ϵ, and η, independent of k, such that

$$\left|\partial_t^i v(x,t)\right| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega,k}(|x|), \tag{4.13}$$
$$\left|\partial_t^i w(x,t)\right| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t) |x|^2 U_+^{\omega,k}(|x|) \tag{4.14}$$

for all $(x,t) \in D_\epsilon(T)$ and all $i \in \boldsymbol{N}_0$ with $2i \le l+1$.

PROOF. Let $i \in \boldsymbol{N}_0$ and put $v_i = \partial_t^i v$. Let T and ϵ be positive constants given in Lemma 4.1. Let W be the function constructed in Lemma 4.1 with $\gamma = N/2p + i$. For any $\eta_1 > 0$, we put

$$\overline{v}_i(x,t) = \eta_1^{\alpha_k} W(x,t)$$

for all $(x,t) \in D_\epsilon(T)$. Then, taking a sufficiently large T and η_1 if necessary, by Lemma 2.3, we have

$$|v_i(x,t)| \le \overline{v}_i(x,t) \quad \text{on} \quad \Gamma_\epsilon(T).$$

So, by the comparison principle, we have

$$|v_i(x,t)| \le \overline{v}_i(x,t) \quad \text{in} \quad D_\epsilon(T).$$

This inequality together with (2.8), (3.7), and (4.4) implies

$$\begin{aligned}|v_i(x,t)| &\le \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_k(|x|)\\ &\asymp \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U^{\omega,k}_{\mu,L}(|x|) \preceq \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U^{\omega,k}_{+}(|x|)\end{aligned}$$

for all $(x,t)\in D_\epsilon(T)$, and we obtain the inequality (4.13). On the other hand, since

$$(\partial_t^i w)(x,t) = F_L^{V_k}\left[(\partial_t^{i+1} v)(\cdot,t)\right](|x|) \tag{4.15}$$

for all $(x,t)\in \Omega_L\times(0,\infty)$, by (3.8), (3.25) and (4.13), we have (4.14), and the proof of Lemma 4.2 is complete. □

Furthermore we have the following lemma on the time derivatives of $\partial_r v$ and $\partial_r w$.

Lemma 4.3. *Assume the same assumptions as in Lemma* 4.2. *Then there exist positive constants T, η, and ϵ, independent of k, such that*

$$\left|\partial_t^i \partial_r v(x,t)\right| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t)|x|^{-1} U^{\omega,k}_{+}(|x|), \tag{4.16}$$

$$\left|\partial_t^i \partial_r w(x,t)\right| \le \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t)|x| U^{\omega,k}_{+}(|x|) \tag{4.17}$$

for all $(x,t)\in D_\epsilon(T)$ and all $i\in \boldsymbol{N}_0$ with $2i\le l+1$.

Proof. By (3.8), (3.26), (4.13), and (4.15), we have (4.17). So we prove (4.16). Put $v_i=\partial_t^i v$ and $w_i=\partial_t^i w$. Then v_i and w_i satisfy

$$\partial_t v_i = \Delta w_i - V_k(|x|) w_i$$

by the definition of $F_L^{V_k}$. By the uniqueness of the initial value problem for the ordinary differential equation, there exists a function $\zeta(t)$ in $(0,\infty)$ such that

$$v_i(x,t) = \zeta(t) U^{V_k}_{\mu,L}(|x|) + w_i(x,t) \tag{4.18}$$

for all $(x,t)\in\Omega_L\times(0,\infty)$. Furthermore, by (3.8), (3.25), (4.13), (4.14), and (4.18), there exist constants C_1, C_2, T, η_1, and ϵ such that

$$\begin{aligned}|\zeta(t)|U_k(\epsilon(1+t)^{1/2}) &\le |v_i(x,t)|\Big|_{|x|=\epsilon(1+t)^{1/2}} + |w_i(x,t)|\Big|_{|x|=\epsilon(1+t)^{1/2}}\\ &\le C_1 t^{-\frac{N}{2p}-i} + C_1\eta_1^{\alpha_k} t^{-\frac{N}{2p}-i-1} g_k(t)|x|^2 U^{\omega,k}_{+}(|x|)\Big|_{|x|=\epsilon(1+t)^{1/2}}\\ &\le C_1 t^{-\frac{N}{2p}-i} + C_2\eta_1^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U^{\omega,k}_{+}(\epsilon(1+t)^{1/2})\end{aligned}$$

for all $t \geq T$. This together with (4.10) implies that there exists a constant η_2 such that

$$|\zeta(t)| \leq \eta_2^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t), \quad t \geq T, \ k \in \boldsymbol{N}_0. \tag{4.19}$$

In addition, by (3.6), (4.17), and (4.18), there exists a constant η_3 such that

$$\begin{aligned}|(\partial_r v_i)(x,t)| &\leq |\zeta(t)| \big(\partial_r U_{\mu,L}^{V_k}\big)(|x|) + |\partial_r w_i(|x|,t)| \\ &\preceq \eta_2^{\alpha_k}(k+1) t^{-\frac{N}{2p}-i} g_k(t) \left(\frac{|x|}{L}\right)^{\alpha_k - 1} + \eta_1^{\alpha_k} t^{-\frac{N}{2p}-i-1} g_k(t) |x| U_+^{\omega,k}(|x|) \\ &\leq \eta_3^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) U_+^{\omega,k}(|x|)|x|^{-1}\end{aligned}$$

for all $(x,t) \in D_\epsilon(T)$ and $k \in \boldsymbol{N}_0$. So we obtain (4.16), and the proof of Lemma 4.3 is complete. □

We give upper estimates on the spatio-temporal derivatives of v and w.

LEMMA 4.4. *Assume the same assumptions as in Lemma 4.2. Then there exist positive constants T, η, and ϵ, independent of k, such that*

$$\left|\partial_t^i \partial_r^j v(x,t)\right| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-i} g_k(t) |x|^{-j} U_+^{\omega,k}(|x|), \tag{4.20}$$

$$\left|\partial_t^i \partial_r^j w(x,t)\right| \leq \eta^{\alpha_k} t^{-\frac{N}{2p}-1-i} g_k(t) |x|^{2-j} U_+^{\omega,k}(|x|) \tag{4.21}$$

for all $(x,t) \in D_\epsilon(T)$, $i \in \boldsymbol{N}_0$ with $2(i+1) \leq l+1$, and $j = 2, \ldots, l+2$.

PROOF. Let $i \in \boldsymbol{N}_0$ with $2(i+1) \leq l+1$. As in the proof of Lemma 4.3, put $v_i = \partial_t^i v$. Then, by (V_ω^l)–(iii) and (P_μ^k), v_i satisfies

$$\begin{aligned}\left|\partial_r^2 v_i\right| &= \left| -\frac{N-1}{r}\partial_r v_i + V(|x|)v_i + \frac{\omega_k}{|x|^2} v_i + \partial_t v_i \right| \\ &\leq \frac{N-1}{r}|\partial_r v_i| + \frac{C}{r^2}|v_i| + \frac{(1+t)}{|x|^2}|v_{i+1}|\end{aligned} \tag{4.22}$$

for all $(x,t) \in D_1(T)$, where C is a positive constant. This inequality together with Lemmas 4.2 and 4.3 implies the inequality (4.20) with $j=2$. Furthermore, since v_i satisfies

$$\partial_r^3 v_i = \frac{N-1}{r^2}\partial_r v_i - \frac{N-1}{r}\partial_r^2 v_i + \left(V' - 2\frac{\omega_k}{r^3}\right) v_i + V \partial_r v_i + \partial_r \partial_t v_i \tag{4.23}$$

in $\Omega_L \times (0,\infty)$, we may obtain the inequality (4.20) with $j=3$. Repeating this argument, we obtain the inequality (4.20). Furthermore, by (3.24), (4.14), (4.17), and (4.20), we obtain (4.21), and the proof of Lemma 4.4 is complete. □

Finally, we give estimates on the derivatives of v for the case (1.12).

Lemma 4.5. *Assume that ψ is a radial function such that $\|\psi\|_{L^p(\Omega_L)} = 1$ with $p \ge 1$. Let v be the solution of (P_μ^k) with $v(\cdot, 0) = \psi(\cdot)$ and $k = 0$, under the condition* (1.12)*. Then, for any $j \in \boldsymbol{N}_0^N$ with $|j| \ge n+1$ and $i \in \boldsymbol{N}_0$, there exist positive constants C, T, and ϵ such that*

$$\left|\partial_t^i \nabla_x^j v(x,t)\right| \le C t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}} \tag{4.24}$$

for all $(x,t) \in D_\epsilon(T)$.

Proof. By (1.12), we have

$$U_{\mu,L}^V(x) = c\left(\sum_{i=1}^N x_i^2\right)^{n'}, \quad U_+^\omega(r) = \left(\frac{r}{L}\right)^n, \quad g(t:\omega) = (1+t)^{-\frac{n}{2}},$$

where $n = 2n'$ and c is a positive constant. (See also Proposition 3.3). Put $v_i(x,t) = \partial_t^i v(x,t)$ and $w_i(x,t) = F_L^V[v_{i+1}](|x|)$. Let $j \in \boldsymbol{N}_0^N$ with $|j| \ge n+1$. Then $\nabla_x^j U_{\mu,L}^V(|x|) \equiv 0$ in Ω_L, and by (4.18), we have $\nabla_x^j v_i(x,t) = \nabla_x^j w_i(x,t)$ for all $(x,t) \in \Omega_L \times (0,\infty)$. Therefore, by the radial symmetry of w_i and the inequality (4.21) with $k=0$, there exist positive constants T and ϵ such that

$$\begin{aligned}\left|(\nabla_x^j v_i)(x,t)\right| &\preceq \sum_{m=1}^{|j|} \frac{|(\partial_r^m w_i)(x,t)|}{|x|^{|j|-m}} \preceq t^{-\frac{N}{2p}-1-i-\frac{n}{2}}|x|^{n+2-|j|} \\ &\preceq t^{-\frac{N}{2p}-1-i-\frac{n}{2}}|x| \preceq t^{-\frac{N}{2p}-\frac{1}{2}-i-\frac{n}{2}}\end{aligned}$$

for all $(x,t) \in D_\epsilon(T)$, and the proof of lemma 4.5 is complete. □

Remark 4.1. If the L^p-norm of the initial value is not 1, then all the right-hand terms in the estimates in Lemmas 4.2, 4.3 and 4.4 must be multiplied by $\|\psi\|_{L^p(\Omega_L)}$.

5. Upper bounds of derivatives of solutions.

In this section, we prove the following two propositions, which are mentioned in Section 1 as upper estimates, by using lemmas given in the previous sections.

Proposition 5.1. *Assume the same assumptions as in Theorem* 1.1*. Then, for any $p \ge 1$ and $j \in \boldsymbol{N}_0^N$ with $|j| \le l+1$,*

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha(\omega),|j|\}}{2}} \tag{5.1}$$

for all sufficiently large t.

Proposition 5.2. *Assume the same assumptions as in Theorem* 1.2*. Then, for* *any* $p \geq 1$ *and* $j \in \boldsymbol{N}_0^N$ *with* $|j| \geq n+1$,

$$\left\| \nabla_x^j G_\mu^V(t) \right\|_{p \to \infty} \preceq t^{-\frac{N}{2p} - \frac{\alpha(\omega_n + \omega_1)}{2}} \tag{5.2}$$

for all sufficiently large t.

Proof of Proposition 5.1. Let u_μ^V be the solution of (1.1) with $\phi \in C_0(\Omega_L)$. By the same arguments as in [**3**] and [**4**], ϕ can be expanded in the Fourier series, that is, there exist radial functions $\{\phi_{k,i}\} \subset L^2(\Omega_L)$ such that

$$\phi(x) = \sum_{k=0}^{\infty} \sum_{i=1}^{l_k} \phi_{k,i}(|x|) Q_{k,i}\left(\frac{x}{|x|}\right) \quad \text{in} \quad L^2(\Omega_L). \tag{5.3}$$

Let $u_\mu^{k,i}$ be a solution of (1.1) with the initial data $\phi_{k,i}(|x|)Q_{k,i}(x/|x|)$ and $v_\mu^{k,i}$ a radial solution of (P_μ^k) with the initial data $\phi_{k,i}$. By the uniqueness of the solution of (1.1), we see that

$$u_\mu^{k,i}(x,t) = v_\mu^{k,i}(x,t) Q_{k,i}\left(\frac{x}{|x|}\right), \quad (x,t) \in \Omega_L \times (0,\infty), \tag{5.4}$$

where $k \in \boldsymbol{N}_0$ and $i = 1, \ldots, l_k$. On the other hand, by the standard elliptic regularity theorem and $\|Q_{k,i}\|_{L^2(\boldsymbol{S}^{N-1})} = 1$, for any $n \in \boldsymbol{N}$, we have

$$\begin{aligned} \|Q_{k,i}\|_{C^{2n}(\boldsymbol{S}^{N-1})} &\preceq (1+\omega_k) \|\Delta_{\boldsymbol{S}^{N-1}} Q_{k,i}\|_{C^{2(n-1)}(\boldsymbol{S}^{N-1})} \\ &\preceq (1+\omega_k)^{n+1} \asymp (k+1)^{2n+2} \end{aligned} \tag{5.5}$$

for all $k \in \boldsymbol{N}_0$ and $i = 1, \ldots, l_k$. Furthermore the eigenspace of $\Delta_{\boldsymbol{S}^{N-1}}$ corresponding to ω_l is spanned by the functions $\nabla_x^j |x|$ for $j \in \boldsymbol{N}_0^N$ with $|j| = l$, and we have

$$l_k \leq N^k. \tag{5.6}$$

By the orthogonality of $\{Q_{k,i}\}_{k,i}$, we have

$$\int_{\Omega_L} u_\mu^{k_1,i_1}(x,t) u_\mu^{k_2,i_2}(x,t)\,dx = 0 \tag{5.7}$$

for all $t \geq 0$ if $(k_1,i_1) \neq (k_2,i_2)$. On the other hand, for any $t > 0$,

$$u_\mu^V(x,t) = \lim_{m \to \infty} \sum_{k=0}^{m} \sum_{i=1}^{l_k} v_\mu^{k,i}(x,t) Q_{k,i}\left(\frac{x}{|x|}\right) \tag{5.8}$$

holds uniformly for all $x \in \Omega_L$. Hence we have

$$\int_{\partial B(0,|x|)} u_\mu^V(x,t) Q_{k,i}\left(\frac{x}{|x|}\right) d\sigma = v_\mu^{k,i}(x,t) \int_{\partial B(0,|x|)} \left| Q_{k,i}\left(\frac{x}{|x|}\right)\right|^2 d\sigma$$
$$= |x|^{N-1} v_\mu^{k,i}(x,t)$$

for all $(x,t) \in \Omega_L \times (0,\infty)$. Then, by (5.5) and the Jensen inequality, we have

$$|x|^{N-1} \left| v_\mu^{k,i}(x,t)\right|^p \le |x|^{(N-1)(1-p)} \left(\int_{\partial B(0,|x|)} \left|u_\mu^V(x,t)\right| \left| Q_{k,i}\left(\frac{x}{|x|}\right)\right| d\sigma \right)^p$$
$$\preceq (k+1)^{2p} \int_{\partial B(0,|x|)} \left| u_\mu^V(x,t)\right|^p d\sigma$$

for all $(x,t) \in \Omega_L \times (0,\infty)$ and $k \in \boldsymbol{N}_0$. So, by (2.12), we have

$$\left\| v_\mu^{k,i}(\cdot,t)\right\|_{L^p(\Omega_L)} \preceq \left(\int_L^\infty r^{N-1} \left| v_\mu^{k,i}(r,t)\right|^p dr \right)^{1/p}$$
$$\preceq (k+1)^2 \left\| u_\mu^V(\cdot,t)\right\|_{L^p(\Omega_L)} \preceq (k+1)^2 \|\phi\|_{L^p(\Omega_L)} \tag{5.9}$$

for all $t > 0$ and $k \in \boldsymbol{N}_0$.

Let $j \in \boldsymbol{N}_0^N$ with $|j| \le l+1$. Let $k \in \boldsymbol{N}$ and $i = 1, \dots, l_k$. By (1.6), (5.4), and (5.5), we have

$$\left| \nabla_x^j u_\mu^{k,i}(x,t)\right| \preceq (k+1)^{l+3} \sum_{m=0}^{|j|} \frac{|\partial_r^m v_\mu^{k,i}(x,t)|}{|x|^{|j|-m}}, \quad (x,t) \in \Omega_L \times (0,\infty). \tag{5.10}$$

Since $D_{\epsilon_1}(T) \subset D_{\epsilon_2}(T)$ if $\epsilon_1 \le \epsilon_2$, by Lemmas 4.2, 4.3, 4.4, Remark 4.1 and (5.9), there exist positive constants η_1, η_2, η_3, T_*, and ϵ_* such that

$$\frac{|\partial_r^m v_\mu^{k,i}(x,t+t_0)|}{|x|^{|j|-m}} \preceq \eta_1^{\alpha_k} t^{-\frac{N}{2p}} g_k(t) U_+^{\omega,k}(|x|) |x|^{-|j|} \left\| v_\mu^{k,i}(\cdot,t_0)\right\|_{L^p(\Omega_L)}$$
$$\preceq (k+1)^2 \eta_2^{\alpha_k} t^{-\frac{N}{2p}-\frac{\alpha_k}{2}} |x|^{\alpha_k-|j|} \|\phi\|_{L^p(\Omega_L)}$$
$$\preceq (k+1)^2 \epsilon^{[\alpha_k-|j|]_+} \eta_3^{\alpha_k} t^{-\frac{N}{2p}-\frac{\alpha_k}{2}} t^{\frac{[\alpha_k-|j|]_+}{2}} \|\phi\|_{L^p(\Omega_L)}$$
$$\preceq (k+1)^2 \epsilon^{[\alpha_k-|j|]_+} \eta_3^{\alpha_k} t^{-\frac{N}{2p}-\frac{\min\{\alpha_k,|j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in D_\epsilon(T_*)$ with $0 < \epsilon \le \epsilon_*$, $t_0 > 0$, and $m = 0, 1, \dots, |j|$, where $\alpha_k = \alpha(\omega+\omega_k)$. Letting $t_0 \to 0$, we obtain

$$\frac{|\partial_r^m v_\mu^{k,i}(x,t)|}{|x|^{|j|-m}} \preceq (k+1)^2 \epsilon^{[\alpha_k-|j|]_+} \eta_3^{\alpha_k} t^{-\frac{N}{2p}-\frac{\min\{\alpha_k,|j|\}}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $(x,t) \in D_\epsilon(T_*)$ with $0 < \epsilon \le \epsilon_*$ and $m = 0, 1, \dots, |j|$. This inequality together with (5.10) implies that

$$\left|\nabla_x^j u_\mu^{k,i}(x,t)\right| \preceq (k+1)^{l+5} \epsilon^{[\alpha_k - |j|]_+} \eta_3^{\alpha_k} t^{-\frac{N}{2p} - \frac{\min\{\alpha_k, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.11}$$

for all $(x,t) \in D_\epsilon(T_*)$ with $0 < \epsilon \le \epsilon_*$. Let $0 < \epsilon \le \epsilon_*$ and T_ϵ be a positive constant such that $T_\epsilon > T_*$ and $\epsilon(1+T_\epsilon)^{1/2} \ge L+2$. By (2.9) and (5.11), taking a sufficiently small ϵ if necessary, we see

$$\left|\nabla_x^j u_\mu^{k,i}(x,t)\right| \preceq \frac{1}{2^k N^k} t^{-\frac{N}{2p} - \frac{\min\{\alpha_k, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.12}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$, $k \in \boldsymbol{N}$, and $i = 1, \dots, l_k$. Similarly, for the case $k = 0$, we have

$$\begin{aligned} \left|\nabla_x^j u_\mu^{0,1}(x,t)\right| = \left|\nabla_x^j v_\mu^{0,1}(x,t)\right| &\preceq \sum_{m=1}^{|j|} \frac{|(\partial_r^m v_\mu^{0,1})(x,t)|}{|x|^{|j|-m}} \\ &\preceq t^{-\frac{N}{2p} - \frac{\min\{\alpha_0, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \end{aligned} \tag{5.13}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. By (5.6), (5.12), and (5.13), we obtain

$$\begin{aligned} \left|(\nabla_x^j u_\mu^V)(x,t)\right| &\le \limsup_{m\to\infty} \sum_{k=0}^m \sum_{i=1}^{l_k} \left|(\nabla_x^j u_\mu^{k,i})(x,t)\right| \\ &\preceq t^{-\frac{N}{2p} - \frac{\min\{\alpha_0, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \sum_{k=0}^{\infty} \sum_{i=1}^{l_k} \frac{1}{2^k N^k} \\ &\preceq t^{-\frac{N}{2p} - \frac{\min\{\alpha_0, |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \end{aligned} \tag{5.14}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. On the other hand, by Lemma 2.3, we have

$$\left|(\nabla_x^j u_\mu^V)(x,t)\right| \preceq t^{-\frac{N}{2p} - \frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.15}$$

for all $(x,t) \notin D_\epsilon(T_\epsilon)$. Therefore, by (5.14) and (5.15), we obtain

$$\left|(\nabla_x^j u_\mu^V)(x,t)\right| \preceq t^{-\frac{N}{2p} - \frac{\min\{\alpha(\omega), |j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.16}$$

for all $(x,t) \in \Omega_L$ with $t \ge T_\epsilon$, where $\phi \in C_0(\Omega_L)$. Since $C_0(\Omega_L)$ is a dense subset of $L^p(\Omega_L)$, the inequality (5.16) holds for all $\phi \in L^p(\Omega_L)$, and the proof of Proposition 5.1 is complete. □

Proof of Proposition 5.2. By (1.12), V satisfies the condition (V_ω^l) with $\omega = \omega_n$ and $l = 0, 1, 2, \dots$. Let $j \in \boldsymbol{N}_0^N$ with $|j| \ge n+1 = 2n'+1$. Let u_μ^V be the solution of

(1.1) with $\phi \in C_0(\Omega_L)$ and $u_\mu^{k,i}$ a function given in the proof of Proposition 5.1. By the same argument as in the proof of (5.13) and Lemma 4.5, for any sufficiently small $\epsilon > 0$, there exists a positive constant T_ϵ such that

$$\left|(\nabla_x^j u_\mu^{0,1})(x,t)\right| \preceq t^{-\frac{N}{2p}-\frac{n+1}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.17}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$.

On the other hand, as discussed in Section 1 (see (1.20)), by the same argument as in the proof of (5.14), taking a sufficiently small $\epsilon > 0$ if necessary, we have

$$\limsup_{m\to\infty} \sum_{k=1}^{m} \sum_{i=1}^{l_k} \left|(\nabla_x^j u_\mu^{k,i})(x,t)\right| \preceq t^{-\frac{N}{2p}-\frac{\min\{\alpha(\omega_n+\omega_1),|j|\}}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.18}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Since $\alpha(\omega_n + \omega_1)$ is the nonnegative root of the equation, $\alpha(\alpha + N - 2) = n(N + n - 2) + N - 1$, we see that $\alpha(\omega_n + \omega_1) \leq \alpha(\omega_n) + 1 = n + 1$. Therefore, by (5.17), (5.18), and $|j| \geq n + 1$, we have

$$\left|(\nabla_x^j u_\mu^{V})(x,t)\right| \preceq t^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.19}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$. Furthermore, by (5.15) and (5.19), taking a sufficiently small ϵ if necessary, we have

$$\left|(\nabla_x^j u_\mu^{V})(x,t)\right| \preceq t^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}} \|\phi\|_{L^p(\Omega_L)} \tag{5.20}$$

for all $(x,t) \in \Omega_L \times (T_\epsilon, \infty)$, where $\phi \in C_0(\Omega_L)$. Furthermore, since $C_0(\Omega_L)$ is a dense subset of $L^p(\Omega_L)$, we have the inequality (5.20) for all $\phi \in L^p(\Omega_L)$, and the proof of Proposition 5.2 is complete. □

6. Proofs of Theorems 1.1 and 1.2.

In this section we consider the asymptotic behavior of the derivatives of the radial solution v of (1.1) for some initial data $\psi \in C_0(\Omega_L)$ and complete proofs of Theorems 1.1 and 1.2.

Proposition 6.1. *Let $R > 0$, $\omega \geq 0$, and $\psi(\not\equiv 0)$ be a nonnegative, radial function belonging to $C_0(\Omega_R)$. Let v be a radial solution of*

$$\begin{cases} \partial_t v = \Delta v - \dfrac{\omega}{|x|^2} v & \text{in } \Omega_R \times (0,\infty), \\ v(x,t) = 0 & \text{on } \partial\Omega_R \times (0,\infty), \\ v(x,0) = \psi(x) & \text{in } \Omega_R. \end{cases} \tag{6.1}$$

Then, for any $p \in [1,\infty]$,

$$\|v(\cdot,t)\|_{L^p(\Omega_R)} \asymp t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{\alpha(\omega)}{2}} \tag{6.2}$$

holds for all sufficiently large t. Furthermore there exists a positive constant ϵ_* such that, for any $0<\epsilon\le\epsilon_*$,

$$v(x,t)\Big|_{|x|=\epsilon(1+t)^{1/2}} \asymp \epsilon^{\alpha(\omega)} t^{-\frac{N+\alpha(\omega)}{2}}, \quad t>T \tag{6.3}$$

holds with suitably chosen $T=T(\epsilon)$.

Proof. Put

$$z(y,s)=(1+t)^{\frac{N+\alpha}{2}}v(x,t), \quad y=(1+t)^{-\frac{1}{2}}x, \quad s=\log(1+t), \tag{6.4}$$

where $\alpha=\alpha(\omega)$. Then the function z satisfies

$$\begin{cases} \partial_s z=\dfrac{1}{\rho}\mathrm{div}\,(\rho\nabla_y z)+\dfrac{N+\alpha}{2}z-\dfrac{\omega}{|y|^2}z & \text{in } W,\\ z=0 & \text{on } \partial W,\\ z(y,0)=\psi(y) & \text{in } \Omega_R, \end{cases} \tag{6.5}$$

where $\rho(y)=\exp(|y|^2/4)$ and

$$\Omega(s)=e^{-s/2}\Omega_R, \quad W=\bigcup_{0<s<\infty}(\Omega(s)\times\{s\}), \quad \partial W=\bigcup_{0<s<\infty}(\partial\Omega(s)\times\{s\}).$$

Put

$$\varphi(y)=c_0|y|^{\alpha(\omega)}\exp(-|y|^2/4),$$

where c_0 is a positive constant such that $\|\varphi\|_{L^2(\boldsymbol{R}^N,\rho dy)}=1$. Then, since

$$\int_{\Omega_R} v(x,t)U^{\omega}_{1,R}(|x|)dx=\int_{\Omega_R}\phi(x)U^{\omega}_{1,R}(|x|)dx>0, \quad t\ge 0,$$

by the same argument as in the proof of Lemma 6.1 in [**4**], we see that

$$a\equiv\int_{\Omega_R}\phi(x)U^{\omega}_{1,R}(|x|)dx=\lim_{s\to\infty}\int_{\Omega(s)}z(y,s)\varphi(y)\rho(y)dy>0. \tag{6.6}$$

Furthermore, by the same argument as in the proof of Lemmas 3.3 and 3.4 in [**4**], for any r_1 and r_2 with $0<r_1<r_2$, we have

$$\sup_{s>0} \|z(\cdot,s)\|_{L^2(\Omega(s),\rho dy)} < \infty, \tag{6.7}$$

$$\sup_{s>0} \|z(\cdot,s)\|_{L^\infty(\{y\,:\,|y|\ge r_1\})} < \infty, \tag{6.8}$$

$$\lim_{s\to\infty} \|z(\cdot,s) - a\varphi\|_{C(\{y\,:\,r_1\le|y|\le r_2\})} = 0. \tag{6.9}$$

By (6.6), (6.7) and (6.9), we have $\|z(\cdot,s)\|_{L^1(\Omega(s))} \asymp 1$ for all sufficiently large s. So, by (6.8) and (6.9), for any $p \in [1,\infty]$, we have $\|z(\cdot,s)\|_{L^p(\Omega(s))} \asymp 1$ for all sufficiently large s, and obtain (6.2).

On the other hand, by the same argument as in (4.18), there exists a function ζ in $(0,\infty)$ such that

$$v(x,t) = \zeta(t)U^V_{0,R}(|x|) + F^V_L[(\partial_t v)(\cdot,t)](|x|) \tag{6.10}$$

for all $(x,t) \in \Omega_R \times (0,\infty)$ with $V = \omega/r^2$. By (6.2) with $p = \infty$, we may apply the same arguments as in the proof of Lemma 4.2 with $\gamma = (N + \alpha(\omega))/2$ to v. Then we see that there exist positive constants ϵ_* and T_* such that

$$\left|F^V_L[(\partial_t v)(\cdot,t)](|x|)\right| \preceq t^{-\frac{N}{2}-\alpha(\omega)-1}|x|^{\alpha(\omega)+2} \tag{6.11}$$

for all $(x,t) \in D_{\epsilon_*}(T_*)$. Therefore, by (3.9), (6.9), (6.10), (6.11), and the same arguments as in the proof of (4.19), we may take a sufficiently small $\tilde\epsilon$ so that

$$\begin{aligned}\zeta(t) &= \left[U^V_{0,R}(\tilde\epsilon(1+t)^{1/2})\right]^{-1}\left[v(x,t) - F^V_L[\partial_t v](|x|)\right]\Big|_{|x|=\tilde\epsilon(1+t)^{1/2}}\\ &\asymp \tilde\epsilon^{-\alpha} t^{-\frac{\alpha}{2}}\left[t^{-\frac{N+\alpha}{2}} + O(\tilde\epsilon^{\alpha+2})t^{-\frac{N+\alpha}{2}}\right] \asymp t^{-\frac{N}{2}-\alpha}\end{aligned} \tag{6.12}$$

for all sufficiently large t. Then, by (6.10)–(6.12) and the similar argument as in (6.12), we have (6.3), and the proof of Proposition 6.1 is complete. □

Proof of Theorem 1.1. Assume (V^l_ω). Let $\tilde\omega$ be a constant such that $\tilde\omega > \omega$ and

$$\alpha(\tilde\omega) < \alpha(\omega) + 1. \tag{6.13}$$

Then, by (V^l_ω)–(i), we may take a sufficiently large R so that

$$V(r) \le \frac{\tilde\omega}{r^2}, \quad r \ge R.$$

Let $p \ge 1$ and $\psi(\not\equiv 0)$ be a nonnegative, radial function belonging to $C_0(\Omega_R)$. Let v be a solution of (6.1) with ω replaced by $\tilde\omega$. For any $T > 0$, let u^V_T be a solution of (1.1) with the initial data $\phi(\cdot) = v(\cdot,T)/\|v(\cdot,T)\|_{L^p(\Omega_R)}$. Here we remark that

$$\left\|u_T^V(\cdot,0)\right\|_{L^p(\Omega_L)} = 1. \tag{6.14}$$

By the comparison principle, (6.2), and (6.3), for any sufficiently small $\epsilon > 0$, there exists a positive constant T_ϵ such that

$$\begin{aligned} u_T^V(x,T) &\geq \frac{v(x,2T)}{\|v(\cdot,T)\|_{L^p(\Omega_R)}} \asymp T^{\frac{N}{2}(1-\frac{1}{p})+\frac{\alpha(\tilde{\omega})}{2}} v(x,2T) \\ &\succeq \epsilon^{\alpha(\tilde{\omega})} T^{-\frac{N}{2p}} \end{aligned} \tag{6.15}$$

for all $(x,T) \in \Omega_L \times (T_\epsilon,\infty)$ with $|x| = \epsilon(1+2T)^{1/2} > \max\{R, 2L+2\}$.

On the other hand, there exists a function $\zeta_V(t)$ such that

$$u_T^V(x,t) = \zeta_V(t) U_{\mu,L}^V(|x|) + F_L^V\left[\partial_t u_T^V\right](|x|) \tag{6.16}$$

for all $x \in \Omega_L$. By Lemmas 4.2–4.4 and (6.14), taking a sufficiently small ϵ and sufficiently large T_ϵ if necessary, we have

$$\left|\partial_r^j F_L^V[\partial_t u_T^V](|x|)\right| \preceq t^{-\frac{N}{2p}-1-\frac{\alpha(\omega)}{2}} |x|^{2-|j|+\alpha(\omega)} \tag{6.17}$$

for all $(x,t) \in D_\epsilon(T_\epsilon)$ and $j \in \boldsymbol{N}_0$ with $|j| \leq l+2$. Furthermore, by (6.13) and (6.15)–(6.17), there exist positive constants C_1, C_2, and C_3 such that

$$\begin{aligned} \zeta_V(T) U_{\mu,L}^V(|x|) &\geq u_T^V(x,T) - \left|F_L^V[\partial_t u_T^V](|x|)\right| \\ &\geq C_1 \epsilon^{\alpha(\tilde{\omega})} T^{-\frac{N}{2p}} - C_2 \epsilon^{\alpha(\omega)+2} T^{-\frac{N}{2p}} \succeq \epsilon^{\alpha(\omega)+1} T^{-\frac{N}{2p}} \end{aligned}$$

for all $x \in \Omega_L$ with $L+1 < |x| = \epsilon(1+2T)^{1/2}/2 < \epsilon(1+T)^{1/2}$ and $T \geq T_\epsilon$. Therefore, by (2.5) and (3.7), we have

$$\zeta_V(T) \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \tag{6.18}$$

for all sufficiently large T. Therefore, by (6.16)–(6.18), there exist positive constants C_3 and C_4 such that

$$\begin{aligned} \left|\nabla_x^j u_T^V(x,T)\right| &\geq \zeta_V(T)\left|\nabla_x^j U_{\mu,L}^V(x)\right| - C_3 \sum_{m=1}^{|j|} \left|\partial_r^m F_L^V[\partial_t u_T^V](|x|)\right||x|^{m-|j|} \\ &\geq C_4 T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \left|\nabla_x^j U_{\mu,L}^V(x)\right| - C_4 T^{-\frac{N}{2p}-1-\frac{\alpha(\omega)}{2}} |x|^{2+\alpha(\omega)-|j|} \end{aligned} \tag{6.19}$$

for all $L < |x| \leq \epsilon(1+T)^{1/2}$, $T \geq T_\epsilon$, and $j \in \boldsymbol{N}_0^N$ with $|j| \leq l$.

Let $j \in \boldsymbol{N}_0^N$ with $|j| \leq l$. By the assumption of Theorem 1.1 and Proposition 3.3, there exists a point $x_0 \in \Omega_L$ such that $(\nabla_x^j U_{\mu,L}^V)(x_0) \neq 0$. Then, by (6.19), there exist positive constants C_5 and C_6 such that

$$\left|(\nabla_x^j u_T^V)(x_0,T)\right| \geq C_5 T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} - C_6 T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}-1} \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \tag{6.20}$$

for all sufficiently large T. This inequality together with (6.14) implies

$$\left\|\nabla_x^j G_\mu^V(T)\right\|_{p\to\infty} \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \tag{6.21}$$

for all sufficiently large T. This together with Proposition 5.1 implies (1.10) and (1.11), and the proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.2. Let $u_T^{V_1}$ be a function given in the proof of Theorem 1.1 with $V(r) = (\omega_n + \omega_1)/r^2$. Put

$$\tilde{u}_T^V(x,t) = u_T^{V_1}(x,t)\frac{x_1}{|x|}.$$

Then $\tilde{u}_T^V$ is a solution of (1.1) with $V(r) = \omega_n/r^2$.

Let $j = (j_1, \dots, j_N) \in \boldsymbol{N}_0^N$ with $|j| \geq n+1$. Put $j' = (j_1+1, j_2, \dots, j_N)$ and

$$\tilde{U}_{\mu,L}^{\omega_n+\omega_1}(r) = \int_L^r U_{\mu,L}^{\omega_n+\omega_1}(s)ds$$

Then, by (2.5), we see that $\tilde{U}_{\mu,L}^{\omega_n+\omega_1}(r) \asymp r^{\alpha(\omega_n+\omega_1)+1}$ for all sufficiently large r. If $\nabla_x^{j'}\tilde{U}_\mu^{\omega_n+\omega_1}(|x|) \equiv 0$ in Ω_L, then, by the same argument as in the proof of (3.33), we see that $\tilde{U}_{\mu,L}^{\omega_n+\omega_1}(r)$ is a polynomial. This contradicts $\alpha(\omega_n+\omega_1) \notin \boldsymbol{N}$ if $n \geq 1$. If $n = 0$, by (1.12),

$$U_{\mu,L}^{\omega_n+\omega_1}(r) = U_{0,L}^{\omega_1}(r) = \frac{1}{N}\left(\frac{r}{L}\right)^{-(N-1)} + \frac{N-1}{LN}r,$$

and $\tilde{U}_\mu^{\omega_n+\omega_1}(r)$ is not a polynomial. So we have

$$\nabla_x^{j'}\tilde{U}_\mu^{\omega_n+\omega_1}(|x|) = \nabla_x^j\left[U_\mu^{\omega_n+\omega_1}(|x|)\frac{x_1}{|x|}\right] \not\equiv 0 \quad \text{in} \quad \Omega_L.$$

By the similar arguments in (6.16)–(6.20) and $\omega = \omega_n + \omega_1$, there exist positive constants C_1 and C_2 such that

$$\begin{aligned}\left|(\nabla_x^j \tilde{u}_T^V)(x_0,T)\right| &\geq C_1 T^{-\frac{N}{2}-\frac{\alpha(\omega_n+\omega_1)}{2}} - C_2 T^{-\frac{N}{2}-\alpha(\omega_n+\omega_1)-1} \\ &\succeq T^{-\frac{N}{2}-\frac{\alpha(\omega_n+\omega_1)}{2}}\end{aligned}$$

for all sufficiently large T. Furthermore, since $\|\tilde{u}_T^V(\cdot,0)\|_{L^p(\Omega_L)} \asymp 1$, we obtain

$$\left\|\nabla_x^j G_\mu^V(T)\right\|_{p\to\infty} \succeq T^{-\frac{N}{2p}-\frac{\alpha(\omega_n+\omega_1)}{2}}$$

for all sufficiently large T. Therefore, this inequality together with Propositions 5.1 and 5.2 implies (1.13) and (1.14), and the proof of Theorem 1.2 is complete. □

Next, we give a result on the estimates of the decay rates of L^∞_{loc}-norm of the derivatives of the solutions to (1.1). For any $R > L$, we put

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{R:p\to\infty} = \sup\left\{\left\|(\nabla_x^j u_\mu^V)(\cdot,t:\phi)\right\|_{L^\infty(B(0,R)\cap\Omega_L)} : \|\phi\|_{L^p(\Omega_L)} = 1\right\}.$$

Then we have the following result on the decay rate of $\|\nabla_x^j G_\mu^V(t)\|_{R:p\to\infty}$ as $t\to\infty$.

THEOREM 6.1. *Let $N \geq 3$ and consider the initial-boundary value problem* (1.1) *under the condition* (V_ω^l) *with $\omega \geq 0$ and $l \in \boldsymbol{N}$. Let $p \geq 1$ and $R > L$.*

(i) *Assume* (1.9) *for any $n' \in \boldsymbol{N}_0$ with $2n' \leq l+2$. Then, for any $j \in \boldsymbol{N}_0^N$ with $|j| \leq l+2$,*

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{R:p\to\infty} \asymp t^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \tag{6.22}$$

for all sufficiently large t.

(ii) *Assume* (1.12). *Then, for any $j \in \boldsymbol{N}_0^N$,*

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{R:p\to\infty} \asymp t^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \qquad \text{if } |j| \leq 2n', \tag{6.23}$$

$$\|\nabla_x^j G_\mu^V(t)\|_{R:p\to\infty} \asymp t^{-\frac{N}{2p}-\frac{\alpha(\omega+\omega_1)}{2}} \qquad \text{if } |j| > 2n'+1 \tag{6.24}$$

for all sufficiently large t.

PROOF. By the condition (V_ω^l), $U_{\mu,L}^V \in C^{l+2}([1,\infty))$. Then, by (2.9), in a similar way to the proof of Theorem 1.1, for any $R > L$, there exist constants η and T such that

$$\begin{aligned}
\left|\nabla_x^j u_\mu^{k,i}(x,t)\right| &\preceq (k+1)^{l+5}\eta^{\alpha_k(\omega)} t^{-\frac{N}{2p}-\frac{\alpha_k(\omega)}{2}} \|\phi\|_{L^p(\Omega_L)} \\
&\preceq (k+1)^{l+5}\eta^{\alpha_k(\omega)} t^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} T^{-\frac{\alpha_k(\omega)-\alpha(\omega)}{2}} \|\phi\|_{L^p(\Omega_L)} \\
&\preceq \frac{1}{2^k N^k} t^{-\frac{N}{2p}-\frac{\alpha(\omega)}{2}} \|\phi\|_{L^p(\Omega_L)}
\end{aligned}$$

for all $x \in \Omega_L \cap B(0,R)$, $t \geq T$, $k \in \boldsymbol{N}$, and $i = 1,\dots,l_k$, instead of (5.11). Therefore, by the same argument as in the proof of Proposition 5.1, we have (6.22) with $\asymp$ replaced by $\preceq$. Furthermore, by the same argument as in the proof of (6.21), we have (6.22) with $\asymp$ replaced by $\succeq$, and obtain (6.22). Similarly, we obtain (6.23) and (6.24), and the proof of Theorem 6.1 is complete. □

Next, for the solution u_μ^V of (1.1), we put

$$\hat{u}_\mu^V(r,\theta,t) = u_\mu^V(x,t), \quad r = |x|, \quad \theta = \frac{x}{|x|}, \tag{6.25}$$

and consider the decay rate of $r^{-|j|}\nabla_\theta^j \hat{u}_\mu^V$ as $t \to \infty$. For any $k = 0, 1, \dots$ and $i = 1, \dots, l_k$, $Q_{k,i}$ is a polynomial in the variable θ of the degree k, and so we see that there exists $J \in \boldsymbol{N}_0^N$ with $|J| = k$ such that $\nabla_\theta^J Q_{k,i} \not\equiv 0$ in $\boldsymbol{S}^{N-1}$ and that $\nabla_\theta^j Q_{k,i} \equiv 0$ on $\boldsymbol{S}^{N-1}$ for all $j \in \boldsymbol{N}_0^N$ with $|j| \ge k+1$. Therefore, in a similar way to the proof of Theorem 1.2, we have the following result.

Theorem 6.2. *Let $N \ge 3$ and consider the initial-boundary value problem* (1.1) *under the condition* (V_ω^l) *with $\omega \ge 0$ and $l \in \boldsymbol{N}$. Let $p \ge 1$. Then, for any $j \in \boldsymbol{N}_0^N$ with $|j| \le l+1$,*

$$\left\| r^{-|j|}\nabla_\theta^j G_\mu^V(t) \right\|_{p\to\infty} \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}}$$

for all sufficiently large t. Here

$$\left\| r^{-|j|}\nabla_\theta^j G_\mu^V(t) \right\|_{p\to\infty} = \sup\left\{ \left\| r^{-|j|}(\nabla_\theta^j \hat{u}_\mu^V)(r,\theta,t:\phi) \right\|_{L^\infty((L,\infty)\times \boldsymbol{S}^{N-1})} : \|\phi\|_{L^p(\Omega_L)} = 1 \right\}.$$

By Theorem 6.2, we have

$$\sup_{x_* = (x_1, 0, \dots, 0) \in \Omega_L} \left| (\nabla_x^j u_\mu^V)(x_*, t:\phi) \right| \preceq t^{-\frac{N}{2p}-\frac{|j|}{2}} \|\phi\|_{L^p(\Omega_L)}$$

for all $j = (0, j_2, \dots, j_N)$ with $|j| \le l+1$.

7. Decay rate of the derivatives of the solution for the case $N = 2$.

In this section, we treat the two dimensional case. We first consider the cases either

$$N = 2 \quad \text{and} \quad \omega > 0 \tag{7.1}$$

or

$$N = 2, \quad \mu = 0, \quad \text{and} \quad V \equiv 0 \quad \text{on} \quad [L, \infty). \tag{7.2}$$

For these cases, by the same arguments as in the proof of Proposition 3.1, we see that

$$U_{\mu,L}^V(r) \asymp \frac{\alpha - \mu\alpha - L\mu}{\alpha+\beta}\left(\frac{r}{L}\right)^{-\beta} + \frac{L\mu - \beta\mu + \beta}{\alpha+\beta}\left(\frac{r}{L}\right)^{\alpha}$$

for all $r \ge L$, where $\alpha = \alpha(\omega)$ and $\beta(\omega) = N - 2 + \alpha(\omega) = \alpha(\omega)$. Furthermore, applying the same arguments as in the previous sections, we have the following theorems.

THEOREM 7.1. *Assume either* (7.1) *or* (7.2). *Then Theorems* 1.1, 1.2, 6.1, *and* 6.2 *hold true.*

Next, we consider the cases either

$$(N, \omega) = (2, 0) \quad \text{and} \quad \mu > 0 \tag{7.3}$$

or

$$(N, \omega, \mu) = (2, 0, 0) \quad \text{and} \quad V \not\equiv 0 \quad \text{on} \quad [L, \infty). \tag{7.4}$$

Then we see that

$$U^0_{\mu,L}(r) = 1 - \mu + \mu \log\left(\frac{r}{L}\right).$$

For these cases either (7.3) or (7.4), we say that the function V satisfies the condition $(\tilde{V}^l_\omega)$ if V satisfies the condition (V^l_ω) with the condition (ii) replaced by

$$\int_L^\infty \left| V(r) - \frac{\omega}{r^2} \right| r \log\left(\frac{2r}{L}\right) dr = \int_L^\infty V(r) r \log\left(\frac{2r}{L}\right) dr < \infty.$$

We assume $(\tilde{V}^l_\omega)$ instead of (V^l_ω), and study the decay rate of the derivatives of the solution of (1.1). We first prove the following proposition, instead of Proposition 3.1 for the case $k = 0$.

PROPOSITION 7.1. *Consider the cases either* (7.3) *or* (7.4). *Assume* $(\tilde{V}^l_\omega)$. *Then, for any* $0 \le \mu \le 1$,

$$0 \le \frac{d}{dr} U^V_{\mu,L}(r) \preceq r^{-1}, \tag{7.5}$$

$$U^V_{\mu,L}(r) \asymp 1 - \mu + \log\left(\frac{r}{L}\right) \tag{7.6}$$

for all $r > L$.

PROOF. For the case (7.3), we have $U^V_{\mu,L}(r) = U^0_{\mu,L}(r)$ on $[L, \infty)$, and, by $\mu > 0$, we obtain (7.5) and (7.6). So we consider the case (7.4), and assume that $V \not\equiv 0$ on $[L, \infty)$. Put

$$U^0_+(r) = \log\left(\frac{r}{L}\right), \quad U^0_-(r) = 1, \quad \iota(r) = r \log\left(\frac{2r}{L}\right),$$

instead of (2.1). Then, by the same arguments as in the proof of Lemma 3.1, we see that Lemma 3.1 holds with (3.25) and (3.26) replaced by

$$\left|H_R^{0,0}[g]'(r)\right| \le C_1 A r^{-1} \int_R^r \iota(\tau)|h_0(\tau)|d\tau, \tag{7.7}$$

$$\left|H_R^{0,0}[g](r)\right| \le C_1 A \log\left(\frac{r}{R}\right) \int_R^r \iota(\tau)|h_0(\tau)|d\tau, \tag{7.8}$$

respectively, where $R \ge L$ and g is a continuous function satisfying

$$|g(r)| \le A h_0(r)\left(1 - \mu + \log\left(\frac{r}{R}\right)\right).$$

On the other hand, by Lemma 3.1–(i), (ii), and $V \ge 0$, $U^V_{\mu,L}(r) \ge 0$ on $[L,\infty)$, and we have

$$U^V_{\mu,L}(r) = U^0_{\mu,L}(r) + H_L^{0,0}\big[h_0 U^V_{\mu,L}\big](r) \ge U^0_{\mu,L}(r) \tag{7.9}$$

for all $r \ge L$. Then we have

$$\begin{aligned} U^V_{\mu,L}(r) &\ge U^0_{\mu,L}(r) + H_L^{0,0}\big[h_0 U^0_{\mu,L}\big](r) \\ &\ge 1 - \mu + \mu\log\left(\frac{r}{L}\right) + (1-\mu)\int_L^r s^{-1}\left(\int_L^s \tau V(\tau)d\tau\right)ds \\ &\ge 1 - \mu + \mu\log\left(\frac{r}{L}\right) > 0 \end{aligned} \tag{7.10}$$

for all $r > L$. Let S be a positive constant to be chosen later such that $S > L$. By the same arguments as in the proof of (2.4) and (3.13), for any $0 \le \mu \le 1$, we have

$$U^V_{\mu,L}(r) \asymp U^0_{\mu,L}(r) \asymp 1 - \mu + \log\left(\frac{r}{L}\right), \quad L \le r \le S. \tag{7.11}$$

Furthermore, since $H_L^{0,0}[h_0 U^0_{\mu,L}]'(r) \ge 0$ on $[L,\infty)$, by (7.7), (7.9), and (7.11), we obtain

$$0 \le \partial_r U^V_{\mu,L}(r) \preceq r^{-1}, \quad L \le r \le S. \tag{7.12}$$

Let ϵ' be a sufficiently small positive constant. By $V \not\equiv 0$ on $[L,\infty)$ and $(\tilde V^l_\omega)$, we may take a constant $S > L$ such that

$$\int_S^\infty \iota(\tau)|h_0(\tau)|d\tau = \int_S^\infty \iota(\tau)V(\tau)d\tau < \epsilon', \tag{7.13}$$

$$\int_L^S \tau V(\tau)d\tau > 0. \tag{7.14}$$

For $r \ge S$, put

$$i(r) = \int_S^r \iota(\tau)V(\tau)d\tau.$$

Put

$$U_1(r) = U^0_{\mu,S}(r), \quad U_{j+1}(r) = U^0_{\mu,S}(r) + H^{0,0}_S[VU_j](r), \; j = 1, 2, \dots.$$

Then, by (7.8) and (7.13), there exists a constant C_2 such that

$$\left|U_2(r) - U^0_{\mu,S}(r)\right| \le C_2 i(r) \log \frac{r}{S} \le C_2 i(r) U^0_{1,S}(r).$$

Furthermore we have

$$\left|U_3(r) - U^0_{\mu,S}(r)\right| \le \left[C_2 i(r) + (C_2 i(r))^2\right] \log \frac{r}{S}.$$

By the same argument as in the proof of (3.21), taking a sufficiently small ϵ' if necessary, we have

$$\left|U^V_{\mu,S}(r) - U^0_{\mu,S}(r)\right| \le C_2 i(r) \log \frac{r}{S} \preceq \log\left(\frac{r}{S}\right), \quad r \ge S. \tag{7.15}$$

By (7.11), (7.12), and (7.15), there exists a positive constant C_3 such that

$$U^V_{\mu,L}(S)U^V_{0,S}(r) \preceq \left(1 - \mu + \log \frac{S}{L}\right)\left(1 + C_3 \log \frac{r}{S}\right) \preceq 1 - \mu + \log \frac{r}{L},$$
$$\left(\partial_r U^V_{\mu,L}\right)(S)U^V_{1,S}(r) \preceq S^{-1}\left(\log \frac{r}{S} + C_3 \log \frac{r}{S}\right) \preceq \log \frac{r}{L}$$

for all $r \ge S$. So we have

$$\begin{aligned} U^V_{\mu,L}(r) &= U^V_{\mu,L}(S)U^V_{0,S}(r) + (\partial_r U^V_{\mu,L})(S)U^V_{1,S}(r) \\ &\preceq 1 - \mu + \log \frac{r}{L} \end{aligned} \tag{7.16}$$

for all $r \ge S$. Furthermore, by (7.10) and (7.14), we have

$$U^V_{\mu,L}(r) \succeq 1 - \mu + \log \frac{r}{L} \tag{7.17}$$

for all $r \ge S$. Therefore, by (7.11), (7.16), and (7.17), we have (7.6). Furthermore, by (7.6), (7.7), and (7.9), we have (7.5), and the proof of Proposition 7.1 is complete. □

Furthermore, we see that Proposition 3.1 for the case $k \ge 1$ and Proposition 3.2 hold. Therefore, by the same arguments as in the proof of Theorems 1.1, 6.1, and 6.2, we have the following theorem.

THEOREM 7.2. *Let $N = 2$ and consider the initial-boundary value problem* (1.1) *under the condition* $(\tilde{V}^l_\omega)$ *with* $\omega = 0$ *and* $l \in \boldsymbol{N}$. *Let* $p \geq 1$, *and* $R > L$. *Assume either* (7.3) *or* (7.4). *Then, for any* $j \in \boldsymbol{N}_0^N$ *with* $|j| \leq l + 1$,

$$\left\|\nabla_x^j G_\mu^V(t)\right\|_{p\to\infty} \asymp \left\|\nabla_x^j G_\mu^V(t)\right\|_{R:p\to\infty} \asymp t^{-\frac{1}{p}}(\log t)^{-1},$$
$$\left\|r^{-|j|}\nabla_\theta^j G_\mu^V(t)\right\|_{p\to\infty} \preceq t^{-\frac{1}{p}-\frac{|j|}{2}}$$

for all sufficiently large t.

Finally, by Theorems 7.1 and 7.2, we see that Corollary 1.1 holds true for the case $N = 2$.

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