J. Math. Soc. Japan
Vol. 58, No. 4 (2006) pp. 1163–1184

The homotopy of spaces of maps between real projective spaces

By Kohhei Yamaguchi

(Received Nov. 30, 2005)

Abstract. We study the homotopy groups of spaces of continuous maps between real projective spaces and we generalize the results given in [**5**], [**8**] and [**12**]. In particular, we determine the rational homotopy types of these spaces and compute their fundamental groups explicitly.

1. Introduction.

For connected spaces X and Y, we denote by $\mathrm{Map}(X,Y)$ (resp. $\mathrm{Map}^*(X,Y)$) the space consisting of all continuous maps (resp. based continuous maps) $f : X \to Y$ with compact-open topology. For $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$, let $i_{m,n} : \boldsymbol{K}\mathrm{P}^m \to \boldsymbol{K}\mathrm{P}^n$ denote the inclusion map given by $i_{m,n}([x_0 : \cdots : x_m]) = [x_0 : \cdots : x_m : 0 : \cdots : 0]$ for each pair of integers $1 \leq m \leq n$. Let $\mathrm{Map}_1(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n)$ denote the path component of $\mathrm{Map}(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n)$ which contains the inclusion $i_{m,n}$. We choose $\mathbf{e}_k = [1 : 0 : \cdots : 0] \in \boldsymbol{K}\mathrm{P}^k$ as the base point of $\boldsymbol{K}\mathrm{P}^k$, and let $\mathrm{Map}_1^*(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n)$ be the subspace defined by $\mathrm{Map}_1^*(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n) = \mathrm{Map}^*(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n) \cap \mathrm{Map}_1(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n)$. We define the group $G_{\boldsymbol{K}}(n)$ by $G_{\boldsymbol{K}}(n) = O(n)$, $U(n)$ for $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$, and define the map $f_{m,n}^{\boldsymbol{K}} : G_{\boldsymbol{K}}(n) \to \mathrm{Map}_1^*(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n)$ by the right matrix multiplication

$$f_{m,n}^{\boldsymbol{K}}(A)([x_0 : \cdots : x_m]) = [x_0 : \cdots : x_m : 0 : \cdots : 0] \cdot \begin{bmatrix} 1 & \mathbf{0}_n \\ {}^t\mathbf{0}_n & A \end{bmatrix}$$

for $A \in G_{\boldsymbol{K}}(n)$, where $\mathbf{0}_n = (0, 0, \ldots, 0) \in \boldsymbol{K}^n$. Since the subgroup of $G_{\boldsymbol{K}}(n)$ which fixes $\boldsymbol{K}\mathrm{P}^m$ is $G_{\boldsymbol{K}}(n-m)$, it induces the map

$$\alpha_{m,n}^{\boldsymbol{K}} : Z_{n,m}^{\boldsymbol{K}} \to \mathrm{Map}_1^*(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n), \tag{1}$$

where $Z_{n,m}^{\boldsymbol{K}} = G_{\boldsymbol{K}}(n)/G_{\boldsymbol{K}}(n-m)$ denotes the $\boldsymbol{K}$-Stiefel manifold of orthogonal m-frames in $\boldsymbol{K}^n$. We usually write $Z_{n,m}^{\boldsymbol{R}} = V_{n,m}$ if $\boldsymbol{K} = \boldsymbol{R}$, and it is called *the real Stiefel manifold of orthogonal m-frames in $\boldsymbol{R}^n$*. Similarly, we define the map $g_{m,n}^{\boldsymbol{K}} : G_{\boldsymbol{K}}(n+1) \to \mathrm{Map}_1(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n)$ by

$$g_{m,n}^{\boldsymbol{K}}(A)([x_0 : \cdots : x_m]) = [x_0 : \cdots : x_m : 0 : \cdots : 0] \cdot A \quad \text{for } A \in G_K(n+1).$$

2000 *Mathematics Subject Classification.* Primary 55P15, 55P62; Secondly 55P10.

Key Words and Phrases. rational homotopy equivalence, group action, fundamental group.

Partially supported by Grant-in-Aid for Scientific Research (No. 16540056 (C)(2)), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Because the subgroup $\Delta_{m+1} \times G_{\boldsymbol{K}}(n-m) \subset G_{\boldsymbol{K}}(n+1)$ is fixed under the map $g^{\boldsymbol{K}}_{m,n}$, it also induces the map

$$\beta^{\boldsymbol{K}}_{m,n} : \mathrm{P}Z^{\boldsymbol{K}}_{n+1,m+1} \to \mathrm{Map}_1(\boldsymbol{K}\mathrm{P}^m, \boldsymbol{K}\mathrm{P}^n), \tag{2}$$

where $\Delta_k \subset G_{\boldsymbol{K}}(k)$ is the center of $G_{\boldsymbol{K}}(k)$, $\mathrm{P}Z^{\boldsymbol{K}}_{n+1,m+1}$ denotes the space given by $\mathrm{P}Z^{\boldsymbol{K}}_{n+1,m+1} = G_{\boldsymbol{K}}(n+1)/(\Delta_{m+1} \times G_{\boldsymbol{K}}(n-m)) \cong \mathrm{P}_{\boldsymbol{K}}(Z^{\boldsymbol{K}}_{n+1,m+1})$, and $\mathrm{P}_{\boldsymbol{K}}(Z^{\boldsymbol{K}}_{n+1,m+1})$ is the space consisting of all $\boldsymbol{K}^*$-projective classes of the Stiefel manifold $Z^{\boldsymbol{K}}_{n+1,m+1}$. We write $\mathrm{P}Z^{\boldsymbol{R}}_{n+1,m+1} = \mathrm{P}V_{n+1,m+1}$ if $\boldsymbol{K} = \boldsymbol{R}$, and it is called *the real projective Stiefel manifold of orthogonal* $(m+1)$*-frames in* $\boldsymbol{R}^{n+1}$.

The principal motivation for this paper derives from the work of S. Sasao [**8**] (cf. [**3**], [**7**]), in which he studies the homotopy groups of $\mathrm{Map}_1(\boldsymbol{C}\mathrm{P}^m, \boldsymbol{C}\mathrm{P}^n)$ and $\mathrm{Map}^*_1(\boldsymbol{C}\mathrm{P}^m, \boldsymbol{C}\mathrm{P}^n)$ by using the maps $\alpha^{\boldsymbol{C}}_{m,n}$ and $\beta^{\boldsymbol{C}}_{m,n}$. Because the case $\boldsymbol{K} = \boldsymbol{C}$ has been studied well, from now on we shall study the case $\boldsymbol{K} = \boldsymbol{R}$. Now we recall the following result.

THEOREM 1.1 ([**5**], [**12**]). *Let $1 \leq m \leq n$ be integers.*

(i) *The map $\beta_{m,n} = \beta^{\boldsymbol{R}}_{m,n} : \mathrm{P}V_{n+1,m+1} \to \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ is a homotopy equivalence up to dimension $2(n-m)-1$.*

(ii) $$\pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } n \equiv 1 \pmod 2 \text{ and } k = n, \text{ or} \\ & n \equiv 0 \pmod 2 \text{ and } k = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK. A map $f : X \to Y$ is called *a homotopy equivalence up to dimension* D if $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism when $k < D$ and an epimorphism when $k = D$. Similarly, a map $f : X \to Y$ is called *a rational homotopy equivalence through the dimension* D if $f_* \otimes 1 : \pi_k(X) \otimes \boldsymbol{Q} \to \pi_k(Y) \otimes \boldsymbol{Q}$ is an isomorphism whenever $k \leq D$. In particular, a map $f : X \to Y$ is called *a rational homotopy equivalence through the maximal dimension* D, if $f_* \otimes 1 : \pi_k(X) \otimes \boldsymbol{Q} \to \pi_k(Y) \otimes \boldsymbol{Q}$ is an isomorphism for any $k \leq D$ and it is not an isomorphism for $k = D+1$.

Sasao [**8**] also shows that $\alpha^{\boldsymbol{C}}_{m,n}$ and $\beta^{\boldsymbol{C}}_{m,n}$ are rational homotopy equivalences, and one may suppose that the maps $\alpha^{\boldsymbol{R}}_{m,n}$ and $\beta^{\boldsymbol{R}}_{m,n}$ might be also rational homotopy equivalences. However, recently N. Okazaki pointed out that the map $\beta^{\boldsymbol{R}}_{m,n}$ might not be a rational homotopy equivalence, and one may suppose that it might be not so useful to use these maps for studying the homotopy of the spaces $\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ and $\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$. However, this is not true. In fact, the main purpose of this paper is to show how these maps are useful for studying the homotopy of these spaces. We shall investigate what extent these maps approximate their rational homotopy groups, and show that it is very useful to study them for computing the *integral* homotopy groups of $\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ and $\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ (see Theorem 1.4 below). More precisely, the main results of this paper are stated as follows.

THEOREM 1.2. *Let $1 \leq m < n$ be integers.*

(i) *The map* $\alpha_{m,n}=\alpha^{\boldsymbol{R}}_{m,n}:V_{n,m}\to \mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)$ *is a homotopy equivalences up to dimension* $D_{\boldsymbol{R}}(m,n)=2(n-m)-1$.

(ii) *The maps*

$$\begin{cases}\alpha_{m,n}=\alpha^{\boldsymbol{R}}_{m,n}:V_{n,m}\to \mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\\ \beta_{m,n}=\beta^{\boldsymbol{R}}_{m,n}:\mathrm{P}V_{n+1,m+1}\to \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\end{cases}$$

are rational homotopy equivalences through the maximal dimension $D(m,n)$, *where* $D(m,n)$ *denotes the number defined by*

$$D(m,n)=\begin{cases}2n-3 & \text{if } n\equiv 0 \pmod 2 \text{ and } m=1,\\ 2(n-m)-1-(-1)^{m+n} & \text{otherwise.}\end{cases}$$

Proposition 1.3. *Let* $1\le m<n$ *be integers.*

(i) *If* $n\equiv 1 \pmod 2$ *and* $m\equiv 0 \pmod 2$, $\pi_k\big(\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong 0$ *for any* k.

(ii) *If* $n\equiv 1 \pmod 2$ *and* $m\equiv 1 \pmod 2$,

$$\pi_k\big(\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \text{if } k=n-m,\\ 0 & \text{otherwise.}\end{cases}$$

(iii) *If* $n\equiv 0$ *and* $m\equiv 0 \pmod 2$,

$$\pi_k\big(\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \text{if } k=n-1,\ k=n-m,\\ 0 & \text{otherwise.}\end{cases}$$

(iv) *If* $n\equiv 0$ *and* $m\equiv 1 \pmod 2$,

$$\pi_k\big(\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \text{if } k=n-1,\ k=2n-m-1,\\ 0 & \text{otherwise.}\end{cases}$$

(v) *In particular, if* $n\ge 2$,

$$\pi_k\big(\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \text{if } k=n-1 \text{ and } n\equiv 0 \pmod 2,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 1.4. *Let* $1\le m\le n$ *be integers.*

(i) *The induced homomorphisms*

$$\begin{cases}\alpha_{m,n*}=\alpha^{\boldsymbol{R}}_{m,n*}:\pi_1(V_{n,m})\to \pi_1\big(\mathrm{Map}^*_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\\ \beta_{m,n*}=\beta^{\boldsymbol{R}}_{m,n*}:\pi_1(\mathrm{P}V_{n+1,m+1})\to \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\end{cases}$$

are isomorphisms when $1\le m<n$ *or* $1\le m=n\le 2$, *and split monomorphisms when* $m=n\ge 3$.

(ii) *If* $m<n$, *there are isomorphisms*

$$\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\cong\begin{cases}\boldsymbol{Z} & \textit{if } (m,n)=(1,2),\\ 0 & \textit{if } m=1 \textit{ and } n\geq 3,\ 2\leq m\leq n-2,\\ \boldsymbol{Z}/2 & \textit{if } m=n-1\geq 2.\end{cases}$$

$$\pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\cong\begin{cases}\boldsymbol{Z}/4 & \textit{if } (m,n)=(1,2),\ m=n-1\geq 2,\\ \boldsymbol{Z}/2 & \textit{if } m=1 \textit{ and } n\geq 3,\ 2\leq m\leq n-2.\end{cases}$$

(iii) *If $m=n$, there are isomorphisms*

$$\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)\cong\begin{cases}0 & \textit{if } n=1,\\ \boldsymbol{Z} & \textit{if } n=2,\\ \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & \textit{if } n\geq 3.\end{cases}$$

$$\pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)\cong\begin{cases}\boldsymbol{Z} & \textit{if } n=1,\\ \boldsymbol{Z}/4 & \textit{if } n=2,\\ \boldsymbol{Z}/4\oplus\boldsymbol{Z}/2 & \textit{if } n\geq 3.\end{cases}$$

COROLLARY 1.5. *If $n\equiv 1 \pmod 2$ and $m\equiv 0 \pmod 2$ with $2\leq m<n$, there is a rational homotopy equivalence $\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\simeq_{\boldsymbol{Q}}\{*\}$.*

COROLLARY 1.6. *Let $1\leq m<n$ be integers.*

(i) *If $n\equiv 1 \pmod 2$ and $m\equiv 0 \pmod 2$,*

$$\pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \textit{if } k=n,\\ 0 & \textit{otherwise.}\end{cases}$$

(ii) *If $n\equiv 1 \pmod 2$ and $m\equiv 1 \pmod 2$,*

$$\pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \textit{if } k=n-m,\ k=n,\\ 0 & \textit{otherwise.}\end{cases}$$

(iii) *If $n\equiv 0$ and $m\equiv 0 \pmod 2$,*

$$\pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \textit{if } k=2n-1,\ k=n-m,\\ 0 & \textit{otherwise.}\end{cases}$$

(iv) *If $n\equiv 0$ and $m\equiv 1 \pmod 2$,*

$$\pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\big)\otimes\boldsymbol{Q}\cong\begin{cases}\boldsymbol{Q} & \textit{if } k=2n-1,\ k=2n-m-1,\\ 0 & \textit{otherwise.}\end{cases}$$

This paper is organized as follows. In section 2, we study the basic properties of the maps $\alpha_{m,n}^{\boldsymbol{R}}$ and $\beta_{m,n}^{\boldsymbol{R}}$. Next, we compute the fundamental groups $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n))$ and $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n))$ when $m<n$, and we investigate the rational homotopy stability of these maps. In section 3, and in section 4, we determine the rational homotopy of the spaces $\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)$ and $\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)$, explicitly. Finally, in section 5, we compute the fundamental groups $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n))$ and $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n))$

when $m = n$.

2. The maps $\alpha_{m,n}$ and $\beta_{m,n}$.

We write $\alpha_{m,n} = \alpha^{\boldsymbol{R}}_{m,n}$ and $\beta_{m,n} = \beta^{\boldsymbol{R}}_{m,n}$. Let $ev : \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \to \boldsymbol{R}\mathrm{P}^n$ denote the evaluation map defined by $ev(f) = f(\mathbf{e}_m)$. Then we have the evaluation fibration sequence

$$\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \longrightarrow \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \xrightarrow{ev} \boldsymbol{R}\mathrm{P}^n. \tag{3}$$

If we define the map $r_m : \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \to \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{m-1}, \boldsymbol{R}\mathrm{P}^n)$ by $r_m(f) = f \mid \boldsymbol{R}\mathrm{P}^{m-1}$, we also obtain the restriction fibration sequence

$$\Omega^m_1 S^n \longrightarrow \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \xrightarrow{r_m} \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{m-1}, \boldsymbol{R}\mathrm{P}^n), \tag{4}$$

where $\Omega^m_1 S^n$ denotes the path component of $\Omega^m S^n$ consisting of all based maps $f : S^m \to S^n$ of degree one; we note that $\Omega^m_1 S^n = \Omega^m S^n$ if $m < n$, because $\Omega^m S^n$ is connected in this case.

Similarly, we also obtain the restriction fibration sequence

$$\Omega^m_1 S^n \longrightarrow \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \xrightarrow{r_m} \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{m-1}, \boldsymbol{R}\mathrm{P}^n). \tag{5}$$

PROPOSITION 2.1. *The map $\alpha_{m,n} : V_{n,m} \to \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ is a homotopy equivalence up to dimension $2(n-m)-1$.*

PROOF. First, if we recall the commutative diagram

$$\begin{tikzcd}
\{*\} \arrow[r] \arrow[d] & \{\pm 1\} \arrow[r, "="] \arrow[d] & \{\pm 1\} \arrow[d] \\
V_{n,m} \arrow[r] \arrow[d, equal] & V_{n+1,m+1} \arrow[r] \arrow[d] & O(n+1)/O(n) \cong S^n \arrow[d] \\
V_{n,m} \arrow[r] & \mathrm{P}V_{n+1,m+1} \arrow[r] & \boldsymbol{R}\mathrm{P}^n
\end{tikzcd}$$

by using [**1**, Lemma 2.1], we have the fibration sequence

$$V_{n,m} \longrightarrow \mathrm{P}V_{n+1,m+1} \longrightarrow \boldsymbol{R}\mathrm{P}^n.$$

Now consider the commutative diagram

$$\begin{tikzcd}
V_{n,m} \arrow[r] \arrow[d, "\alpha_{m,n}"'] & \mathrm{P}V_{n+1,m+1} \arrow[r] \arrow[d, "\beta_{m,n}"'] & \boldsymbol{R}\mathrm{P}^n \arrow[d, equal] \\
\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \arrow[r] & \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \arrow[r, "ev"] & \boldsymbol{R}\mathrm{P}^n
\end{tikzcd} \tag{6}$$

where two horizontal sequences are fibration sequences. Because $\beta_{m,n}$ is a homotopy equivalence up to dimension $2(n-m)-1$ by Theorem 1.1, the map $\alpha_{m,n}$ is so. $\square$

Remark. If we use the method given in [**12**, (†)], we obtain the homotopy commutative diagram

$$\begin{tikzcd} S^{n-m} \arrow[r] \arrow[d, "E^m"'] & V_{n,m} \arrow[r] \arrow[d, "\alpha_{m,n}"'] & V_{n,m-1} \arrow[d, "\alpha_{m-1,n}"'] \\ \Omega^m S^n \arrow[r] & \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n) \arrow[r, "r_m"] & \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{m-1}, \boldsymbol{R}\mathrm{P}^n) \end{tikzcd} \qquad (*)_m$$

where two horizontal sequences are fibration sequences.

Proposition 2.2. (i) $\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^n)\big) \cong \begin{cases} \boldsymbol{Z} & \text{if } n=2, \\ 0 & \text{otherwise.} \end{cases}$

(ii) $\pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^n)\big) \cong \begin{cases} \boldsymbol{Z} & \text{if } n=1, \\ \boldsymbol{Z}/2 & \text{if } n\geq 3. \end{cases}$

(iii) *If* $2\leq m\leq n-2$, $\begin{cases} \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)\big) \cong 0, \\ \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)\big) \cong \boldsymbol{Z}/2. \end{cases}$

(iv) *If* $n\geq 3$, $\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)\big) \cong \boldsymbol{Z}/2$.

(v) *If* $n\geq 2$, $\pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)\big) \cong \boldsymbol{Z}/4$.

Proof. (i) Since $\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^n) \simeq \Omega S^n$, the assertion clearly holds.

(ii) We note that $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1))=0$ by (i). If we consider the exact sequence induced from (3)

$$0\to \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1)\big) \xrightarrow{ev_*} \pi_1(\boldsymbol{R}\mathrm{P}^1)\cong \boldsymbol{Z} \to \pi_0\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1)\big)=0,$$

we have an isomorphism $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1))\cong \boldsymbol{Z}$. If $n\geq 3$, by using Theorem 1.1, $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^n))\cong \pi_1(\mathrm{P}V_{n+1,2})\cong \boldsymbol{Z}/2$.

(iii) Since $2\leq m\leq n-2$, by using Proposition 2.1 and Theorem 1.1, we have the isomorphisms

$$\begin{cases} \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)\big) \cong \pi_1(V_{n,m})\cong 0, \\ \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)\big) \cong \pi_1(\mathrm{P}V_{n+1,m+1})\cong \boldsymbol{Z}/2. \end{cases}$$

(iv) We assume that $n\geq 3$, and we note that $\alpha_{n-2,n_*}:\pi_1(V_{n,n-2}) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2}, \boldsymbol{R}\mathrm{P}^n))$ is an isomorphism (by Proposition 2.1). If we remark that $V_{n,2}=SO(n)/SO(2)$, $V_{n,n-1}=SO(n)$ and $SO(2)=S^1$, and consider the commutative diagram of the exact sequences induced from $(*)_{n-1}$

$$\begin{array}{ccccc}
\pi_2(V_{n,n-2}) & \xrightarrow{\partial_1} & \pi_1(S^1) & \longrightarrow & \pi_1(SO(n)) \longrightarrow 0 \\
{\scriptstyle \alpha_{n-2,n_*}}\downarrow\cong & & {\scriptstyle E^{n-1}}\downarrow\cong & & {\scriptstyle \alpha_{n-1,n_*}}\downarrow \\
\pi_2\bigl(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2},\boldsymbol{R}\mathrm{P}^n)\bigr) & \xrightarrow{\partial} & \pi_n(S^n) & \longrightarrow & \pi_1\bigl(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)\bigr) \to 0
\end{array}$$

by using the Five Lemma,

$$\alpha_{n-1,n_*} : \boldsymbol{Z}/2 = \pi_1(V_{n,n-1}) = \pi_1(SO(n)) \xrightarrow{\cong} \pi_1\bigl(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)\bigr)$$

is an isomorphism, and the assertion (iv) follows.

(v) First, we show that $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)) = \boldsymbol{Z}/4$ if $n \geq 3$.
If $n \geq 3$, because $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)) = \boldsymbol{Z}/2$ and $\pi_2(\boldsymbol{R}\mathrm{P}^n) = 0$, by using the evaluation fibration (3), we have the short exact sequence

$$0 \to \boldsymbol{Z}/2 \to \pi_1\bigl(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)\bigr) \xrightarrow{ev_*} \pi_1(\boldsymbol{R}\mathrm{P}^n) = \boldsymbol{Z}/2 \to 0.$$

Hence, $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n))$ is isomorphic to $\boldsymbol{Z}/4$ or $\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2$. Because $\beta_{n-1,n_*} : \boldsymbol{Z}/4 = \pi_1(\mathrm{P}V_{n+1,n}) \to \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n))$ is surjective by Theorem 1.1, $\beta_{n-1,n_*} : \boldsymbol{Z}/4 = \pi_1(\mathrm{P}V_{n+1,n}) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n))$ is an isomorphism. Hence, $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)) \cong \boldsymbol{Z}/4$ if $n \geq 3$.

Finally, consider the case $n = 2$. Since the induced homomorphism $\alpha_{1,2_*} : \pi_1(SO(2)) \to \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1,\boldsymbol{R}\mathrm{P}^2))$ is surjective (by Proposition 2.1) and $\pi_1(SO(2)) \cong \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1,\boldsymbol{R}\mathrm{P}^2)) \cong \boldsymbol{Z}$,

$$\alpha_{1,2_*} : \boldsymbol{Z} = \pi_1(SO(2)) = \pi_1(V_{2,1}) \xrightarrow{\cong} \pi_1\bigl(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1,\boldsymbol{R}\mathrm{P}^2)\bigr)$$

is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_2(\boldsymbol{R}\mathrm{P}^2) & \xrightarrow{\partial} & \pi_1(SO(2)) & \longrightarrow & \pi_1(V_{3,2}) & \longrightarrow & \pi_1(\boldsymbol{R}\mathrm{P}^2) \to 0 \\
\| & & {\scriptstyle \alpha_{1,2_*}}\downarrow\cong & & {\scriptstyle \beta_{1,2_*}}\downarrow & & \| \\
\pi_2(\boldsymbol{R}\mathrm{P}^2) & \xrightarrow{\partial'} & \pi_1(\mathrm{Map}_1^*) & \longrightarrow & \pi_1\bigl(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1,\boldsymbol{R}\mathrm{P}^2)\bigr) & \xrightarrow{ev_*} & \pi_1(\boldsymbol{R}\mathrm{P}^2) \to 0
\end{array}$$

where $\mathrm{Map}_1^* = \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1,\boldsymbol{R}\mathrm{P}^2)$ and two horizontal sequences are exact. Then it follows from the Five Lemma that $\beta_{1,2_*} : \boldsymbol{Z}/4 = \pi_1(V_{3,2}) \xrightarrow{\cong} \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1,\boldsymbol{R}\mathrm{P}^2))$ is an isomorphism. □

COROLLARY 2.3. *The induced homomorphisms*

$$\begin{cases}\alpha_{m,n_*} : \pi_1(V_{n,m}) \to \pi_1\bigl(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\bigr) \\ \beta_{m,n_*} : \pi_1(\mathrm{P}V_{n+1,m+1}) \to \pi_1\bigl(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m,\boldsymbol{R}\mathrm{P}^n)\bigr)\end{cases}$$

are isomorphisms if $1 \leq m < n$ *or if* $m = n = 1$, *and split monomorphisms if* $m = n \geq 2$.

REMARK. We remark that α_{n,n_*} and β_{n,n_*} is an isomorphism for the case $n=2$, too, which will be proved in section 5 (see Theorem 5.1).

PROOF. If $1 \leq m \leq n-2$, it follows from Proposition 2.1 and Theorem 1.1 that α_{m,n_*} and β_{m,n_*} are isomorphisms. Moreover, we show that α_{n-1,n_*} and β_{n-1,n_*} are isomorphisms if $n \geq 2$ in the proof of Proposition 2.2.

If $m=n=1$, since $\pi_1(V_{1,1}) = \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1)) = 0$, $\alpha_{1,1_*}$ is trivially isomorphism. Next, we take $\mathrm{Map}_1^*(1) = \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1)$ and consider the commutative diagram

$$\begin{CD}
0=\pi_1(SO(1)) @>>> \pi_1(\mathrm{P}V_{2,2}) @>{\cong}>> \pi_1(\boldsymbol{R}\mathrm{P}^1) @>>> 0\\
@. @V{\beta_{1,1_*}}VV @| @.\\
0=\pi_1\big(\mathrm{Map}_1^*(1)\big) @>>> \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^1)\big) @>{ev_*}>{\cong}> \pi_1(\boldsymbol{R}\mathrm{P}^1) @>>> 0
\end{CD}$$

where two horizontal sequences are exact. Because ev_* is an isomorphism by the proof of Proposition 2.2, $\beta_{1,1_*}$ is also an isomorphism. So it remains to show that α_{n,n_*} and β_{n,n_*} are split monomorphisms if $n \geq 2$. First, consider the homomorphism α_{n,n_*}. Because $SO(1)$ is a trivial group, we can identify $V_{n,n} = V_{n,n-1} = SO(n)$, and we obtain the commutative diagram

$$\begin{CD}
\pi_1(V_{n,n}) @>{=}>> \pi_1(V_{n,n-1})\\
@V{\alpha_{n,n_*}}VV @V{\alpha_{n-1,n_*}}V{\cong}V\\
\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) @>{r_{n*}}>> \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)\big)
\end{CD}$$

where $\pi_1(V_{n,n-1}) = \boldsymbol{Z}$ if $n=2$, or $\boldsymbol{Z}/2$ if $n \geq 3$. Then because α_{n-1,n_*} is an isomorphism, $\alpha_{n,n_*} : \pi_1(V_{n,n}) \to \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$ is a split monomorphism. If we recall that β_{n-1,n_*} is an isomorphism and consider the commutative diagram

$$\begin{CD}
\pi_1(\mathrm{P}V_{n+1,n+1}) @>>{\cong}> \pi_1(\mathrm{P}V_{n+1,n}) = \boldsymbol{Z}/4\\
@V{\beta_{n,n_*}}VV @V{\beta_{n-1,n_*}}V{\cong}V\\
\pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) @>{r_{n*}}>> \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)\big) = \boldsymbol{Z}/4
\end{CD}$$

it is easy to see that β_{n,n_*} is also a split monomorphism. □

REMARK. Because $\pi_1(X)$ is not always an abelian group, the rational homotopy group $\pi_1(X) \otimes \boldsymbol{Q}$ is not well defined, in general. However, since $\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)$ and $\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)$ are H-spaces, $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$, and $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n))$ are abelian groups if $m=n$.

Since $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n))$ and $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n))$ are also abelian groups even if $m < n$ (by Proposition 2.2), we can consider the rational homotopy groups $\pi_*(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \otimes \boldsymbol{Q}$ and $\pi_*(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \otimes \boldsymbol{Q}$ for each $1 \leq m \leq n$.

LEMMA 2.4. *Let $1 \leq m < n$ be integers and let $E^m : S^{n-m} \to \Omega^m S^n$ denote the m-fold suspension.*

(i) *If $n \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 2$, $E^m : S^{n-m} \to \Omega^m S^n$ is a rational homotopy equivalence.*
(ii) *If $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$, $E^m : S^{n-m} \to \Omega^m S^n$ is a rational homotopy equivalence through the maximal dimension $2(n-m)-2$.*
(iii) *If $n \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 2$, $E^m : S^{n-m} \to \Omega^m S^n$ is a rational homotopy equivalence through the maximal dimension $2(n-m)-2$.*
(iv) *If $n \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$, $E^m : S^{n-m} \to \Omega^m S^n$ is a rational homotopy equivalence through the maximal dimension $2n-m-2$.*

PROOF. We note that $\pi_{2k-1}(S^k) = \boldsymbol{Z} \cdot [\iota_k, \iota_k] \oplus (\text{finite group})$ if $k \equiv 0 \pmod 2$, $\pi_k(S^k) = \boldsymbol{Z} \cdot \iota_k$ and that $\pi_k(S^n)$ is a finite group except these above two cases. Then the assertion easily follows from mod $\mathscr{C}$ Serre Theorem [**9**]. □

THEOREM 2.5. *Let $1 \leq m < n$ be integers.*

(i) *If $n \equiv 1 \pmod 2$, the map $\alpha_{m,n} : V_{n,m} \to \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ is a rational homotopy equivalence through the dimension $D_1(m,n)$, where we take $D_1(m,n) = D(m,n) = 2(n-m)-1+(-1)^m$.*
(ii) *If $n \equiv 0 \pmod 2$, the map $\alpha_{m,n} : V_{n,m} \to \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)$ is a rational homotopy equivalence through the dimension $D_2(m,n)$, where we take*

$$D_2(m,n) = D(m,n) = \begin{cases} 2(n-m)-1-(-1)^m & \text{if } m \geq 2 \\ 2n-3 & \text{if } m = 1. \end{cases}$$

PROOF. (i) We assume $n \equiv 1 \pmod 2$. The proof is based on the induction over m. If $m = 1$, the map $\alpha_{1,n}$ can be identified with the suspension $E : V_{n,1} = S^{n-1} \to \Omega S^n \simeq \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^n)$ (up to homotopy equivalences), and the assertion (i) follows from Lemma 2.4. Next, we assume $m = 2$ and consider the diagram $(*)_m$ for $m = 2$. Because $\alpha_{1,n}$ is a rational homotopy equivalence through the dimension $2n-4$ and E^2 is a rational homotopy equivalence by Lemma 2.4, $\alpha_{2,n}$ is a rational homotopy equivalence through the dimension $2n-4 = D_1(2,n)$. Hence, (i) also holds when $m = 2$.

Now we assume that the assertion (i) holds for the case $m-1$. If $m \equiv 1 \pmod 2$, E^m is a rational homotopy equivalence through the dimension $2(n-m)-2$ (by Lemma 2.4) and $\alpha_{m-1,n}$ is a rational homotopy equivalence through the dimension $D_1(m-1,n) = 2(n-m+1)-1+(-1)^{m-1} = 2(n-m)+2$, by using the diagram $(*)_m$, $\alpha_{m,n}$ is a rational homotopy equivalence through the dimension $2(n-m)-2 = 2(n-m)-1+(-1)^m = D_1(m,n)$. So (i) is true for the map $\alpha_{m,n}$, too.

Similarly, if $m \equiv 0 \pmod 2$, E^m is a rational homotopy equivalence (by Lemma 2.4) and $\alpha_{m-1,n}$ is a rational homotopy equivalence through the dimension $D_1(m-1,n) = 2(n-m+1)-1+(-1)^{m-1} = 2(n-m)$. So $\alpha_{m,n}$ is a rational homotopy equivalence through the dimension $2(n-m) = 2(n-m)-1+(-1)^m = D_1(m,n)$ and (i) is satisfied in this case, too. Therefore, the assertion (i) is proved.

(ii) We assume $n \equiv 0 \pmod 2$. First, consider the case $m = 1$. Because $\alpha_{1,m}$ is identified with the suspension $E^1 : S^{n-1} \to \Omega S^n$, it follows from Lemma 2.4 that $\alpha_{1,n}$ is a rational homotopy equivalence through the dimension $2n-3 = D_2(1,n)$. So the assertion (ii) is proved when $m=1$.

Next, we suppose $m = 2$ and consider the diagram $(*)_2$. Then because E^2 is a rational homotopy equivalence through the dimension $2(n-2)-2 = 2n-6$ (by Lemma 2.4) and $\alpha_{1,n}$ is a rational homotopy equivalence through the dimension $2n-3$, $\alpha_{2,n}$ is a rational homotopy equivalence through the dimension $2n-6 = 2(n-2)-1-(-1)^m = D_2(2,n)$. So the assertion (ii) holds when $m=2$, too. Thirdly, we assume $m=3$, and consider the diagram $(*)_3$. Then because E^3 is a rational homotopy equivalence through the dimension $2n-3-2 = 2n-5$ and $\alpha_{2,n}$ is a rational homotopy equivalence through the dimension $2n-6$, $\alpha_{3,n}$ is a rational homotopy equivalence through the dimension $2n-6 = D_2(3,n)$. Hence, (ii) is proved for $1 \le m \le 3$.

Now we shall prove (ii) for the general case by using the induction over m. We assume that the assertion (ii) holds for the case $m-1$ with $m \ge 2$, and consider the diagram $(*)_m$. It follows from Lemma 2.4 that E^m is a rational homotopy equivalence through the dimension D', where D' denotes the number

$$D' = \begin{cases} 2(n-m)-2 & \text{if } m \equiv 0 \pmod 2, \\ 2n-m-2 & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

Then because $\alpha_{m-1,n}$ is a rational homotopy equivalence through the dimension $D_2(m-1,n) = 2(n-m)+1+(-1)^m$, the map $\alpha_{m,n}$ is a rational homotopy equivalence through the dimension $\min(D_2(m-1,n), D')$. However, because

$$\min\big(D_2(m-1,n), D'\big) = \begin{cases} 2(n-m)-2 & \text{if } m \equiv 0 \pmod 2 \\ 2(n-m) & \text{if } m \equiv 1 \pmod 2 \end{cases} = D_2(m,n),$$

the assertion (ii) holds for the case m, too. □

Next, we compute the rational homotopy group $\pi_k(V_{n,m}) \otimes \boldsymbol{Q}$. For this purpose, first we recall the following result.

Lemma 2.6. *Let $1 \le m < n$ be integers and we take $|e_k| = k$.*

(i) *If $n \equiv 1 \pmod 2$ and $m \equiv 0 \pmod 2$,*

$H^*(V_{n,m}, \boldsymbol{Q}) = E\big[e_{2(n-m)+1}, e_{2(n-m)+5}, e_{2(n-m)+9}, \dots, e_{2n-3}\big].$

(ii) *If $n \equiv 1 \pmod 2$ and $m \equiv 1 \pmod 2$,*

$H^*(V_{n,m}, \boldsymbol{Q}) = E\big[e_{n-m}, e_{2(n-m)+3}, e_{2(n-m)+7}, e_{2(n-m)+11}, \dots, e_{2n-3}\big].$

(iii) *If $n \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 2$,*

$H^*(V_{n,m}, \boldsymbol{Q}) = E\big[e_{n-m}, e_{n-1}, e_{2(n-m)+3}, e_{2(n-m)+7}, \dots, e_{2n-5}\big].$

(iv) *If $n \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$,*

$H^*(V_{n,m}, \boldsymbol{Q}) = E\big[e_{n-m}, e_{2(n-m)+1}, e_{2(n-m)+5}, e_{2(n-m)+9}, \dots, e_{2n-5}\big].$

Proof. This result is well known (for example, see [**6**]). □

Proposition 2.7. *Let $1 \leq m < n$ be integers.*

(i) *If $n \equiv 1 \pmod 2$ and $m = 2l$, there is a rational homotopy equivalence*

$$V_{n,m} \simeq_{\boldsymbol{Q}} \prod_{k=0}^{l-1} S^{2(n-m)+1+4k} = S^{2(n-m)+1} \times S^{2(n-m)+5} \times \cdots \times S^{2n-3}.$$

(ii) *If $n \equiv 1 \pmod 2$ and $m = 2l+1$, there is a rational homotopy equivalence*

$$\begin{aligned} V_{n,m} &\simeq_{\boldsymbol{Q}} S^{n-m} \times \prod_{k=1}^{l-1} S^{2(n-m)-1+4k} \\ &= \begin{cases} S^{n-1} & \text{if } l = 0 \\ S^{n-m} \times S^{2(n-m)+3} \times S^{2(n-m)+7} \times \cdots \times S^{2n-3} & \text{if } l \geq 1. \end{cases} \end{aligned}$$

(iii) *If $n \equiv 0 \pmod 2$ and $m = 2l$, there is a rational homotopy equivalence*

$$\begin{aligned} V_{n,m} &\simeq_{\boldsymbol{Q}} S^{n-m} \times S^{n-1} \times \prod_{k=1}^{l-1} S^{2(n-m)-1+4k} \\ &= \begin{cases} S^{n-2} \times S^{n-1} & \text{if } l = 1 \\ S^{n-m} \times S^{n-1} \times S^{2(n-m)+3} \times S^{2(n-m)+7} \times \cdots \times S^{2n-5} & \text{if } l \geq 2. \end{cases} \end{aligned}$$

(iv) *If $n \equiv 0 \pmod 2$ and $m = 2l+1$, there is a rational homotopy equivalence*

$$\begin{aligned} V_{n,m} &\simeq_{\boldsymbol{Q}} S^{n-1} \times \prod_{k=0}^{l-1} S^{2(n-m)+1+4k} \\ &= \begin{cases} S^{n-1} & \text{if } l = 0 \\ S^{n-1} \times S^{2(n-m)+1} \times S^{2(n-m)+5} \times \cdots \times S^{2n-5} & \text{if } l \geq 1. \end{cases} \end{aligned}$$

Proof. The assertions easily follow from Lemma 2.6 and the standard rational homotopy theory (cf. [**4**]). □

Corollary 2.8. *Let $1 \leq m < n$ be integers.*

(i) *If $n \equiv 1 \pmod 2$ and $m = 2l$,*

$$\pi_k(V_{n,m}) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = 2(n-m)+1+4s \quad (0 \leq s < l), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If* $n \equiv 1 \pmod 2$ *and* $m = 2l+1$,

$$\pi_k(V_{n,m}) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-m, \ \text{or } k = 2(n-m)-1+4s \quad (0 \leq s < l), \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If* $n \equiv 0 \pmod 2$ *and* $m = 2l$,

$$\pi_k(V_{n,m}) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-m, \ k = n-1 \text{ or} \\ & k = 2(n-m)-1+4s \quad (0 \leq s < l), \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *If* $n \equiv 0 \pmod 2$ *and* $m = 2l+1$,

$$\pi_k(V_{n,m}) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-1, \ \text{or } k = 2(n-m)+1+4s \quad (0 \leq s < l), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This easily follows from Proposition 2.7. □

3. Rational homotopy when $n \equiv 1 \pmod 2$.

In section 3 and section 4, we shall compute the rational parts of the homotopy groups

$$\begin{cases} A_k(m,n) = \pi_k\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)\big) \otimes \boldsymbol{Q}, \\ B_k(m,n) = \pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)\big) \otimes \boldsymbol{Q}, \end{cases}$$

explicitly. In this section, we consider the case $n \equiv 1 \pmod 2$.

Lemma 3.1. *Let* $n \geq 2$ *be an integer.*

(i) *If* $n \equiv 1 \pmod 2$, $A_k(1,n) \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$

(ii) *If* $n \equiv 0 \pmod 2$, $A_k(1,n) \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-1, \ 2n-2, \\ 0 & \text{otherwise.} \end{cases}$

Proof. The assertion easily follows from $\pi_k(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^1, \boldsymbol{R}\mathrm{P}^n)) \cong \pi_{k+1}(S^n)$, and we omit the detail. □

Proposition 3.2. *Let* $n \geq 2$ *be an integer.*

(i) *If* $n \equiv 1 \pmod 2$, $A_k(n,n) = 0$ *for any* k.

(ii) *If* $n \equiv 0 \pmod 2$, $A_k(n,n) \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases}$

Proof. (i) Because $n \equiv 1 \pmod 2$, by Theorem 1.1, we have

$$B_k(n,n) = \pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

If we consider the homotopy exact sequence induced from (3) for $m = n$

$$\cdots \to \pi_{k+1}(\boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \xrightarrow{\partial} A_k(n,n) \to B_k(n,n) \xrightarrow{ev_* \otimes 1} \pi_k(\boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \xrightarrow{\partial} \cdots,$$

we obtain the following two assertions.

(i-a) $A_k(n,n) = 0$ for any $k \notin \{n, n-1\}$.
(i-b) The sequence $0 \to A_n(n,n) \to \boldsymbol{Q} \to \boldsymbol{Q} \to A_{n-1}(n,n) \to 0$ is exact.

It suffices to show that $A_n(n,n) \cong A_{n-1}(n,n) \cong 0$. If this does not holds, $A_n(n,n) \cong A_{n-1}(n,n) \cong \boldsymbol{Q}$. By using the homotopy exact sequence induced from (4) for $m = n$

$$\to \pi_n(\Omega^n \boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \to A_n(n,n) \to A_n(n-1,n) \xrightarrow{\partial} \pi_{n-1}(\Omega^n \boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q},$$

we have $A_n(n-1,n) \cong \boldsymbol{Q}$. Similarly, by using the homotopy exact sequence induced from (4) for $m = n-1$, we have $A_n(n-2,n) \cong \boldsymbol{Q}$. If we repeat these computations, we obtain the equality

$$A_n(n,n) \cong A_{n-1}(n-1,n) \cong A_n(n-2,n) \cong \cdots\cdots \cong A_n(1,n) \cong \boldsymbol{Q}.$$

On the other hand, by Lemma 3.1, $A_n(1,n) = 0$ and this is a contradiction. Hence, $A_n(n,n) \cong A_{n-1}(n,n) \cong 0$ and the assertion (i) follows.

(ii) We assume $n \equiv 0 \pmod 2$. By Theorem 1.1, we have

$$B_k(n,n) = \pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = 2n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then if we consider the homotopy exact sequence of (3) for $m = n$

$$\to \pi_{k+1}(\boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \xrightarrow{\partial} A_k(n,n) \to B_k(n,n) \xrightarrow{ev_* \otimes 1} \pi_k(\boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \xrightarrow{\partial}$$

and we recall $\pi_k(\boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \cong \boldsymbol{Q}$ if $k \in \{n, 2n-1\}$ or 0 otherwise, we easily obtain the following two assertions.

(ii-a) $A_{n-1}(n,n) \cong \boldsymbol{Q}$, and $A_k(n,n) = 0$ if $k \notin \{n-1, 2n-1, 2n-2\}$.
(ii-b) The sequence $0 \to A_{2n-1}(n,n) \to \boldsymbol{Q} \to \boldsymbol{Q} \xrightarrow{\partial} A_{2n-2}(n,n) \to 0$ is exact.

It remains to show that $A_{2n-1}(n,n) = A_{2n-2}(n,n) = 0$. If this does not holds, $A_{2n-1}(n,n) \cong A_{2n-2}(n,n) \cong \boldsymbol{Q}$. Consider the homotopy exact sequence of the restriction fibration (4) for $m = n$,

$$\pi_{2n}(\Omega^n \boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} \xrightarrow{\partial} A_{2n-1}(n,n) \to A_{2n-1}(n-1,n) \to \pi_{2n-1}(\Omega^n \boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q}.$$

Then because $\pi_k(\Omega^n \boldsymbol{R}\mathrm{P}^n) \otimes \boldsymbol{Q} = 0$ for $k \neq n-1$, we have $A_{2n-1}(n-1,n) \cong \boldsymbol{Q}$. If we consider the homotopy exact sequence induced from (4) for $m = n-1$, similarly we have $A_{2n-1}(n-2,n) \cong \boldsymbol{Q}$. If we repeat this argument, we have

$$A_{2n-1}(n,n) \cong A_{2n-1}(n-1,n) \cong \cdots \cong A_{2n-1}(1,n) \cong \boldsymbol{Q}.$$

However, by Lemma 3.1, $A_{2n-1}(1,n) = 0$ and this is a contradiction. □

Lemma 3.3. *If $n \geq 3$ is an odd integer and $2 \leq m \leq n-1$, the sequence*

$$\begin{aligned} 0 \to A_{n-m+1}(m,n) \to A_{n-m+1}(m-1,n) \to \boldsymbol{Q} \\ \to A_{n-m}(m,n) \to A_{n-m}(m-1,n) \to 0 \end{aligned}$$

is exact, and there is an isomorphism $A_k(m,n) \cong A_k(m-1,n)$ for any $k \notin \{n-m+1, n-m\}$.

Proof. If we consider the exact sequence induced from the fibration (4), the assertion easily follows from

$$\pi_k(\Omega_1^m S^n) \otimes \boldsymbol{Q} \cong \pi_{k+m}(S^n) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k = n-m, \\ 0 & \text{otherwise,} \end{cases}$$ □

Lemma 3.4. *If $n \geq 3$ and $n \equiv 1 \pmod 2$, $A_k(2,n) = 0$ for any k.*

Proof. By using Lemma 3.1 and Lemma 3.3 for $m = 2$, there is an exact sequence $0 \to A_{n-1}(2,n) \to \boldsymbol{Q} \to \boldsymbol{Q} \to A_{n-2}(2,n) \to 0$ with $A_k(2,n) = 0$ if $k \neq n-1, n-2$. So we have the equality $A_{n-1}(2,n) \cong A_{n-2}(2,n)$. It remains to show that $A_{n-1}(2,n) = A_{n-2}(2,n) = 0$. If we use Lemma 3.3 for $m = 3, 4, \cdots, n$, then we have $A_{n-1}(2,n) \cong A_{n-1}(3,n) \cong \cdots \cong A_{n-1}(n,n)$. However, because $A_{n-1}(n,n) = 0$ by Proposition 3.2, we have $A_{n-1}(2,n) = A_{n-2}(2,n) = 0$ and this completes the proof. □

Proposition 3.5. *Let $1 \leq m < n$ be integers with $n \equiv 1 \pmod 2$.*

(i) *If $m \equiv 0 \pmod 2$, $A_k(m,n) = 0$ for any k.*

(ii) *If $m \equiv 1 \pmod 2$,* $A_k(m,n) \cong \begin{cases} \boldsymbol{Q} & \textit{if } k = n-m, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. The proof is based on the induction over m. If $m = 1$ or $m = 2$, the assertion follows from Lemma 3.1 and Lemma 3.4. Now we suppose that Proposition 3.5 holds for the case $m-1$ with $m \geq 3$.

First, consider the case $m \equiv 0 \pmod 2$. Then by the induction hypothesis, $A_k(m-1,$

$n) \cong \boldsymbol{Q}$ if $k = n-m+1$ and $A_k(m-1, n) = 0$ otherwise. Then it follows from Lemma 3.3 and the above equality that there is an exact sequence $0 \to A_{n-m+1}(m,n) \to \boldsymbol{Q} \to \boldsymbol{Q} \to A_{n-m}(m,n) \to 0$, and $A_k(m,n) = 0$ for any $k \notin \{n-m+1, n-m\}$. In particular, it is easy to see that $A_{n-m+1}(m,n) \cong A_{n-m}(m,n)$. However, by using Proposition 3.2 and Lemma 3.3 for $m = 3, 4, \ldots, n$, we have

$$A_{n-m+1}(m,n) \cong A_{n-m+1}(m+1,n) \cong \cdots \cong A_{n-m+1}(n,n) \cong 0.$$

Hence, $A_{n-m+1}(m,n) = A_{n-m}(m,n) = 0$ and so that $A_k(m,n) = 0$ for any k. Therefore, Proposition 3.5 holds for the case $m \equiv 0 \pmod 2$.

Next, consider the case $m \equiv 1 \pmod 2$. Then by the induction hypothesis, $A_k(m-1, n) = 0$ for any k. Hence, by using Lemma 3.3, we can easily show that $A_k(m,n) \cong \boldsymbol{Q}$ if $k - n - m$ and $A_k(m,n) = 0$ otherwise. So that Proposition 3.5 also holds for the case m when $m \equiv 1 \pmod 2$, and this completes the proof. □

4. Rational homotopy when $n \equiv 0 \pmod 2$.

In this section we consider rational homotopy groups $A_k(m,n)$ for the case $n \equiv 0 \pmod 2$. Because the proofs are almost similar as those for the case $m \equiv 1 \pmod 2$, we do not explain the detail.

Lemma 4.1. *$A_k(1,2) \cong \boldsymbol{Q}$ if $k = 1$ or $k = 2$, and $A_k(1,2) = 0$ otherwise.*

Proof. This follows from Lemma 3.1. □

Lemma 4.2. *If $n \geq 4$ and $2 \leq m < n$ be integers such that $n \equiv 0 \pmod 2$, the sequences*

$$\begin{aligned} 0 \to A_{n-m+1}(m,n) &\to A_{n-m+1}(m-1,n) \to \boldsymbol{Q} \\ &\to A_{n-m}(m,n) \to A_{n-m}(m-1,n) \to 0, \ \textit{and} \\ 0 \to A_{2n-m}(m,n) &\to A_{2n-m}(m-1,n) \to \boldsymbol{Q} \\ &\to A_{2n-m-1}(m,n) \to A_{2n-m-1}(m-1,n) \to 0 \end{aligned}$$

are exact, and there is an isomorphism $A_k(m,n) \cong A_k(m-1,n)$ for any $k \notin \{n-m+1, n-m, 2n-m, 2n-m-1\}$.

Proof. The assertion follows from the induced exact sequence from (4) and

$$\pi_k(\Omega_1^m S^n) \otimes \boldsymbol{Q} \cong \begin{cases} \boldsymbol{Q} & \text{if } k \in \{n-m, 2n-m-1\}, \\ 0 & \text{otherwise.} \end{cases}$$ □

Lemma 4.3. *If $n \equiv 0 \pmod 2$ and $n \geq 4$, $A_k(2,n) \cong \boldsymbol{Q}$ if $k = n-1$ or $k = n-2$, and $A_k(2,n) = 0$ otherwise.*

Proof. If we use Lemma 4.2, we can prove the assertion in a similar way as the proof of Lemma 3.4. □

Proposition 4.4. *Let $1 \leq m < n$ be integers such that $n \equiv 0 \pmod 2$.*

(i) *If $m \equiv 0 \pmod 2$, $A_k(m,n) \cong \begin{cases} \boldsymbol{Q} & \text{if } k \in \{n-1, n-m\}, \\ 0 & \text{otherwise.} \end{cases}$*

(ii) *If $m \equiv 1 \pmod 2$, $A_k(m,n) \cong \begin{cases} \boldsymbol{Q} & \text{if } k \in \{n-1, 2n-m-1\}, \\ 0 & \text{otherwise.} \end{cases}$*

Proof. If $n = 2$, the assertion follows from Lemma 4.1. So we assume $n \geq 4$. The proof is based on the induction over m. If $m = 1$, the assertion follows from Lemma 3.1. When $m = 2$, the assertion is already proved in Lemma 4.2. So we suppose that the assertions (i), (ii) hold for some number $m-1$ with $m \geq 3$. In this situation, we can prove that the assertion hold for the case m by using the complete analogous way as in the proof of Proposition 3.5. The only different point is to use Lemma 4.2 instead of Lemma 3.3. □

Finally in this section, we give the proofs of Theorem 1.2, Proposition 1.3, Corollary 1.5 and Corollary 1.6.

Proof of Theorem 1.2. (i) The assertion (i) follows from Proposition 2.1.

(ii) First, we show that $\alpha_{m,n}$ is a rational homotopy equivalence through the maximal dimension $D(m,n)$. Because the proof is similar, we only give the proof when $n \equiv 0 \pmod 2$. If $m = 1$, $\alpha_{1,n}$ is identified with the suspension $E : S^{n-1} \to \Omega S^n$ (up to homotopy equivalence) and the assertion clearly holds. So we assume $m \geq 2$. Then if we take $N = D(m,n)+1$, it follows from Corollary 2.8 and Proposition 4.4 that $\pi_N(V_{n,m}) \otimes \boldsymbol{Q} \cong \boldsymbol{Q}$ and $\pi_N(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^m, \boldsymbol{R}\mathrm{P}^n)) \otimes \boldsymbol{Q} = 0$. So $\alpha_{m,n_*} \otimes \boldsymbol{Q}$ is not an isomorphism when $k = D(m,n)+1$, and the assertion follows from Theorem 2.5.

Next, we show that $\beta_{m,n}$ is a rational homotopy equivalence through the maximal dimension $D(m,n)$. Consider the commutative diagram (6) given in the proof of Proposition 2.1. Because $\alpha_{m,n}$ is a rational homotopy equivalence through the maximal dimension $D(m,n)$, it follows from the Five Lemma that the map $\beta_{m,n}$ is so. □

Proofs of Proposition 1.3, Corollary 1.5 and Corollary 1.6. Proposition 1.3 follow from Proposition 3.2, Proposition 3.5 and Proposition 4.4. Similarly, Corollary 1.5 easily follows from (i) of Proposition 1.3, and Corollary 1.6 from the evaluation fibration (3) and Theorem 1.2. □

5. Fundamental groups.

In this section, we compute the fundamental groups $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$ and $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$, and we give the proof of Theorem 1.4.

We note that $\mathrm{P}V_{n+1,n+1} = \mathrm{P}O(n+1)$ and $V_{n,n} = SO(n)$. For a map $f \in \mathrm{Map}(X,Y)$, let $f_\# : \mathrm{Map}(Z,X) \to \mathrm{Map}(Z,Y)$ and $f^\# : \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z)$ denote the maps defined by $f_\#(g) = f \circ g$ and $f^\#(h) = h \circ f$.

Let $\gamma_n : S^n \to \boldsymbol{R}\mathrm{P}^n$ denotes the Hopf fibering, and consider the cofiber sequence

$\boldsymbol{R}\mathrm{P}^{n-1} \xrightarrow{i_n} \boldsymbol{R}\mathrm{P}^n \xrightarrow{q_n} S^n$. Then we have the commutative diagram

$$\begin{array}{ccccc} \Omega_1^n S^n & \xrightarrow{\subset} & \mathrm{Map}_1(S^n, S^n) & \xrightarrow{ev} & S^n \\ {\scriptstyle q_n^\#}\downarrow & & {\scriptstyle q_n^\#}\downarrow & & \| \\ \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, S^n) & \xrightarrow{\subset} & \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, S^n) & \xrightarrow{ev} & S^n \\ {\scriptstyle \gamma_{n\#}}\downarrow & & {\scriptstyle \gamma_{n\#}}\downarrow & & {\scriptstyle \gamma_n}\downarrow \\ \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n) & \xrightarrow{\subset} & \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n) & \xrightarrow{ev} & \boldsymbol{R}\mathrm{P}^n \end{array} \tag{7}$$

where three horizontal sequences are fibration sequences. We note that these maps induce the short exact sequence

$$0 \to \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, S^n)\big) \xrightarrow{\gamma_{n\#*}} \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \to \boldsymbol{Z}/2 \to 0 \tag{8}$$

and isomorphisms

$$\begin{cases} \gamma_{n\#*} : \pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, S^n)\big) \xrightarrow{\cong} \pi_k\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) & \text{for } k \geq 2, \\ \gamma_{n\#*} : \pi_k\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, S^n)\big) \xrightarrow{\cong} \pi_k\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) & \text{for } k \geq 1. \end{cases} \tag{9}$$

THEOREM 5.1. *The induced homomorphisms*

$$\begin{cases} \alpha_{2,2*} : \pi_1(SO(2)) \xrightarrow{\cong} \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \\ \beta_{2,2*} : \pi_1(\mathrm{P}O(3)) \xrightarrow{\cong} \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \end{cases}$$

are isomorphisms, and there are isomorphisms

$$\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \cong \boldsymbol{Z}, \text{ and } \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \cong \boldsymbol{Z}/4.$$

PROOF. Because $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, S^2)) = \boldsymbol{Z}/2$ [**2**, Theorem 2], by using the exact sequence (8), $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2))$ is isomorphic to $\boldsymbol{Z}/4$ or $\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2$. However, because $\beta_{n,n*} : \boldsymbol{Z}/4 = \pi_1(\mathrm{P}O(3)) \to \pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2))$ is a split monomorphism (by Corollary 2.3), in fact, $\beta_{n,n*}$ is an isomorphism and we also have the isomorphism $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \cong \boldsymbol{Z}/4$.

Next, for computing $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2))$, consider the exact sequence

$$\begin{aligned} &\pi_2\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \stackrel{ev_*}{\to} \pi_2(\boldsymbol{R}\mathrm{P}^2) \stackrel{\partial}{\to} \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \to \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \\ &\quad = \boldsymbol{Z}/4 \stackrel{ev_*}{\to} \pi_1(\boldsymbol{R}\mathrm{P}^2) = \boldsymbol{Z}/2 \to 0. \end{aligned}$$

Because $\pi_2(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \otimes \boldsymbol{Q} = 0$ (by Theorem 1.1), $\pi_2(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2))$ is a torsion group. Hence, $ev_* : \pi_2(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \to \pi_2(\boldsymbol{R}\mathrm{P}^2) = \boldsymbol{Z} \cdot \gamma_2$ is trivial and the above exact sequence reduces to the short exact sequence

$$0 \to \pi_2(\boldsymbol{R}\mathrm{P}^2) \xrightarrow{\partial} \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \to \boldsymbol{Z}/2 \to 0. \tag{10}$$

Since $\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)$ is a H-space, $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2))$ is an abelian group. So $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2))$ is isomorphic to $\boldsymbol{Z}$ or to $\boldsymbol{Z} \oplus \boldsymbol{Z}/2$.

On the other hand, it follows from (7), (8) and (9) that there is a commutative diagram

$$\begin{tikzcd}
\pi_2(S^2) \arrow[r, "\partial_1"] \arrow[d, equal] & \pi_1(\Omega_1^2 S^2) \arrow[r] \arrow[d, "q_2^\#{}_*"] & \pi_1\big(\mathrm{Map}_1(S^2, S^2)\big) \arrow[d, "q_2^\#{}_*"] \\
\pi_2(S^2) \arrow[r, "\partial_2"] \arrow[d, "\gamma_{2*}", "\cong"'] & \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, S^2)\big) \arrow[r] \arrow[d, "\gamma_2^\#{}_*", "\cong"'] & \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, S^2)\big) \arrow[d, "\gamma_2^\#{}_*"] \\
\pi_2(\boldsymbol{R}\mathrm{P}^2) \arrow[r, "\partial"] & \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \arrow[r] & \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big)
\end{tikzcd}$$

where three horizontal sequences are exact. If we identify ∂_1 with the homomorphism $\pi_2(S^2) \to \pi_1(\Omega_1^2 S^2) \cong \pi_3(S^2) = \boldsymbol{Z} \cdot \eta_2$, by using [**11**], it is given by $\partial_1(\iota_2) = [\iota_2, \iota_2] = 2\eta_2$. Hence, by using the above commutative diagram, we have

$$\partial\big(\pi_2(\boldsymbol{R}\mathrm{P}^2)\big) \subset 2\pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big). \tag{11}$$

However, if $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$, ∂ must be a split monomorphisms and this contradicts to (11). Hence, $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)) \cong \boldsymbol{Z}$.

It remains to show that $\alpha_{2,2_*}$ is an isomorphism. However, we know that $\alpha_{2,2_*}$ is a monomorphism (by Corollary 2.3) and it suffices to show that $\alpha_{2,2_*}$ is an epimorphism. Consider the commutative diagram

$$\begin{tikzcd}
\pi_2(\boldsymbol{R}\mathrm{P}^2) \arrow[r, "\partial'"] \arrow[d, equal] & \pi_1(SO(2)) \arrow[r] \arrow[d, "\alpha_{2,2_*}"] & \pi_1(\mathrm{P}O(3)) \arrow[r] \arrow[d, "\beta_{2,2_*}", "\cong"'] & \pi_1(\boldsymbol{R}\mathrm{P}^2) \arrow[d, equal] \\
\pi_2(\boldsymbol{R}\mathrm{P}^2) \arrow[r, "\partial"] & \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)\big) \arrow[r] & \pi_1(\mathrm{Map}_1(2)) \arrow[r, "ev_*"] & \pi_1(\boldsymbol{R}\mathrm{P}^2)
\end{tikzcd}$$

where we take $\mathrm{Map}_1(2) = \mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^2, \boldsymbol{R}\mathrm{P}^2)$ and two horizontal sequences are exact. Then by using the diagram chasing, it is easy to see that $\alpha_{2,2_*}$ is an epimorphism. □

From now on, we assume $n \geq 3$, and consider the restriction fibration sequences

$(\dagger)_1$ $\Omega^{n-1} S^n \xrightarrow{j_1} \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n) \xrightarrow{r_{n-1}} \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2}, \boldsymbol{R}\mathrm{P}^n)$,

$(\dagger)_2$ $\Omega_1^n S^n \xrightarrow{j} \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n) \xrightarrow{r_n} \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)$.

Let F_n denote the homotopy fiber of the restriction map

$$r = r_{n-1} \circ r_n : \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n) \to \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2}, \boldsymbol{R}\mathrm{P}^n).$$

Then it follows from [**1**, Lemma 2.1] that we obtain the following homotopy commutative diagram

$$\begin{array}{ccccc} \Omega_1^n S^n & \xrightarrow{j'} & F_n & \xrightarrow{r'} & \Omega^{n-1}S^n \\ \| & & \downarrow & & \downarrow{\scriptstyle j_1} \\ \Omega_1^n S^n & \xrightarrow{j} & \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n) & \xrightarrow{r_n} & \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n) \\ \downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r_{n-1}} \\ * & \longrightarrow & \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2}, \boldsymbol{R}\mathrm{P}^n) & \xrightarrow{=} & \mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2}, \boldsymbol{R}\mathrm{P}^n) \end{array} \tag{12}$$

where all horizontal and vertical sequences are fibration sequences. So we also obtain the fibration sequence

$$\Omega_1^n S^n \xrightarrow{j'} F_n \longrightarrow \Omega^{n-1}S^n \tag{13}$$

LEMMA 5.2. *Let $n \geq 3$ be an integer.*

(i) $j'_* : \pi_1(\Omega_1^n S^n) \to \pi_1(F_n)$ *is a monomorphims.*

(ii) $j_{1*} : \pi_2(\Omega^{n-1}S^n) \cong \pi_{n+1}(S^n) \xrightarrow{\cong} \pi_2(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)) \cong \boldsymbol{Z}/2$ *is an isomorphism.*

PROOF. (i) Consider the Serre spectral sequence

$$E^2_{s,t} = H_s\big(\Omega^{n-1}S^n, \mathscr{H}_t(\Omega_1^n S^n, \boldsymbol{Z})\big) \Rightarrow H_{s+t}(F_n, \boldsymbol{Z})$$

associated to the fibration sequence (13). We note that we can identify $E^2_{s,0} = H_s(\Omega^{n-1}S^n, \boldsymbol{Z})$ and $E^2_{0,t} = H_t(\Omega^{n-1}S^n, \boldsymbol{Z})/Q_t$ for any (s,t), where we take $Q_t = \{\mathscr{H}_t(\gamma)(u) - u : \gamma \in \pi_1(\Omega^{n-1}S^n), u \in H_t(\Omega_1^n S^n, \boldsymbol{Z})\}$. By the dimensional reason, $E^\infty_{1,0} = E^2_{1,0} = H_1(\Omega^{n-1}S^n, \boldsymbol{Z}) = \boldsymbol{Z}$. Moreover, because $\mathrm{Aut}(\boldsymbol{Z}/2) = \{1\}$, $\mathscr{H}_1(\gamma)(u) = u$ for any $(\gamma, u) \in \pi_1(\Omega^{n-1}S^n) \times H_1(\Omega_1^n S^n, \boldsymbol{Z})$. Hence, $Q_1 = 0$ and $E^2_{0,1} = \boldsymbol{Z}/2$. Since $E^2_{2,0} = H_2(\Omega^{n-1}S^n, \boldsymbol{Z}) = 0$, $E^\infty_{0,1} = E^2_{0,1} = \boldsymbol{Z}/2$. Hence, there is an isomorphism $H_1(F_n, \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}/2$.

Now we assume that $j'_* : \pi_1(\Omega_1^n S^n) \to \pi_1(F_n)$ is not a monomorphism. Then, clearly $j'_* = 0$, and if we recall the exact sequence

$$\boldsymbol{Z}/2 \cong \pi_1(\Omega_1^n S^n) \xrightarrow{j'_*} \pi_1(F_n) \to \pi_1(\Omega^{n-1}S^n) \cong \boldsymbol{Z} \to 0,$$

we have $\pi_1(F_n) \cong \boldsymbol{Z}$. However, by using the Hurewicz Theorem, there is an isomorphism $H_1(F_n, \boldsymbol{Z}) \cong \pi_1(F_n) \cong \boldsymbol{Z}$, which is a contradiction. Hence, j'_* is a monomorphism.

(ii) By Proposition 2.1, $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2}, \boldsymbol{R}\mathrm{P}^n)) \cong \pi_1(\mathrm{P}V_{n,n-2}) \cong 0$. So it follows from the proof of Proposition 2.2 and the diagram $(*)_{n-1}$ that there is a commutative diagram of the exact sequences

$$\begin{array}{ccccc}
\pi_2(V_{n,n-2}) & \xrightarrow{\partial'''} & \pi_1(S^1) & \longrightarrow & \pi_1(SO(n)) \longrightarrow 0 \\
{\scriptstyle \alpha_{n-2,n_*}}\downarrow\cong & & {\scriptstyle E^{n-1}}\downarrow\cong & & {\scriptstyle \alpha_{n-1,n_*}}\downarrow\cong \\
\pi_2\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2},\boldsymbol{R}\mathrm{P}^n)\big) & \xrightarrow{\partial''} & \pi_n(S^n) & \longrightarrow & \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)\big) \to 0 \\
\| & & \| & & \| \\
\boldsymbol{Z} & & \boldsymbol{Z} & & \boldsymbol{Z}/2
\end{array}$$

Since ∂'' is a monomorphism, it follows from the fibration sequence $(\dagger)_1$ that r_{n-1_*} : $\pi_2(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)) \to \pi_2(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2},\boldsymbol{R}\mathrm{P}^n))$ is trivial. Hence, by using $(\dagger)_1$, $j_{1*} : \pi_2(\Omega^{n-1}S^n) \to \pi_2(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n))$ is an epimorphism. It remains to show that j_{1*} is a monomorphism. If we recall that

$$\alpha_{n-2,n_*} : \pi_3(V_{n,n-2}) \to \pi_3\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2},RP^n)\big)$$

is an epimorphism (by Proposition 2.1) and consider the commutative diagram

$$\begin{array}{ccc}
\pi_3(V_{n,n-2}) & \xrightarrow{\partial'} & \pi_2(S^1)=0 \\
{\scriptstyle \alpha_{n-2,n_*}}\downarrow & & {\scriptstyle E^{n-1}}\downarrow \\
\pi_3\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-2},\boldsymbol{R}\mathrm{P}^n)\big) & \xrightarrow{\partial_1} & \pi_2(\Omega^{n-1}\boldsymbol{R}\mathrm{P}^n)\cong\pi_{n+1}(S^n),
\end{array}$$

∂_1 is trivial. Hence, by using the fibration sequence $(\dagger)_1$, $j_{1*} : \pi_2(\Omega^{n-1}S^n) \to \pi_2(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n))$ is also a monomorphism. □

Theorem 5.3. *If $n\geq 3$, there are isomorphisms*

$$\begin{cases}\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)\cong\boldsymbol{Z}/2\oplus\boldsymbol{Z}/2,\\ \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)\cong\boldsymbol{Z}/4\oplus\boldsymbol{Z}/2.\end{cases}$$

Proof. Consider the commutative diagram induced from the diagram (12)

$$\begin{array}{ccccc}
\pi_2(\Omega^{n-1}S^n) & \xrightarrow{\partial'} & \pi_1(\Omega_1^nS^n) & \xrightarrow{j'_*} & \pi_1(F_n) \\
{\scriptstyle j_{1*}}\downarrow\cong & & \| & & \downarrow \\
\pi_2\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)\big) & \xrightarrow{\partial} & \pi_1(\Omega_1^nS^n) & \xrightarrow{j_*} & \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)
\end{array}$$

where two horizontal sequences are exact. Because j'_* is a monomorphism and j_{1*} is an isomorphism (by Lemma 5.2), it follows from the above commutative diagram that $\partial=0$ and j_* is a monomorphism. Hence, we have the short exact sequence

$$0\to\pi_1(\Omega_1^nS^n)\xrightarrow{j_*}\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n,\boldsymbol{R}\mathrm{P}^n)\big)\xrightarrow{r_{n*}}\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1},\boldsymbol{R}\mathrm{P}^n)\big)\to 0.$$

Because $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^{n-1}, \boldsymbol{R}\mathrm{P}^n)) \cong \boldsymbol{Z}/2 \cong \pi_1(\Omega_1^n S^n)$, $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$ is isomorphic to $\boldsymbol{Z}/4$ or to $\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2$.

However, because $\alpha_{n,n_*} : \boldsymbol{Z}/2 = \pi_1(SO(n)) \to \pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$ is a splitting monomorphism (by Corollary 2.3), $\pi_1(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$ contains $\boldsymbol{Z}/2$ as a direct summand and we obtain the isomorphism

$$\pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \cong \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2. \tag{14}$$

Next, if we recall $\pi_2(\boldsymbol{R}\mathrm{P}^n) = 0$ and consider the homotopy exact sequence induced from (3), we have the short exact sequence

$$0 \to \pi_1\big(\mathrm{Map}_1^*(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \to \pi_1\big(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)\big) \xrightarrow{ev_*} \pi_1(\boldsymbol{R}\mathrm{P}^n) \to 0.$$

Then, it follows from Corollary 2.3 and (14) that $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n))$ is an abelian group of order 8 and it contains $\boldsymbol{Z}/4 = \pi_1(\mathrm{P}O(n+1))$ as a direct summand. Hence, $\pi_1(\mathrm{Map}_1(\boldsymbol{R}\mathrm{P}^n, \boldsymbol{R}\mathrm{P}^n)) \cong \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2$. □

Proof of Theorem 1.4. The assertion (i) follows from Proposition 2.2, Corollary 2.3 and Theorem 5.1. The assertion (ii) follows from Proposition 2.2, Theorem 5.1 and Theorem 5.3. □

Acknowledgements. The author would like to take this opportunity to thank Dr. N. Okazaki for teaching him his idea concerning rational homotopy equivalences. He is also indebted to Professors A. Kozlowski, M. Mimura, and Y. Kotani for numerous helpful conversations concerning the rational homotopy theory of mapping spaces.

References

[1] F. R. Cohen, J. C. Moore and J. A. Neisendorfer, The double suspension and exponents of the homotopy groups of spheres, Ann. Math., **110** (1979), 549–565.

[2] V. L. Hansen, On the space of maps of a closed surface into the 2-sphere, Math. Scand., **35** (1974), 140–158.

[3] A. Kozlowski and K. Yamaguchi, Spaces of holomorphic maps between complex projective spaces of degree one, Topology Appl., **132** (2003), 139–145.

[4] M. Mimura, G. Nishida and H. Toda, Localization of CW complexes and its applications, J. Math. Soc. Japan, **23** (1971), 593–624.

[5] W. Meier and R. Strebel, Homotopy group of acyclic spaces, Quart. J. Math. Oxford, **32** (1981), 81–95.

[6] M. Mimura and H. Toda, Topology of Lie groups, I, II, Translation of Math. Monographs, **91**, Amer. Math. Soc. Providence, 1991.

[7] J. M. Møller, On spaces of maps between complex projective spaces, Proc. Amer. Math. Soc., **91** (1984), 471–476.

[8] S. Sasao, The homotopy of Map($\boldsymbol{C}\mathrm{P}^m, \boldsymbol{C}\mathrm{P}^n$), J. London Math., **8** (1974), 193–197.

[9] J. P. Serre, Groupes d'homotopie et classes de groupes abéliens, Ann. Math., **58** (1953), 258–294.

[10] H. Toda, Composition methods in homotopy groups of spheres, Annals of Math. Studies, **49**, Princeton Univ. Press, 1962.

[11] G. W. Whitehead, On products in homotopy groups, Ann. Math., **47** (1946), 460–475.

[12] K. Yamaguchi, The topology of spaces of maps between real projective spaces, J. Math. Kyoto Univ., **43** (2003), 503–507.

[13] A. Kozlowski and K. Yamaguchi, Spaces of algebraic maps between real projective spaces, preprint.

Kohhei Yamaguchi
Department of Computer Science
and Information Mathematics
University of Electro-Communications
E-mail: kohhei@im.uec.ac.jp