J. Math. Soc. Japan
Vol. 59, No. 1 (2007) pp. 179–184

Homotopy minimal periods for expanding maps on infra-nilmanifolds

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(Received Mar. 29, 2006)

Abstract. We prove that the sets of homotopy minimal periods for expanding maps on n-dimensional infra-nilmanifolds are uniformly cofinite, i.e., there exists a positive integer m_0, which depends only on n, such that for any integer $m \geq m_0$, for any n-dimensional infra-nilmanifold M, and for any expanding map f on M, any self-map on M homotopic to f has a periodic point of least period m, namely, $[m_0, \infty) \subset \mathrm{HPer}(f)$. This extends the main result, Theorem 4.6, of [**13**] from periods to homotopy periods.

1. Introduction.

Given a self map $f : X \to X$, the fixed point set $\mathrm{Fix}(f) := \{x \in X |\ x = f(x)\}$ of f and the periodic point set $P_n(f) := \mathrm{Fix}(f^n) - \bigcup_{k<n} \mathrm{Fix}(f^k)$ of f with least period n are interesting in many fields of mathematics.

A well-known invariant from algebraic topology is the Lefschetz number $L(f)$ of f. If $L(f)$ is non-zero, then any map homotopic to f has at least one fixed point. We naturally ask for an invariant, an analogy of $L(f)$, to ensure the existence of periodic points with given period, i.e., $P_n(f) \neq \varnothing$.

As a classical branch in algebraic topology, Nielsen fixed point theory mainly deals with the estimation for number of fixed points of self maps, and is already applied to the estimation of the number of periodic points of maps. In [**1**], Nielsen fixed point theory was successfully employed in deciding whether $P_n(f)$ is non-empty for a self-map f on the torus. Because only topological invariants are involved, the resulting estimation is unchanged during any homotopy. What is obtained in [**1**] is the information about the so-called *homotopy minimal periods* for torus maps f

$$\mathrm{HPer}(f) := \bigcap_{g \simeq f} \{n \in N \mid P_n(g) \neq \varnothing\},$$

where $g : X \to X$ ranges over all maps homotopic to f. It is worth mentioning that such an invariant reflects the very rigid part of the map in dynamical system point of view, because a small perturbation of the map is still homotopic to itself as long as X is a nice space, for example: a compact smooth manifold.

2000 *Mathematics Subject Classification.* Primary 55M20; Secondary 57S30.

Key Words and Phrases. essentially reducible, expanding maps, homotopy minimal periods, infra-nilmanifolds.

The first author was supported in part by Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (H00021) and the second author was partially supported by natural science foundation of Beijing (1032002).

It is our purpose of this paper to provide a description of the homotopy minimal periods of expanding maps on infra-nilmanifolds. Toward this end, we follow the approach used in [**7**], [**1**] for torus maps, in which they determine $\mathrm{HPer}(f)$ only from the knowledge of the sequence of Nielsen numbers $\{N(f^k)\}_{k\geq 1}$. Since the Nielsen numbers for continuous maps on infra-nilmanifolds are easily computable using the averaging formula [**11**], [**13**], these provide a good tool for investigating the homotopy minimal periods of expanding maps on infra-nilmanifolds.

2. Essential reducibility.

It is known that any fixed point class of f^n is naturally contained in a fixed point class of f^{kn}. In general, such an inclusion does not preserve the essentiality. So, we have

DEFINITION 2.1 ([**8**, Definition 4.1]). A self-map $f\colon X \to X$ is said to be *essentially reducible* if any fixed point class of f^k being contained in an essential fixed point class of f^{kn} is essential, for any positive integers k and n. The space X is *essentially reducible* if every self-map on X is essentially reducible.

By definition, we obtain immediately that

COROLLARY 2.2. *A self-map $f\colon X \to X$ is essentially reducible if and only if for any positive integers k and n, any inessential fixed point class of f^k is contained in an inessential fixed point class of f^{kn}.*

Recall a result in [**1**], in which the terminology "index assumption" instead of "essential reducibility" is used.

PROPOSITION 2.3 ([**1**, Proposition 2.2]). *Let $f\colon X \to X$ be an essentially reducible self-map. If*

$$\sum_{\frac{m}{k}:\text{prime}} N(f^k) < N(f^m),$$

then any self map homotopic to f has a periodic point of least period m, i.e. $m \in \mathrm{HPer}(f)$.

Under a finite regular covering projection $\bar{p}\colon \bar{X} \to X$, we decide when a self-map $f : X \to X$ is essentially reducible.

PROPOSITION 2.4. *Let $\bar{p}\colon \bar{X} \to X$ be a finite regular covering projection. Suppose that $\bar{X}$ is essentially reducible. Then a self-map $f\colon X \to X$ is essentially reducible provided that it admits a lifting with respect to the covering projection $\bar{p}\colon \bar{X} \to X$.*

PROOF. Let $\tilde{X}$ denote the universal covering space of $\bar{X}$ and hence of X. Let Π and H be the groups of covering transformations for the covering projections $p : \tilde{X} \to X$ and $p' : \tilde{X} \to \bar{X}$, respectively. Then H is a finite index normal subgroup of Π and the finite quotient group Π/H is the group of covering transformations for $\bar{p} : \bar{X} \to X$. Fix a lifting $\bar{f}$ of f with respect to $\bar{p} : \bar{X} \to X$ and a lifting $\tilde{f}$ of $\bar{f}$ with respect to the universal covering projection $p' : \tilde{X} \to \bar{X}$. Then $\tilde{f}$ is a lifting of f, so the following diagram is commuting:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ \downarrow{p'} & & \downarrow{p'} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{X} \\ \downarrow{\bar{p}} & & \downarrow{\bar{p}} \\ X & \xrightarrow{f} & X \end{array}$$

Let $\boldsymbol{F}$ be an inessential fixed point class of f^k. Then $\boldsymbol{F} = p(\mathrm{Fix}(\alpha\tilde{f}^k))$ for some $\alpha \in \Pi$. For any positive integer n, there exists a unique fixed point class $\boldsymbol{F}'$ of f^{kn} such that $\boldsymbol{F} \subset \boldsymbol{F}'$. We shall show that $\boldsymbol{F}'$ is inessential.

Note that $\boldsymbol{F}' = p(\mathrm{Fix}(\beta\tilde{f}^{kn}))$ for some $\beta \in \Pi$. Let $\varphi : \Pi \to \Pi$ be the homomorphism defined by the rule $\tilde{f}\alpha = \varphi(\alpha)\tilde{f}$ for $\alpha \in \Pi$. Clearly $\tilde{f}^m\alpha = \varphi^m(\alpha)\tilde{f}^m$. If $x \in \boldsymbol{F} = p(\mathrm{Fix}(\alpha\tilde{f}^k))$, then there is $\tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = x$ and $\alpha\tilde{f}^k(\tilde{x}) = \tilde{x}$. Now it is easy to show that

$$\alpha\varphi^k(\alpha)\varphi^{2k}(\alpha)\cdots\varphi^{(n-1)k}(\alpha)\tilde{f}^{kn}(\tilde{x}) = \tilde{x}.$$

This implies that we can take $\beta = \alpha\varphi^k(\alpha)\varphi^{2k}(\alpha)\cdots\varphi^{(n-1)k}(\alpha)$. Note further that since $\varphi(H) \subset H$, φ restricts to a homomorphism $\varphi' : H \to H$ and in turn induces a homomorphism $\bar{\varphi} : \Pi/H \to \Pi/H$ so that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & \Pi & \longrightarrow & \Pi/H & \longrightarrow & 1 \\ & & \downarrow{\varphi'} & & \downarrow{\varphi} & & \downarrow{\bar{\varphi}} & & \\ 1 & \longrightarrow & H & \longrightarrow & \Pi & \longrightarrow & \Pi/H & \longrightarrow & 1 \end{array}$$

If we denote the image of $\alpha \in \Pi$ under the canonical homomorphism $\Pi \to \Pi/H$ by $\bar{\alpha} \in \Pi/H$, then we have

$$\bar{\beta} = \big(\alpha\varphi^k(\alpha)\varphi^{2k}(\alpha)\cdots\varphi^{(n-1)k}(\alpha)\big)^{-} = \bar{\alpha}\bar{\varphi}^k(\bar{\alpha})\bar{\varphi}^{2k}(\bar{\alpha})\cdots\bar{\varphi}^{(n-1)k}(\bar{\alpha}).$$

Since the fixed point class $\boldsymbol{F} = p(\mathrm{Fix}(\alpha\tilde{f}^k))$ of f^k is inessential, by [**11**, Remark 2.7], the corresponding fixed point class $\bar{\boldsymbol{F}} := p'(\mathrm{Fix}(\alpha\tilde{f}^k))$ of $\bar{\alpha}\bar{f}^k : \bar{X} \to \bar{X}$ is also inessential. Since the self map $\bar{\alpha}\bar{f}^k : \bar{X} \to \bar{X}$ is essentially reducible, the unique fixed point class $\bar{\boldsymbol{F}}'$ of $(\bar{\alpha}\bar{f}^k)^n$ such that $\bar{\boldsymbol{F}} \subset \bar{\boldsymbol{F}}'$ is inessential. Applying the same argument as above, we can see easily that $\bar{\boldsymbol{F}}' = p'(\mathrm{Fix}((\alpha\tilde{f}^k)^n))$. Since $(\bar{\alpha}\bar{f}^k)^n = \bar{\alpha}\bar{\varphi}^k(\bar{\alpha})\bar{\varphi}^{2k}(\bar{\alpha})\cdots\bar{\varphi}^{(n-1)k}(\bar{\alpha})\bar{f}^{kn} = \bar{\beta}\bar{f}^{kn}$, we have $\bar{\boldsymbol{F}}' = p'(\mathrm{Fix}(\beta\tilde{f}^{kn}))$, which corresponds to the fixed point class $\boldsymbol{F}' = p(\mathrm{Fix}(\beta\tilde{f}^{kn}))$ of f^{kn}. By [**11**, Remark 2.7] again, $\boldsymbol{F}'$ is inessential. Thus our lemma is proved. □

3. Main results.

Let G be a connected and simply connected nilpotent Lie group. Let Π be a torsion-free, discrete, cocompact subgroup of $G \rtimes C \subset \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G)$, where C is a maximal compact subgroup of $\mathrm{Aut}(G)$. Then such a group Π is called *an almost Bieberbach group* of G and the closed manifold $\Pi\backslash G$ is called *an infra-nilmanifold*. It is known that these are exactly the class of *almost flat* Riemannian manifolds.

Let $M = \Pi\backslash G$ be an infra-nilmanifold. Let $\Gamma = \Pi \cap G$. Then it is L. Auslander's result that (see, for example, [**14**]) Γ is a lattice of G, and is the unique maximal normal nilpotent subgroup of Π. The group $\Psi = \Pi/\Gamma$ is the *holonomy group* of Π or M. It sits naturally in $\mathrm{Aut}(G)$.

LEMMA 3.1 ([**13**, Lemma 3.1]). *Any infra-nilmanifold M admits a finite regular covering projection $q\colon N \to M$ such that N is a nilmanifold and every self-map on M has a lifting with respect to this covering projection.*

PROOF. Let $M = \Pi\backslash G$ be an infra-nilmanifold. Then $\Pi \subset G \rtimes \mathrm{Aut}(G)$. Let $\Gamma = \Pi \cap G$. Then we have a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Psi \longrightarrow 1$$

where $\Psi = \Pi/\Gamma$ is a finite group which naturally sits in $\mathrm{Aut}(G)$. The nilmanifold $N_0 = \Gamma\backslash G$ is the nil-covering of M with the group of covering transformations Ψ.

Let $f : M \to M$ be a self-map. Then it is not necessarily true that f has a lifting $\bar{f} : N_0 \to N_0$. However by [**13**, Lemma 3.1] we know that Π has a fully invariant subgroup Λ of finite index and $\Lambda \subset \Gamma$. Therefore $\Lambda \subset \Gamma \subset G$ and $N = \Lambda\backslash G$ is a nilmanifold which covers N_0 and hence M. Since Λ is a fully invariant subgroup of Π, it follows that $f : M \to M$ has a lifting $\bar{f} : N \to N$, and N is a regular covering of M. □

We say that such a nilmanifold N is a *universal nil-covering* of M. Thus every infra-nilmanifold admits a universal nil-covering.

Let M be a closed manifold. A smooth map $f : M \to M$ is called an *expanding map* if there exist constants $C > 0$ and $\mu > 1$ such that $\|Df^n(v)\| \geq C\mu^n\|v\|$, $\forall v \in T(M)$, for some Riemannian metric $\|\ \|$ on M. In [**6**], it was shown that any expanding map on a closed manifold is topologically conjugate to an expanding map on an infra-nilmanifold.

It is known that every infra-nilmanifold of homogeneous type admits an expanding map [**2**]. However, there are examples of infra-nilmanifolds which do not admit expanding maps [**3**]. We refer the reader to [**2**], [**3**], [**5**], [**6**], [**12**], [**13**], [**15**] for examples of papers dealing with expanding maps.

Our main result is the following:

THEOREM 3.2. *Let n be a positive integer. Then there exists a positive integer m_0, which depends only on n, such that for any integer $m \geq m_0$, for any n-dimensional infra-nilmanifold M, and for any expanding map f on M, we have $[m_0, \infty) \subset \mathrm{HPer}(f)$.*

PROOF. Let M be an n-dimensional infra-nilmanifold and let f be an expanding map on M. By Lemma 3.1, M admits a universal nil-covering N and thus $f : M \to M$

has a lifting $\bar{f} : N \to N$. By [**8**, Corollary 4.5], the nilmanifold N is essentially reducible. Hence by Proposition 2.4, f is essentially reducible.

Next we show that there exists m_0, which depends only on n, such that for all $m \geq m_0$ the following strict inequality holds

$$\sum_{\frac{m}{k}:\text{prime}} N(f^k) < N(f^m).$$

For the expanding map $f : M \to M$, it is known in [**15**] that all the f^m are also expanding maps. Moreover, the set of fixed points, $\mathrm{Fix}(f^m)$, is non-empty and finite. By [**10**, Proposition 1], the path components in $\mathrm{Fix}(f^m)$ are the fixed point classes in the infra-nilmanifold M. Since the isolated fixed point classes are essential with local fixed point index ± 1, we have $N(f^m) = |\,\mathrm{Fix}(f^m)|$. On the other hand, the main part of the proof of [**13**, Theorem 4.6] shows that there exists $m_0 = m_0(n)$ such that for $m \geq m_0$,

$$\sum_{k|m,\, k<m} |\,\mathrm{Fix}(f^k)| < |\,\mathrm{Fix}(f^m)|.$$

In particular, the required strict inequality $\sum_{\frac{m}{k}:\text{prime}} N(f^k) < N(f^m)$ holds.

In all, we have observed that f satisfies the conditions of Proposition 2.3 for all $m \geq m_0$. Therefore, for any $m \geq m_0$, $m \in \mathrm{HPer}(f)$. □

A similar result for self-maps on NR-solvmanifolds can be found in [**9**].

Acknowledgement. The authors would like to thank the referee for making careful corrections of a few expressions in their original version.

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