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Amenable discrete quantum groups

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Abstract. Z.-J. Ruan has shown that several amenability conditions are all equivalent in the case of discrete Kac algebras. In this paper, we extend this work to the case of discrete quantum groups. That is, we show that a discrete quantum group, where we do not assume its unimodularity, has an invariant mean if and only if it is strongly Voiculescu amenable.

1. Introduction.

In this paper, we study amenability of non-Kac type discrete quantum groups. We use notions of discrete quantum groups or its dual compact quantum groups introduced in [**7**], [**16**], for example. Amenability is defined as a generalization of the group case, that is, by the existence of an invariant mean. In a discrete group case, it is known that amenability is characterized by several conditions (see [**12**] for its survey). The first step to its generalization for discrete quantum groups has been made by Z.-J. Ruan [**14**] under the tracial condition of the Haar weight, that is, the Kac algebra condition. In particular, he has shown that amenability is equivalent to strong Voiculescu amenability. For general quantum groups, a generalization of Ruan's theorem has been investigated by E. Bédos, R. Conti, G. J. Murphy and L. Tuset in [**2**], [**3**], [**4**], [**5**]. They have shown that strong Voiculescu amenability implies amenability. Our main theorem (Theorem 3.8) says that both the notions are equivalent for general discrete quantum groups. We should mention that all the implications except for the above converse one in Theorem 3.8 have been already known in the pioneering works [**2**], [**3**], [**4**], [**5**], however we give a proof in order to give another proof of nuclearity of dual compact quantum groups. After this work was done, we learned from S. Vaes that E. Blanchard and he also have proved equivalence between amenability and strong Voiculescu amenability.

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2. Notations for quantum groups.

On symbols of tensor products (minimal tensor product or tensor product von Neumann algebra), the same notation $\otimes$ is used throughout this paper. A flip unitary on tensor product Hilbert spaces $H\otimes H$ is denoted by Σ. For an weight θ on a von Neumann algebra N, define the left ideal n_θ and $*$-subalgebra m_θ by $n_\theta = \{x \in N \mid \theta(x^*x) < \infty\}$

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and $m_\theta = n_\theta^* n_\theta$. m_θ^+ means $m_\theta \cap N_+$.

2.1. locally compact quantum groups and their duals.

We adopt the definition of locally compact quantum groups advocated in [**11**] as follows.

Definition 2.1. A pair (M, Δ) is called a (von Neumann algebraic) locally compact quantum group when it satisfies the following conditions.

(1) M is a von Neumann algebra and $\Delta : M \longrightarrow M \otimes M$ is a unital normal $*$-homomorphism satisfying the coassociativity relation: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
(2) There exist two faithful normal semifinite weights φ and ψ which satisfy $\varphi((\omega \otimes \iota)\Delta(x)) = \omega(1)\varphi(x)$ for all $x \in m_\varphi^+$, $\omega \in M_*^+$ and $\psi((\iota \otimes \omega)\Delta(x)) = \omega(1)\psi(x)$ for all $x \in m_\psi^+$, $\omega \in M_*^+$.

The von Neumann algebra M is realized in $B(H)$ via the GNS representation associated to φ, $\{H, \Lambda\}$ where Λ is a map from the left ideal $n_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}$ to the Hilbert space H. On the tensor product $H \otimes H$, the multiplicative unitary W is defined as follows, for $x, y \in n_\varphi$,

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1)).$$

It satisfies the pentagonal equality $W_{12}W_{13}W_{23} = W_{23}W_{12}$. We often use a C^*-subalgebra A of M which is defined as the norm closure of the linear space $\{(\mathrm{id} \otimes \omega)(W) \mid \omega \in B(H)_*\}$. It will be considered a continuous function part of M.

This unitary W also plays a role in defining the dual locally compact quantum group $(\hat{M}, \hat{\Delta})$. The von Neumann algebra $\hat{M}$ is the σ-weak closure of $\{\lambda(\omega) \mid \omega \in B(H)_*\}$, where $\lambda(\omega) = (\omega \otimes \mathrm{id})(W)$. Set $\widehat{W} = \Sigma W^* \Sigma$ and its coproduct is given by $\hat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W}$. As above, a C^*-subalgebra $\hat{A}$ of $\hat{M}$ is also defined by the norm closure of the linear space $\{(\mathrm{id} \otimes \omega)(\widehat{W}) \mid \omega \in B(H)_*\}$.

The left invariant weight $\hat{\varphi}$ on $\hat{M}$ is characterized by the following property. For $\omega \in M_*$, if a vector ξ meets $\omega(x^*) = \langle \xi \mid \Lambda(x) \rangle$ for all $x \in n_\varphi$, then $\hat{\varphi}(\lambda(\omega)^*\lambda(\omega)) = \|\xi\|^2$. Note that H also becomes a GNS Hilbert space for $\hat{\varphi}$ via $\hat{\Lambda}(\lambda(\omega)) = \xi$. Denote the set of such ω by $\mathscr{I}$, that is a dense subspace of M_*.

Let J and $\hat{J}$ be the modular conjugation for φ and $\hat{\varphi}$. Then the following useful equalities hold,

$$(\hat{J} \otimes J)W(\hat{J} \otimes J) = W^*, \quad (J \otimes \hat{J})\widehat{W}(J \otimes \hat{J}) = \widehat{W}^*.$$

The modular operator and the modular automorphism of φ are denoted by Δ_φ and σ^φ, respectively. The autopolar of φ, $\mathscr{P}_\varphi^\natural$ is defined by the norm closure of $\{xJ\Lambda(x) \mid x \in n_\varphi \cap n_\varphi^*\}$. Then any normal state on M is of the form ω_ξ with $\xi \in \mathscr{P}_\varphi^\natural$ and such a vector is unique ([**10**]).

2.2. Discrete and compact quantum groups.

A locally compact quantum group (M, Δ) is called *discrete* if $\hat{\varphi}(1) < \infty$. Then its dual $(\hat{M}, \hat{\Delta})$ is called *compact*. In this case, the state condition $\hat{\varphi}(1) = 1$ is always assumed. $\hat{\varphi}$ has the left and right invariance. About them, the basic references are [**7**], [**15**] and [**16**]. Then M becomes a direct sum von Neumann algebra of matrix algebras, say $M = \oplus_{\alpha\in I}\boldsymbol{M}_{n_\alpha}(\boldsymbol{C})$. Then its left invariant weight φ is decomposed as $\oplus_\alpha \operatorname{Tr}_\alpha h_\alpha$, where Tr_α are usual non-normalized traces. For α, fix a matrix unit $\{e(\alpha)_{i,j}\}_{i,j\in I}$ which diagonalize h_α as $h_\alpha = \sum_{i\in I}\nu(\alpha)_i\, e(\alpha)_{i,i}$. The positive affiliated operator $h' = \sum_{\alpha\in I}\operatorname{Tr}_\alpha(h_\alpha)^{-1}h_\alpha$ is group-like, i.e. $\Delta(h') = h'\otimes h'$. Hence for the modular automorphism σ^φ and the coproduct Δ, we have for $t\in\boldsymbol{R}$,

$$\Delta\circ\sigma_t^\varphi = \left(\sigma_t^\varphi\otimes\sigma_t^\varphi\right)\circ\Delta.$$

Note that the above equality does not hold for general cases.

3. Amenability of quantum groups.

We begin with the following well-known definition.

Definition 3.1. Let (M,Δ) be a locally compact quantum group.

(1) A state m of M is called a left invariant mean if $m((\omega\otimes\iota)(\Delta(x))) = \omega(1)m(x)$ for all $\omega\in M_*$ and $x\in M$.
(2) A state m of M is called a right invariant mean if $m((\iota\otimes\omega)(\Delta(x))) = \omega(1)m(x)$ for all $\omega\in M_*$ and $x\in M$.
(3) A state m of M is called an invariant mean if m is a left and right invariant mean.

Remark 3.2. If (M,Δ) has a left invariant mean, it also has an invariant mean (see [**6**, Proposition 3] for its proof).

The following definition is due to Z.-J. Ruan [**14**, Theorem 1.1].

Definition 3.3. Let (M,Δ) be a locally compact quantum group. We say that it is strongly Voiculescu amenable if there exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H such that for any vector η in H, $\|W^*(\eta\otimes\xi_j)-\eta\otimes\xi_j\|$ converges to 0.

Lemma 3.4. *Let (M,Δ) be a locally compact quantum group. Then the following conditions are equivalent.*

(1) *It is strongly Voiculescu amenable.*
(2) *There exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H with $\lim_j\|\lambda(\omega)\xi_j-\omega(1)\xi_j\| = 0$ for any functional $\omega\in M_*$.*
(3) *There exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on $\hat{M}$ such that $\{(\iota\otimes\omega_j)(W)\}_{j\in\mathscr{J}}$ is a σ-weakly approximate unit of A.*
(4) *There exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on $\hat{M}$ such that $\mathrm{id}:\hat{A}\longrightarrow\hat{A}$ is pointwise-weakly approximated by the net of unital completely positive maps $\{(\mathrm{id}\otimes\omega_j)\circ\hat{\Delta}\}_{j\in\mathscr{J}}$ and $\{(\omega_j\otimes\mathrm{id})\circ\hat{\Delta}\}_{j\in\mathscr{J}}$.*
(5) *There exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on $\hat{M}$ such that $\mathrm{id}:\hat{A}\longrightarrow\hat{A}$ is pointwise-norm approximated by the net of unital completely positive maps $\{(\mathrm{id}\otimes\omega_j)\circ\hat{\Delta}\}_{j\in\mathscr{J}}$ and $\{(\omega_j\otimes\mathrm{id})\circ\hat{\Delta}\}_{j\in\mathscr{J}}$.*

Proof. (1) ⇒ (2). It suffices to prove the statement in the case that ω is a normal state on M by considering linear combinations. Since M is standardly represented, ω is written as $\omega = \omega_\eta$ with a unit vector $\eta \in H$. For any vector $\zeta \in H$, we have

$$|\langle \lambda(\omega_\eta)\xi_j - \omega_\eta(1)\xi_j | \zeta \rangle| \leq \|\zeta\|\|\eta\|\|W(\eta \otimes \xi_j) - \eta \otimes \xi_j\|,$$

so the inequality $\|\lambda(\omega_\eta)\xi_j - \omega_\eta(1)\xi_j\| \leq \|\eta\|\|W(\eta\otimes\xi_j) - \eta\otimes\xi_j\|$ holds. Hence $\|\lambda(\omega_\eta)\xi_j - \omega_\eta(1)\xi_j\|$ converges to 0.

(2) ⇒ (3). Put $\omega_j = \omega_{\xi_j}$ for any j in $\mathscr{J}$. Then for any operator $a \in A$ and normal functional $\theta \in M_*$, we have

$$\begin{aligned} |\theta(a(\iota \otimes \omega_j)(W) - a)| &= |\langle \lambda(\theta a)\xi_j - \theta(a)\xi_j | \xi_j \rangle| \\ &\leq \|\lambda(\theta a)\xi_j - (\theta a)(1)\xi_j\|. \end{aligned}$$

Therefore, $a(\iota \otimes \omega_j)(W) - a$ converges to 0 σ-weakly. Similarly we see $(\iota \otimes \omega_j)(W)a - a$ converges to 0 σ-weakly.

(3) ⇒ (4). Take a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ of $\hat{M}_*$ which satisfies the third condition. Let ω be a normal functional on M. By applying Cohen's factorization theorem ([**1**, Theorem 10, p. 61]) to the left A-module M_*, we get a in A and ω' in M_* such that $\omega = a\omega'$. Then for any functional $\theta \in \hat{A}^*$, we have

$$\begin{aligned} \theta((\omega_j \otimes \mathrm{id}) \circ \hat{\Delta}(\lambda(\omega))) &= \omega((\iota \otimes \theta)(W)(\iota \otimes \omega_j)(W)) \\ &= (\omega'(\iota \otimes \theta)(W))((\iota \otimes \omega_j)(W))a), \end{aligned}$$

which converges to $\theta(\lambda(\omega))$. Since the linear subspace $\{\lambda(\omega); \omega \in M_*\}$ is norm dense in $\hat{A}$ and $\{\theta \circ (\omega_j \otimes \mathrm{id}) \circ \hat{\Delta}\}_{j\in\mathscr{J}}$ is a norm bounded family, $\theta((\omega_j \otimes \mathrm{id}) \circ \hat{\Delta}(x))$ converges to $\theta(x)$ for any operator $x \in \hat{A}$. Similarly we can see that $\theta((\mathrm{id} \otimes \omega_j) \circ \hat{\Delta}(x))$ converges to $\theta(x)$ for $x \in \hat{A}$.

(4) ⇒ (5). Take a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on $\hat{M}$ which satisfies the fourth condition. Let $\mathscr{F}$ be the set of finite subsets of $\hat{A}$. Take $F = \{a_1, a_2, \ldots, a_k\}$ in $\mathscr{F}$ and n in $\boldsymbol{N}$. Consider the product Banach space $\hat{A}_F = l_\infty\text{-}\sum_{x\in\mathscr{F}} \hat{A} \times \hat{A}$ and its dual Banach space $\hat{A}_F^* = l_1\text{-}\sum_{x\in\mathscr{F}} (\hat{A} \times \hat{A})^*$. Denote the following element of $\hat{A}_F$ by $x_F(\omega)$,

$$\begin{gathered} \big((\omega \otimes \mathrm{id}) \circ \hat{\Delta}(a_1) - a_1,\ (\mathrm{id} \otimes \omega) \circ \hat{\Delta}(a_1) - a_1, \\ (\omega \otimes \mathrm{id}) \circ \hat{\Delta}(a_2) - a_2,\ (\mathrm{id} \otimes \omega) \circ \hat{\Delta}(a_2) - a_2, \\ \vdots \\ (\omega \otimes \mathrm{id}) \circ \hat{\Delta}(a_k) - a_k,\ (\mathrm{id} \otimes \omega) \circ \hat{\Delta}(a_k) - a_k\big). \end{gathered}$$

Then $x_F(\omega_j)$ converges to 0 weakly. Hence the norm closure of the convex hull of $\{x_F(\omega_j); j \in \mathscr{J}\}$ contains 0. So there exists a normal state $\omega_{(F,n)}$ on $\hat{M}$ such that $\big\|(\omega_{(F,n)} \otimes \mathrm{id}) \circ \hat{\Delta}(a) - a\big\| < \frac{1}{n}$, and $\big\|(\mathrm{id} \otimes \omega_{(F,n)}) \circ \hat{\Delta}(a) - a\big\| < \frac{1}{n}$ for any element $a \in F$.

This new net $\{\omega_{(F,n)}\}_{(F,n)\in\mathscr{F}\times\boldsymbol{N}}$ is a desired one.

(5) ⇒ (4). It is trivial.

(4) ⇒ (3). Easy to prove by reversing the proof of (3) ⇒ (4).

(3) ⇒ (1). Take such a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on M. Since $\hat{M}$ is standardly represented, there exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H with $\omega_j = \omega_{\xi_j}$. Take a vector η in H. Now by Cohen's factorization theorem, there exist an operator $a \in A$ and a vector $\zeta \in H$ with $\eta = a\zeta$. Then we have

$$\begin{aligned}\|W(\eta\otimes\xi_j)-\eta\otimes\xi_j\|^2 &= 2\|a\zeta\|^2 - 2\operatorname{Re}(\langle W(a\zeta\otimes\xi_j)|a\zeta\otimes\xi_j\rangle)\\ &= 2\|a\zeta\|^2 - 2\operatorname{Re}(\langle(\iota\otimes\omega_j)(W)a\zeta|a\zeta\rangle).\end{aligned}$$

This converges to 0. □

Lemma 3.5. *Let (M,Δ) be a locally compact quantum group. The following conditions are equivalent.*

1. *There exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H with $\lim_j \|(\pi\otimes\iota)(W)^*(\eta\otimes\xi_j) - \eta\otimes\xi_j\| = 0$ for any representation $\{\pi, H_\pi\}$ of A and $\eta\in H_\pi$.*
2. *There exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H with $\lim_j \|(\pi\otimes\iota)(W)^*(\eta\otimes\xi_j) - (\eta\otimes\xi_j)\| = 0$ for any cyclic representation $\{\pi, H_\pi\}$ of A and $\eta\in H_\pi$.*
3. *There exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H with $\lim_j \|\lambda(\omega)\xi_j - \omega(1)\xi_j\| = 0$ for any functional $\omega\in A^*$.*
4. *There exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ in $\hat{M}_*$ such that $\{(\iota\otimes\omega_j)(W)\}_{j\in\mathscr{J}}$ is a weakly approximate unit of A.*
5. *There exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ in $\hat{M}_*$ such that $\{(\iota\otimes\omega_j)(W)\}_{j\in\mathscr{J}}$ is a norm approximate unit of A.*
6. *There exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ in $\hat{M}_*$ such that* $\mathrm{id} : \hat{A} \longrightarrow \hat{A}$ *is approximated in the pointwise norm topology by the net of unital completely positive maps $\{(\mathrm{id}\otimes\omega_j)\circ\hat{\Delta}\}_{j\in\mathscr{J}}$ and $\{(\omega_j\otimes\mathrm{id})\circ\hat{\Delta}\}_{j\in\mathscr{J}}$.*
7. *There exists a character ϱ on $\hat{A}$ with $(\iota\otimes\varrho)(W) = 1$.*
8. *The C^*-algebra $\hat{A}$ has a character.*
9. *There exists a state ϱ on $\hat{M}$ such that ϱ is an $\hat{A}$-linear map and satisfies $(\iota\otimes\varrho)(W) = 1$.*

Proof. (1) ⇒ (2). It is trivial.

(2) ⇒ (3). Take such a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H. Let ω be a state on A and $\{H_\omega, \pi_\omega, \xi_\omega\}$ be its GNS representation. Then for any ζ in H, we have

$$|\langle\lambda(\omega)\xi_j - \omega(1)\xi_j|\zeta\rangle| \le \|\zeta\|\|(\pi_\omega\otimes\iota)(W)(\xi_\omega\otimes\xi_j) - \xi_\omega\otimes\xi_j\|.$$

So we get $\|\lambda(\omega)\xi_j - \omega(1)\xi_j\| \le \|(\pi_\omega\otimes\iota)(W)(\xi_\omega\otimes\xi_j) - \xi_\omega\otimes\xi_j\|$. Hence for any $\omega\in A^*$, $\|\lambda(\omega)\xi_j - \omega(1)\xi_j\|$ converges to 0.

(3) ⇒ (4). Put $\omega_j = \omega_{\xi_j}$ for any j in $\mathscr{J}$. Then for any operator $a\in A$ and any functional $\theta\in A^*$,

$$|\theta(a(\iota\otimes\omega_j)(W) - a)| = |\langle\lambda(\theta a)\xi_j - \theta(a)\xi_j|\xi_j\rangle| \le \|\lambda(\theta a)\xi_j - (\theta a)(1)\xi_j\|.$$

Therefore, $a(\iota\otimes\omega_j)(W)-a$ converges to 0 weakly. Similarly, we can see $(\iota\otimes\omega_j)(W)a-a$ converges to 0 weakly.

(4) $\Rightarrow$ (5). Take such a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ of $\hat{M}_*$. Let $\mathscr{F}$ be the set of finite subsets of A. Take $F=\{a_1,a_1,\dots,a_k\}$ in $\mathscr{F}$ and n in $\boldsymbol{N}$. Consider the product Banach space $A_F=l_\infty\text{-}\sum_{x\in\mathscr{F}}A$ and its dual Banach space $A_F^*=l_1\text{-}\sum_{x\in\mathscr{F}}A^*$. Denote the following element in A_F by $x_F(\omega)$,

$$\big((\iota\otimes\omega)(W)a_1-a_1,(\iota\otimes\omega)(W)a_2-a_2,\dots,(\iota\otimes\omega)(W)a_k-a_k\big).$$

Now the net of the elements in A_F, $x_F(\omega_j)$ converges to 0 weakly. Hence the norm closure of the convex hull of $\{x_F(\omega_j); j\in\mathscr{J}\}$ contains 0. So there exists a normal state of $\hat{M}_*$, $\omega_{(F,n)}$ such that $\|(\iota\otimes\omega_{(F,n)})(W)a-a\|<\frac{1}{n}$ for any a in F. Then we get a new net of normal states of $\hat{M}_*$, $\{\omega_{(F,n)}\}_{(F,n)\in\mathscr{F}\times\boldsymbol{N}}$ and it is easy to see this net is a desired one.

(5) $\Rightarrow$ (6). This is shown in a similar way to the proof of (3) $\Rightarrow$ (4) $\Rightarrow$ (5) in Lemma 3.4.

(6) $\Rightarrow$ (7). Take such a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ of $\hat{M}_*$. Let ϱ be a weak$*$-accumulating state in $\hat{A}^*$ of the net. Then for any normal functional ω on $\hat{M}$, we have

$$\begin{aligned}(\omega_j\otimes\mathrm{id})\big(\hat{\Delta}(\lambda(\omega))\big)&=(\omega_j\otimes\iota\otimes\omega)\big(\hat{W}_{23}^*\hat{W}_{13}^*\big)\\&=(\iota\otimes\omega)\big(\hat{W}^*(\omega_j\otimes\iota)(\hat{W}^*)\big).\end{aligned}$$

This converges to $\lambda(\omega)=(\iota\otimes\omega)(\hat{W}^*(\varrho\otimes\iota)(\hat{W}^*))$. Therefore we obtain $(\iota\otimes\varrho)(W)=1$ and easily see ϱ is a character of $\hat{A}$.

(7) $\Rightarrow$ (8). It is trivial.

(8) $\Rightarrow$ (7). Take a character ϱ on $\hat{A}$. Let u be a unitary $(\iota\otimes\varrho)(W)$ in M. Then we have

$$\begin{aligned}\Delta(u)&=(\iota\otimes\iota\otimes\varrho)((\Delta\otimes\iota)(W))=(\iota\otimes\iota\otimes\varrho)(W_{13}W_{23})\\&=u\otimes u.\end{aligned}$$

Therefore, for any normal functional $\omega\in M_*$, we get

$$\begin{aligned}\lambda(u^*\omega)&=(\omega\otimes\iota)(W(u^*\otimes1))=(\omega\otimes\iota)((1\otimes u^*)W(1\otimes u))\\&=u^*\lambda(\omega)u.\end{aligned}$$

Then we obtain

$$\begin{aligned}|\omega(1)|&=|(u^*\omega)(u)|=|\varrho(\lambda(u^*\omega))|\\&\le\|\lambda(u^*\omega)\|=\|u^*\lambda(\omega)u\|\\&=\|\lambda(\omega)\|.\end{aligned}$$

Hence we can define the character χ on $\hat{A}$ with $\chi(\lambda(\omega))=\omega(1)$ for $\omega\in M_*$.

(7) ⇒ (9). Take such a character ϱ on $\hat{A}$ and extend it to the state on $\hat{M}$. Denote the extended state by $\bar{\varrho}$. The $\hat{A}$-linearity of $\bar{\varrho}$ easily follows from Stinespring's theorem.

(9) ⇒ (3). Take such a state ϱ on $\hat{M}$ and let $\{\omega_j\}_{j\in\mathscr{J}}$ be a net of normal states on $\hat{M}$ which weakly$*$ converges to ϱ. Then for any functional θ on A and for any normal functional ω on $\hat{M}$, we have $\theta((\iota\otimes\omega_j)(W)) = \omega_j((\theta\otimes\iota)(W))$. This converges to $\varrho((\theta\otimes\iota)(W)) = \theta(1)$.

(3) ⇒ (2). Take a state ω of A and let $\{H_\omega, \pi_\omega, \xi_\omega\}$ be the GNS representation of ω. Take a vector η in H_ω. By Cohen's factorization theorem, there exist an operator $a \in A$ and a vector $\zeta \in H$ with $\eta = \pi_\omega(a)\zeta$. Then, we have

$$\begin{aligned}
&\|(\pi_\omega\otimes\iota)(W)\eta\otimes\xi_j - \eta\otimes\xi_j\|^2 \\
&\quad = 2\|\pi_\omega(a)\zeta\|^2 - \mathrm{Re}(\langle(\pi_\omega\otimes\iota)(W)\pi_\omega(a)\zeta\otimes\xi_j|\pi_\omega(a)\zeta\otimes\xi_j\rangle) \\
&\quad = 2\|\pi_\omega(a)\zeta\|^2 - \mathrm{Re}(\langle\pi_\omega(a^*(\iota\otimes\omega_j)(W)a)\zeta|\zeta\rangle).
\end{aligned}$$

This converges to 0.

(2) ⇒ (1). Let $\{\pi, H_\pi\}$ be a representation of A. We may assume that this representation is nondegenerate. Then π is decomposed to the direct sum of cyclic representation. This observation derives the statement of 1. □

By the previous two lemmas, we obtain the following result.

Corollary 3.6. *Let (M,Δ) be a locally compact quantum group. Then the following statements are equivalent.*

(1) *It is strongly Voiculescu amenable.*
(2) *There exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H satisfying $\lim_j \|(\pi\otimes\iota)(W)^*(\eta\otimes\xi_j) - \eta\otimes\xi_j\| = 0$ for any representation $\{\pi, H_\pi\}$ of A and $\eta \in H_\pi$.*

Proof. (2) ⇒ (1). It is trivial.

(1) ⇒ (2). The latter statement is equivalent to the statement (6) in Lemma 3.5 and it is the same as the statement (5) in Lemma 3.4 which is equivalent to strong Voiculescu amenability. □

The following results have been already known in various settings [**3**, Theorem 4.2].

Corollary 3.7. *Let (M,Δ) be a locally compact quantum group. Then the following statements are equivalent.*

(1) *It is strongly Voiculescu amenable.*
(2) *The C^*-algebra $\hat{A}$ has a character ϱ with $(\iota\otimes\varrho)(W) = 1$.*
(3) *The C^*-algebra $\hat{A}$ has a character.*

We begin to treat the discrete quantum groups from now. The discreteness of (M,Δ) is always assumed. The following theorem is the main result of this paper.

Theorem 3.8. *If (M,Δ) is a discrete quantum group, the following statements are equivalent.*

(1) *It has an invariant mean.*
(2) *It is strongly Voiculescu amenable.*
(3) *The C^*-algebra $\hat{A}$ is nuclear and has a character.*
(4) *The von Neumann algebra $\hat{M}$ is injective and has an $\hat{A}$-linear state.*

We remark that the equivalence (2) ⇔ (3) ⇔ (4) have been already proved by the combination of [**2**, Theorem 5.2], [**3**, Corollary 2.9], [**4**, Theorem 4.8], [**5**, Theorem 1.1, Theorem 4.2, Corollary 4.3], however we give a different proof of deriving nuclearity by using completely positive approximation property. Before proving it, we state the following corollary, where a subgroup means a C^*-algebraic compact quantum group $(C(K),\Delta_K)$ with a compact group H which is an image space of $*$-homomorphism $r:\hat{A}\longrightarrow C(K)$ with $\Delta_K\circ r=(r\otimes r)\circ\hat{\Delta}$.

Corollary 3.9. *Let $(\hat{A},\hat{\Delta})$ be a compact quantum group which has a subgroup. Then the dual discrete quantum group (A,Δ) is amenable.*

A character is constructed by the composition of the restriction map and a character on continuous function algebra of the subgroup. Therefore, if a compact quantum group is deformed by some parameters from a ordinary group and a subgroup (e.g. maximal torus) becomes a non-deformed subgroup, then its dual discrete quantum group is amenable. Let $(M,\Delta)=\oplus_{\alpha\in I}Mz_\alpha$ be a matrix decomposition as in Section 2, where $Mz_\alpha\cong\boldsymbol{M}_{n_\alpha}(\boldsymbol{C})$. For a finite subset F in I, let us denote $z_F=\sum_{\alpha\in F}z_\alpha$. Note that $z_FH=\Lambda(Mz_F)$ is finite dimensional subspace.

Lemma 3.10. *If F is a finite subset of I, then the linear subspace $K_F=\{\lambda(\omega z_F);\omega\in M_*\}\subset\hat{A}$ is finite dimensional.*

Proof. Take a normal functional ω in $\mathscr{I}$ as introduced in Section 2 and an operator x in n_φ. Then we have

$$\begin{aligned}\omega(z_Fx^*)&=\omega((z_Fx)^*)\\&=\langle\lambda(\omega)|z_F\Lambda(x)\rangle\\&=\langle z_F\lambda(\omega)\xi_{\hat{\varphi}}|\Lambda(x)\rangle.\end{aligned}$$

So ωz_F is in $\mathscr{I}$ and we obtain $\lambda(\omega z_F)\xi_{\hat{\varphi}}=z_F\lambda(\omega)\xi_{\hat{\varphi}}\in z_FH$. Since $\xi_{\hat{\varphi}}$ is a separating vector for $\hat{M}$ and $\mathscr{I}$ is norm dense in M_*, the statement follows. □

We give a proof of Theorem 3.8 partly first.

Proof of Theorem 3.8. ((2) ⇒ (3) ⇒ (4) ⇒ (1)). Fix a complete orthonormal system $\{e_p\}_{p\in\mathscr{P}}$ of H.

(2) ⇒ (3). We show that $\hat{A}$ has completely positive approximation property. Set $T_\omega=(\iota\otimes\omega)\circ\hat{\Delta}$ for $\omega\in\hat{M}_*$. By assumption, there exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on $\hat{M}$ satisfying the statement (6) in Lemma 3.5, that is, T_{ω_j} converges to the identity map of $\hat{A}$ in the pointwise norm topology. Since $\hat{M}$ is standardly represented, each ω_j is a vector state defined by a unit vector $\xi_j\in H$. Take a finite subset F of I. Then for any operator $x\in\hat{A}$, we have

$$\begin{aligned}
T_{\omega_{z_F\xi_j}}(x) &= (\iota \otimes \omega_{z_F\xi_j})(\hat{\Delta}(x)) \\
&= \sum_{p\in\mathscr{P}} (\iota \otimes \omega_{z_F\xi_j, e_p})(\hat{W})^* (\iota \otimes \omega_{z_F\xi_j, e_p})((1\otimes x)\hat{W}) \\
&= \sum_{p\in\mathscr{P}} \lambda((\omega_{e_p, \xi_j})z_F)\lambda((\omega_{x^*e_p, \xi_j})z_F)^*.
\end{aligned}$$

Since $\lambda((\omega_{e_p, \xi_j})z_F)\lambda((\omega_{x^*e_p, \xi_j})z_F)^*$ is in the finite dimensional linear subspace $K_FK_F^* = \mathrm{span}\{ab^*; a, b \in K_F\} \subset \hat{A}$, so is $T_{\omega_{z_F\xi_j}}(x)$. Hence the completely positive map $T_{\omega_{z_F\xi_j}}$ has finite rank. For $n \in \boldsymbol{N}$ and $j \in \mathscr{J}$, take an finite subset $F(n,j)$ of I with $\|T_{\omega_{z_{F(n,j)}\xi_j}} - T_{\omega_{\xi_j}}\| < \frac{1}{n}$. Then we get a new net of completely positive maps $\{T_{\omega_{z_{F(n,j)}\xi_j}}\}_{(n,j)\in\boldsymbol{N}\times\mathscr{J}}$ of finite rank. This net gives a completely positive approximation of the identity map of $\hat{A}$, hence $\hat{A}$ is a nuclear C^*-algebra.

(3) $\Rightarrow$ (4). Nuclearity $\hat{A}$ implies injectivity $\hat{M}$. Extend the character on $\hat{A}$ to the state on $\hat{M}$. We can easily see the $\hat{A}$-linearity of this state by Stinespring's theorem.

(4) $\Rightarrow$ (1). Let ϱ be an $\hat{A}$-linear state on $\hat{M}$. There exists a conditional expectation E from $B(H)$ onto $\hat{M}$. We set a state m on M by $m = \varrho \circ E|_M$. Then for any vector $\xi \in H$ and for any operator $x \in M$, we have

$$\begin{aligned}
\omega_\xi * m(x) &= m((\omega_\xi \otimes \iota)(\Delta(x))) \\
&= m\Big(\sum_{p\in\mathscr{P}} (\omega_{\xi, e_p} \otimes \iota)(W)^* x (\omega_{\xi, e_p} \otimes \iota)(W)\Big) \\
&= \sum_{p\in\mathscr{P}} m\big((\omega_{\xi, e_p} \otimes \iota)(W)^* x (\omega_{\xi, e_p} \otimes \iota)(W)\big) \\
&= \sum_{p\in\mathscr{P}} \varrho\big((\omega_{\xi, e_p} \otimes \iota)(W)^* E(x) (\omega_{\xi, e_p} \otimes \iota)(W)\big) \\
&= \sum_{p\in\mathscr{P}} \varrho\big((\omega_{\xi, e_p} \otimes \iota)(W)^*\big) \varrho(E(x)) \varrho\big((\omega_{\xi, e_p} \otimes \iota)(W)\big) \\
&= \sum_{p\in\mathscr{P}} \varrho\big((\omega_{\xi, e_p} \otimes \iota)(W)^*\big) \varrho\big((\omega_{\xi, e_p} \otimes \iota)(W)\big) \varrho(E(x)) \\
&= \sum_{p\in\mathscr{P}} \varrho\big((\omega_{\xi, e_p} \otimes \iota)(W)^* (\omega_{\xi, e_p} \otimes \iota)(W)\big) \varrho(E(x)) \\
&= \varrho(\omega_\xi(1))\varrho(E(x)) \\
&= \omega_\xi(1)m(x),
\end{aligned}$$

where we have used the norm convergence of $\sum_{p\in\mathscr{P}} (\omega_{\xi, e_p} \otimes \iota)(W)^* x (\omega_{\xi, e_p} \otimes \iota)(W)$ in the third equality. Therefore, m is a left invariant mean on M. □

Now we are going to prove the implication (1) $\Rightarrow$ (2) of Theorem 3.8 after proving the several lemmas. As usual, $L^\infty(\boldsymbol{R})$ means the von Neumann algebra which consists of essentially bounded measurable functions with respect to the Lebesgue measure.

Lemma 3.11. *Let $m_{\boldsymbol{R}}$ be an invariant mean of $L^\infty(\boldsymbol{R})$. For any ω in M_* define the σ^φ-invariant functional ω' by $\omega'(x) = m_{\boldsymbol{R}}(\{t \mapsto \omega(\sigma_t^\varphi(x))\})$ for x in M. Then ω' is a normal functional with $\|\omega'\| \le \|\omega\|$.*

Proof. For ω in M_* set $f_{\omega,x}(t) = \{t \mapsto \omega(\sigma_t^\varphi(x))\}$ in $C^b(\boldsymbol{R})$. For any finite subset F in I, $|f_{\omega,x}(t) - f_{\omega z_F,x}(t)| \le \|\omega - \omega z_F\|\|x\|$, hence $\|\omega' - (\omega z_F)'\| \le \|\omega - \omega z_F\|$. By the normality of ω, $\lim_F (\omega z_F)' = \omega'$. Notice that $M z_F$ is finite dimensional, so $(\omega z_F)' = (\omega z_F)' z_F$ is a normal functional. □

Note that this averaging procedure can be also done by considering a conditional expectation from M to M_φ. Recall that we have fixed a matrix unit $\{e(\alpha)_{kl}\}_{1\le k,\,l\le n_\alpha}$ of $M z_\alpha$ for each α in I, such that they are diagonalizing $h z_\alpha$ as $h z_\alpha = \sum_{k=1}^{n_\alpha} \nu(\alpha)_k e(\alpha)_{kk}$, where $\nu(\alpha)_k$ denotes a positive real number.

Lemma 3.12. *If (M,Δ) has an invariant mean m, there exists a net of normal states $\{\omega_j\}_{j\in\mathscr{J}}$ on M, which satisfies the following two conditions.*

(1) $\lim_j \|\omega * \omega_j - \omega(1)\omega_j\| = 0$ *for any ω in M_*.*
(2) $\omega_j \circ \sigma_t^\varphi = \omega_j$ *for any $t \in \boldsymbol{R}$.*

Proof. Firstly we show the existence of a net satisfying the first condition. Since the convex hull of the vector states is weak$*$-dense in the state space of M, there exists a net of normal states $\{\chi_j\}_{j\in\mathscr{J}}$ in M_* such that $m = \text{w*-}\lim_j \chi_j$. Let $\mathscr{F}$ be the set of finite subsets of M_*. For $F = \{\omega_1, \omega_2, \dots, \omega_k\} \in \mathscr{F}$, consider the Banach space $(M_*)_F = l_1\text{-}\sum_{\omega\in F} M_*$ and its dual Banach space $M_F = l_\infty\text{-}\sum_{\omega\in F} M$. Set

$$x_F(\chi) = (\omega_1 * \chi - \omega_1(1)\chi,\, \omega_k * \chi - \omega_k(1)\chi,\, \dots,\, \omega_k * \chi - \omega_k(1)\chi),$$

for χ in M_*. Then $x_F(\chi_j)$ converges to 0 weakly. So the norm closure of the convex hull of $\{x_F(\chi_j); j \in \mathscr{J}\}$ contains 0. Hence for any n in $\boldsymbol{N}$, there exists $\chi_{(F,n)}$ such that $\|\omega * \chi_{(F,n)} - \omega(1)\chi_{(F,n)}\| < \frac{1}{n}$ for an ω in F. The new net $\{\chi_{(F,n)}\}_{(F,n)\in\mathscr{F}\times\boldsymbol{N}}$ is a desired one. Next we show existence of a net satisfying the both conditions. Let $\{\omega_j\}_{j\in\mathscr{J}}$ be a net satisfying the first condition. By the previous Lemma, ω_j' is normal. We show that the net $\{\omega_j'\}_{j\in\mathscr{J}}$ satisfies the first condition. For $\omega = \omega_{\Lambda(e(\alpha)_{kl}),\Lambda(e(\alpha)_{mn})}$, we have $\omega \circ \sigma_{-t}^\varphi = \nu_k(\alpha)^{it}\nu_l(\alpha)^{-it}\nu_m(\alpha)^{-it}\nu_n(\alpha)^{it}\omega$. Then we obtain

$$\begin{aligned}
&\left|\omega * \omega_j'(x) - \omega(1)\omega_j'(x)\right| \\
&\quad = \left|m_{\boldsymbol{R}}\left(\left\{t \mapsto (\omega \otimes \omega_j \circ \sigma_t^\varphi)(\Delta(x)) - \omega(1)\omega_j(\sigma_t^\varphi(x))\right\}\right)\right| \\
&\quad = \left|m_{\boldsymbol{R}}\left(\left\{t \mapsto (\omega \circ \sigma_{-t}^\varphi \otimes \omega_j)(\Delta(\sigma_t^\varphi(x))) - \omega(\sigma_{-t}^\varphi(1))\omega_j(\sigma_t^\varphi(x))\right\}\right)\right| \\
&\quad \le \sup_{t\in\boldsymbol{R}} \left\{\left|(\omega \circ \sigma_{-t}^\varphi) * \omega_j(\sigma_t^\varphi(x)) - \omega(\sigma_{-t}^\varphi(1))\omega_j(\sigma_t(x))\right|\right\} \\
&\quad \le \sup_{t\in\boldsymbol{R}} \left\{\left\|(\omega \circ \sigma_{-t}^\varphi) * \omega_j - \omega \circ \sigma_{-t}^\varphi(1)\,\omega_j\right\|\right\}\|x\| \\
&\quad = \sup_{t\in\boldsymbol{R}} \left\{\left\|\nu_k(\alpha)^{it}\nu_l(\alpha)^{-it}\nu_m(\alpha)^{-it}\nu_n(\alpha)^{it}(\omega * \omega_j - \omega(1)\omega_j)\right\|\right\}\|x\| \\
&\quad \le \|\omega * \omega_j - \omega(1)\omega_j\|\|x\|.
\end{aligned}$$

So for this ω, we have $\|\omega*\omega'_j-\omega(1)\omega'_j\| \le \|\omega*\omega_j-\omega(1)\omega_j\|$ and therefore $\|\omega*\omega'_j-\omega(1)\omega'_j\|$ converges to 0. By taking the linear combination for ω, we see that $\|\omega*\omega'_j-\omega(1)\omega'_j\|$ converges to 0 for any normal functional with $\omega=\omega z_\alpha$. Take a normal functional ω and a positive ε. Then there exist a finite subset F of I and j_0 in $\mathscr{J}$ such that $\|\omega z_F-\omega\|<\varepsilon$ and $\|\omega z_F*\omega'_j-\omega z_F(1)\omega'_j\|<\varepsilon$ for $j\ge j_0$. Then we have

$$\begin{aligned}&\left\|\omega*\omega'_j-\omega(1)\omega'_j\right\|\\&\quad\le\left\|(\omega-\omega z_F)*\omega'_j\right\|+\left\|\omega z_F*\omega'_j-\omega z_F(1)\omega'_j\right\|+\left\|(\omega z_F(1)-\omega(1))\omega'_j\right\|\\&\quad<\varepsilon+\varepsilon+\varepsilon=3\varepsilon,\end{aligned}$$

for $j\ge j_0$. Therefore, $\|\omega*\omega'_j-\omega(1)\omega'_j\|$ converges to 0. □

Lemma 3.13. *If (M,Δ) has an invariant mean, there exists a net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in H which satisfies the following four conditions.*

(1) $\lim_j\|\omega*\omega_{\xi_j}-\omega(1)\omega_{\xi_j}\|=0$ *for any ω in M_*.*
(2) $\Delta_\varphi^{it}\xi_j=\xi_j$ *for any t in $\boldsymbol{R}$.*
(3) *For any j in $\mathscr{J}$, there exists a finite subset F_j of I with $z_{F_j}\xi_j=\xi_j$.*
(4) *For any j in $\mathscr{J}$, the vector ξ_j is in the convex cone $\mathscr{P}_\varphi^\natural$.*

Proof. There exists a net of normal state $\{\omega_j\}_{j\in\mathscr{J}}$ which satisfies the two conditions of the previous lemma. If necessary, by cutting and normalizing, we may assume that there exists a finite subset F_j of I such that $\omega_j z_{F_j}=\omega_j$ for any j in $\mathscr{J}$. The von Neumann algebra M is standardly represented, so there exists a unique net of unit vectors $\{\xi_j\}_{j\in\mathscr{J}}$ in $\mathscr{P}_\varphi^\natural$ such that $\omega_j=\omega_{\xi_j}$ and $z_{F_j}\xi_j=\xi_j$. From the assumption $\omega_j=\omega_j\circ\sigma_{-t}^\varphi=\omega_{\Delta_\varphi^{it}\xi_j}$, the uniqueness of ξ_j implies $\Delta_\varphi^{it}\xi_j=\xi_j$. □

Let $\{\xi_j\}_{j\in\mathscr{J}}$ be a net of unit vectors in H in the previous Lemma. Since $z_{F_j}H=\Lambda(Mz_{F_j})$, there exists x_j in M such that $\xi_j=\Lambda(x_j)$. The operator $x_j=x_jz_{F_j}$ is in M_φ and satisfies $\varphi(x_j^*x_j)=1$ and $x_j=x_j^*$. We prepare some notations. For an operator X in $M\otimes M$, $X(\alpha)$ means the operator $X(z_\alpha\otimes1)$ in $Mz_\alpha\otimes M$. An operator Y in $Mz_\alpha\otimes M$ is written as $Y=\sum_{k,l=1}^{n_\alpha}e(\alpha)_{kl}\otimes Y_{kl}$, where $\{Y_{kl}\}_{kl}$ are operators in M.

Lemma 3.14. *Let $x=x^*$ be in $M_\varphi\cap M_{z_F}$, where F is a finite subset of I, and α be in I. Then the following inequality holds.*

$$\begin{aligned}&\left\|\omega_{\hat{J}\Lambda(e(\alpha)_{k1}),\,\hat{J}\Lambda(e(\alpha)_{l1})}*\omega_{\Lambda(x)}-\omega_{\hat{J}\Lambda(e(\alpha)_{k1}),\,\hat{J}\Lambda(e(\alpha)_{l1})}(1)\omega_{\Lambda(x)}\right\|\\&\quad\ge\nu(\alpha)_k^{-\frac12}\nu(\alpha)_l^{\frac12}\varphi(e(\alpha)_{11})\varphi(|X(\alpha)_{kl}|),\end{aligned}$$

where $X=\Delta(x^2)-(1\otimes x^2)$.

Proof. We simply write e_{kl}, ν_k and X_{kl} for $e(\alpha)_{kl}$, $\nu(\alpha)_k$ and $X(\alpha)_{kl}$ respectively. Since $X(\alpha)=\sum_{1\le k,\,l\le n_\alpha}(e_{kl}\otimes X_{kl})$ is in $(M\otimes M)_{\varphi\otimes\varphi}$ and $\sigma_t^\varphi(e_{kl})=\nu_k^{it}\nu_l^{-it}e_{kl}$, we have $\sigma_t^\varphi(X_{kl})=\nu_k^{-it}\nu_l^{it}X_{kl}$. Let $X_{kl}=v_{kl}|X_{kl}|$ be the polar decomposition of X_{kl}. Put $a_{kl}=v_{kl}^*$. Then $\sigma_t^\varphi(a_{kl})=\nu_k^{it}\nu_l^{-it}a_{kl}$. Then we have

$$
\begin{aligned}
&\big(\omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})} * \omega_{\Lambda(x)} - \omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})}(1)\omega_{\Lambda(x)}\big)(a_{kl}) \\
&\quad = \big\langle \Delta(a_{kl})(\hat{J}\Lambda(e_{k1}) \otimes \Lambda(x)) | \hat{J}\Lambda(e_{l1}) \otimes \Lambda(x)\big\rangle - \langle \Lambda(e_{l1}) | \Lambda(e_{k1})\rangle \langle a_{kl}\Lambda(x) | \Lambda(x)\rangle \\
&\quad = \big\langle (1 \otimes a_{kl}) W(\hat{J} \otimes J)(\Lambda(e_{k1}) \otimes \Lambda(x)) | W(\hat{J} \otimes J)(\Lambda(e_{l1}) \otimes \Lambda(x))\big\rangle \\
&\qquad - \langle \Lambda(e_{l1}) | \Lambda(e_{k1})\rangle \langle a_{kl}\Lambda(x) | \Lambda(x)\rangle \\
&\quad = \big\langle (1 \otimes J a_{kl}^* J) W^*(\Lambda(e_{l1}) \otimes \Lambda(x)) | W^*\Lambda(e_{k1}) \otimes \Lambda(x)\rangle \\
&\qquad - \langle \Lambda(e_{l1}) | \Lambda(e_{k1})\rangle \langle J a_{kl}^* J \Lambda(x) | \Lambda(x)\rangle \\
&\quad = \Big\langle (\Lambda \otimes \Lambda)\big(\Delta(x)(e_{l1} \otimes 1)\big(1 \otimes \big(\nu_k^{\frac{1}{2}} \nu_l^{-\frac{1}{2}} a_{kl}\big)\big)\big) | (\Lambda \otimes \Lambda)(\Delta(x)(e_{k1} \otimes 1))\Big\rangle \\
&\qquad - \langle \Lambda(e_{l1}) | \Lambda(e_{k1})\rangle \Big\langle \Lambda\big(x \nu_k^{\frac{1}{2}} \nu_l^{-\frac{1}{2}} a_{kl}\big) | \Lambda(x)\Big\rangle \\
&\quad = \nu_k^{\frac{1}{2}} \nu_l^{-\frac{1}{2}} \big\{ (\varphi \otimes \varphi)\big((e_{1k} \otimes 1)\Delta(x^2)(e_{l1} \otimes 1)(1 \otimes a_{kl})\big) \\
&\qquad\qquad - (\varphi \otimes \varphi)\big((e_{1k} \otimes 1)(1 \otimes x^2)(e_{l1} \otimes 1)(1 \otimes a_{kl})\big)\big\} \\
&\quad = \nu_k^{\frac{1}{2}} \nu_l^{-\frac{1}{2}} (\varphi \otimes \varphi)(e_{11} \otimes X_{kl} a_{kl}) \\
&\quad = \nu_k^{\frac{1}{2}} \nu_l^{-\frac{1}{2}} \varphi(e_{11}) \varphi(X_{kl} a_{kl}) \\
&\quad = \nu_k^{\frac{1}{2}} \nu_l^{-\frac{1}{2}} \nu_k^{-1} \nu_l \varphi(e_{11}) \varphi(a_{kl} X_{kl}) \\
&\quad = \nu_k^{-\frac{1}{2}} \nu_l^{\frac{1}{2}} \varphi(e_{11}) \varphi(|X_{kl}|).
\end{aligned}
$$

Therefore, we obtain

$$
\big\|\omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})} * \omega_{\Lambda(x)} - \omega_{\hat{J}\Lambda(e_{k1}),\,\hat{J}\Lambda(e_{l1})}(1)\omega_{\Lambda(x)}\big\| \geq \nu_k^{-\frac{1}{2}} \nu_l^{\frac{1}{2}} \varphi(e_{11}) \varphi(|X_{kl}|). \qquad \square
$$

Lemma 3.15. *Let N be a von Neumann algebra and θ be an n.s.f. weight on N. Let $\boldsymbol{M}_n(\boldsymbol{C})$ be a matrix algebra with the matrix unit $\{e_{ij}\}_{1 \leq i,\, j \leq n}$ and $\chi = \mathrm{Tr}_h$ be an n.s.f. weight on $\boldsymbol{M}_n(\boldsymbol{C})$ with $h = \sum_{i=1}^n \lambda_i e_{ii}$, $\lambda_i > 0$. If $\{A_j\}_{j \in \mathscr{J}}$ is a net in $(\boldsymbol{M}_n(\boldsymbol{C}) \otimes N)_{\chi \otimes \theta}$ such that $\theta(|(A_j)_{kl}|) < \infty$ and $\lim_j \theta(|(A_j)_{kl}|) = 0$ for any $k, l = 1, 2, \ldots, n$, then $\lim_j \theta(|A_j|_{kl}) = 0$ for any $k, l = 1, 2, \ldots, n$.*

Proof. Let $A_j = V_j |A_j|$ be the polar decomposition in $(\boldsymbol{M}_n(\boldsymbol{C}) \otimes N_\theta)_{\chi \otimes \theta}$. Then for each k, l, we obtain $\sigma_t^\theta((V_j)_{kl}) = \lambda_k^{-it} \lambda_l^{it} (V_j)_{kl}$ and $\sigma_t^\theta((A_j)_{kl}) = \lambda_k^{-it} \lambda_l^{it} (A_j)_{kl}$. Since $|A_j|_{kl} = \sum_{m=1}^n (V_j^*)_{km} (A_j)_{ml}$ and each $(V_j)_{km}$ is analytic, $\theta(|A_j|_{kl})$ is well-defined. Let $(A_j)_{ml} = v_{j,m,l} |(A_j)_{ml}|$ be the polar decomposition. Then we have

$$
\begin{aligned}
|\theta(|A_j|_{kl})| &\leq \sum_{m=1}^n \big|\theta((V_j^*)_{km} (A_j)_{ml})\big| \\
&= \sum_{m=1}^n \Big|\Big\langle \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}}\big) \big| \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}} v_{j,m,l}^* (V_j)_{mk}\big)\Big\rangle\Big|
\end{aligned}
$$

$$= \sum_{m=1}^{n} \left| \left\langle \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}}\big) \middle| J_\theta \sigma^\theta_{\frac{i}{2}}\big(v^*_{j,m,l}(V_j)_{mk}\big)^* J_\theta \Lambda_\theta\big(|(A_j)_{ml}|^{\frac{1}{2}}\big) \right\rangle \right|$$

$$\leq \sum_{m=1}^{n} \left\| \sigma^\theta_{\frac{i}{2}}(v^*_{j,m,l}(V_j)_{mk}) \right\| \theta(|(A_j)_{ml}|)$$

$$= \lambda_k^{-\frac{1}{2}} \lambda_l^{\frac{1}{2}} \sum_{m=1}^{n} \theta(|(A_j)_{ml}|).$$

Hence $\lim_j \theta(|A_j|_{kl}) = 0$. □

LEMMA 3.16. *Let $x = x^*$ be in $M_\varphi \cap M_{z_F}$, where F is a finite subset of I such that $\varphi(x^2) = 1$ and $\Lambda(x)$ is in $\mathscr{P}^\natural_\varphi$. Let α be in I. Then the following inequality holds.*

$$\left\| W^*(\Lambda(e(\alpha)_{k1}) \otimes \Lambda(x)) - \Lambda(e(\alpha)_{k1}) \otimes \Lambda(x) \right\|^2$$
$$\leq 2 \max_{1 \leq k \leq n_\alpha} \left\{ \nu(\alpha)_k^{-1} \nu(\alpha)_1 \right\} \cdot \varphi(z_\alpha)^{\frac{1}{2}} (\varphi \otimes \varphi)(|X(\alpha)|)^{\frac{1}{2}},$$

where $X = \Delta(x^2) - 1 \otimes x^2$.

PROOF. We use the notations in Lemma 3.14. We have

$$\left\| W^*\big(\Lambda(e_{k1}) \otimes \Lambda(x)\big) - \Lambda(e_{k1}) \otimes \Lambda(x) \right\|^2$$
$$= 2\varphi(e_{11}) - 2 \operatorname{Re} \left\langle W^*(\Lambda(e_{k1}) \otimes \Lambda(x)) | \Lambda(e_{k1}) \otimes \Lambda(x) \right\rangle$$
$$= 2\varphi(e_{11}) - 2 \operatorname{Re} (\varphi \otimes \varphi)((e_{1k} \otimes 1)\Delta(x)(e_{k1} \otimes x^*))$$
$$= 2\varphi(e_{11}) - 2 \operatorname{Re} \nu_k^{-1} \nu_1 (\varphi \otimes \varphi)((e_{kk} \otimes 1)(1 \otimes x^*)\Delta(x))$$
$$= 2\nu_k^{-1} \nu_1 \operatorname{Re}\{\varphi(e_{kk}) - (\varphi \otimes \varphi)((e_{kk} \otimes 1)(1 \otimes x^*)\Delta(x))\}$$
$$= 2\nu_k^{-1} \nu_1 \operatorname{Re}\{(\varphi \otimes \varphi)((e_{kk} \otimes 1)(1 \otimes x^*)(1 \otimes x - \Delta(x)))\}$$
$$\leq 2 \max_{1 \leq k \leq n_\alpha} \left\{ \nu_k^{-1} \nu_1 \right\} \cdot \sum_{k=1}^{n_\alpha} \operatorname{Re}\{(\varphi \otimes \varphi)((e_{kk} \otimes x^*)(1 \otimes x - \Delta(x)))\}$$
$$= 2 \max_{1 \leq k \leq n_\alpha} \left\{ \nu_k^{-1} \nu_1 \right\} \cdot \operatorname{Re}\{(\varphi \otimes \varphi)((z_\alpha \otimes x^*)(1 \otimes x - \Delta(x)))\}$$
$$\leq 2 \max_{1 \leq k \leq n_\alpha} \left\{ \nu_k^{-1} \nu_1 \right\} \cdot |(\varphi \otimes \varphi)((z_\alpha \otimes 1)(1 \otimes x^*)(1 \otimes x - \Delta(x)))|$$
$$\leq 2 \max_{1 \leq k \leq n_\alpha} \left\{ \nu_k^{-1} \nu_1 \right\} \cdot (\varphi \otimes \varphi)(z_\alpha \otimes x^* x)^{\frac{1}{2}}$$
$$\cdot (\varphi \otimes \varphi)\big((z_\alpha \otimes 1)(1 \otimes x - \Delta(x))^*(1 \otimes x - \Delta(x))\big)^{\frac{1}{2}}$$
$$= 2 \max_{1 \leq k \leq n_\alpha} \left\{ \nu_k^{-1} \nu_1 \right\} \cdot \varphi(z_\alpha)^{\frac{1}{2}} \left\| (\Lambda \otimes \Lambda)(z_\alpha \otimes x) - (\Lambda \otimes \Lambda)((z_\alpha \otimes 1)\Delta(x)) \right\|.$$

From the assumption: $\Lambda(x) = z_F\Lambda(x) \in \mathscr{P}_\varphi^\natural$, there exists a sequence $\{y_n\}_{n\in\boldsymbol{N}} \in Mz_F$ with $\lim_n y_n J\Lambda(y_n) = \Lambda(x)$. Then we obtain

$$\begin{aligned}(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x)) &= W^*(\Lambda\otimes\Lambda)(z_\alpha\otimes x)\\ &= \lim_n W^*(\Lambda\otimes\Lambda)\big(z_\alpha\otimes y_n\sigma^\varphi_{\frac{i}{2}}(y_n)^*\big)\\ &= \lim_n(\Lambda\otimes\Lambda)\big(\Delta(y_n)\big(\sigma^\varphi_{\frac{i}{2}}\otimes\sigma^\varphi_{\frac{i}{2}}\big)(\Delta(y_n))^*(z_\alpha\otimes 1)\big)\\ &= \lim_n \Delta(y_n)(z_\alpha\otimes 1)(J\otimes J)(\Lambda\otimes\Lambda)(\Delta(y_n)(z_\alpha\otimes 1)).\end{aligned}$$

Therefore, we see $(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x)) \in \mathscr{P}^\natural_{\varphi\otimes\varphi}$. By using the Powers-Størmer inequality (see, for example, [**10**], [**13**]), we obtain

$$\begin{aligned}&\|(\Lambda\otimes\Lambda)(z_\alpha\otimes x) - (\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))\|\\ &\quad\le \|\omega_{(\Lambda\otimes\Lambda)(z_\alpha\otimes x)} - \omega_{(\Lambda\otimes\Lambda)((z_\alpha\otimes 1)\Delta(x))}\|^{\frac{1}{2}}\\ &\quad= (\varphi\otimes\varphi)(|z_\alpha\otimes x^2 - (z_\alpha\otimes 1)\Delta(x^2)|)^{\frac{1}{2}}\\ &\quad= (\varphi\otimes\varphi)(|X(\alpha)|)^{\frac{1}{2}}.\end{aligned}$$ □

Proof of (1) ⇒ (2) of Theorem 3.8. By the assumption, we can pick up a net $\{x_j\}_{j\in\mathscr{J}}$ in M_φ which satisfies the conditions of Lemma 3.13. Now we apply the Lemma 3.14 to this net for fixed $\alpha\in I$, then $\varphi(|X_j(\alpha)_{kl}|)$ converges to 0 for any $k, l = 1, 2, \ldots, n_\alpha$, where $X_j = \Delta(x_j^2) - 1\otimes x_j^2$. By Lemma 3.15, it implies that $\varphi(|X_j(\alpha)|_{kl})$ converges to 0 for any $k, l = 1, 2, \ldots, n_\alpha$. Since we have

$$(\varphi\otimes\varphi)(|X_j(\alpha)|) = \sum_{1\le k,\, l\le n_\alpha}\varphi(e(\alpha)_{kl})\varphi(|X_j(\alpha)|_{kl}),$$

we see $(\varphi\otimes\varphi)(|X_j(\alpha)|)$ converges to 0. By Lemma 3.16, we see $\|W^*(\Lambda(e(\alpha)_{k1})\otimes\Lambda(x_j)) - \Lambda(e(\alpha)_{k1})\otimes\Lambda(x_j)\|$ converges to 0 for any $k = 1, 2, \ldots, n_\alpha$. Then we have

$$\begin{aligned}&\big\|W^*(\Lambda(e(\alpha)_{kl})\otimes\Lambda(x_j)) - \Lambda(e(\alpha)_{kl})\otimes\Lambda(x_j)\big\|\\ &\quad= \big\|\big(J\sigma^\varphi_{\frac{i}{2}}(e(\alpha)_{1l})^*J\otimes 1\big)\big(W^*(\Lambda(e(\alpha)_{k1})\otimes\Lambda(x_j)) - \Lambda(e(\alpha)_{k1})\otimes\Lambda(x_j)\big)\big\|\\ &\quad\le \big\|\big(J\sigma^\varphi_{\frac{i}{2}}(e(\alpha)_{1l})^*J\otimes 1\big)\big\|\big\|W^*(\Lambda(e(\alpha)_{k1})\otimes\Lambda(x_j)) - \Lambda(e(\alpha)_{k1})\otimes\Lambda(x_j)\big\|.\end{aligned}$$

This implies that $\|W^*(\Lambda(e(\alpha)_{kl})\otimes\Lambda(x_j)) - \Lambda(e(\alpha)_{kl})\otimes\Lambda(x_j)\|$ converges to 0 for any $k, l = 1, 2, \ldots, n_\alpha$. By taking a linear combination, $\|W^*(z_\alpha\eta\otimes\Lambda(x_j)) - z_\alpha\eta\otimes\Lambda(x_j)\|$ converges to 0 for any vector $\eta\in H$. Take a vector $\eta\in H$. For any $\varepsilon > 0$, there exists a finite subset F of I such that $\|\sum_{\alpha\in F} z_\alpha\eta - \eta\| < \varepsilon$. By the above arguments, we can take j_0 in $\mathscr{J}$ such that $\sum_{\alpha\in F}\|W^*(z_\alpha\eta\otimes\Lambda(x_j)) - z_\alpha\eta\otimes\Lambda(x_j)\| < \varepsilon$ for $j\ge j_0$. Then we have

$$
\begin{aligned}
&\big\|W^*(\eta\otimes\Lambda(x_j))-\eta\otimes\Lambda(x_j)\big\|\\
&=\bigg\|W^*\bigg(\bigg(\eta-\sum_{\alpha\in F}z_\alpha\eta\bigg)\otimes\Lambda(x_j)\bigg)-\bigg(\eta-\sum_{\alpha\in F}z_\alpha\eta\bigg)\otimes\Lambda(x_j)\\
&\qquad+W^*\bigg(\sum_{\alpha\in F}z_\alpha\eta\otimes\Lambda(x_j)\bigg)-\sum_{\alpha\in F}z_\alpha\eta\otimes\Lambda(x_j)\bigg\|\\
&\le 2\bigg\|\sum_{\alpha\in F}z_\alpha\eta-\eta\bigg\|+\sum_{\alpha\in F}\big\|W^*(z_\alpha\eta\otimes\Lambda(x_j))-z_\alpha\eta\otimes\Lambda(x_j)\big\|\\
&<2\varepsilon+\varepsilon=3\varepsilon,
\end{aligned}
$$

for $j\geq j_0$. Therefore, $\|W^*\eta\otimes\Lambda(x_j)-\eta\otimes\Lambda(x_j)\|$ converges to 0 for any vector $\eta\in H$. This completes the proof of Theorem 3.8. □

Upon ending this paper, we mention a part of Ruan's Theorem [**14**, Theorem4.5] as a corollary of Theorem 3.8 for Kac algebras, i.e. the invariant weight $\hat{\varphi}$ of $(\hat{M},\hat{\Delta})$ is a normal tracial state. In this case, φ is also a trace ([**9**]).

Corollary 3.17. *Let (M,Δ) be a discrete Kac algebra. Then the following statements are equivalent.*

(1) *It has an invariant mean.*
(2) *It is strongly Voiculescu amenable.*
(3) *The C^*-algebra $\hat{A}$ is nuclear.*
(4) *The von Neumann algebra $\hat{M}$ is injective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). This has been already proved in Theorem 3.8.

(3) $\Rightarrow$ (4). It is trivial.

(4) $\Rightarrow$ (1). Let E be a conditional expectation from $B(H)$ onto $\hat{M}$. Note that $\hat{\varphi}$ is a normal trace on $\hat{M}$. Take a complete orthonormal system $\{e_p\}_{p\in\mathscr{P}}$. Then for any operator $x\in M$ and for any vector $\xi\in H$, we have

$$
\begin{aligned}
\omega_\xi*m(x)&=\hat{\varphi}\bigg(E\bigg(\sum_{p\in\mathscr{P}}(\omega_{\xi,\,e_p}\otimes\iota)(W)^*x(\omega_{\xi,\,e_p}\otimes\iota)(W)\bigg)\bigg)\\
&=\sum_{p\in\mathscr{P}}\hat{\varphi}\big((\omega_{\xi,\,e_p}\otimes\iota)(W)^*E(x)(\omega_{\xi,\,e_p}\otimes\iota)(W)\big)\\
&=\sum_{p\in\mathscr{P}}\hat{\varphi}\big((\omega_{\xi,\,e_p}\otimes\iota)(W)(\omega_{\xi,\,e_p}\otimes\iota)(W)^*E(x)\big)\\
&=\sum_{p\in\mathscr{P}}\hat{\varphi}\big((\omega_{\hat{J}\xi,\,\hat{J}e_p}\otimes\iota)(W)^*(\omega_{\hat{J}\xi,\,\hat{J}e_p}\otimes\iota)(W)E(x)\big)\\
&=\omega_{\hat{J}\xi}(1)\hat{\varphi}(E(x))\\
&=\omega_\xi(1)m(x).
\end{aligned}
$$

Therefore, m is a left invariant mean on M. □

As we have seen, nuclearity of a compact Kac algebra leads amenability of the dual discrete Kac algebra, however, it is now open whether it holds in the case of a compact quantum group or not.

References

[1] F. F. Bonsall and J. Duncan, Complete Normed Algebras, Ergeb. Math. Grenzgeb., Band 80, Springer, New York-Heidelberg, 1973, pp. x+301.
[2] E. Bédos, R. Conti and L. Tuset, On amenability and co-amenability of algebraic quantum groups and their corepresentations, Canad. J. Math., **57** (2005), 17–60.
[3] E. Bédos, G. Murphy and L. Tuset, Co-amenability of compact quantum groups, J. Geom. Phys., **40** (2001), 130–153.
[4] E. Bédos, G. Murphy and L. Tuset, Amenability and coamenability of algebraic quantum groups, Int. J. Math. Math. Sci., **31** (2002), 577–601.
[5] E. Bédos, G. Murphy and L. Tuset, Amenability and coamenability of algebraic quantum groups II, J. Funct. Anal., **201** (2003), 303–340.
[6] P. Desmedt, J. Quaegebeur and S. Vaes, Amenability and the bicrossed product construction, Illinois J. Math., **46** (2002), 1259–1277.
[7] E.-G. Effros and Z.-J. Ruan, Discrete quantum groups. I. The Haar measure, Internat. J. Math., **5** (1994), 681–723.
[8] M. Enock and J.-M. Schwartz, Algèbres de Kac moyennables, [Amenable Kac algebras], Pacific J. Math., **125** (1986), 363–379.
[9] M. Enock and J.-M. Schwartz, Kac Algebras and Duality of Locally Compact Groups, Springer, 1992.
[10] U. Haagerup, The standard form of von Neumann algebras, Math. Scand., **37** (1975), 271–283.
[11] J. Kustermans and S. Vaes, Locally compact quantum groups, Ann. Sci. Ecole Norm. Sup. (4), **33** (2000), 837–934.
[12] S.-H. Kye, Notes on operator algebras, Lecture Notes Ser., **7**. Seoul National Univ., Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993, pp. ii+179.
[13] R. T. Powers and E. Størmer, Free states of the canonical anticommutation relations, Comm. Math. Phys., **16** (1970), 1–33.
[14] Z.-J. Ruan, Amenability of Hopf von Neumann algebras and Kac algebras, J. Funct. Anal., **139** (1996), 466–499.
[15] A. Van Daele, Discrete quantum groups, J. Algebra, **180** (1996), 431–444.
[16] S. L. Woronowicz, Compact quantum groups, In: Symetries quantiques, (Les Houches, 1995), North-Holland, Amsterdam, 1998, 845–884.

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