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Relations between principal functions of *p*-hyponormal operators

Dedicated to Professor Sin-Ei Takahasi on his sixtieth birthday

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Abstract. Let $T = U|T|$ be a bounded linear operator with the associated polar decomposition on a separable infinite dimensional Hilbert space. For $0 < t < 1$, let $T_t = |T|^t U|T|^{1-t}$ and g_T and g_{T_t} be the principal functions of T and T_t, respectively. We show that, if T is an invertible semi-hyponormal operator with trace class commutator $[|T|, U]$, then $g_T = g_{T_t}$ almost everywhere on $\boldsymbol{C}$. As a biproduct we reprove Berger's theorem and index properties of invertible p-hyponormal operators.

1. Introduction.

An operator below means a bounded linear operator on a separable infinite dimensional Hilbert space $\mathscr{H}$ and $\mathscr{C}_1$ stands for the trace class operators on $\mathscr{H}$. The commutator of two operators A, B is denoted by $[A,B] = AB - BA$. In the analysis of an operator T with self-commutator $[T^*, T] \in \mathscr{C}_1$, the principal function g of T is one of the most important tools. It is known that the principal function g gives much information about the structure of T (see, for example, [**5**], [**9**], [**11**], [**14**], [**17**]). The construction of g depends on the Cartesian decomposition $T = X + iY$. There exists another approach to the principal function g_T related to the polar decomposition $T = U|T|$ (see, for example, [**5**], [**8**], [**15**], [**17**]). In the theory of operator inequalities, the generalized Aluthge transformations $T_t = |T|^t U|T|^{1-t}$ $(0 < t < 1)$ are proved to be very useful. It is a natural problem to consider relations between the functions g, g_T and g_{T_t}. In this article we discuss this problem.

An operator T is called p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ (see [**1**]). If $p = 1$ and $1/2$, then T is called hyponormal and semi-hyponormal, respectively. An invertible operator T is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ (see [**2**], [**16**]). If T is hyponormal, then the principal function $g(x,y)$ is obtained from the Cartesian decomposition $T = X + iY$ (see, for example, [**17**]). If T is semi-hyponormal, then the principal function $g_T(e^{i\theta}, r)$ is obtained from the polar decomposition $T = U|T|$ (see, for example, [**17**]). By Löwner-Heinz inequality, if $0 < q \leq p \leq 1$ and T is p-hyponormal, then T is q-hyponormal. We often use the following result: For $0 < p \leq 1/2$ and $0 < t < 1$, if $T = U|T|$ is p-hyponormal, then $T_t = |T|^t U|T|^{1-t}$ is q-hyponormal, where $q = p + \min\{t, 1-t\}$ ([**10**, §3.4.1, Theorem 2]).

Following [**17**], we introduced the principal functions for p-hyponormal operators and log-hyponormal operators ([**6**]). In this paper, we show that, if T is an invertible semi-hyponormal operator with trace class commutator $[|T|, U] \in \mathscr{C}_1$, then $g_T = g_{T_t}$ almost everywhere on $\boldsymbol{C}$. Moreover, we show that, if $T = U|T|$ is hyponormal with unitary U, then $g = g_T$ almost everywhere

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on $\boldsymbol{C}$. As applications of this result, we extend Berger's theorem in the case of p-hyponormal operators. Throughout this paper, t will satisfy $0 < t < 1$.

2. Relations with principal functions associated with polar decompositions.

We denote by $\mathscr{A}$ the linear space of all Laurent polynomials $\mathscr{P}(r,z)$ with polynomial coefficients such that $\mathscr{P}(r,z) = \sum_{k=-N}^{N} p_k(r)z^k$, where N is a non-negative integer and $p_k(r)$ is a polynomial. For $T = U|T|$ with unitary operator U, put $\mathscr{P}(|T|,U) = \sum_{k=-N}^{N} p_k(|T|)U^k$. We denote by $J(\phi,\psi)$ the Jacobian of functions $\phi(r,z)$, $\psi(r,z)$ defined on $\boldsymbol{R}\times\boldsymbol{C}$, that is,

$$J(\phi,\psi)(r,e^{i\theta}) = \phi_r(r,e^{i\theta})\cdot\psi_z(r,e^{i\theta}) - \phi_z(r,e^{i\theta})\cdot\psi_r(r,e^{i\theta}).$$

THEOREM A ([**17**, Chapter 7, Theorem 3.3], [**8**, Theorem 9]). *Let an operator $T = U|T|$ be semi-hyponormal with unitary U. Assume $[|T|,U] \in \mathscr{C}_1$. Then there exists a summable function g_T such that, for $\mathscr{P}(r,z), \mathscr{Q}(r,z) \in \mathscr{A}$,*

$$\mathrm{Tr}([\mathscr{P}(|T|,U),\mathscr{Q}(|T|,U)]) = \frac{1}{2\pi}\iint J(\mathscr{P},\mathscr{Q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)\,dr\,d\theta.$$

DEFINITION 1. The function g_T in Theorem A is called the *principal function* of T. Let $T = U|T|$ be a p-hyponormal operator with unitary U such that $[|T|^{2p},U] \in \mathscr{C}_1$. Put $S = U|T|^{2p}$. Then S is semi-hyponormal. By Theorem A, there exists the principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta},r) = g_S(e^{i\theta},r^{1/(2p)})$$

(see [**8**, Definition 3]).

We begin with a well-known important property of commutators (see, for example, [**17**, Chapter 7, 1.2 and 3.1]).

LEMMA 1. *If operators A,B,C satisfy $[A,C],[B,C] \in \mathscr{C}_1$, then we have $[AB,C] \in \mathscr{C}_1$.*

Let $\|A\|_1 = \mathrm{Tr}(|A|)$ for $A \in \mathscr{C}_1$, that is, $\|A\|_1$ is the trace norm of A.

THEOREM 2. *If a positive invertible operator A and an operator D satisfy $[A,D] \in \mathscr{C}_1$, then, for any real number α, we have*

$$[A^{\alpha},D] \in \mathscr{C}_1.$$

PROOF. We use the following expansion known as the binomial series: For z ($|z| < 1$),

$$(1+z)^{\alpha} = \sum_{m=0}^{\infty}\binom{\alpha}{m}z^m,$$

where $\binom{\alpha}{m} = (\alpha(\alpha-1)\cdots(\alpha-m+1))/(m!)$.

Considering $\|\beta A\| < 1$ with some positive number β, we may assume that $\|A\| < 1$. Since A is an invertible positive operator and $\|A\| < 1$, we have $\|A - I\| < 1$ and

$$A^{\alpha} = (I + (A - I))^{\alpha} = \lim_{n\to\infty} \sum_{m=0}^{n} \binom{\alpha}{m} (A - I)^m. \tag{1}$$

Let $A_n = [\sum_{m=0}^{n} \binom{\alpha}{m} (A - I)^m, D]$ for $n = 1, 2, 3, \cdots$. Then $\lim_{n\to\infty} A_n = [A^{\alpha}, D]$ with respect to the operator norm. By [**11**, p. 158 (3.3)], for a positive integer m, it holds that

$$\|[(A - I)^m, D]\|_1 \le m\|A - I\|^{m-1} \|[(A - I), D]\|_1,$$

so that, for n,

$$\|A_n\|_1 \le \left(\sum_{m=1}^{n} \left| \binom{\alpha}{m} \right| m\|A - I\|^{m-1} \right) \|[A, D]\|_1.$$

Since $\|A - I\| < 1$, (1) converges absolutely and hence

$$\left(\sum_{m=1}^{\infty} \left| \binom{\alpha}{m} \right| m\|A - I\|^{m-1} \right) < \infty.$$

Therefore, $\{A_n\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_1$. Let B denote the limit of the sequence $\{A_n\}$ in $\mathscr{C}_1$. For any unit vector $\xi \in \mathscr{H}$, we define an operator C on $\mathscr{H}$ by $C\eta = (\eta, \xi)\xi$ $(\eta \in \mathscr{H})$. Let $\{e_j\}$ be a complete orthonormal basis of $\mathscr{H}$ such that $e_1 = \xi$. Since $\mathrm{Tr}(SC) = \sum_{j=1}^{\infty} (SCe_j, e_j) = (S\xi, \xi)$, then

$$(B\xi, \xi) = \mathrm{Tr}(BC) = \lim_{n\to\infty} \mathrm{Tr}(A_n C) = \lim_{n\to\infty} (A_n \xi, \xi) = ([A^{\alpha}, D]\xi, \xi).$$

Since ξ is an arbitrary vector, it follows that

$$[A^{\alpha}, D] = B \in \mathscr{C}_1. \qquad \square$$

THEOREM 3. *Let $T = U|T|$ be an invertible operator. Put $T_t = |T|^t U |T|^{1-t}$. If $[|T|, U] \in \mathscr{C}_1$, then $T_t^* T_t - T_t T_t^* \in \mathscr{C}_1$.*

PROOF. Since $|T|$ is invertible, by Theorem 2 we have, for any $\alpha > 0$,

$$[|T|^{\alpha}, U] \in \mathscr{C}_1,$$

so that

$$U^*|T|^{2t} - |T|^{2t} U^* \in \mathscr{C}_1 \text{ and } U|T|^{2(1-t)} U^* - |T|^{2(1-t)} \in \mathscr{C}_1.$$

Therefore,

$$\begin{aligned} T_t^*T_t - T_tT_t^* &= (|T|^{1-t}U^*|T|^{2t}U|T|^{1-t} - |T|^2) - (|T|^tU|T|^{2(1-t)}U^*|T|^t - |T|^2) \\ &= |T|^{1-t}(U^*|T|^{2t} - |T|^{2t}U^*)U|T|^{1-t} - |T|^t(U|T|^{2(1-t)}U^* - |T|^{2(1-t)})|T|^t \\ &\in \mathscr{C}_1. \end{aligned}$$

□

THEOREM 4. *Let $T = U|T|$ be an invertible operator and $T_t = |T|^tU|T|^{1-t}$. For the polar decomposition $T_t = V|T_t|$ of T_t, if $[|T|,U] \in \mathscr{C}_1$, then, for every real number α,*

$$[|T_t|^\alpha, V] \in \mathscr{C}_1.$$

PROOF. Since $T_t = V|T_t|$ is the polar decomposition, by Theorem 3 we have

$$|T_t|^2 - V|T_t|^2V^* = T_t^*T_t - T_tT_t^* \in \mathscr{C}_1.$$

Since T is invertible, so is T_t. Hence the operator $|T_t|$ is invertible and V is unitary. Therefore, by the above we have

$$[|T_t|^2, V] = (|T_t|^2 - V|T_t|^2V^*)V \in \mathscr{C}_1.$$

Since $|T_t|$ is an invertible positive operator, by Theorem 2 we obtain, for every real number α,

$$[|T_t|^\alpha, V] \in \mathscr{C}_1.$$

□

THEOREM 5. *Let $T = U|T|$ be an invertible operator and $T_t = |T|^tU|T|^{1-t}$. For the polar decomposition $T_t = V|T_t|$ of T_t, if $[|T|,U] \in \mathscr{C}_1$, then, for a positive integer n, it holds that*

$$\mathrm{Tr}([U^n|T|^n, |T|^2]) = \mathrm{Tr}([V^n|T_t|^n, |T_t|^2]),$$
$$\mathrm{Tr}([U^{-n}|T|^n, |T|^2]) = \mathrm{Tr}([V^{-n}|T_t|^n, |T_t|^2])$$

and

$$\mathrm{Tr}([U^*|T|, U|T|]) = \mathrm{Tr}([V^*|T_t|, V|T_t|]).$$

PROOF. If operators A,B,C,D and E satisfy $[ABCD,E] \in \mathscr{C}_1$, $[ACBD,E] \in \mathscr{C}_1$ and $[B,C] \in \mathscr{C}_1$, then we have

$$\mathrm{Tr}([ABCD,E]) = \mathrm{Tr}([ACBD,E]). \tag{2}$$

By Theorem 4, we get $[|T_t|,V] \in \mathscr{C}_1$. Then Lemma 1 yields that $[(V|T_t|)^mV^{n-m}|T_t|^{n-m}, |T_t|^2] \in \mathscr{C}_1$ and $[V^m, |T_t|] \in \mathscr{C}_1$ $(m = 0, 1, \cdots, n)$. And by equality (2) it follows that

$$\begin{aligned}\mathrm{Tr}([V^n|T_t|^n,|T_t|^2]) &= \mathrm{Tr}([VV^{n-1}|T_t||T_t|^{n-1},|T_t|^2]) \\ &= \mathrm{Tr}([V|T_t|V^{n-1}|T_t|^{n-1},|T_t|^2]) \\ &= \mathrm{Tr}([T_tV^{n-1}|T_t|^{n-1},|T_t|^2]) \\ &\vdots \\ &= \mathrm{Tr}([T_t^n,|T_t|^2]).\end{aligned}$$

Since $[|T|^s,U] \in \mathscr{C}_1$ for $s > 0$, similarly we have

$$\begin{aligned}\mathrm{Tr}([T_t^n,|T_t|^2]) &= \mathrm{Tr}([|T|^tU|T|U\cdots U|T|^{1-t},|T|^{1-t}(U^*|T|^{2t})U|T|^{1-t}]) \\ &= \mathrm{Tr}([|T|^tU|T|U\cdots U|T|^{1-t},|T|^{1-t}(|T|^{2t}U^*)U|T|^{1-t}]) \\ &= \mathrm{Tr}([(|T|^tU)|T|U\cdots U|T|^{1-t},|T|^2]) \\ &= \mathrm{Tr}([(U|T|^t)|T|U\cdots U|T|^{1-t},|T|^2]) \\ &\vdots \\ &= \mathrm{Tr}([U|T|U\cdots U|T|^t|T|^{1-t},|T|^2]) \\ &= \mathrm{Tr}([U^n|T|^n,|T|^2]).\end{aligned}$$

Therefore, we obtain

$$\mathrm{Tr}([U^n|T|^n,|T|^2]) = \mathrm{Tr}([V^n|T_t|^n,|T_t|^2]).$$

We also have

$$\mathrm{Tr}([V^{-n}|T_t|^n,|T_t|^2]) = \mathrm{Tr}([|T_t|^nV^{-n},|T_t|^2]) = \mathrm{Tr}([|T_t|^2,V^n|T_t|^n]^*).$$

By the above result, we get

$$\mathrm{Tr}([|T_t|^2,V^n|T_t|^n]^*) = \mathrm{Tr}([|T|^2,U^n|T|^n]^*),$$

so that

$$\mathrm{Tr}([|T|^2,U^n|T|^n]^*) = \mathrm{Tr}([U^{-n}|T|^n,|T|^2]).$$

Hence, we obtain

$$\mathrm{Tr}([V^{-n}|T_t|^n,|T_t|^2]) = \mathrm{Tr}([U^{-n}|T|^n,|T|^2]).$$

Similarly, we have

$$\mathrm{Tr}([U^*|T|,U|T|]) = \mathrm{Tr}([|T|U^*,U|T|]) = \mathrm{Tr}([T^*,T])$$

and

$$\begin{aligned}\mathrm{Tr}([V^*|T_t|,V|T_t|]) &= \mathrm{Tr}([|T_t|V^*,V|T_t|]) = \mathrm{Tr}([T_t^*,T_t]) \\ &= \mathrm{Tr}([|T|^{1-t}U^*|T|^t,|T|^tU|T|^{1-t}]) = \mathrm{Tr}([|T|U^*,U|T|]) \\ &= \mathrm{Tr}([T^*,T]). \end{aligned}\qquad \square$$

Let $T = U|T|$ be an invertible p-hyponormal operator such that $[|T|,U] \in \mathscr{C}_1$. Let $T_t = |T|^tU|T|^{1-t}$ and $T_t = V|T_t|$ be the polar decomposition. If $0 < p \leq 1/2$, put $q = p + \min\{t, 1-t\}$; if $1/2 < p \leq 1$, put $q = 1/2 + \min\{t, 1-t\}$. By [**10**, §3.4.1, Theorem 2], T_t is q-hyponormal. By Theorem 2 we have $[|T_t|^{2q},V] \in \mathscr{C}_1$. Hence, by Definition 1 there exists the principal function g_{T_t} of T_t. If T_t is q-hyponormal, then T_t is s-hyponormal $(0 < s < q)$. Hence we can consider the principal function of T_t with respect to s-hyponormality. The trace formula implies the uniqueness of the principal function of $T_t = V|T_t|$ (cf. [**14**, Chapter X, §3]).

THEOREM 6. *Let $T = U|T|$ be an invertible semi-hyponormal operator such that $[|T|,U] \in \mathscr{C}_1$. For $T_t = |T|^tU|T|^{1-t}$, let g_T and g_{T_t} be the principal functions of T and T_t, respectively. Then we have*

$$g_T = g_{T_t}$$

almost everywhere on $\boldsymbol{C}$.

PROOF. Let $T_t = V|T_t|$ be the polar decomposition of T_t. For a non-zero integer n, let $p_n(r,z) = r^2$, $q_n(r,z) = z^nr^{|n|}$, $p_0(r,z) = z^{-1}r$ and $q_0(r,z) = zr$. Then by Theorem 5 we have

$$\mathrm{Tr}([p_n(|T|,U),q_n(|T|,U)]) = \mathrm{Tr}([|T|^2,U^n|T|^{|n|}]) = \mathrm{Tr}([p_n(|T_t|,V),q_n(|T_t|,V)])$$

and

$$\mathrm{Tr}([p_0(|T|,U),q_0(|T|,U)]) = \mathrm{Tr}([U^*|T|,U|T|]) = \mathrm{Tr}([p_0(|T_t|,V),q_0(|T_t|,V)]).$$

By Theorem A, we have

$$\mathrm{Tr}([p_n(|T|,U),q_n(|T|,U)]) = \frac{1}{2\pi}\iint 2ne^{i(n-1)\theta}r^{|n|+1}e^{i\theta}g_T(e^{i\theta},r)drd\theta$$

and

$$\mathrm{Tr}([p_0(|T|,U),q_0(|T|,U)]) = \frac{1}{2\pi}\iint 2re^{-i\theta}e^{i\theta}g_T(e^{i\theta},r)drd\theta.$$

Since T_t is invertible, we can choose a positive integer m such that T_t is $1/(2m)$-hyponormal and $[|T_t|^{1/m},V] \in C$ by Theorem 2. By [**8**, Theorem 10], we have

$$\mathrm{Tr}([p_n(|T_t|,V),q_n(|T_t|,V)]) = \frac{1}{2\pi}\iint 2ne^{i(n-1)\theta}r^{|n|+1}e^{i\theta}g_{T_t}(e^{i\theta},r)drd\theta$$

and

$$\mathrm{Tr}([p_0(|T_t|,V),q_0(|T_t|,V)]) = \frac{1}{2\pi}\iint 2re^{-i\theta}e^{i\theta}g_{T_t}(e^{i\theta},r)drd\theta,$$

so that

$$\iint r^{|n|}(e^{i\theta})^n rg_T(e^{i\theta},r)drd\theta = \iint r^{|n|}(e^{i\theta})^n rg_{T_t}(e^{i\theta},r)drd\theta$$

and

$$\iint rg_T(e^{i\theta},r)drd\theta = \iint rg_{T_t}(e^{i\theta},r)drd\theta.$$

Since n is arbitrary, we obtain $g_T = g_{T_t}$ almost everywhere on $\boldsymbol{C}$. □

Next we recall the principal functions for log-hyponormal operators.

DEFINITION 2. Let $T = U|T|$ be log-hyponormal with $\log|T| \geq 0$ such that $[\log|T|,U] \in \mathscr{C}_1$. Put $S = U\log|T|$. Then S is semi-hyponormal with unitary U. Hence there exists the principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta},r) = g_S(e^{i\theta},\log r)$$

(see [**6**, Definition 4]).

It is known that, if $T = U|T|$ is log-hyponormal, then the Aluthge transform $T_{1/2} = |T|^{1/2}U|T|^{1/2}$ is semi-hyponormal (see [**16**]). Hence there exists the principal function $g_{T_{1/2}}$ of $T_{1/2}$.

THEOREM 7. *Let $T = U|T|$ be a* log-*hyponormal operator such that* $\log|T| \geq 0$ *and* $[\log|T|,U] \in \mathscr{C}_1$. *For $T_{1/2} = |T|^{1/2}U|T|^{1/2} = V|T_{1/2}|$, let g_T and $g_{T_{1/2}}$ be the principal functions of T and $T_{1/2}$, respectively. Then we have*

$$g_T = g_{T_{1/2}}$$

almost everywhere on $\boldsymbol{C}$.

PROOF. Since $\|[U,|T|]\|_1 = \|[U,e^{\log|T|}]\|_1 \leq \|[U,\log|T|]\|_1 e^{\|\log|T|\|}$, we have $[U,|T|] \in \mathscr{C}_1$. Hence we have

$$\begin{aligned}|T_{1/2}|^2 - V|T_{1/2}|^2V^* &= T_{1/2}^*T_{1/2} - T_{1/2}T_{1/2}^* = |T|^{1/2}U^*|T|U|T|^{1/2} - |T|^{1/2}U|T|U^*|T|^{1/2} \\ &= |T|^{1/2}(U^*|T|U - U|T|U^*)|T|^{1/2} = |T|^{1/2}\big(U^*[|T|,U] + [|T|,U]U^*\big)|T|^{1/2} \\ &\in \mathscr{C}_1.\end{aligned}$$

Therefore, by Theorem 2, we have

$$|T_{1/2}| - V|T_{1/2}|V^* \in \mathscr{C}_1.$$

By [**10**, §3.4.2, Theorem 2], since $T_{1/2}$ is semi-hyponormal, there exists the principal function $g_{T_{1/2}}$ of $T_{1/2}$. For a non-zero integer n, let $p_n(r,z) = r^2$, $q_n(r,z) = z^n r^{|n|}$, $p_0(r,z) = z^{-1}r$ and $q_0(r,z) = zr$. By the same argument of the proof of Theorem 5, we obtain, for an integer m,

$$\mathrm{Tr}([p_m(|T|,U), q_m(|T|,U)]) = \mathrm{Tr}([p_m(|T_{1/2}|,V), q_m(|T_{1/2}|,V)]).$$

By [**6**, Theorem 8], we have

$$\mathrm{Tr}([p_n(|T|,U), q_n(|T|,U)]) = \frac{1}{2\pi}\iint 2ne^{i(n-1)\theta} r^{|n|+1} e^{i\theta} g_T(e^{i\theta}, r)\, dr\, d\theta$$

and

$$\mathrm{Tr}([p_0(|T|,U), q_0(|T|,U)]) = \frac{1}{2\pi}\iint 2r g_T(e^{i\theta}, r)\, dr\, d\theta.$$

Since $T_{1/2}$ is semi-hyponormal, by Theorem A we have

$$\mathrm{Tr}([p_n(|T_{1/2}|,V), q_n(|T_{1/2}|,V)]) = \frac{1}{2\pi}\iint 2ne^{i(n-1)\theta} r^{|n|+1} e^{i\theta} g_{T_{1/2}}(e^{i\theta}, r)\, dr\, d\theta$$

and

$$\mathrm{Tr}([p_0(|T_{1/2}|,V), q_0(|T_{1/2}|,V)]) = \frac{1}{2\pi}\iint 2r g_{T_{1/2}}(e^{i\theta}, r)\, dr\, d\theta,$$

so that

$$\iint r^{|m|}(e^{i\theta})^m r g_T(e^{i\theta}, r)\, dr\, d\theta = \iint r^{|m|}(e^{i\theta})^m r g_{T_{1/2}}(e^{i\theta}, r)\, dr\, d\theta$$

for any integer m. Since m is arbitrary, this implies $g_T = g_{T_{1/2}}$ almost everywhere on $\boldsymbol{C}$. □

Now we generalize Theorem 6 as follows.

THEOREM 8. *Let $T = U|T|$ be an invertible p-hyponormal operator such that $[|T|,U] \in \mathscr{C}_1$. For $T_t = |T|^t U|T|^{1-t}$, let g_T and g_{T_t} be the principal functions of T and T_t, respectively. Then we have $g_T = g_{T_t}$ almost everywhere on $\boldsymbol{C}$.*

PROOF. Let $T_t = V|T_t|$ be the polar decomposition of T_t. For a non-zero integer n, let $p_n(r,z) = r^2$, $q_n(r,z) = z^n r^{|n|}$, $p_0(r,z) = z^{-1}r$ and $q_0(r,z) = zr$. Then by Theorem 5 we have for any integer m,

$$\mathrm{Tr}([p_m(|T|,W), q_m(|T|,W)]) = \mathrm{Tr}([p_m(|T_t|,V), q_m(|T_t|,V)]).$$

By Theorem 2, we have $[|T|^{2p}, U] \in \mathscr{C}_1$. By [**10**, §3.4.1. Theorem 2], T_t is q-hyponormal $(0 < q \leq 1)$. We choose a positive integer m such that $1/2m < q$. Then T_t is $1/2m$-hyponormal. By Theorem 4, we have $[|T_t|^{1/m}, V] \in \mathscr{C}_1$. It follows from [**8**, Theorem 10] that

$$\mathrm{Tr}([p_n(|T|,W), q_n(|T|,W)]) = \frac{1}{2\pi}\iint 2ne^{i(n-1)\theta} r^{|n|+1} e^{i\theta} g_T(e^{i\theta}, r)\, dr\, d\theta,$$

$$\mathrm{Tr}([p_n(|T_t|,V), q_n(|T_t|,V)]) = \frac{1}{2\pi}\iint 2ne^{i(n-1)\theta} r^{|n|+1} e^{i\theta} g_{T_t}(e^{i\theta}, r)\, dr\, d\theta,$$

$$\mathrm{Tr}([p_0(|T|,W), q_0(|T|,W)]) = \frac{1}{2\pi}\iint 2r g_T(e^{i\theta}, r)\, dr\, d\theta,$$

and

$$\mathrm{Tr}([p_0(|T_t|,V), q_0(|T_t|,V)]) = \frac{1}{2\pi}\iint 2r g_{T_t}(e^{i\theta}, r)\, dr\, d\theta.$$

Hence, we have, for any integer m,

$$\iint r^{|m|}(e^{i\theta})^m r g_T(e^{i\theta}, r)\, dr\, d\theta = \iint r^{|m|}(e^{i\theta})^m r g_{T_t}(e^{i\theta}, r)\, dr\, d\theta.$$

Since m is arbitrary, we obtain $g_T = g_{T_t}$ almost everywhere on $\boldsymbol{C}$. □

3. Relation with principal functions associated with two decompositions.

Next, we show the following theorem (cf. [**5**, Theorem 7.1]).

THEOREM 9. *Let $T = X + iY = U|T|$ be hyponormal with unitary U. Suppose that $[|T|, U] \in \mathscr{C}_1$. Let g and g_T be the principal functions corresponding to the Cartesian and the polar decompositions of T, respectively. For $x + iy = re^{i\theta}$, let $g_T(x,y) = g_T(e^{i\theta}, r)$. Then $g = g_T$ almost everywhere on $\boldsymbol{C}$.*

PROOF. Since $[|T|, U] \in \mathscr{C}_1$, by Lemma 1 we have $[|T|^2, U] \in \mathscr{C}_1$. Hence

$$T^*T - TT^* = |T|^2 - U|T|^2U^* = [|T|^2, U]U^* \in \mathscr{C}_1.$$

Theorem A yields that, for a polynomial $q(x,y) = y$ and an arbitrary polynomial $p(x,y)$,

$$\begin{aligned}\mathrm{Tr}([p(X,Y), q(X,Y)]) &= \frac{1}{2\pi i}\iint_{\sigma(T)} J(p,q) g(x,y)\, dx\, dy \\ &= \frac{1}{2\pi i}\iint_{\sigma(T)} p_x(x,y) g(x,y)\, dx\, dy \\ &= \frac{1}{2\pi i}\iint_{\mathrm{M}} p_x(r\cos\theta, r\sin\theta) g(r\cos\theta, r\sin\theta) r\, dr\, d\theta,\end{aligned}$$

where $\mathrm{M} = \{(r,\theta) : re^{i\theta} \in \sigma(T),\ 0 \le \theta < 2\pi\}$. On the other hand, we have

$$\mathrm{Tr}([p(X,Y),q(X,Y)]) = \mathrm{Tr}\left(\left[p\left(\frac{U|T|+|T|U^{-1}}{2}, \frac{U|T|-|T|U^{-1}}{2i}\right), \frac{U|T|-|T|U^{-1}}{2i}\right]\right).$$

Let

$$\tilde{p}(r,z) = p\left(\frac{zr+rz^{-1}}{2}, \frac{zr-rz^{-1}}{2i}\right) \text{ and } \tilde{q}(r,z) = \frac{zr-rz^{-1}}{2i}.$$

Then we get

$$\begin{aligned} J(\tilde{p},\tilde{q}) = &\left(p_x \cdot \frac{z+z^{-1}}{2} + p_y \cdot \frac{z-z^{-1}}{2i}\right)\left(\frac{r}{2i}\left(1+\frac{1}{z^2}\right)\right) \\ &- \frac{r}{2}\left\{p_x \cdot \left(1-\frac{1}{z^2}\right) + \frac{1}{i}p_y \cdot \left(1+\frac{1}{z^2}\right)\right\}\frac{z-z^{-1}}{2i}. \end{aligned}$$

Therefore,

$$\begin{aligned} J(\tilde{p},\tilde{q})(r,e^{i\theta}) \cdot e^{i\theta} &= (p_x \cdot \cos\theta + p_y \cdot \sin\theta)(-ir\cos\theta) - r(ip_x \cdot \sin\theta - ip_y \cdot \cos\theta)\sin\theta \\ &= -irp_x. \end{aligned}$$

Hence we have

$$\begin{aligned} &\mathrm{Tr}\left(\left[p\left(\frac{U|T|+|T|U^{-1}}{2}, \frac{U|T|-|T|U^{-1}}{2i}\right), \frac{U|T|-|T|U^{-1}}{2i}\right]\right) \\ &= \frac{1}{2\pi}\iint_{\mathrm{M}} J(\tilde{p},\tilde{q})(r,e^{i\theta})e^{i\theta}g_T(e^{i\theta},r)drd\theta \\ &= \frac{1}{2\pi}\iint_{\mathrm{M}} -irp_x(r\cos\theta,\, r\sin\theta)g_T(e^{i\theta},r)drd\theta, \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{2\pi i}\iint_{\mathrm{M}} p_x(r\cos\theta,\, r\sin\theta)g(r\cos\theta,\, r\sin\theta)rdrd\theta \\ &\quad = \frac{1}{2\pi}\iint_{\mathrm{M}} -irp_x(r\cos\theta,\, r\sin\theta)g_T(e^{i\theta},r)drd\theta. \end{aligned}$$

Since p is arbitrary, we obtain

$$rg_T(e^{i\theta},r) = rg(r\cos\theta,\, r\sin\theta) \qquad \text{a.e.}$$

Hence, $g_T = g$ almost everywhere on $\boldsymbol{C}$. □

Though the results above can be generalized to operators with trace-class self-commutator, we confine ourselves to deal only with the p-hyponormal case (cf. [**5**]).

4. Application: Berger's Theorem and index.

In this section, we apply previous results to Berger's Theorem [**3**] and an index property [**11**]. First we show the following:

LEMMA 10. *Let operators $S = V|S|$ and $T = U|T|$ be invertible. Assume that $|S| - |S^*|$, $|T| - |T^*| \in \mathscr{C}_1$ and there exists a trace class operator A such that $SA = AT$ and* $\ker(A) = \ker(A^*) = \{0\}$. *Then, for $S_{1/2} = |S|^{1/2}V|S|^{1/2}$ and $T_{1/2} = |T|^{1/2}U|T|^{1/2}$, there exists $B \in \mathscr{C}_1$ such that $S_{1/2}B = BT_{1/2}$ and* $\ker(B) = \ker(B^*) = \{0\}$.

PROOF. Let $B = |S|^{1/2}A|T|^{-1/2}$. Then it is clear that $B \in \mathscr{C}_1$ and $\ker(B) = \ker(B^*) = \{0\}$. Since $S_{1/2} = |S|^{1/2}V|S|^{1/2}$ and $T_{1/2} = |T|^{1/2}U|T|^{1/2}$, we have

$$\begin{aligned} S_{1/2}B = S_{1/2}|S|^{1/2}A|T|^{-1/2} &= |S|^{1/2}SA|T|^{-1/2} = |S|^{1/2}AT|T|^{-1/2} \\ &= |S|^{1/2}A|T|^{-1/2}T_{1/2} = BT_{1/2}. \end{aligned}$$ □

THEOREM 11. *Let S and T be invertible semi-hyponormal operators. Assume that $|S| - |S^*|, |T| - |T^*| \in \mathscr{C}_1$ and there exists a trace class operator A such that $SA = AT$ and* $\ker(A) = \ker(A^*) = \{0\}$. *Then $g_S \le g_T$ almost everywhere on $\boldsymbol{C}$.*

PROOF. Let g and h be the principal functions of $S_{1/2}$ and $T_{1/2}$ related to the Cartesian decompositions of $S_{1/2}$ and $T_{1/2}$, respectively. Since $S_{1/2}$ and $T_{1/2}$ are invertible hyponormal operators with trace class self-commutators, by Lemma 10 and Theorem 2 of [**3**] (or Theorem X.4.3 of [**14**]), we have $g \le h$ almost everywhere on $\boldsymbol{C}$. Moreover, Theorem 9 implies

$$g = g_{S_{1/2}} \quad \text{and} \quad h = g_{T_{1/2}} \quad \text{almost everywhere on } \boldsymbol{C}.$$

And Theorem 6 implies $g_S = g_{S_{1/2}}$ and $g_T = g_{T_{1/2}}$ almost everywhere on $\boldsymbol{C}$, so that, $g_S \le g_T$ almost everywhere on $\boldsymbol{C}$. □

COROLLARY 12. *Let S and T be invertible p-hyponormal or* log*-hyponormal operators. Assume that $|S| - |S^*|, |T| - |T^*| \in \mathscr{C}_1$ and there exists a trace class operator A such that $SA = AT$ and* $\ker(A) = \ker(A^*) = \{0\}$. *Then $g_S \le g_T$ almost everywhere on $\boldsymbol{C}$.*

PROOF. Since $S_{1/2}$ and $T_{1/2}$ are invertible semi-hyponormal operators, by Theorem 11 we have $g_{S_{1/2}} \le g_{T_{1/2}}$ almost everywhere on $\boldsymbol{C}$. Moreover, Theorems 7 or 8 imply $g_{S_{1/2}} = g_S$ and $g_{T_{1/2}} = g_T$ almost everywhere on $\boldsymbol{C}$. □

THEOREM 13. *Let $T = U|T|$ be an invertible cyclic p-hyponormal operator. Assume $[|T|, U] \in \mathscr{C}_1$. Then*

$$g_T \le 1 \quad \textit{almost everywhere on } \boldsymbol{C}.$$

PROOF. If $T = U|T|$ has a cyclic vector, then $T_{1/2}$ also has a cyclic vector by Lemma 3 of [**7**]. Hence, let $p \geq 1/2$. Since then $T_{1/2}$ is a cyclic hyponormal operator, by Corollary 4.4 of [**14**] we have $g_{T_{1/2}} \leq 1$. By Theorem 8, it follows that $g_T \leq 1$. Similarly, the theorem holds for $0 \leq p \leq 1/2$. □

Let $\mathbf{Rat}(\sigma)$ be the set of all rational functions with poles off σ.

DEFINITION 3. The *rational multiplicity* of $T \in B(\mathscr{H})$ is the smallest cardinal number m with the property which there exists a set $\{x_n\}_{n=1}^m$ of m-vectors in $\mathscr{H}$ such that

$$\bigvee\{\, f(T)x_i \,;\ f \in \mathbf{Rat}(\sigma(T)),\ 1 \leq i \leq m \,\} = \mathscr{H}.$$

THEOREM 14. *Let $T = U|T|$ be an invertible cyclic p-hyponormal operator with finite rational cyclic multiplicity m. Assume $[|T|, U] \in \mathscr{C}_1$. Then*

$$g_T \leq m \quad \textit{almost everywhere on } \boldsymbol{C}.$$

PROOF. Let $T_{1/2} = |T|^{1/2}U|T|^{1/2} = V|T_{1/2}|$ (the polar decomposition of $T_{1/2}$). Since, by Theorem 4, we have $[|T_{1/2}|, V] \in \mathscr{C}_1$, first we show that $T_{1/2}$ has an operator with finite rational cyclic multiplicity m. It is easy to see that $p(T_{1/2})|T|^{1/2} = |T|^{1/2}p(T)$ for every polynomial p and $\sigma(T) = \sigma(T_{1/2})$. If $\{x_1, \cdots, x_m\}$ is a system of vectors such that $\bigvee\{\, f(T)x_i \,;\ f \in \mathbf{Rat}(\sigma(T)),\ 1 \leq i \leq m\,\} = \mathscr{H}$, then $\{|T|^{1/2}x_1, \cdots, |T|^{1/2}x_m\}$ is a system of vectors such that $\bigvee\{\, f(T_{1/2})|T|^{1/2}x_i \,;\ f \in \mathbf{Rat}(\sigma(T_{1/2})),\ 1 \leq i \leq m\,\} = \mathscr{H}$. Hence, the operator $T_{1/2}$ has a finite rational cyclic multiplicity m. If $p \geq 1/2$, then $T_{1/2}$ is a hyponormal operator with finite rational cyclic multiplicity m. Hence by Theorem 8 and Proposition 4.6 of [**14**] we have $g_T = g_{T_{1/2}} \leq m$ almost everywhere on $\boldsymbol{C}$. Similarly, the theorem holds for $0 < p \leq 1/2$. □

Finally, we show index properties. Let $\sigma_e(T)$ be the essential spectrum of T and $\mathrm{ind}(T)$ the index of T; i.e.,

$$\mathrm{ind}(T) = \dim\ker(T) - \dim\ker(T^*).$$

Then it is known the following result. Let T be a pure hyponormal operator and $g(z)$ be the principal function of T. Then it holds that, for $z \notin \sigma_e(T)$,

$$g(z) = -\mathrm{ind}(T - z)$$

[**11**, Theorem] (see also [**4**, Theorem 4]).

An operator T is called *pure* if it has no nontrivial reducing subspace on which it is normal. Then we need the following

LEMMA B (Lemma 4 of [**7**]). *For an operator $T = U|T|$, let $T_{1/2} = |T|^{1/2}U|T|^{1/2}$. Assume that T is an invertible p-hyponormal operator. If T is pure, then $T_{1/2}$ is also pure.*

THEOREM 15. *Let $T = U|T|$ be a pure invertible semi-hyponormal operator. If $0 \neq z \notin \sigma_e(T)$, then $g_T(e^{i\theta}, r) = -\mathrm{ind}(T - z)$, where $z = re^{i\theta}$.*

PROOF. Let $T_{1/2} = |T|^{1/2}U|T|^{1/2}$. Then, by Lemma B, $T_{1/2}$ is a pure invertible hyponormal operator. Since $\sigma_e(T_{1/2}) = \sigma_e(T)$ by Theorem 1.5 of [**12**], we have $z \notin \sigma_e(T_{1/2})$. Hence

$$g_{T_{1/2}}(z) = -\text{ind}(T_{1/2} - z).$$

Theorem 1.10 of [**13**] implies that

$$\text{ind}(T_{1/2} - z) = \text{ind}(T - z).$$

By Theorem 6, we have $g_T = g_{T_{1/2}}$. Therefore, we obtain that $g_T(e^{i\theta}, r) = -\text{ind}(T - z)$. □

COROLLARY 16. *Let $T = U|T|$ be a pure invertible p-hyponormal operator $(0 < p < 1/2)$. If $0 \neq z \notin \sigma_e(T)$, then $g_T(e^{i\theta}, r) = -\text{ind}(T - z)$, where $z = re^{i\theta}$.*

PROOF. By Lemma B, $T_{1/2} = |T|^{1/2}U|T|^{1/2}$ is a pure invertible semi-hyponormal operator. Hence a similar argument of the proof of Theorem 15 gives a proof of Corollary 16. □

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