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The initial value problem for the 1-D semilinear Schrödinger equation in Besov spaces

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Abstract. We define a class of Besov type spaces which is a generalization of that defined by Kenig-Ponce-Vega ([**4**], [**5**]) in their study on KdV equation and nonlinear Schrödinger equation. Using these spaces, we prove the following results: the 1-dimendional semilinear Schrödinger equation with the nonlinear term $c_1u^2 + c_2\bar{u}^2$ has a unique local-in-time solution for the initial data $\in B_{2,1}^{-3/4}$, and that with $cu\bar{u}$ has a unique local-in-time solution for the initial data $\in B_{2,1}^{-1/4,\#}$. Note that $B_{2,1}^{-1/4,\#}(\boldsymbol{R}) \supset B_{2,1}^{-1/4}(\boldsymbol{R}) \supset H^s(\boldsymbol{R})$ for any $s > -1/4$.

1. Introduction and definition of Besov-type norms.

Kenig, Ponce and Vega ([**5**]) reported an excellent result which states that there exists a unique local-in-time solution to the semilinear Schrödinger equation

$$\partial_t u = i\partial_x^2 u + N(u, \bar{u}), \quad x, t \in \boldsymbol{R}, \tag{1.1}$$

with the initial value $u(x,0) = u_0(x) \in H^s(\boldsymbol{R})$, $s > -3/4$, where $N(u,\bar{u}) = u^2$ or $N(u,\bar{u}) = \bar{u}^2$. The main tool in [**5**] is the bilinear estimate

$$\|c_1 fg + c_2 \bar{f}\bar{g}\|_{X_{s,b-1}} \leq C\|f\|_{X_{s,b}}\|g\|_{X_{s,b}}, \tag{1.2}$$

where the space $X_{s,b}$ is defined by the norm (introduced by Bourgain [**3**])

$$\|f\|_{X_{s,b}} = \|(1+|\xi|)^s(1+|\tau-\xi^2|)^b\hat{f}(\xi,\tau)\|_{L^2(\boldsymbol{R}^2)}, \tag{1.3}$$

and $\hat{f}$ denotes the Fourier transform of f. They proved that (1.2) holds for any $s > -3/4$ and some $b > 1/2$, and fails for any $s < -3/4$ and any $b \in \boldsymbol{R}$. Nakanishi-Takaoka-Tsutsumi [**8**] showed that (1.2) fails also when $s = -3/4$.

From these results we see that the Sobolev version of the Fourier restriction norm does not work in the critical case $s = -3/4$, and experience suggests us some kind of its Besov version would work. But we could not prove the estimate (1.2) with $X_{s,b}$ replaced by $B_{2,1,-\xi^2}^{(-3/4,1/2)}$ (see the definition below). The fact that b should be $1/2$ is also indicated by the calculations in [**5**]. In estimating the norm of fg there appear terms analogous to the sum

$$\sum_{j=0}^{\infty}(j+1)2^{sj}\|f_j\|_{L^2}, \tag{1.4}$$

where $\hat{f}_j(\xi) = \varphi_j(|\xi|)\hat{f}(\xi)$ and $\{\varphi_j\}_{j=0,1,\dots}$ is a set of C^∞-functions satisfying

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$$(1.5)\qquad \begin{cases} \sum_{j=0}^{\infty}\varphi_j(z)=1,\ \operatorname{supp}\varphi_0\subset\{z;|z|<2\},\ \operatorname{supp}\varphi_1\subset\{z;1<|z|<4\},\\ \varphi_k(z)=\varphi_1(2^{-k+1}z)\ \text{ for }\ k\ge 1, \varphi_j(z)=\varphi_j(-z)\ge 0,\end{cases}$$

namely, the sequence which gives the Littlewood-Payley decomposition. The quantity (1.4) is equivalent with the norm of f in $B_{2,1}^{\rho}$ with $\rho(z)=z^s\log(2+z)$ (a special case of Besov spaces with 'function' parameter which is a good generalization of the usual ones, see [**7**]). Thus, we are forced to use Besov type spaces of 'function' order differentiability. Namely,

DEFINITION 1. For a weight ρ on $\boldsymbol{R}_+$, $b\in\boldsymbol{R}$, $1\le p\le\infty$, $1\le q\le\infty$ and a real-valued C^∞-function $P(\xi)$ the space $B_{p,q,P}^{(\rho,b)}(\boldsymbol{R}^{d+1})$ is the space of tempered distributions f such that the norm

$$(1.6)\qquad \|f\|_{B_{p,q,P}^{(\rho,b)}}:=\|\{\rho(2^j)2^{bk}\|f_{jk,P}(x,t)\|_{L^p(\boldsymbol{R}^{d+1})}\}\|_{\ell^q(\bar{\boldsymbol{N}}\times\bar{\boldsymbol{N}})}$$

is finite. The space is written by $B_{p,q,P}^{(s,b)}(\boldsymbol{R}^{d+1})$ when $\rho(z)=z^s$. Here, $\bar{\boldsymbol{N}}:=\boldsymbol{N}\cup\{0\}$,

$$(1.7)\qquad \hat{f}_{jk,P}(\xi,\tau)=\varphi_j(|\xi|)\varphi_k(\tau-P(\xi))\hat{f}(\xi,\tau),$$

and $\{\varphi_j\}_{j=0,1,\dots}$ is a sequence of C^∞-functions satisfying (1.5).

We omit the subscript P when $P=0$.

$B_{p,q}^{\rho}(\boldsymbol{R}^d)$ is the space of tempered distributions f such that the norm

$$(1.8)\qquad \|f\|_{B_{p,q}^{\rho}}:=\|\{\rho(2^j)\|f_j(x)\|_{L^p(\boldsymbol{R}^d)}\}\|_{\ell^q(\bar{\boldsymbol{N}})}$$

is finite, where $\hat{f}_j(\xi):=\varphi_j(\xi)\hat{f}(\xi)$. ($B_{p,q}^{s}(\boldsymbol{R}^d)=B_{p,q}^{\rho}(\boldsymbol{R}^d)$ with $\rho(z)=z^s$.)

$B_{p,q,P}^{(\rho,b)}(\boldsymbol{R}^{d+1})$ is a generalization of the space of the Fourier restriction norm $X_{s,b}$, since $X_{s,b}=B_{2,2,\xi^2}^{(s,b)}(\boldsymbol{R}^2)$, and by making use of this norm we can prove that

$$\|c_1fg+c_2\bar{f}\bar{g}\|_{B_{2,1,-\xi^2}^{(\rho,-1/2)}}\le C\|f\|_{B_{2,1,-\xi^2}^{(\rho,1/2)}}\|g\|_{B_{2,1,-\xi^2}^{(s,1/2)}}$$

(Theorem 2.3), which leads to, with the help of the usual successive approximation method, the existence of solutions to (1.1) with the initial condition $u(0,x)=u_0\in B_{2,1}^{\rho}(\boldsymbol{R})$ in the case $N(u,\bar{u})=c_1u^2+c_2\bar{u}^2$. But, since the 0-th approximation solution $e^{it\partial^2}u_0$ ($e^{it\partial^2}$ denotes the Schrödinger group) belongs to the space $B_{2,1,-\xi^2}^{(s,b)}$ for any b if $u_0\in B_{2,1}^{s}(\boldsymbol{R})$ and the estimate

$$\|c_1fg+c_2\bar{f}\bar{g}\|_{B_{2,1,-\xi^2}^{(\rho,-1/2)}}\le C\{\|f\|_{B_{2,1,-\xi^2}^{(s,b)}}\|g\|_{B_{2,1,-\xi^2}^{(s,1/2)}}+\|f\|_{B_{2,1,-\xi^2}^{(s,1/2)}}\|g\|_{B_{2,1,-\xi^2}^{(s,b)}}\}$$

holds for any $b>1/2$, by solving the equation with respect to $v:=u-e^{it\partial^2}u_0$ we can construct solutions $u=e^{it\partial^2}u_0+v$, $v\in B_{2,1,-\xi^2}^{(\rho,1/2)}$ for any initial data $u_0\in B_{2,1}^{s}(\boldsymbol{R})$.

In improving Kenig-Ponce-Vega's results for the case where the nonlinearity $N(u,\bar{u})=u\bar{u}$ there arises another difficulty which force us to use the slightly complicated space $B_{2,1,-\xi^2}^{(s,1/2),\#}$ which is defined as follows:

DEFINITION 2. For $s,b\in\boldsymbol{R}$, $1\le q\le\infty$ and a real-valued C^∞-function $P(\xi)$ the space $B_{2,q,P}^{(s,b),\#}(\boldsymbol{R}^{d+1})$ is the completion of $\mathscr{S}(\boldsymbol{R}^{d+1})$ with the norm

$$(1.9)\quad \|f\|_{B_{2,q,P}^{(s,b),\#}}:=\|\{2^{bk}\|f_{0k,P}^{\#}(x,t)\|_{L^2(\boldsymbol{R}^{d+1})}\}\|_{\ell^q(\bar{\boldsymbol{N}})}+\|\{2^{sj+bk}\|f_{jk,P}(x,t)\|_{L^2(\boldsymbol{R}^{d+1})}\}\|_{\ell^q(\boldsymbol{N}\times\bar{\boldsymbol{N}})}$$

where $\hat{f}_{0k,P}^{\#}(\xi,\tau)=\varphi_0(|\xi|)(1+|\log|\xi||)^{-2}\varphi_k(\tau-P(\xi))\hat{f}(\xi,\tau)$.

We omit the subscript P when $P = 0$.

The space $B^{s,\#}_{2,q}(\boldsymbol{R}^d)$ is the completion of $\mathscr{S}(\boldsymbol{R}^d)$ with the norm

$$\|u\|_{B^{s,\#}_{2,q}(\boldsymbol{R}^d)} := \|u_0^{\#}\|_{L^2(\boldsymbol{R}^d)} + \|\{2^{sj}\|u_j\|_{L^2(\boldsymbol{R}^d)}\}\|_{\ell^q(\boldsymbol{N})} \tag{1.10}$$

where $\hat{u}_0^{\#}(\xi) = \varphi_0(|\xi|)(1 + |\log|\xi||)^{-2}\hat{u}(\xi)$, $\hat{u}_j(\xi) = \varphi_j(|\xi|)\hat{u}(\xi)$.

To prove the uniqueness, according to Bekiranov-Ogawa-Ponce [**2**] pp. 380–382, we must use the 'localized' norm: that is,

Definition 3. For a function space $X(\boldsymbol{R}^{d+1})$ and an open set Ω in $\boldsymbol{R}^{d+1}$ the space $X(\Omega)$ is the set of all distributions f which have an extension $\tilde{f} \in X(\boldsymbol{R}^{d+1})$, and its norm is defined by

$$\|f\|_{X(\Omega)} := \inf\{\|\tilde{f}\|_{X(\boldsymbol{R}^{d+1})}; f = \tilde{f}|_{\Omega}\}. \tag{1.11}$$

This paper is arranged as follows: The definition of the Besov type spaces which we use is given in §1. Main Theorem together with key estimates is stated in §2. §3 is concerned with the definition of bilinear operators with kernel K and the properties of their norm. In §4 we give the estimates of norm of special bilinear operators which are essential tools in our proof of Theorem 2.3 Part (I) in §5 and that of Part (II) in §6. Proof of Theorem 2.4 is given in §7 and that of Theorem 2.5 is given in §8. Main Theorem is proved in the last section. Proof of Theorem 2.1 and Theorem 2.2 is given in Appendix.

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2. Main results.

Basic properties of Besov type spaces defined in §1 are stated as follows:

Theorem 2.1. *Assume that there exists a real number* $v \geq 1$ *such that*

$$|\partial_\xi^\alpha P(\xi)| \leq c_\alpha(1 + |\xi|)^{v-|\alpha|} \text{ holds for any } \alpha \text{ with some } c_\alpha,$$

and $\rho(z) \leq cz^\sigma$ *for some* σ *and* $c > 0$. *Then,* $B^{(\rho,b)}_{p,q,P}(\boldsymbol{R}^d \times \boldsymbol{R})$ *and* $B^{(\rho,b),\#}_{2,q,P}(\boldsymbol{R}^d \times \boldsymbol{R})$ *are Banach spaces, and* $\mathscr{S}(\boldsymbol{R}^{d+1})$ *is dense in these spaces if* p *and* q *are finite.*

The fact that the initial value makes sense in our spaces is a consequence of the following

Theorem 2.2 (Imbedding theorem). $B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^{d+1})$ (*or* $B^{(s,1/2),\#}_{2,1,P}(\boldsymbol{R}^{d+1})$) *is continuously imbedded in the space of* $B^{\rho}_{2,1}(\boldsymbol{R}^d)$*-valued* (*or* $B^{s,\#}_{2,1}(\boldsymbol{R}^d)$*-valued*) *bounded continuous functions in* $t \in \boldsymbol{R}$.

Our bilinear estimates which correspond to (1.2) are stated as follows:

Theorem 2.3. *Let* $P(\xi) = \pm\xi^2$.

(I) *Let* $\rho(t) = \log(2 + t)t^s$ *and assume that* $-3/4 \leq s < 0$. *Then,*

$$\|c_1 fg + c_2 \bar{f}\bar{g}\|_{B^{(\rho,-1/2)}_{2,1,P}(\boldsymbol{R}^2)} \leq C\|f\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^2)}\|g\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}^2)} \tag{2.1}$$

holds for any $f, g \in B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}^2)$, *and*

$$\|c_1 fg + c_2 \bar{f}\bar{g}\|_{B_{2,1,P}^{(\rho,-1/2)}(\boldsymbol{R}^2)} \le C\{\|f\|_{B_{2,1,P}^{(s,b)}(\boldsymbol{R}^2)}\|g\|_{B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}^2)} + \|f\|_{B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}^2)}\|g\|_{B_{2,1,P}^{(s,b)}(\boldsymbol{R}^2)}\} \tag{2.2}$$

holds for any $f, g \in B_{2,1,P}^{(s,b)}(\boldsymbol{R}^2)$ *if* $b > 1/2$.

(II) *Let* $\rho(z) = \log(2+z)z^s$, $s = -1/4$. *Then,*

$$\|f\bar{g}\|_{B_{2,1,P}^{(\rho,-1/2)}(\boldsymbol{R}^2)} \le C\|f\|_{B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}^2)}\|g\|_{B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}^2)} \tag{2.3}$$

holds for any $f, g \in B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}^2)$, *and*

$$\|f\bar{g}\|_{B_{2,1,P}^{(s,-1/2),\#}(\boldsymbol{R}^2)} \le C\|f\|_{B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}^2)}\|g\|_{B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}^2)} \tag{2.4}$$

holds for any $f, g \in B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}^2)$. *Here* C *is a constant independent of* f *and* g.

As usual, in constructing solutions to (1.1) with the initial data u_0 we use the integral formulation, namely

$$u(x) = W(t)u_0(x) + \int_0^t W(t-t')N(u,\bar{u})(x,t')\,dt', \tag{2.5}$$

where $W(t)$ is defined by $\{W(t)f\}(x,t) := \mathscr{F}_x^{-1}e^{itP(\xi)}\mathscr{F}_x f(x,t)$ ($\mathscr{F}_x$ denotes the Fourier transform with respect to x), and so the key to prove existence of solutions is the following:

THEOREM 2.4. *Let* $I = (-a,a)$, $a > 0$, $P(\xi) = \pm|\xi|^2$.

(I) *Let* $\rho(t) = t^s\log(2+t)$, $s \ge -3/4$, *and define*

$$B(f,g) := \int_0^t W(t-t')\{c_1 f(x,t')g(x,t') + c_2\overline{f(x,t')g(x,t')}\}\,dt'. \tag{2.6}$$

Then

$$\|B(f,g)\|_{B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}\times I)} \le C\|f\|_{B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}\times I)}\|g\|_{B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}\times I)} \tag{2.7}$$

holds for any $f \in B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}\times I)$, $g \in B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}\times I)$, *and*

$$\|B(f,g)\|_{B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}\times I)} \le C\{\|f\|_{B_{2,1,P}^{(s,b)}(\boldsymbol{R}\times I)}\|g\|_{B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}\times I)} + \|f\|_{B_{2,1,P}^{(s,1/2)}(\boldsymbol{R}\times I)}\|g\|_{B_{2,1,P}^{(s,b)}(\boldsymbol{R}\times I)}\} \tag{2.8}$$

holds for any $f, g \in B_{2,1,P}^{(s,b)}(\boldsymbol{R}\times I)$ *if* $1/2 < b < 1$.

(II) *Let* $s \ge -1/4$. *Then*

$$\left\|\int_0^t W(t-t')f(x,t')\overline{g(x,t')}\,dt'\right\|_{B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}\times I)} \le C\|f\|_{B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}\times I)}\|g\|_{B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}\times I)}. \tag{2.9}$$

Here C *is a constant independent of* a, f *and* g.

Recall that the definition of the spaces $B_{2,1,P}^{(\rho,1/2)}(\boldsymbol{R}\times I)$ and $B_{2,1,P}^{(s,1/2),\#}(\boldsymbol{R}\times I)$ is given in Definition 3 combined with Definitions 1, 2. These estimates are obtained from our bilinear estimates with the aid of some properties of the Besov type norms stated in Theorem 7.1.

Our proof of the uniqueness relies on the following

Theorem 2.5. *Let* $1/2 \le b < 1$, $I = (-a, a)$, $a > 0$, $f \in B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d \times I)$ *and assume that* $f(x,0) = 0$. *Then* $\|f|_{\boldsymbol{R}^d \times(-\delta,\delta)}\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d \times(-\delta,\delta))} \to 0$ *as* $\delta \to +0$.

The same fact holds for the space $B^{(\rho,b),\#}_{2,1,P}$.

Thus, we can obtain the following conclusion:

Main theorem

(I) *If* $N(u,\bar u) = c_1 u^2 + c_2 \bar u^2$ *and* $u_0 \in B^{-3/4}_{2,1}(\boldsymbol{R})$, *then there exist* $T = T(\|u_0\|_{B^{-3/4}_{2,1}(\boldsymbol{R})}) > 0$ *and a unique solution* $u(x,t)$ *to* (1.1) *in* $\boldsymbol{R} \times I_T$ *with* $u(x,0) = u_0(x)$ *of the form*

$$u(x,t) = W(t)u_0(x) + v(x,t), \quad v \in B^{(\rho,1/2)}_{2,1,-|\xi|^2}(\boldsymbol{R} \times I_T). \tag{2.10}$$

(II) *If* $N(u,\bar u) = c_3 u\bar u$ *and* $u_0 \in B^{-1/4,\#}_{2,1}(\boldsymbol{R})$, *then there exist* $T = T(\|u_0\|_{B^{-1/4,\#}_{2,1}(\boldsymbol{R})}) > 0$ *and a unique solution* $u(x,t) \in B^{(-1/4,1/2),\#}_{2,1,-|\xi|^2}(\boldsymbol{R} \times I_T)$ *to* (1.1) *in* $\boldsymbol{R} \times I_T$ *with* $u(x,0) = u_0(x)$.

Here $I_T := (-T, T)$.

3. Norm of integral operators.

The following lemma is a convenient tool in estimating the norm of integral operators:

Lemma 3.1. *Let* (Ω_j, μ_j), $j = 1, 2$, *be σ-finite measure spaces*, $1 \le p \le q \le \infty$, X *and* Y *be Banach spaces and let* $K(x,y)$ *be a strongly measurable* $\mathscr{L}(X,Y)$*-valued function on* $(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$. *Assume that there exist non-negative measurable functions* $H_1(x,y)$ *and* $H_2(x,y)$ *such that*

$$\|K(x,y)\|_{\mathscr{L}(X,Y)} \le H_1(x,y)H_2(x,y), \tag{3.1}$$

$$\operatorname*{ess.\,sup}_{y\in\Omega_2} \|H_1(x,y)\|_{L^q(\Omega_1,\mu_1)} = C_1 < \infty, \tag{3.2}$$

$$\operatorname*{ess.\,sup}_{x\in\Omega_1} \|H_2(x,y)\|_{L^{p'}(\Omega_2,\mu_2)} = C_2 < \infty. \tag{3.3}$$

Here $1/p + 1/p' = 1$ ($p' = \infty$ *if* $p = 1$ *and* $p' = 1$ *if* $p = \infty$), *and* $\mathscr{L}(X,Y)$ *denotes the space of bounded linear operators from* X *into* Y. *Define the operator* T *by*

$$Tf(x) = \int_{\Omega_2} K(x,y)f(y)\,d\mu_2(y). \tag{3.4}$$

Then T *is a bounded operator from* $L^p(\Omega_2,\mu_2;X)$ *into* $L^q(\Omega_1,\mu_1;Y)$ *and* $\|T\| \le C_1C_2$.

In particular, $\|K * f\|_{L^p} \le \|K\|_{L^1} \cdot \|f\|_{L^p}$ *holds for any* $K \in L^1(\boldsymbol{R}^d)$ *and* $f \in L^p(\boldsymbol{R}^d)$.

For the proof of this lemma see [**1**, Theorem 6.3, p. 239] or [**7**, p. 38].

Definition 4. For a measurable function $K(x,y)$ defined on $\boldsymbol{R}^d \times \boldsymbol{R}^d$ the bilinear operator $B(K; f, g)$ is defined by

$$B(K;f,g)(x) := \int K(x,y)f(y)g(x-y)\,dy, \tag{3.5}$$

and its norm as an operator from $L^2 \times L^2$ into L^2 is denoted by $N_{bl}(K)$.

LEMMA 3.2. *For a measurable function* $K(x,y)$ *on* $\boldsymbol{R}^d \times \boldsymbol{R}^d$ *we have* $N_{bl}(K) \le C = \min\{C_1, C_2\}$, *where* $C_1 = \operatorname{ess.sup}_{y \in \boldsymbol{R}^d} (\int |K(x,y)|^2 \, dx)^{1/2}$, $C_2 = \operatorname{ess.sup}_{x \in \boldsymbol{R}^d} (\int |K(x,y)|^2 \, dy)^{1/2}$.

PROOF. Put $H_1(x,y) = |K(x,y)|$, $H_2(x,y) = |g(x-y)|$, and apply Lemma 3.1. Then, we have $\|B(K; f, g)\|_{L^2} \le C_1 \|f\|_{L^2} \|g\|_{L^2}$, which means $N_{bl}(K) \le C_1$. Also, putting $H_1(x,y) = |g(x-y)|$, $H_2(x,y) = |K(x,y)|$, Lemma 3.1 gives that $N_{bl}(K) \le C_2$. □

The norm of bilinear operators has the following properties:

LEMMA 3.3. *Let* $K(x,y)$ *be a measurable function on* $\boldsymbol{R}^d \times \boldsymbol{R}^d$.

(a) *Put* $K_1(x,y) = K(y,x)$, $K_2(x,y) = K(x, x-y)$, $K_3(x,y) = K(y, y-x)$, $K_4(x,y) = K(x-y, -y)$, $K_5(x,y) = K(x-y, x)$. *Then we have* $N_{bl}(K) = N_{bl}(K_1) = N_{bl}(K_2) = N_{bl}(K_3) = N_{bl}(K_4) = N_{bl}(K_5)$.

(b) *If* $M(x,y)$ *is a non-negative measurable function such that* $|K(x,y)| \le M(x,y)$ *for almost everywhere, then* $N_{bl}(K) \le N_{bl}(M)$.

PROOF. It follows from Fubini's theorem that

$$(B(K; f, g), \psi)_{L^2 \times L^2} = \iint K(x,y) f(y) g(x-y) \overline{\psi(x)} \, dy dx = (B(K_1; \bar{\psi}, \check{g}), \bar{f})_{L^2 \times L^2},$$

where $\check{g}(x) = g(-x)$, which implies that $|(B(K; f, g), \psi)_{L^2 \times L^2}| \le N_{bl}(K_1) \|f\|_{L^2} \|g\|_{L^2} \|\psi\|_{L^2}$ holds for any $\psi \in L^2$. This means by duality that $\|B(K; f, g)\|_{L^2} \le N_{bl}(K_1) \|f\|_{L^2} \|g\|_{L^2}$, which implies that $N_{bl}(K) \le N_{bl}(K_1)$. Since $K_1(y,x) = K(x,y)$, we also have $N_{bl}(K_1) \le N_{bl}(K)$. Hence we have $N_{bl}(K) = N_{bl}(K_1)$. This also gives $N_{bl}(K_2) = N_{bl}(K_3)$ because of the fact that $K_2(y,x) = K_3(x,y)$.

$N_{bl}(K) = N_{bl}(K_2)$ follows from the identity

$$\begin{aligned} B(K; f, g)(x) &= \int K(x,y) f(y) g(x-y) \, dy \\ &= \int K(x, x-y) f(x-y) g(y) \, dy = B(K_2; g, f)(x). \end{aligned}$$

This implies $N_{bl}(K_3) = N_{bl}(K_4)$ and $N_{bl}(K_1) = N_{bl}(K_5)$, since $K_3(x, x-y) = K_4(x,y)$, $K_1(x, x-y) = K_5(x,y)$.

Part (b) follows from the inequality $|B(K; f, g)(x)| \le B(M; |f|, |g|)(x)$. □

4. Special class of bilinear operators.

The three lemmas in this section are concerned with the norm of the special class of bilinear operators, which give the key estimates to prove Theorem 2.3.

When $j > 0$ γ_j denotes the defining function of the set $[2^{j-1}, 2^{j+1}] \cup [-2^{j+1}, -2^{j-1}]$, and γ_0 denotes that of the interval $[-2, 2]$. For real-valued functions P, Q, R we define

$$H_{j,\ell m}^{[P]}(\xi, \tau, \eta, \sigma) := \gamma_j(\eta) \gamma_\ell(\xi - \eta) \gamma_m(\tau - \sigma - P(\xi - \eta)), \tag{4.1}$$

$$H_{jk,\ell m}^{[P,Q]}(\xi, \tau, \eta, \sigma) := \gamma_k(\sigma - P(\eta)) H_{j,\ell m}^{[Q]}(\xi, \tau, \eta, \sigma), \tag{4.2}$$

$$H^{[P,Q]}_{h,jk,\ell m}(\xi,\tau,\eta,\sigma) := \gamma_h(\xi) H^{[P,Q]}_{jk,\ell m}(\xi,\tau,\eta,\sigma), \tag{4.3}$$

$$H^{[P,Q,R]}_{hn,jk,\ell m}(\xi,\tau,\eta,\sigma) := \gamma_n(\tau - P(\xi)) H^{[Q,R]}_{h,jk,\ell m}(\xi,\tau,\eta,\sigma). \tag{4.4}$$

Note first

$$N_{bl}(H^{[P,Q]}_{jk,\ell m}) = N_{bl}(H^{[Q,P]}_{\ell m,jk}) \le \begin{cases} N_{bl}(H^{[Q]}_{j,\ell m}), \\ N_{bl}(H^{[P]}_{\ell,jk}). \end{cases} \tag{4.5}$$

$$N_{bl}(H^{[P,Q]}_{h,jk,\ell m}) = N_{bl}(H^{[Q,P]}_{h,\ell m,jk}) \le \begin{cases} N_{bl}(H^{[P,Q]}_{jk,\ell m}), \\ N_{bl}(H^{[-\check{Q}]}_{h,\ell m}), \\ N_{bl}(H^{[-\check{P}]}_{h,jk}). \end{cases} \tag{4.6}$$

$$\begin{aligned} N_{bl}(H^{[P,Q,R]}_{hn,jk,\ell m}) &= N_{bl}(H^{[P,R,Q]}_{hn,\ell m,jk}) = N_{bl}(H^{[Q,P,-\check{R}]}_{jk,hn,\ell m}) = N_{bl}(H^{[R,P,-\check{Q}]}_{\ell m,hn,jk}) \\ &\le N_{bl}(H^{[Q,R]}_{h,jk,\ell m}). \end{aligned} \tag{4.7}$$

Here $\check{P}(\xi) = P(-\xi)$. In fact, with the help of the identity $H^{[P,Q]}_{jk,\ell m}(\xi,\tau,\xi-\eta,\tau-\sigma) = H^{[Q,P]}_{\ell m,jk}(\xi,\tau,\eta,\sigma)$ and the inequality $0 \le H^{[P,Q]}_{jk,\ell m}(\xi,\tau,\eta,\sigma) \le H^{[Q]}_{j,\ell m}(\xi,\tau,\eta,\sigma)$ Lemma 3.3 gives (4.5), with the help of the inequalities $0 \le H^{[P,Q]}_{h,jk,\ell m}(\xi,\tau,\eta,\sigma) \le H^{[P,Q]}_{jk,\ell m}(\xi,\tau,\eta,\sigma)$, $0 \le H^{[P,Q]}_{h,jk,\ell m}(\eta,\sigma,\xi,\tau) \le H^{[-\check{Q}]}_{h,\ell m}(\xi,\tau,\eta,\sigma)$ it gives (4.6), and with the help of the identity $H^{[P,Q,R]}_{hn,jk,\ell m}(\eta,\sigma,\xi,\tau) = H^{[Q,P,-\check{R}]}_{jk,hn,\ell m}(\xi,\tau,\eta,\sigma)$ and the inequality $0 \le H^{[P,Q,R]}_{hn,jk,\ell m}(\xi,\tau,\eta,\sigma) \le H^{[Q,R]}_{h,jk,\ell m}(\xi,\tau,\eta,\sigma)$ it gives (4.7).

We also use the notations $j \wedge \ell := \min\{j,\ell\}$ and $j \vee \ell := \max\{j,\ell\}$.

Lemma 4.1. *Let P, Q and R be real-valued C^∞-functions. Then*

$$N_{bl}(H^{[P,Q]}_{jk,\ell m}) \le 2^{(k\wedge m + j\wedge \ell + 4)/2}, \tag{4.8}$$

$$N_{bl}(H^{[P,Q]}_{h,jk,\ell m}) \le 2^{(k\wedge m + \min(h,j,\ell) + 4)/2}, \tag{4.9}$$

$$N_{bl}(H^{[P,Q,R]}_{hn,jk,\ell m}) \le 2^{\{\min(n,k,m) + \min(h,j,\ell) + 4\}/2}. \tag{4.10}$$

Proof. First note that Lemma 3.2 and the inequality

$$\iint |H^{[P]}_{j,\ell m}(\xi,\tau,\eta,\sigma)|^2\, d\eta d\sigma = \int \gamma_j(\eta)\gamma_\ell(\xi-\eta)\, d\eta \int \gamma_m(\tau-\sigma-P(\xi-\eta))\, d\sigma \le 2^{j\wedge\ell+m+4}$$

imply the estimate

$$N_{bl}(H^{[P]}_{j,\ell m}) \le 2^{(j\wedge\ell+m+4)/2}. \tag{4.11}$$

The estimate (4.8) is a direct consequence of (4.5) and (4.11), the estimate (4.9) follows from (4.6), (4.8) and (4.11), and finally the estimate (4.10) follows from (4.7) and (4.9). □

LEMMA 4.2. *Let* $P(\xi) = \pm\xi^2$. *Then*

$$N_{bl}(H^{[P,P]}_{jk,\ell m}) \le 2^{(k\wedge m/2)+(k\vee m/4+2)}, \tag{4.12}$$

$$N_{bl}(H^{[P,-P]}_{h,jk,\ell m}) \le 2^{(k+m-h+5)/2}, \tag{4.13}$$

$$N_{bl}(H^{[P,P]}_{jk,\ell m}) \le 2^{(k+m-j\vee\ell+5)/2} \quad \textit{when } |j-\ell| \ge 3, \tag{4.14}$$

$$N_{bl}(H^{[P,P]}_{h,jk,\ell m}) \le 2^{(k+m-h\vee\ell+5)/2} \quad \textit{when } |h-1-\ell| \ge 3. \tag{4.15}$$

PROOF. Change variables $\sigma' = \sigma - P(\eta)$, $\zeta = \tau - \sigma' - P(\eta) - P(\xi-\eta)$. Then we have

$$\begin{aligned}
&\int \gamma_m(\tau - \sigma' - P(\eta) - P(\xi-\eta))\,d\eta \\
&\quad \le \int_{|\eta-\xi/2|<2^{m/2-1}} d\eta + \int_{|\eta-\xi/2|\ge 2^{m/2-1}} \gamma_m(\tau-\sigma'-P(\eta)-P(\xi-\eta))\,d\eta \\
&\quad \le 2^{m/2} + 2^{-m/2-1}\int \gamma_m(\zeta)\,d\zeta \le 2^{m/2} + 2^{-m/2+m+1} \le 2^{m/2+2},
\end{aligned}$$

since $|d\zeta/d\eta| = |2\xi - 4\eta|$. Hence

$$\iint |H^{[P,P]}_{jk,\ell m}(\xi,\tau,\eta,\sigma)|^2\,d\eta d\sigma \le \int \gamma_k(\sigma')\,d\sigma' \int \gamma_m(\tau-\sigma'-P(\eta)-P(\xi-\eta))\,d\eta \le 2^{k+(m/2)+4},$$

which implies, with the aid of Lemma 3.2, that $N_{bl}(H^{[P,P]}_{jk,\ell m}) \le 2^{(k/2)+(m/4)+2}$. With the help of (4.5) this estimate gives (4.12).

Next consider (4.13). We may assume that $h > 0$, because (4.13) with $h = 0$ follows from (4.9). We change variables $\sigma' = \sigma - P(\eta)$, $\zeta = \tau - \sigma' + P(\xi-\eta) - P(\eta)$. Then

$$\begin{aligned}
\iint H^{[P,-P]}_{h,jk,\ell m}(\xi,\tau,\eta,\sigma)\,d\eta d\sigma &\le \int \gamma_k(\sigma')\,d\sigma' \int \gamma_h(\xi)\gamma_m(\tau-\sigma'+P(\xi-\eta)-P(\eta))\,d\eta \\
&\le 2^{-h}\int \gamma_k(\sigma')\,d\sigma' \int \gamma_m(\zeta)\,d\zeta \le 2^{k+m-h+4},
\end{aligned}$$

since $|d\zeta/d\eta| = |2\xi| \ge 2^h$. Therefore we have (4.13) by Lemma 3.2.

Thirdly, assume that $|j-\ell| \ge 3$ and consider (4.14). By (4.5) we may assume that $j \ge \ell + 3$. Since $2^{j-1} < |\eta|$, $|\xi - \eta| < 2^{\ell+1}$ if $\gamma_j(\eta)\gamma_\ell(\xi-\eta) \ne 0$, we see that $|2\xi - 4\eta| \ge 2|\eta| - 2|\xi-\eta| > 2^j - 2^{\ell+2} \ge 2^{j-1}$. Therefore $|d\zeta/d\eta| = |2\xi - 4\eta| \ge 2^{j-1}$ when we change variables $\zeta = \tau - \sigma' - P(\eta) - P(\xi-\eta)$, which gives that

$$\begin{aligned}
\iint |H^{[P,P]}_{jk,\ell m}(\xi,\tau,\eta,\sigma)|^2\,d\eta d\sigma &\le \int d\sigma' \int \gamma_k(\sigma')\gamma_m(\tau-\sigma'-P(\eta)-P(\xi-\eta))\,d\eta \\
&\le 2^{-j+1}\int \gamma_k(\sigma')\,d\sigma' \int \gamma_m(\zeta)\,d\zeta \le 2^{k+m-j+5}.
\end{aligned}$$

So we have (4.14) by Lemma 3.2.

Finally, assume that $|h-1-\ell| \geq 3$. It follows from the fact $2^{h-1} < |\xi| < 2^{h+1}$, $2^{\ell-1} < |\xi-\eta| < 2^{\ell+1}$ when $\gamma_h(\xi)\gamma_\ell(\xi-\eta) \neq 0$ that

$$|2\xi - 4\eta| \geq \begin{cases} 2|\xi| - 4|\xi-\eta| > 2^h - 2^{\ell+3} \geq 2^{h-1} & \text{for } h \geq \ell+4, \\ 4|\xi-\eta| - 2|\xi| > 2^{\ell+1} - 2^{h+2} \geq 2^{\ell} & \text{for } \ell \geq h+2. \end{cases}$$

Hence $|d\zeta/d\eta| = |2\xi - 4\eta| \geq 2^{(h-1)\vee\ell}$ when we change variables $\zeta = \tau - \sigma' - P(\xi-\eta) - P(\eta)$. This fact and the same consideration as above give the estimate (4.15). □

Lemma 4.3. *Let $P(\xi) = \pm\xi^2$, and let Q be a real-valued C^∞ function. Then we have*

$$N_{bl}(H^{[P,Q,P]}_{hn,jk,\ell m}) \leq 2^{(n+m-j+5)/2}, \tag{4.16}$$

$$N_{bl}(H^{[P,P,P]}_{hn,jk,\ell m}) \leq 2^{(n+(m-j)\wedge(k-\ell)+5)/2}, \tag{4.17}$$

$$N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) \leq 2^{(m\wedge n/2)+(m\vee n/4)+2}, \tag{4.18}$$

$$N_{bl}(H^{[P,-P,-P]}_{hn,jk,\ell m}) \leq 2^{n/4+(k\wedge m)/2+2}, \tag{4.19}$$

$$N_{bl}(H^{[P,P,-P]}_{hn,jk,\ell m}) \leq 2^{(n+k-\ell+5)/2}, \tag{4.20}$$

$$N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) \leq 2^{(n+m-j\vee\ell+5)/2} \quad \textit{when } |j-1-\ell| \geq 3, \tag{4.21}$$

$$N_{bl}(H^{[P,P]}_{h,jk,\ell m}) \leq 2^{(k+m-j\vee\ell+5)/2} \quad \textit{when } h \leq j\vee\ell - 2, \tag{4.22}$$

$$N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) \leq 2^{(n+m-j\vee\ell+5)/2} \quad \textit{when } h \leq j\vee\ell - 4, \tag{4.23}$$

$$N_{bl}(H^{[P,-P,-P]}_{hn,jk,\ell m}) \leq 2^{(n+k\wedge m-j\vee\ell+5)/2}, \quad \textit{when } h \leq j\vee\ell - 4. \tag{4.24}$$

Proof. Since (4.7) implies that $N_{bl}(H^{[P,Q,P]}_{hn,jk,\ell m}) = N_{bl}(H^{[Q,P,-P]}_{jk,hn,\ell m}) \leq N_{bl}(H^{[P,-P]}_{j,hn,\ell m})$, (4.16) follows from (4.13), and (4.17) follows from (4.16) and (4.7). Also, (4.18) and (4.19) follow from (4.12) and (4.7). Since we have $N_{bl}(H^{[P,P,-P]}_{hn,jk,\ell m}) = N_{bl}(H^{[-P,P,-P]}_{\ell m,hn,jk}) \leq N_{bl}(H^{[P,-P]}_{\ell,hn,jk})$ by (4.7), the estimate (4.13) implies (4.20). Similarly, (4.21) follows from $N_{bl}(H^{[P,Q,-P]}_{n,jk,\ell m}) = N_{bl}(H^{[Q,P,P]}_{jk,hn,\ell m}) \leq N_{bl}(H^{[P,P]}_{j,hn,\ell m})$ and (4.15).

When $h \leq \ell - 2$, (4.15) implies the estimate $N_{bl}(H^{[P,P]}_{h,jk,\ell m}) \leq 2^{(k+m-\ell+5)/2}$, and it also gives $N_{bl}(H^{[P,P]}_{h,jk,\ell m}) \leq 2^{(k+m-j+5)/2}$ when $h \leq j-2$, since $N_{bl}(H^{[P,P]}_{h,jk,\ell m}) = N_{bl}(H^{[P,P]}_{h,\ell m,jk})$.

Consider next (4.23). When $h \leq \ell - 4$, the estimate $N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) \leq 2^{(n+m-\ell+5)/2}$ follows from (4.14) and $N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) = N_{bl}(H^{[Q,P,P]}_{jk,hn,\ell m}) \leq N_{bl}(H^{[P,P]}_{hn,\ell m})$, and when $h \leq j-4$, the estimate $N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) \leq 2^{(n+m-j+5)/2}$ follows from (4.15) and $N_{bl}(H^{[P,Q,-P]}_{hn,jk,\ell m}) = N_{bl}(H^{[Q,P,P]}_{jk,\ell m,hn}) \leq N_{bl}(H^{[P,P]}_{j,\ell m,hn})$. Finally, (4.7) and (4.23) imply (4.24). □

5. Proof of Theorem 2.3 (I).

Since $\|\bar{f}\|_{B^{(\rho,-1/2)}_{2,1,P}} = \|f\|_{B^{(\rho,-1/2)}_{2,1,-P}}$, Theorem 2.3 Part (I) is reduced to the following:

THEOREM 5.1. *Let* $s \geq -3/4$, $\rho(t) = \log(2+t)t^s$, $P(\xi) = \pm\xi^2$, *and let* Q *be* $\pm P$. *Then,*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\|f\|_{B^{(\rho,1/2)}_{2,1,Q}} \|g\|_{B^{(s,1/2)}_{2,1,Q}}, \tag{5.1}$$

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\{\|f\|_{B^{(s,b)}_{2,1,Q}} \|g\|_{B^{(s,1/2)}_{2,1,Q}} + \|f\|_{B^{(s,1/2)}_{2,1,Q}} \|g\|_{B^{(s,b)}_{2,1,Q}}\} \tag{5.2}$$

hold. Here we assume that $b > 1/2$.

In this section we put $\varphi^{[P]}_{hn}(\xi,\tau) := \varphi_h(\xi)\varphi_n(\tau - P(\xi))$, and write $\hat{f}_{jk,Q}(\xi,\tau) := \varphi^{[Q]}_{jk}(\xi,\tau)\hat{f}(\xi,\tau)$. Then we have $f = \sum_{j,k} f_{jk,Q}$, $g = \sum_{\ell,m} g_{\ell m,R}$, hence

$$fg = \sum_{j,k,\ell,m} f_{jk,Q}\, g_{\ell m,R}. \tag{5.3}$$

To prove the theorem, first note the following facts:

LEMMA 5.1. *Assume that* P *and* Q *be real-valued* C^∞*-functions.*
If $\|\varphi_h(\xi)\hat{f}_{jk,P} * \hat{g}_{\ell m,Q}(\xi,\tau)\|_{L^2} \neq 0$, *then* $h \leq j \vee \ell + 2$.
Moreover, $h \geq j \vee \ell - 2$ *when* $|j-\ell| \geq 3$ *and* $|j-\ell| \leq 2$ *when* $h \leq j \vee \ell - 3$.

PROOF. If $\|\varphi_h(\xi)\hat{f}_{jk,P} * \hat{g}_{\ell m,Q}(\xi,\tau)\|_{L^2} \neq 0$, then there exist ξ and η such that

$$2^{h-1} < |\xi| < 2^{h+1}, \quad 2^{j-1} < |\eta| < 2^{j+1}, \quad 2^{\ell-1} < |\xi-\eta| < 2^{\ell+1}.$$

This gives $2^{h-1} < |\xi| \leq |\eta| + |\xi-\eta| < 2^{j+1} + 2^{\ell+1} \leq 2^{j\vee\ell+2}$, which implies $h \leq j \vee \ell + 2$.

When $\ell \leq j-3$ we have $2^{h+1} > |\xi| \geq |\eta| - |\xi-\eta| > 2^{j-1} - 2^{\ell+1} \geq 2^{j-2}$, which implies $h \geq j-2$. In the same way we see that $h \geq \ell - 2$ when $j \leq \ell - 3$.

If $h \leq j-3$, $j \geq \ell$, then we have $2^{\ell+1} > |\xi-\eta| \geq |\eta| - |\xi| > 2^{j-1} - 2^{h+1} \geq 2^{j-2}$, which implies that $\ell \geq j-2$. In the same way we see that $j \geq \ell - 2$ if $h \leq \ell - 3$, $\ell \geq j$. □

LEMMA 5.2. *Let* $P(\xi) = \pm\xi^2$, *and assume that* $j > 0$, $\ell > 0$.
(a) *If* $\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{jk,P} * \hat{g}_{\ell m,P}(\xi,\tau)\|_{L^2} \neq 0$, *then* $\max\{k,m,n\} \geq j+\ell-3$.
(b) *If* $\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{jk,-P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \neq 0$, *then* $\max\{k,m,n\} \geq 2(j\vee\ell) - 2$.

PROOF. (a) Assume that $\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{jk,P} * \hat{g}_{\ell m,P}(\xi,\tau)\|_{L^2} \neq 0$, $j > 0$, $\ell > 0$. Then, there exist ξ,τ,η,σ such that $2^{j-1} < |\eta|$, $2^{\ell-1} < |\xi-\eta|$, $|\tau'| < 2^{n+1}$, $|\sigma'| < 2^{k+1}$, $|\tau' - \sigma' + P(\xi) - P(\eta) - P(\xi-\eta)| < 2^{m+1}$, where $\tau' = \tau - P(\xi)$, $\sigma' = \sigma - P(\eta)$. Noting that $|P(\xi) - P(\eta) - P(\xi-\eta)| = |2\eta(\xi-\eta)|$, this implies that $2^{j+\ell-1} < |2\eta(\xi-\eta)| \leq |\tau' - \sigma' + P(\xi) - P(\eta) - P(\xi-\eta)| + |\tau'| + |\sigma'| < 2^{m+1} + 2^{n+1} + 2^{k+1} < 2^{\max\{k,m,n\}+3}$, so that $\max\{k,m,n\} \geq j+\ell-3$.

(b) Assume that $\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{jk,-P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \neq 0$, $j \vee \ell > 0$. Then, there exist ξ,τ,η,σ such that $2^{j-1} < |\eta|$, $2^{\ell-1} < |\xi-\eta|$, $|\tau'| < 2^{n+1}$, $|\sigma'| < 2^{k+1}$, $|\tau' - \sigma' + P(\xi) + P(\eta) + P(\xi-\eta)| < 2^{m+1}$, where $\tau' = \tau - P(\xi)$, $\sigma' = \sigma + P(\eta)$. From the inequality $|P(\xi) + P(\eta) + P(\xi-\eta)| = 2(\xi^2 - \xi\eta + \eta^2) \geq (3/2)\max\{\eta^2, (\xi-\eta)^2\} > 3 \cdot 2^{2(j\vee\ell)-3}$ it follows that $3 \cdot 2^{2(j\vee\ell)-3} < |P(\xi) + P(\eta) + P(\xi-\eta)| \leq |\tau'| + |\sigma'| + |\tau' - \sigma' + P(\xi) + P(\eta) + P(\xi-\eta)| < 3 \cdot 2^{\max\{k,m,n\}}$, so that we have $\max\{k,m,n\} \geq 2(j\vee\ell) - 2$. □

Now we are going to prove Theorem 5.1. We divide fg into 4 terms:

$$fg = \sum_{i=0}^{3} F_i := \left\{ \sum_{j+\ell\le 2} + \sum_{j=0,\ell>2} + \sum_{j>2,\ell=0} + \sum_{j>0,\ell>0,j+\ell>2} \right\} \sum_{k,m} f_{jk,Q} g_{\ell m,Q}. \tag{5.4}$$

The inequality

$$\|\varphi_{hn}^{[P]}(\xi,\tau)\hat{f}_{jk,Q} * \hat{g}_{\ell m,R}(\xi,\tau)\|_{L^2} \le \|B(H_{hn,jk,\ell m}^{[P,Q,R]}; |\hat{f}_{jk,Q}|, |\hat{g}_{\ell m,R}|)\|_{L^2}, \tag{5.5}$$

Lemma 5.1 and (4.10) give that the $\|F_0\|_{B_{2,1,P}^{(\rho,-1/2)}}$ is estimated by

$$c \sum_{j+\ell\le 2} \sum_{h=0}^{4} \rho(2^h) \sum_{k,m} \sum_{n} 2^{(-n+m)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \le c' \|f\|_{B_{2,1,Q}^{(s,1/2)}} \|g\|_{B_{2,1,Q}^{(s,1/2)}},$$

and (5.5), Lemma 5.1, (4.6), (4.7) and (4.14) imply that $\|F_1\|_{B_{2,1,P}^{(\rho,-1/2)}}$ is estimated by

$$c \sum_{\ell\ge 3} \sum_{k,m} \sum_{h=\ell-2}^{\ell+2} \rho(2^h) \sum_{n} 2^{(-n+k+m-\ell)/2} \|f_{0k,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \le c' \|f\|_{B_{2,1,Q}^{(s,1/2)}} \|g\|_{B_{2,1,Q}^{(s,1/2)}}.$$

Since $F_2 = \sum_{\ell>2} \sum_{k,m} g_{0k,Q} f_{\ell m,Q}$, this also gives $\|F_2\|_{B_{2,1,P}^{(\rho,-1/2)}} \le c \|f\|_{B_{2,1,Q}^{(s,1/2)}} \|g\|_{B_{2,1,Q}^{(s,1/2)}}$.

To estimate the norm of F_3 we divide it into 5 parts:

$$\begin{aligned} \|F_3\|_{B_{2,1,P}^{(\rho,-1/2)}} &\le c \sum_{i=1}^{5} G_i \\ &:= c \sum_{j,\ell>0} \left\{ \sum_{h=0}^{j\vee\ell-4} \sum_{k,m} \sum_{n\ge k\vee m} + \sum_{h=0}^{j\vee\ell-4} \sum_{n=0}^{j+\ell} \sum_{k\vee m>n} + \sum_{h=j\vee\ell-3}^{j\vee\ell+2} \sum_{k,m} \sum_{n\ge k\vee m} \right. \\ &\quad \left. + \sum_{h=j\vee\ell-3}^{j\vee\ell+2} \sum_{n=0}^{j+\ell} \sum_{k\vee m>n} + \sum_{h} \sum_{n>j+\ell} \sum_{k\vee m>n} \right\} \rho(2^h) 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,Q} * \hat{g}_{\ell m,Q}\|_{L^2}. \end{aligned} \tag{5.6}$$

At first, by (4.22), (5.5) and (4.7) we have

$$\begin{aligned} G_1 &\le c \sum_{j,\ell>0} \sum_{h=0}^{j\vee\ell-4} \rho(2^h) \sum_{k,m} \sum_{n\ge j+\ell-3} 2^{(-n+k+m-j\vee\ell)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \\ &\le c' \sum_{j,\ell>0} \sum_{k,m} 2^{(k+m-j\vee\ell-j-\ell)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \le c'' \|f\|_{B_{2,1,Q}^{(s,1/2)}} \|g\|_{B_{2,1,Q}^{(s,1/2)}}. \end{aligned}$$

Secondly, by (4.17) and (4.24) we have

$$\begin{aligned} G_2 &\le c \sum_{|j-\ell|\le 2} \sum_{h=0}^{j\vee\ell-4} \sum_{n=0}^{j+\ell} \sum_{k\vee m\ge j+\ell-3} \rho(2^h) 2^{(-n+n+k\wedge m-(j\vee\ell))/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \\ &\le c' \sum_{|j-\ell|\le 2} \sum_{k,m} (j+\ell+1) 2^{(k+m-j\vee\ell-j-\ell)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \\ &\le c'' \min\{\|f\|_{B_{2,1,Q}^{(\rho,1/2)}} \|g\|_{B_{2,1,Q}^{(s,1/2)}}, \|f\|_{B_{2,1,Q}^{(s,1/2)}} \|g\|_{B_{2,1,Q}^{(\rho,1/2)}}\}, \end{aligned}$$

and by Lemma 5.2 and (4.12) we have

$$G_3 \le c \sum_{j,\ell>0} \sum_{h=j\vee\ell-3}^{j\vee\ell+2} \rho(2^h) \sum_{k,m} \sum_{n\ge j+\ell-3} 2^{-n/2+(k\vee m)/4+(k\wedge m)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2}$$

$$\le c' \sum_{j,\ell>0} \sum_{k,m} \rho(2^{j\vee\ell}) 2^{(k+m-j-\ell)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \le c'' \|f\|_{B^{(s,1/2)}_{2,1,Q}} \|g\|_{B^{(s,1/2)}_{2,1,Q}}.$$

When $Q = P$ it follows from (4.17) that $N_{bl}(H^{[P,P,P]}_{n,jk,\ell m}) \le c2^{(n+(k-\ell)\wedge(m-j))/2} \le c'2^{(n+k\wedge m)/2}$, and when $Q = -P$ it follows from (4.19) that $N_{bl}(H^{[P,-P,-P]}_{hn,jk,\ell m}) \le c2^{(n+k\wedge m)/2}$. Therefore

$$G_4 \le c \sum_{j,\ell>0} \sum_{h=j\vee\ell-2}^{j\vee\ell+2} \sum_{n=0}^{j+\ell} \sum_{k\vee m\ge j+\ell-3} \rho(2^h) 2^{(-n+n+k\wedge m)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2}$$

$$\le c' \sum_{j,\ell>0} \sum_{k,m} (j+\ell+1) \rho(2^{j\vee\ell}) 2^{(k+m-j-\ell)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2}$$

$$\le c'' \|f\|_{B^{(s,1/2)}_{2,1,Q}} \|g\|_{B^{(s,1/2)}_{2,1,Q}},$$

and by (4.8) we have

$$G_5 \le c \sum_{j,\ell>0} \sum_{h} \sum_{n>j+\ell} \sum_{k\vee m>j+\ell} \rho(2^h) 2^{(-n+k\wedge m+j\wedge\ell)/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2}$$

$$\le c' \sum_{j,\ell>0} \sum_{k,m} 2^{(k+m+j\wedge\ell)/2-j-\ell} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,Q}\|_{L^2} \le c'' \|f\|_{B^{(s,1/2)}_{2,1,Q}} \|g\|_{B^{(s,1/2)}_{2,1,Q}}. \qquad \square$$

6. Proof of Theorem 2.3 (II).

As Proof of (I), that of Theorem 2.3 (II) is reduced to

THEOREM 6.1. *Let* $s \ge -1/4$, $\rho(t) = \log(2+t)t^s$ *and let* $P(\xi) = \pm\xi^2$. *Then we have*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}(\boldsymbol{R}^2)} \le c \min\{\|f\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^2)} \|g\|_{B^{(s,1/2)}_{2,1,-P}(\boldsymbol{R}^2)}, \|f\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}^2)} \|g\|_{B^{(\rho,1/2)}_{2,1,-P}(\boldsymbol{R}^2)}\}, \tag{6.1}$$

$$\|fg\|_{B^{(s,-1/2),\#}_{2,1,P}(\boldsymbol{R}^2)} \le c \|f\|_{B^{(s,1/2),\#}_{2,1,P}(\boldsymbol{R}^2)} \|g\|_{B^{(s,1/2),\#}_{2,1,-P}(\boldsymbol{R}^2)}. \tag{6.2}$$

First, noting that $|P(\xi) - P(\eta) + P(\xi-\eta)| = 2|\xi(\xi-\eta)|$, in the same way as Lemma 5.2 we obtain

LEMMA 6.1. *Let* $P(\xi) = \pm\xi^2$. *If* $\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \ne 0$, $h > 0$, $\ell > 0$, *then we have* $\max\{k,m,n\} \ge h+\ell-3$.

PROOF OF THE ESTIMATE (6.1).

By (5.3) we have $\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \le c(G_{00} + G_{01} + G_{02} + G_1)$, where

$$G_{00} = \sum_{j,\ell} \sum_{n} \sum_{k,m} \rho(1) 2^{-n/2} \|\varphi^{[P]}_{0n} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2}, \tag{6.3}$$

$$G_{01} = \sum_{\ell} \sum_{h>0} \sum_{n} \sum_{k,m} \rho(2^h) 2^{-n/2} \|\varphi^{[P]}_{hn} \hat{f}_{0k,P} * \hat{g}_{\ell m,-P}\|_{L^2}, \tag{6.4}$$

$$G_{02} = \sum_j \sum_{h>0} \sum_n \sum_{k,m} \rho(2^h) 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,P} * \hat{g}_{0m,-P}\|_{L^2}, \tag{6.5}$$

$$G_1 = \sum_{j,\ell>0} \sum_{h>0} \sum_n \sum_{k,m} \rho(2^h) 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2}. \tag{6.6}$$

By Lemma 5.1, (5.5), (4.10) and (4.23) we have

$$\begin{aligned} G_{00} &= \sum_{k,m} \left(\sum_\ell \sum_{j>0} \sum_{n=0}^{j-1} + \sum_{j,\ell} \sum_{n \ge j} \right) \rho(1) 2^{-n/2} \|\varphi_{0n}^{[P]} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2} \\ &\le c \sum_{k,m} \sum_{|j-\ell| \le 2} \left(\sum_{n=0}^{j-1} 2^{(m-j \vee \ell)/2} + \sum_{n \ge j} 2^{(-n+m)/2} \right) \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c' \sum_{k,m} \sum_{|j-\ell| \le 2} j 2^{(k+m-j)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c'' \min\{\|f\|_{B^{(\rho,1/2)}_{2,1,P}} \|g\|_{B^{(s,1/2)}_{2,1,-P}}, \|f\|_{B^{(s,1/2)}_{2,1,P}} \|g\|_{B^{(\rho,1/2)}_{2,1,-P}}\}. \end{aligned}$$

By Lemma 5.1, (4.8) and (4.13) we have

$$\begin{aligned} G_{01} &\le c \sum_{k,m} \left(\sum_{\ell=0}^{2} \sum_{h=1}^{\ell+2} 2^{k/2} + \sum_{\ell=3}^{\infty} \sum_{h=\ell-2}^{\ell+2} 2^{(k+m-h)/2} \right) \sum_n \rho(2^h) 2^{-n/2} \|f_{0k,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c' \sum_{k,m} \sum_\ell 2^{(k+m-\ell)/2} \|f_{0k,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \le c'' \|f\|_{B^{(s,1/2)}_{2,1,P}} \|g\|_{B^{(s,1/2)}_{2,1,-P}}, \end{aligned}$$

and in the same way we see that G_{02} is estimated by $c\|f\|_{B^{(s,1/2)}_{2,1,P}} \|g\|_{B^{(s,1/2)}_{2,1,-P}}$.

To estimate G_1 we divide it into 3 parts: $G_1 = G_{11} + G_{12} + G_{13}$, where

$$\begin{cases} G_{11} := \displaystyle\sum_{j,\ell>0} \sum_{h=1}^{j \vee \ell - 4} \sum_{k,m} \sum_{n \ge k \vee m} \rho(2^h) 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2}, \\ G_{12} := \displaystyle\sum_{j,\ell>0} \sum_{h=1}^{j \vee \ell - 4} \sum_n \sum_{k \vee m > n} \rho(2^h) 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2}, \\ G_{13} := \displaystyle\sum_{j,\ell>0} \sum_{h=j \vee \ell - 3}^{j \vee \ell + 2} \sum_{k,m} \sum_n \rho(2^h) 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2}. \end{cases} \tag{6.7}$$

With the aid of Lemma 5.1 and Lemma 6.1, by (4.13) we have

$$\begin{aligned} G_{11} &\le c \sum_{j,\ell>0, |j-\ell| \le 2} \sum_{k,m} \sum_{h=1}^{j \vee \ell - 4} \sum_{n \ge h+\ell-3} 2^{-n/2} \rho(2^h) 2^{(k+m-h)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c \sum_{j,\ell>0, |j-\ell| \le 2} \sum_{k,m} \sum_{h=1}^{j \vee \ell - 4} \rho(2^h) 2^{(k+m-2h-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c' \sum_{j,\ell>0, |j-\ell| \le 2} \sum_{k,m} 2^{(k+m-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \le c'' \|f\|_{B^{(s,1/2)}_{2,1,P}} \|g\|_{B^{(s,1/2)}_{2,1,-P}}. \end{aligned}$$

Since $(k \vee m)2^{-(k\vee m)/2}$ is bounded, by (4.23) and (4.20) we have

$$G_{12} \le c \sum_{j,\ell>0,|j-\ell|\le 2} \sum_{h=1}^{j\vee\ell-4} \rho(2^h) \sum_{k,m} \sum_{n=0}^{k\vee m-1} 2^{(k\wedge m-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$

$$\le c' \sum_{|j-\ell|\le 2} \sum_{k,m} (k\vee m) 2^{(k\wedge m-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$

$$\le c'' \sum_{|j-\ell|\le 2} \sum_{k,m} 2^{(k+m-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \le c''' \|f\|_{B^{(s,1/2)}_{2,1,P}} \|g\|_{B^{(s,1/2)}_{2,1,-P}}.$$

Also, by (4.13) we have

$$G_{13} \le c \sum_{j,\ell>0} \sum_{h=j\vee\ell-3}^{j\vee\ell+2} \sum_{k,m} \sum_{n} \rho(2^h) 2^{(-n+k+m-h)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$

$$\le c'' \sum_{j,\ell>0} \sum_{k,m} \rho(2^{j\vee\ell}) 2^{(k+m-j\vee\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \le c'' \|f\|_{B^{(s,1/2)}_{2,1,P}} \|g\|_{B^{(s,1/2)}_{2,1,-P}}.$$

This completes the proof of (6.1). □

PROOF OF THE ESTIMATE (6.2).

From (5.3) it follows that $\|fg\|_{B^{(s,-1/2),\#}_{2,1,P}} = G^{\#}_{00} + G_{01} + G_{02} + G_1$, where

$$G^{\#}_{00} = \sum_{j,\ell} \sum_{n} \sum_{k,m} 2^{-n/2} \left\| \frac{\varphi^{[P]}_{0n}(\xi,\tau)}{(1+|\log|\xi||)^2} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau) \right\|_{L^2}, \tag{6.8}$$

and G_{01}, G_{02}, G_1 are defined by (6.4), (6.5) and (6.6) with $\rho(2^h)$ replaced by 2^{sh}.

ESTIMATE OF $G^{\#}_{00}$. We divide $G^{\#}_{00}$ into 4 parts: $G^{\#}_{00} = G^{\#}_{000} + G^{\#}_{001} + G^{\#}_{002} + G^{\#}_{003}$, where

$$G^{\#}_{000} = \sum_{n} \sum_{k,m} 2^{-n/2} \|(1+|\log|\xi||)^{-2} \varphi^{[P]}_{0n}(\xi,\tau) \hat{f}_{0k,P} * \hat{g}_{0m,-P}(\xi,\tau)\|_{L^2},$$

$$G^{\#}_{001} = \sum_{\ell=1}^{2} \sum_{n} \sum_{k,m} 2^{-n/2} \|(1+|\log|\xi||)^{-2} \varphi^{[P]}_{0n}(\xi,\tau) \hat{f}_{0k,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2},$$

$$G^{\#}_{002} = \sum_{j=1}^{2} \sum_{n} \sum_{k,m} 2^{-n/2} \|(1+|\log|\xi||)^{-2} \varphi^{[P]}_{0n}(\xi,\tau) \hat{f}_{jk,P} * \hat{g}_{0m,-P}(\xi,\tau)\|_{L^2},$$

$$G^{\#}_{003} = \sum_{j,\ell>0,|j-\ell|\le 2} \sum_{n} \sum_{k,m} 2^{-n/2} \|(1+|\log|\xi||)^{-2} \varphi^{[P]}_{0n}(\xi,\tau) \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2}.$$

When $|\xi|/2 \le |\eta| \le 2$ we have $2(1+|\log|\xi||) \ge 1+|\log|\eta||$ and when $|\xi|/2 > |\eta|$, $|\xi-\eta| \le 2$ we have $|\xi|/2 \le |\xi-\eta|$ hence $2(1+|\log|\xi||) \ge 1+|\log|\xi-\eta||$. Therefore, we have the inequality

$$\frac{\gamma_0(\xi)\gamma_0(\eta)\gamma_0(\xi-\eta)}{(1+|\log|\xi||)^2} \le \frac{4\gamma_0(\eta)}{(1+|\log|\eta||)^2} + \frac{4\gamma_0(\xi-\eta)}{(1+|\log|\xi-\eta||)^2},$$

which implies that

$$|\varphi_0(\xi)(1+|\log|\xi||)^{-2}\hat f_{0k,P}*\hat g_{0m,-P}(\xi,\tau)|$$

$$= \left|\varphi_0(\xi)(1+|\log|\xi||)^{-2}\iint \hat f_{0k,P}(\eta,\sigma)\hat g_{0m,-P}(\xi-\eta,\tau-\sigma)\,d\eta d\sigma\right|$$

$$\le 4B(\gamma^{\#}_{k,P}(\eta,\sigma);|\hat f^{\#}_{0k,P}|,|\hat g^{\#}_{0m,-P}|)(\xi,\tau)+4B(\gamma^{\#}_{m,-P}(\xi-\eta,\tau-\sigma);|\hat f^{\#}_{0k,P}|,|\hat g^{\#}_{0m,-P}|)(\xi,\tau),$$

where

$$\gamma^{\#}_{k,Q}(\xi,\tau):=(1+|\log|\xi||)^2\gamma_0(\xi)\gamma_k(\tau-Q(\xi)). \tag{6.9}$$

Combining this with the fact that

$$N_{bl}(\gamma^{\#}_{k,Q}(\eta,\sigma))=N_{bl}(\gamma^{\#}_{k,Q}(\xi-\eta,\tau-\sigma))\le c2^{k/2}, \tag{6.10}$$

which is a consequence of Lemma 3.2 and the estimate $\|\gamma^{\#}_{k,Q}(\xi,\tau)\|_{L^2(\boldsymbol{R}^2)}\le c2^{k/2}$, we have

$$G^{\#}_{000}\le c\sum_n\sum_{k,m}2^{-n/2}\|\varphi_0(\xi)(1+|\log|\xi||)^{-2}\hat f_{0k,P}*\hat g_{0m,-P}(\xi,\tau)\|_{L^2}$$

$$\le c'\sum_n\sum_{k,m}2^{-n/2}(2^{m/2}+2^{k/2})\|f^{\#}_{0k,P}\|_{L^2}\|g^{\#}_{0m,-P}\|_{L^2}\le 2c'\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}.$$

It follows from the identity

$$\hat f_{0k,P}*\hat g_{\ell m,Q}(\xi,\tau)=B(\gamma^{\#}_{k,P}(\eta,\sigma);\hat f^{\#}_{0k,P},\hat g_{\ell m,Q})(\xi,\tau) \tag{6.11}$$

and (6.10) that

$$\|f_{0k,P}g_{\ell m,-P}\|_{L^2}\le c2^{k/2}\|f^{\#}_{0k,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}, \tag{6.12}$$

which implies that

$$G^{\#}_{001}\le c\sum_{\ell=1}^{2}\sum_n\sum_{k,m}2^{(-n+k)/2}\|f^{\#}_{0k,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}\le c\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}.$$

In the same way we have $G^{\#}_{002}\le c\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}$.

To estimate $G^{\#}_{003}$ we note that the identity $\varphi_0(z)+\sum_{k=0}^{\infty}\varphi_1(2^{-k}z)=1$ implies that

$$\varphi_0(z)=\sum_{p=0}^{j-1}\varphi_1(2^pz)+\varphi_0(2^jz). \tag{6.13}$$

It follows from the fact that $(1+|\log|\xi||)^{-2}\le c_0(1+p)^{-2}$ when $\varphi_1(2^p\xi)\ne 0$ and $(1+|\log|\xi||)^{-2}\le c_0(1+j)^{-2}$ when $\varphi_0(2^j\xi)\ne 0$ that

$$G^{\#}_{003}\le c_0\sum_{j,\ell>0,|j-\ell|\le 2}\sum_{p=0}^{j}\sum_n\sum_{k,m}2^{-n/2}(1+p)^{-2}I_{jk\ell mnp},$$

where

$$I_{jk\ell mnp} := \|\varphi_1(2^p\xi)\varphi_n(\tau - P(\xi))\hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \quad \text{for } p < j,$$

$$I_{jk\ell mnj} := \|\varphi_0(2^j\xi)\varphi_n(\tau - P(\xi))\hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2}.$$

By the identity $\varphi_0(2^j\xi)\hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau) = B(\varphi_0(2^j\xi)\gamma_m(\tau - \sigma + P(\xi-\eta)); f_{jk,P}, g_{\ell m,-P})$ we have $\|\varphi_0(2^j\xi)\hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \le c2^{(m-j)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}$, since

$$N_{bl}(\varphi_0(2^j\xi)^2\gamma_m(\tau - \sigma + P(\xi - \eta))) \le 2^{(m-j)/2+2},$$

which follows from

$$\iint \varphi_0(2^j\xi)^2\gamma_m(\tau - \sigma + P(\xi - \eta))\, d\xi d\tau \le 2^{m-j+4}, \tag{6.14}$$

and Lemma 3.2. Therefore, we have

$$\sum_{j,\ell>0,|j-\ell|\le 2}\sum_n\sum_{k,m} 2^{-n/2}I_{jk\ell mnj} \le c\sum_{j,\ell>0,|j-\ell|\le 2}\sum_{k,m} 2^{(m-j)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}$$
$$\le c\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}.$$

Assume now $p < j$. Note that $2^{\ell-p} < |P(\xi) - P(\eta) + P(\xi-\eta)| = 2|\xi(\xi-\eta)| < 2^{\ell-p+4}$ if $\varphi_1(2^p\xi)\gamma_\ell(\xi-\eta) \ne 0$. We write $\tau' = \tau - P(\xi)$, $\sigma' = \sigma - P(\eta)$. Assume moreover that $k \vee m \le \ell - p - 3$. Then, $\varphi_1(2^p\xi)\gamma_\ell(\xi-\eta)\varphi_n(\tau')\gamma_m(\tau' - \sigma' + P(\xi) - P(\eta) + P(\xi-\eta))\gamma_k(\sigma')$ $\ne 0$ implies that

$$2^{n-1} < |\tau'| \le 2|\xi(\xi-\eta)| + |\sigma'| + 2^{m+1} < 2^{\ell-p+4} + 2^{k+1} + 2^{m+1} < 2^{\ell-p+5},$$

and $2^{n+1} > |\tau'| \ge 2|\xi(\xi-\eta)| - |\sigma'| - 2^{m+1} > 2^{\ell-p} - 2^{k+1} - 2^{m+1} \ge 2^{\ell-p-1}$, which imply that $\ell - p - 1 \le n \le \ell - p + 5$. On the other hand from the estimate

$$\iint \varphi_1(2^p\xi)^2\gamma_m(\tau - \sigma - Q(\xi-\eta))\, d\xi d\tau \le 2^{m-p+6} \tag{6.15}$$

for any function Q and Lemma 3.2 it follows that

$$\begin{aligned} I_{jk\ell mnp} &\le \|\varphi_1(2^p\xi)\hat{f}_{jk,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \\ &\le \left\|\iint \varphi_1(2^p\xi)\gamma_m(\tau - \sigma + P(\xi-\eta))\hat{f}_{jk,P}(\eta,\sigma)\hat{g}_{\ell m,-P}(\xi-\eta,\tau-\sigma)\, d\eta d\sigma\right\|_{L^2} \\ &\le c2^{(m-p)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}. \end{aligned}$$

Since $\hat{f}_{jk,P} * \hat{g}_{\ell m,-P} = \hat{g}_{\ell m,-P} * \hat{f}_{jk,P}$, the same calculation shows that $I_{jk\ell mnp}$ is estimated by $c2^{(k-p)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}$. Hence we have

$$I_{jk\ell mnp} \le c2^{(k\wedge m-p)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}. \tag{6.16}$$

Thus we obtain that

$$\sum_{j,\ell>0,|j-\ell|\le 2}\sum_{n}\sum_{k\vee m<\ell-p-2} 2^{-n/2} I_{jk\ell mnp} \le \sum_{j,\ell>0,|j-\ell|\le 2}\sum_{k\vee m<\ell-p-2}\sum_{n=\ell-p-1}^{\ell-p+5} 2^{-n/2} I_{jk\ell mnp}$$

$$\le c \sum_{j,\ell>0,|j-\ell|\le 2}\sum_{k,m} 2^{(m-\ell)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}$$

$$\le c'\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}.$$

It follows from the estimate (6.16) and the identity $k\wedge m + k\vee m = k+m$ that

$$\sum_{j,\ell>0,|j-\ell|\le 2}\sum_{n}\sum_{k\vee m\ge\ell-p-2} 2^{-n/2} I_{jk\ell mnp}$$

$$\le c \sum_{j,\ell>0,|j-\ell|\le 2}\sum_{n} 2^{-n/2}\sum_{k\vee m\ge\ell-p-2} 2^{(k\wedge m-p)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}$$

$$\le c' \sum_{j,\ell>0,|j-\ell|\le 2}\sum_{k,m} 2^{(k\wedge m+k\vee m-p+p-\ell)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}$$

$$\le c'\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}.$$

Combining these estimates, we can conclude that

$$G^{\#}_{003} \le c_0\sum_{p=0}^{j}(1+p)^{-2}\sum_{j,\ell>0,|j-\ell|\le 2}\sum_{n}\sum_{k,m} 2^{-n/2} I_{jk\ell mnp}$$

$$\le c_1\sum_{p=0}^{j}(1+p)^{-2}\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}} \le c\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}.$$

Estimate of G_{01} and G_{02}. By Lemma 5.1 we have $G_{01} = G_{010} + G_{011}$, where

$$G_{010} = \sum_{h=1}^{2}\sum_{n}\sum_{k,m} 2^{sh-n/2}\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{0k,P} * \hat{g}_{0m,-P}(\xi,\tau)\|_{L^2},$$

$$G_{011} = \sum_{\ell=1}^{\infty}\sum_{h=1\vee(\ell-2)}^{\ell+2}\sum_{n}\sum_{k,m} 2^{sh-n/2}\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{0k,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2}.$$

Note that when $|\eta|<2$, $|\xi-\eta|<2$, $|\xi|>1$

$$(1+|\log|\eta||)^2(1+|\log|\xi-\eta||)^2 \le 4(1+|\log|\eta||)^2 + 4(1+|\log|\xi-\eta||)^2.$$

In fact, $1<|\xi|\le|\eta|+|\xi-\eta|$ implies that $|\eta|>1/2$ or $|\xi-\eta|>1/2$. Hence we have

$$|\varphi_h(\xi)\hat{f}_{0k,P} * \hat{g}_{0m,-P}(\xi,\tau)|$$

$$= \left|\iint \varphi_h(\xi)(1+|\log|\eta||)^2(1+|\log|\xi-\eta||)^2\hat{f}^{\#}_{0k,P}(\eta,\sigma)\hat{g}^{\#}_{0m,-P}(\xi-\eta,\tau-\sigma)\,d\eta d\sigma\right|$$

$$\le 4B(\gamma^{\#}_{k,P}(\eta,\sigma);|\hat{f}^{\#}_{0k,P}|,|\hat{g}^{\#}_{0m,-P}|)(\xi,\tau) + 4B(\gamma^{\#}_{m,-P}(\xi-\eta,\tau-\sigma);|\hat{f}^{\#}_{0k,P}|,|\hat{g}^{\#}_{0m,-P}|)(\xi,\tau).$$

From this fact and (6.10) we see that

$$\|\varphi_h(\xi)\hat{f}_{0k,P} * \hat{g}_{0m,-P}(\xi,\tau)\|_{L^2} \le c(2^{k/2}+2^{m/2})\|f^{\#}_{0k,P}\|_{L^2}\|g^{\#}_{0m,-P}\|_{L^2},$$

which implies that

$$\begin{aligned} G_{010} &\le c\sum_{h=1}^{2}\sum_{n}\sum_{k,m} 2^{-n/2}\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{0k,P} * \hat{g}_{0m,-P}(\xi,\tau)\|_{L^2} \\ &\le c\sum_{k,m}(2^{k/2}+2^{m/2})\|f^{\#}_{0k,P}\|_{L^2}\|g^{\#}_{0m,-P}\|_{L^2} \le c'\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}. \end{aligned}$$

Similarly, by (6.12) we have

$$\begin{aligned} G_{011} &= \sum_{\ell=1}^{\infty}\sum_{h=1\vee(\ell-2)}^{\ell+2}\sum_{n}\sum_{k,m} 2^{sh-n/2}\|\varphi^{[P]}_{hn}(\xi,\tau)\hat{f}_{0k,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \\ &\le c\sum_{\ell=1}^{\infty}\sum_{k,m} 2^{s\ell}\|\hat{f}_{0k,P} * \hat{g}_{\ell m,-P}(\xi,\tau)\|_{L^2} \\ &\le c\sum_{\ell=1}^{\infty}\sum_{k,m} 2^{s\ell+k/2}\|f^{\#}_{0k,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2} \le c\|f\|_{B^{(s,1/2),\#}_{2,1,P}}\|g\|_{B^{(s,1/2),\#}_{2,1,-P}}. \end{aligned}$$

We can estimate G_{02} in the same way as G_{01}.

Finally, the arguments in the proof of (6.1) give the estimate of G_1. □

7. Proof of Theorem 2.4.

The auxiliary result to prove Theorem 2.4 is the following theorem:

THEOREM 7.1. *Let ρ be a weight on $\boldsymbol{R}_+$, $b \in \boldsymbol{R}$, $P(\xi)$ a real-valued C^∞-function, and define $W(t)$ by $\{W(t)f\}(x,t) := \mathscr{F}_x^{-1}e^{itP(\xi)}\mathscr{F}_x f(x,t)$.*

(a) *Assume that $\psi \in B^b_{2,1}(\boldsymbol{R})$.*

If $u_0 \in B^\rho_{2,1}(\boldsymbol{R}^d)$, then we have $\psi(t)W(t)u_0 \in B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})$ and

$$\|\psi(t)W(t)u_0\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})} = \|u_0\|_{B^\rho_{2,1}(\boldsymbol{R}^d)}\|\psi\|_{B^b_{2,1}(\boldsymbol{R})}. \tag{7.1}$$

If $u_0 \in B^{s,\#}_{2,1}(\boldsymbol{R}^d)$, then we have $\psi(t)W(t)u_0 \in B^{(s,b),\#}_{2,1,P}(\boldsymbol{R}^{d+1})$ and

$$\|\psi(t)W(t)u_0\|_{B^{(s,b),\#}_{2,1,P}(\boldsymbol{R}^{d+1})} = \|u_0\|_{B^{s,\#}_{2,1}(\boldsymbol{R}^d)}\|\psi\|_{B^b_{2,1}(\boldsymbol{R})}. \tag{7.2}$$

(b) *Assume that $\psi \in \mathscr{S}(\boldsymbol{R})$ and $1/2 \le b \le 1$. Then,*

$$\left\|\psi(t)\int_0^t W(t-t')f(x,t')\,dt'\right\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})} \le c\|f\|_{B^{(\rho,b-1)}_{2,1,P}(\boldsymbol{R}^{d+1})} \tag{7.3}$$

holds for any $f \in B^{(s,b-1)}_{2,1,P}(\boldsymbol{R}^{d+1})$, and

$$\left\|\psi(t)\int_0^t W(t-t')f(x,t')\,dt'\right\|_{B^{(s,b),\#}_{2,1,P}(\boldsymbol{R}^{d+1})} \le c\|f\|_{B^{(s,b-1),\#}_{2,1,P}(\boldsymbol{R}^{d+1})} \tag{7.4}$$

holds for any $f \in B^{(s,b-1),\#}_{2,1,P}(\boldsymbol{R}^{d+1})$, where c is a constant independent of f.

In proving Theorem 7.1 we may assume that $P(\xi)=0$ because of the following identity

$$\|f\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^{d+1})} = \|W(-t)f\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^{d+1})}, \tag{7.5}$$

which is a simple consequence of the formula $\mathscr{F}\{W(-t)f\} = \hat{f}(\xi,\tau+P(\xi))$ and Parseval's identity.

Proof of Theorem 7.1 (a). Since $\mathscr{F}(\psi(t)u_0(x)) = \hat{\psi}(\tau)\hat{u}_0(\xi)$, we have

$$\|\psi(t)u_0(x)\|_{B^{(\rho,b)}_{2,1}} = \sum_{j,k}\rho(2^j)2^{bk}\|\varphi_j(|\xi|)\hat{u}_0(\xi)\varphi_k(\tau)\hat{\psi}(\tau)\|_{L^2(\boldsymbol{R}^{d+1})} = \|u_0\|_{B^{\rho}_{2,1}}\|\psi\|_{B^{b}_{2,1}}.$$

The same arguments as above also work for the case $B^{s,\#}_{2,1}$. □

Proof of Theorem 7.1 (b). Setting $\hat{f}_k(\xi,\tau) := \varphi_k(\tau)\hat{f}(\xi,\tau)$, $\hat{\psi}_m(\tau) := \varphi_m(\tau)\hat{\psi}(\tau)$, by the identities $f=\sum_k f_k$, $\psi = \sum_m \psi_m$, we obtain that

$$F(x,t) := \psi(t)\int_0^t f(x,t')\,dt' = \sum_m\sum_k F_{km}(x,t), \tag{7.6}$$

$$F_{km}(x,t) := \psi_m(t)\int_0^t f_k(x,t')\,dt'. \tag{7.7}$$

To estimate the norm of F we divide F into two parts:

$$F = F_1+F_2,\quad F_1 = \sum_k\sum_{m=k}^{\infty}F_{km},\quad F_2 = \sum_k\sum_{m=0}^{k-1}F_{km}.$$

First we estimate the norm of F_1. It follows from the identity

$$\begin{aligned}\hat{F}_{km}(\xi,\tau) &= \frac{1}{2\pi}\int e^{-i\tau t}\psi_m(t)\,dt\int_0^t dt'\int e^{it'\sigma}\hat{f}_k(\xi,\sigma)\,d\sigma \\ &= \frac{1}{2\pi}\int \hat{f}_k(\xi,\sigma)\,d\sigma\int e^{-i\tau t}\psi_m(t)\,dt\int_0^t e^{i\sigma t'}\,dt'\end{aligned} \tag{7.8}$$

and $\int e^{-i\tau t}\psi_m(t)\,dt\int_0^t e^{i\sigma t'}\,dt' = \int_0^1 d\theta\int e^{-i(\tau-\sigma\theta)t}t\psi(t)\,dt = \sqrt{2\pi}\int_0^1 i\hat{\psi}'_m(\tau-\sigma\theta)\,d\theta$ that

$$\hat{F}_{km}(\xi,\tau) = \frac{i}{\sqrt{2\pi}}\int_0^1 d\theta\int(\hat{\psi}_m)'(\tau-\sigma\theta)\hat{f}_k(\xi,\sigma)\,d\sigma. \tag{7.9}$$

Since $\int|\hat{\psi}'_m(\tau-\sigma\theta)|\,d\tau = \|\hat{\psi}'_m\|_{L^1}$, $\int|\hat{\psi}'_m(\tau-\sigma\theta)|\,d\sigma = (1/\theta)\|\hat{\psi}'_m\|_{L^1}$, by Lemma 3.2 we have

$$\|\varphi_j(|\xi|)\varphi_n(\tau)\hat{F}_{km}(\xi,\tau)\|_{L^2} \le C\int_0^1\frac{d\theta}{\sqrt{\theta}}\|\hat{\psi}'_m\|_{L^1}\|\hat{f}_{jk}\|_{L^2} \le C'2^{m/2}\|\hat{\psi}'_m\|_{L^2}\|f_{jk}\|_{L^2}$$

where $\hat{f}_{jk}(\xi,\tau) := \varphi_j(|\xi|)\varphi_k(\tau)\hat{f}(\xi,\tau)$. On the other hand, if $\varphi_n(\tau)\gamma_k(\sigma)\hat{\psi}'_m(\tau-\sigma\theta)\neq 0$, then $|\tau-\sigma\theta|<2^{m+1}$, $|\sigma|<2^{k+1}$, $|\tau|>2^{n-1}$, which gives $n\le m+2$. Thus we obtain that

$$\|F_1\|_{B^{(\rho,b)}_{2,1}} \le c\sum_j \rho(2^j)\sum_k \sum_{m=k}^{\infty}\sum_{n=0}^{m+2} 2^{bn}2^{m/2}\|\hat{\psi}'_m\|_{L^2}\|f_{jk}\|_{L^2} \tag{7.10}$$

$$\le c'\sum_j \rho(2^j)\sum_k 2^{(b-1)k}\sum_{m=k}^{\infty} 2^{(3/2)m}\|\hat{\psi}'_m\|_{L^2}\|f_{jk}\|_{L^2}$$

$$\le c'\|t\psi(t)\|_{B^{3/2}_{2,1}}\cdot\|f\|_{B^{(\rho,b-1)}_{2,1}}.$$

Next, we estimate the norm of F_2. Since

$$\int e^{-it\tau}\psi_m(t)\,dt\int_0^t e^{i\sigma t'}\,dt' = \frac{1}{i\sigma}\int (e^{-i(\tau-\sigma)t} - e^{-i\tau t})\psi_m(t)\,dt = \sqrt{2\pi}\frac{\hat{\psi}_m(\tau-\sigma)-\hat{\psi}_m(\tau)}{i\sigma}$$

holds when $\sigma \ne 0$, by (7.8) we have

$$\hat{F}_{km}(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int \frac{\hat{\psi}_m(\tau-\sigma)-\hat{\psi}_m(\tau)}{i\sigma}\hat{f}_k(\xi,\sigma)\,d\sigma, \tag{7.11}$$

so that we obtain that $F_2 = F_{21} + F_{22}$, where

$$\hat{F}_{21}(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\sum_k\sum_{m=0}^{k-1}\int \hat{\psi}_m(\tau-\sigma)\frac{f_k(\xi,\sigma)}{i\sigma}\,d\sigma \tag{7.12}$$

$$\hat{F}_{22}(\xi,\tau) = \frac{-1}{\sqrt{2\pi}}\sum_k\sum_{m=0}^{k-1}\int \hat{\psi}_m(\tau)\frac{f_k(\xi,\sigma)}{i\sigma}\,d\sigma. \tag{7.13}$$

To estimate the norm of F_{21} we note that $\varphi_n(\tau)\hat{F}_{21}(\xi,\tau)$ is the sum of terms with $0 \le n \le k+2$, since $|\tau| \le |\tau-\sigma| + |\sigma| < 2^{m+1} + 2^{k+1} < 2^{k+2}$ on the support of the function $\hat{\psi}_m(\tau-\sigma)\gamma_k(\sigma)$ and since $2^{n-1} < |\tau|$ on the support of φ_n. Thus, by Lemma 3.1 and the formula $\int|\hat{\psi}_m(\tau-\sigma)|\,d\tau = \int|\hat{\psi}_m(\tau-\sigma)|\,d\sigma = \|\hat{\psi}_m\|_{L^1}$ we obtain

$$\|F_{21}\|_{B^{(\rho,b)}_{2,1}} \le c\sum_j \rho(2^j)\sum_k\sum_{m=0}^{k-1}\sum_{n=0}^{k+2} 2^{bn}\|\hat{\psi}_m\|_{L^1}2^{-k}\|f_{jk}\|_{L^2}$$

$$\le c'\sum_j\sum_k \rho(2^j)2^{(b-1)k}\sum_m 2^{m/2}\|\psi_m\|_{L^2}\|f_{jk}\|_{L^2} \le c'\|\psi\|_{B^{1/2}_{2,1}}\cdot\|f\|_{B^{(\rho,b-1)}_{2,1}}.$$

Finally consider F_{22}. Since $\varphi_n(\tau)\hat{\psi}_m(\tau) \ne 0$ only when $|m-n| \le 1$, $\varphi_n(\tau)\hat{F}_{22}(\xi,\tau)$ is the sum of the terms with $|m-n| \le 1$. Hence, with the aid of the inequality

$$\int\left|\frac{\varphi_j(|\xi|)\hat{f}_k(\xi,\sigma)}{\sigma}\right|d\sigma \le 2^{1-k/2}\|\hat{f}_{jk}(\xi,\tau)\|_{L^2(\boldsymbol{R}_\tau)}, \tag{7.14}$$

which is a consequence of Schwarz's inequality, we have

$$\|F_{22}\|_{B^{(\rho,b)}_{2,1}} \le c\sum_j \rho(2^j)\sum_k\sum_{m=0}^{k-1}\sum_{n=m-1}^{m+1} 2^{bn-k/2}\|\hat{\psi}_m\|_{L^2}\|f_{jk}\|_{L^2}$$

$$\le c'\sum_j \rho(2^j)\sum_k 2^{(b-1)k}\sum_m 2^{m/2}\|\hat{\psi}_m\|_{L^2}\|f_{jk}\|_{L^2} \le c'\|\psi\|_{B^{1/2}_{2,1}}\cdot\|f\|_{B^{(\rho,b-1)}_{2,1}}.$$

The same argument as above also works for the case $B^{(s,1/2),\#}_{2,1,P}$. □

Now we are ready to *prove Theorem* 2.4. Let $f, g \in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times I)$, and let $\tilde{f}, \tilde{g} \in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^2)$, $\tilde{f}|_{\boldsymbol{R}\times I} = f$, $\tilde{g}|_{\boldsymbol{R}\times I} = g$. We take $\psi(t) \in C_0^\infty$ such that $\psi(t) = 1$ if $|t| \le 1$, and put

$$\tilde{F}(x,t) := \psi(t)\int_0^t W(t-t')\{c_1\tilde{f}(x,t')\tilde{g}(x,t') + c_2\overline{\tilde{f}(x,t')\tilde{g}(x,t')}\}\,dt'.$$

Then, $F(x,t) := \int_0^t W(t-t')\{c_1 f(x,t')g(x,t') + c_2\overline{f(x,t')g(x,t')}\}\,dt' = \tilde{F}(x,t)|_{\boldsymbol{R}\times I}$. By (1.11), Theorem 7.1 (b) and Theorem 2.3 (I), we have

$$\|F\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times I)} \le \|\tilde{F}\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^2)} \le C\|\tilde{f}\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^2)}\|\tilde{g}\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}^2)}.$$

Taking the infimum for all $\tilde{f}$ and $\tilde{g}$ with $\tilde{f}|_{\boldsymbol{R}\times I} = f$, $\tilde{g}|_{\boldsymbol{R}\times I} = g$, this gives (2.7). We can prove (2.8) and (2.9) in the same way, and complete the proof of Theorem 2.4.

8. Proof of Theorem 2.5.

In proving Theorem 2.5 we use the theorem on the equivalent norm of anisotropic Besov spaces. For the sake of completeness we state and prove it here:

THEOREM 8.1. *Let* $0 < b < 1$. *Then the norm of* $B^{(\rho,b)}_{p,q}(\boldsymbol{R}^{d+1})$ *is equivalent to the norm*

$$\|\{\rho(2^j)\|f_j\|_{L^p(\boldsymbol{R}^{d+1})}\}\|_{\ell^q(\overline{\boldsymbol{N}})} + \|[\|\{\|\rho(2^j)|t'|^{-b}\{f_j(x,t+t') - f_j(x,t)\}\|_{L^p}\}\|_{\ell^q(\overline{\boldsymbol{N}})}]\|_{L^q_*(\boldsymbol{R}_{t'})}. \tag{8.1}$$

Also, the norm of the space $B^{(s,b),\#}_{2,q}(\boldsymbol{R}^{d+1})$ *is equivalent to the norm*

$$\begin{aligned}\|\{2^{sj}\|f_j\|_{L^2(\boldsymbol{R})}\}\|_{\ell^q(\boldsymbol{N})} &+ \|[\|\{\|2^{sj}|t'|^{-b}\{f_j(x,t+t') - f_j(x,t)\}\|_{L^2}\}\|_{\ell^q(\boldsymbol{N})}]\|_{L^q_*(\boldsymbol{R}_{t'})}\\ &+ \|f_0^\#\|_{L^2} + \|[\{|t'|^{-b}\|\{f_0^\#(x,t+t') - f_0^\#(x,t)\}\|_{L^2}\}]\|_{L^q_*(\boldsymbol{R}_{t'})}.\end{aligned} \tag{8.2}$$

Where $\hat{f}_j(\xi,\tau) = \varphi_j(|\xi|)\hat{f}(\xi,\tau)$, $\hat{f}_0^\#(\xi,\tau) = \varphi_0(|\xi|)(1 + |\log|\xi||)^{-2}\hat{f}(\xi,\tau)$.

Here we write $L^q_*(\boldsymbol{R}) := L^q(\boldsymbol{R}, dt/t)$.

To prove this theorem we need the following.

LEMMA 8.1. *Let* $H(t) = \min(t,1)$, *and assume that* $m > \theta > 0$. *Then the inequality* $\|\sum_{k=0}^\infty (H(2^k|t|)^m/(2^k|t|)^\theta)a_k\|_{L^q_*(\boldsymbol{R})} \le C\|\{a_k\}\|_{\ell^q(\overline{\boldsymbol{N}})}$ *holds for any* $\{a_k\} \in \ell^q(\overline{\boldsymbol{N}})$, *and the inequality* $\|\{\int_{\boldsymbol{R}}(H(2^k|t|)^m/(2^k|t|)^\theta)f(t)(dt/t)\}\|_{\ell^q(\overline{\boldsymbol{N}})} \le C\|f\|_{L^q_*(\boldsymbol{R})}$ *holds for any* $f \in L^q_*(\boldsymbol{R})$. *Here* C *is a constant.*

PROOF. This lemma is a special case of Lemma 3.1, since

$$\sum_{k=0}^\infty \frac{H(2^k|t|)^m}{(2^k|t|)^\theta} \le C_1 < \infty \quad \text{for any } t \in \boldsymbol{R},$$

$$\int_{\boldsymbol{R}} (H(2^k|t|)^m/(2^k|t|)^\theta)(dt/t) = C_2 < \infty \quad \text{for any } k \in \overline{\boldsymbol{N}}. \qquad \square$$

PROOF OF THEOREM 8.1. Put $\tilde{\varphi}_k(z) = \varphi_k(z/2) + \varphi_k(z) + \varphi_k(2z)$ for $k \geq 1$, $\tilde{\varphi}_0(z) = \varphi_0(z/2)$. Then we have $\varphi_k \tilde{\varphi}_k = \varphi_k$. Hence, $f_j(x,t) = c_1 \sum_{k=0}^{\infty} \tilde{\Phi}_k \underset{t}{*} f_{jk}$, where $c_1 = 1/\sqrt{2\pi}$, $\hat{f}_j(\xi,\tau) = \varphi_j(|\xi|)\hat{f}(\xi,\tau)$, $\hat{f}_{jk}(\xi,\tau) = \varphi_j(|\xi|)\varphi_k(\tau)\hat{f}(\xi,\tau)$, $\tilde{\Phi}_k(t) = \mathscr{F}^{-1}_t \tilde{\varphi}_k(\tau)$. We also see that

$$\tilde{\Phi}_k \underset{t}{*} f_{jk}(x, t+t') - \tilde{\Phi}_k \underset{t}{*} f_{jk}(x,t) = \int_0^{t'} \int \tilde{\Phi}_k'(t-r+u) f_{jk}(x,r)\, dr du,$$

where $\tilde{\Phi}_k'(t) = d\tilde{\Phi}_k/dt$. Hence $\|f_j(x,t+t') - f_j(x,t)\|_{L^p} \leq \sum_{k=0}^{\infty} \|\tilde{\Phi}_k'\|_{L^1} \|f_{jk}\|_{L^p} |t'|$. In view of the identity $\int |\tilde{\Phi}_k'(t)|\, dt = \int |(d/dt) 2^{k-1}\tilde{\Phi}_1(2^{k-1}t)|\, dt = 2^{k-1}\|\tilde{\Phi}_1'\|_{L^1}$ for $k \geq 1$, this implies that $\|f_j(x,t+t') - f_j(x,t)\|_{L^p} \leq c \sum_{k=0}^{\infty} 2^k |t'| \|f_{jk}\|_{L^p}$. On the other hand,

$$\|\tilde{\Phi}_k \underset{t}{*} f_{jk}(x,t+t') - \tilde{\Phi}_k \underset{t}{*} f_{jk}(x,t)\|_{L^p} \leq 2\|\tilde{\Phi}_k\|_{L^1} \|f_{jk}\|_{L^p} = 2\|\tilde{\Phi}_1\|_{L^1}\|f_{jk}\|_{L^p}.$$

Therefore we have $\|f_j(x,t+t') - f_j(x,t)\|_{L^p} \leq c \sum_{k=0}^{\infty} H(2^k|t'|)\|f_{jk}\|_{L^p}$, which gives

$$\rho(2^j)|t'|^{-b} \|f_j(x,t+t') - f_j(x,t)\|_{L^p} \leq c \sum_{k=0}^{\infty} K(t',k)\rho(2^j) 2^{bk} \|f_{jk}\|_{L^p},$$

where $K(t',k) = H(2^k|t'|)(2^k|t'|)^{-b}$. Thus, by Lemma 8.1, we have

$$\|[\|\{\|\rho(2^j)|t'|^{-b}\{f_j(x,t+t') - f_j(x,t)\}\|_{L^p}\}\|_{\ell^q(\bar{\boldsymbol{N}})}]\|_{L^q_*} \leq c\|\{\|\rho(2^j)2^{bk} f_{jk}\|_{L^p}\}\|_{\ell^q(\bar{\boldsymbol{N}}\times\bar{\boldsymbol{N}})}.$$

By Minkowski's inequality and Hölder's inequality we have

$$\begin{aligned}\|\{\rho(2^j)\|f_j\|_{L^p}\}\|_{\ell^q} &\leq \left\| \left\{ \sum_k \rho(2^j)\|f_{jk}\|_{L^p} \right\} \right\|_{\ell^q} \leq \sum_k \|\{\rho(2^j)\|f_{jk}\|_{L^p}\}\|_{\ell^q} \\ &\leq c\|\{2^{bk}\rho(2^j)\|f_{jk}\|_{L^p}\}\|_{\ell^q(\bar{\boldsymbol{N}}\times\bar{\boldsymbol{N}})} = c'\|f\|_{B^{(\rho,b)}_{p,q}}.\end{aligned}$$

We show the reverse inequality. For $k \geq 1$ we have

$$f_{jk}(x,t) = \int \Phi_k(t-t') f_j(x,t')\, dt' = \int \Phi_k(-t')(f_j(x,t+t') - f_j(x,t))\, dt',$$

where $\Phi_k(t) := \mathscr{F}_t^{-1}\varphi_k(\tau)$, which gives that $\|f_{jk}\|_{L^p} \leq \int |\Phi_k(t')|\, \|f_j(x,t+t') - f_j(x,t)\|_{L^p}\, dt'$. We set $K(t',k) = |t'|^{b+1}|\Phi_k(t')|2^{bk} = 2^{-1}|t'|^{b+1}2^{k(b+1)}|\Phi_1(2^{k-1}t')|$. Since $\Phi_1 \in \mathscr{S}$, $|\Phi_1(t)| \leq c\min(1,t^{-2}) = ct^{-2}H(t)^2$, so that $K(t',k) \leq c(2^k|t'|)^{b-1}H(2^k|t'|)^2$. Lemma 8.1 gives

$$\|\{\|\rho(2^j)2^{bk} f_{jk}\|_{L^p}\}\|_{\ell^q(\bar{\boldsymbol{N}}\times\bar{\boldsymbol{N}})} \leq c\|[\|\{\|\rho(2^j)|t'|^{-b}\{f_j(x,t+t') - f_j(x,t)\}\|_{L^p}\}\|_{L^q_*(\boldsymbol{R}_{t'})}]\|_{\ell^q_{(j)}}.$$

For $k = 0$, it is clear that $\|\{\|\{\rho(2^j)f_{j0}\}\|_{L^p}\}\|_{\ell^q} \leq c\|\{\|\{\rho(2^j)f_j\}\|_{L^p}\}\|_{\ell^q}$. Similarly, we have the same consequence for the norm of $B^{(s,b),\#}_{2,q}$. □

Next, we recall the following integral representations:

LEMMA 8.2. *Let $I = (-a,a)$, $a > 0$, $1 \leq p < \infty$, and X a Banach space. Assume that $\omega \in C^\infty(\boldsymbol{R})$, $\int \omega(z)\, dz = 1$, $\operatorname{supp}\omega \subset (-1,1)$, $\omega(z) \geq 0$, and put $\omega_1(t,z) := \omega(z-t)$, $M_1(t,z) := \partial_z\{z\omega_1(t,z)\} = \partial_z\{z\omega(z-t)\}$. Then,*

$$\vec{f}(t) = \int_0^a \frac{d\lambda}{\lambda} \int \frac{1}{\lambda} M_1\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \vec{f}(t_1)\, dt_1 + \vec{f}^{[0]} \tag{8.3}$$

$$= \sum_{i=1}^{3} \int_0^a \frac{d\lambda}{\lambda} \int \frac{1}{\lambda} K_i\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \vec{u}_i(\lambda, t_1)\, dt_1 + \vec{f}^{[0]}, \tag{8.4}$$

$$\vec{f}^{[0]} := \int \frac{1}{a} \omega\left(\frac{-t_1}{a}\right) \vec{f}(t_1)\, dt_1, \tag{8.5}$$

$$\vec{u}_1(\lambda, t_1) := \int_\lambda^a \frac{\lambda}{\mu} \frac{d\mu}{\mu} \int \left(\frac{-\mu}{a}\right) \frac{1}{\mu} L_1\left(\frac{t_1}{a}, \frac{t_1 - t_2}{\mu}\right) \vec{f}(t_2)\, dt_2, \tag{8.6}$$

$$\vec{u}_2(\lambda, t_1) := \int_\lambda^a \frac{\lambda}{\mu} \frac{d\mu}{\mu} \int \frac{1}{\mu} L_2\left(\frac{t_1}{a}, \frac{t_1 - t_2}{\mu}\right) \vec{f}(t_2)\, dt_2, \tag{8.7}$$

$$\vec{u}_3(\lambda, t_1) := \int_0^\lambda \frac{d\mu}{\mu} \int \frac{1}{\mu} L_3\left(\frac{t_1}{a}, \frac{t_1 - t_2}{\mu}\right) \vec{f}(t_2)\, dt_2, \tag{8.8}$$

hold for any $\vec{f} \in L^p(I; X)$, *where* $K_1(t,z) = K_2(t,z) := z\omega(z-t)$, $K_3 := M_1$, $L_1(t,z) := \partial_z\{z\omega'(z-t)\}$, $L_2(t,z) := \partial_z^2\{z\omega(z-t)\}$, $L_3 := M_1$.

Proof (cf. [**6**] p. 331 or [**7**]). Let $\vec{f} \in L^p(I; X)$ and define

$$\vec{f}_1(\lambda, t) := \int \frac{1}{\lambda} \omega_1\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \vec{f}(t_1)\, dt_1, \tag{8.9}$$

$$\vec{U}_1(\lambda, t) := \int \frac{1}{\lambda} M_1\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \vec{f}(t_1)\, dt_1. \tag{8.10}$$

Then, we see that $\|\vec{f}_1(\lambda, t) - \vec{f}(t)\|_{L^p(I;X)} \le \int_{|z| \le 2} \|\vec{f}(t - \lambda z) - \vec{f}(t)\|_{L^p(I;X)}\, dz \to 0$ as $\lambda \to +0$. This and the identity $\partial_\lambda \vec{f}_1(\lambda, t) = -\vec{U}_1(\lambda, t)/\lambda$ give (8.3).

Substitute (8.3) into the right-hand side of (8.10). An integration by parts gives

$$\int \frac{1}{\lambda} M_1\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \left\{\int_\lambda^a \vec{U}_1(\mu, t_1) \frac{d\mu}{\mu}\right\} dt_1 = \int K_1\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \left\{\int_\lambda^a \partial_{t_1} \vec{U}_1(\mu, t_1) \frac{d\mu}{\mu}\right\} dt_1,$$

$$\partial_{t_1} \vec{U}_1(\mu, t_1) = \frac{-1}{a} \int \frac{1}{\mu} L_1\left(\frac{t_1}{a}, \frac{t_1 - t_2}{\mu}\right) \vec{f}(t_2)\, dt_2 + \frac{1}{\mu} \int \frac{1}{\mu} L_2\left(\frac{t_1}{a}, \frac{t_1 - t_2}{\mu}\right) \vec{f}(t_2)\, dt_2.$$

Therefore we have (8.4).

The following lemma is closely related to the above formula (8.4).

Lemma 8.3. *Let* X *be a Banach space*, $1 \le p \le \infty$, $I := (-a, a)$, $a > 0$. *Assume that* $K(t,z)$ *is a bounded continuous function of* (t,z), *and that* $\operatorname{supp}_z K(t,z) \subset (-1,1) + t$. *Define*

$$\vec{U}(\lambda, t) := \int \frac{1}{\lambda} K\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \vec{f}(t_1)\, dt_1. \tag{8.11}$$

(a) *The inequality* $\|\vec{U}(\lambda, t)\|_{L^p(I;X)} \le 4C_0 \|\vec{f}\|_{L^p(I;X)}$ *holds for any* $\vec{f} \in L^p(I; X)$ *and* $0 < \lambda \le a$.

(b) *Assume that* $\int K(t,z)\,dz = 0$. *Then, the inequality*

$$\|\vec{U}(\lambda,t)\|_{L^p(I;X)} \le \frac{C_0}{\lambda}\int \chi\left(\frac{t_1}{2\lambda}\right)\|\vec{f}(t+t_1)-\vec{f}(t)\|_{L^p(I\cap(I-t_1);X)}\,dt_1$$

holds for any $\vec{f} \in L^p(I;X)$ *and* $0 < \lambda \le a$.

Here $C_0 = \sup_{t,z}|K(t,z)|$, *and* χ *denotes the defining function of the interval* $(-1,1)$.

Proof. (a) This follows from Lemma 2.1, since

$$\int\left|\frac{1}{\lambda}K\left(\frac{t}{a},\frac{t-t_1}{\lambda}\right)\right|dt \le 4C_0, \quad \int\left|\frac{1}{\lambda}K\left(\frac{t}{a},\frac{t-t_1}{\lambda}\right)\right|dt_1 \le 4C_0. \tag{8.12}$$

(b) Since $t+t_1 \in I$ if $K(t/a, -t_1/\lambda) \ne 0$, $t \in I$, $0 < \lambda \le a$, we see that

$$\begin{aligned}\|\vec{U}(\lambda,t)\|_{L^p(I;X)} &= \left\|\int\frac{1}{\lambda}K\left(\frac{t}{a},\frac{-t_1}{\lambda}\right)\{\vec{f}(t+t_1)-\vec{f}(t)\}\,dt_1\right\|_{L^p(I;X)}\\ &\le \frac{C_0}{\lambda}\int_{|t_1|\le 2\lambda}\left\|\chi\left(\frac{t+t_1}{a}\right)\{\vec{f}(t+t_1)-\vec{f}(t)\}\right\|_{L^p(I;X)}dt_1\\ &= \frac{C_0}{\lambda}\int\chi\left(\frac{t_1}{2\lambda}\right)\|\{\vec{f}(t+t_1)-\vec{f}(t)\}\|_{L^p(I\cap(I-t_1);X)}\,dt_1. \qquad \square\end{aligned}$$

To prove Theorem 2.5 we need the following

Theorem 8.2. *Let* $I = (-a,a)$, $0 < a \le 1$, $1 \le p < \infty$, $0 < b < 1$, *and let* ρ *be a weight on* $\boldsymbol{R}_+$. *Then, the following formula holds for any* $f \in B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d \times I)$:

$$f(x,t) = \sum_{i=1}^{3}\int_0^a\frac{d\lambda}{\lambda}\int\frac{1}{\lambda}K_i\left(\frac{t}{a},\frac{t-t_1}{\lambda}\right)u_i(\lambda,x,t_1)\,dt_1 + f^{[0]}(x), \tag{8.13}$$

$$f^{[0]}(x) := \int\frac{1}{a}\omega\left(\frac{-t_1}{a}\right)f(x,t_1)\,dt_1, \tag{8.14}$$

$$u_1(\lambda,x,t_1) := \int_\lambda^a\frac{\lambda}{\mu}\frac{d\mu}{\mu}\int\left(\frac{-\mu}{a}\right)\frac{1}{\mu}L_1\left(\frac{t_1}{a},\frac{t_1-t_2}{\mu}\right)f(x,t_2)\,dt_2, \tag{8.15}$$

$$u_2(\lambda,x,t_1) := \int_\lambda^a\frac{\lambda}{\mu}\frac{d\mu}{\mu}\int\frac{1}{\mu}L_2\left(\frac{t_1}{a},\frac{t_1-t_2}{\mu}\right)f(x,t_2)\,dt_2, \tag{8.16}$$

$$u_3(\lambda,x,t_1) := \int_0^\lambda\frac{d\mu}{\mu}\int\frac{1}{\mu}L_3\left(\frac{t_1}{a},\frac{t_1-t_2}{\mu}\right)f(x,t_2)\,dt_2. \tag{8.17}$$

Moreover,

$$\|G_I(t')|t'|^{-b}\|_{L^1_*(\boldsymbol{R})} + \|f\|_{L^p(I;B^\rho_{p,q}(\boldsymbol{R}^d))} \le C_1\|f\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d\times I)}, \tag{8.18}$$

$$\|f\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d\times I)} \le C_2\|G_I(t')|t'|^{-b}\|_{L^1_*(\boldsymbol{R})} + C_2a^{-1/p}\|f\|_{L^p(I;B^\rho_{p,q}(\boldsymbol{R}^d))}, \tag{8.19}$$

$$G_I(t') := \|\{\rho(2^j)\|P_j\{f(x,t+t')-f(x,t)\}\|_{L^p(\boldsymbol{R}^d\times I\cap(I-t'))}\}\|_{\ell^1}, \tag{8.20}$$

where $P_j := \Phi_j \underset{x}{*}$, $\Phi_j(x) = c_d\mathscr{F}_x^{-1}\varphi_j(|\xi|)$, *and* C_1, C_2 *are constants independent of* a.

Proof. Let $\tilde{f} \in B_{p,1}^{(\rho,b)}(\boldsymbol{R}^{d+1})$ be an extension of $f \in B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d \times I)$. Then, by Theorem 8.1 we have $\|G_I(t')|t'|^{-b}\|_{L_*^1(\boldsymbol{R})} \leq \|[\|\{\rho(2^j)\|P_j\{\tilde{f}(x,t+t') - \tilde{f}(x,t)\}\|_{L^p(\boldsymbol{R}^{d+1})}\}\|_{\ell^1}]|t'|^{-b}\|_{L_*^1(\boldsymbol{R})} \leq C\|\tilde{f}\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^{d+1})}$. Thus we have $\|G_I(t')|t'|^{-b}\|_{L_*^1(\boldsymbol{R})} \leq C\|f\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d\times I)}$. Also, from

$$\|f\|_{L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))} \leq \left\| \left[\sum_j \rho(2^j)\|P_j\tilde{f}(x,t)\|_{L^p(\boldsymbol{R}^d)} \right] \right\|_{L^p(\boldsymbol{R})}$$

$$\leq \sum_j \rho(2^j)\|P_j\tilde{f}(x,t)\|_{L^p(\boldsymbol{R}^{d+1})} \leq C\|\tilde{f}\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^{d+1})}$$

we see that $B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d \times I) \subset L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))$ and $\|f\|_{L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))} \leq C\|f\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d\times I)}$.

Next, by Lemma 8.2 we see that (8.13) holds in the topology of $L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))$. We will show that the integrals with respect to λ on the right-hand side of this formula are convergent in $B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d \times I)$.

Take $\psi \in C^\infty(\boldsymbol{R})$ with $\operatorname{supp}\psi \subset (-2,2)$, $\psi(t) = 1$ on $[-1,1]$, $0 \leq \psi(t) \leq 1$, put $\tilde{u}_i(x,t) = u_i(x,t)$ for $t \in I$, $\tilde{u}_i(x,t) = 0$ for $t \notin I$, and define

$$F_i(\lambda,x,t) := \int \frac{1}{\lambda} K_i\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) u_i(\lambda,x,t_1)\,dt_1, \tag{8.21}$$

$$\tilde{F}_i(\lambda,x,t) := \int \frac{1}{\lambda} \psi\left(\frac{t}{a}\right) K_i\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right) \tilde{u}_i(\lambda,x,t_1)\,dt_1, \tag{8.22}$$

for $i = 1,2,3$. From Lemma 3.1 and the inequality

$$\left|\psi\left(\frac{t+t'}{a}\right)K_i\left(\frac{t+t'}{a}, \frac{t+t'-t_1}{\lambda}\right) - \psi\left(\frac{t}{a}\right)K_i\left(\frac{t}{a}, \frac{t-t_1}{\lambda}\right)\right|$$

$$\leq CH\left(\frac{t'}{\lambda}\right)\left\{\chi\left(\frac{t+t'-t_1}{3\lambda}\right) + \chi\left(\frac{t-t_1}{3\lambda}\right)\right\}$$

if $0 < \lambda \leq a$, where $H(z) := \min\{z,1\}$ and C is a constant independent of (a,λ,t,t',t_1), it follows that

$$\|P_j\{\tilde{F}_i(\lambda,x,t+t') - \tilde{F}_i(\lambda,x,t)\}\|_{L^p(\boldsymbol{R}^{d+1})} \leq CH\left(\frac{|t'|}{\lambda}\right)\|P_ju_i(\lambda,x,t_1)\|_{L^p(\boldsymbol{R}^d\times I)}.$$

Also, from Lemma 3.1 and the inequalities similar to (8.12) it follows that

$$\|P_j\tilde{F}_i(\lambda,x,t)\|_{L^p(\boldsymbol{R}^{d+1})} \leq C\|P_ju_i(\lambda,x,t_1)\|_{L^p(\boldsymbol{R}^d\times I)}. \tag{8.23}$$

Hence we have

$$\|\tilde{F}_i(\lambda,x,t)\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^{d+1})} \leq C\int_{\boldsymbol{R}} H\left(\frac{|t'|}{\lambda}\right)|t'|^{-b}\frac{dt'}{t'}\sum_j \rho(2^j)\|P_ju_i(\lambda,x,t_1)\|_{L^p(\boldsymbol{R}^d\times I)}$$

$$+ C\sum_j \rho(2^j)\|P_ju_i(\lambda,x,t_1)\|_{L^p(\boldsymbol{R}^d\times I)}$$

$$\leq C'(\lambda^{-b}+1)\sum_j \rho(2^j)\|P_ju_i(\lambda,x,t_1)\|_{L^p(\boldsymbol{R}^d\times I)}.$$

On the other hand, it follows from Lemma 8.3 that

$$\sum_j \rho(2^j)\left\|\int \frac{1}{\mu} L_i\left(\frac{t_1}{a}, \frac{t_1-t_2}{\mu}\right) P_j f(x,t_2)\,dt_2\right\|_{L^p(\boldsymbol{R}^d\times I)} \le C\int \chi\left(\frac{t_2}{2\mu}\right)\frac{|t_2|}{2\mu} G_I(t_2)\frac{dt_2}{|t_2|} \tag{8.24}$$

holds for $i = 1,2,3$, which gives that

$$\sum_j \rho(2^j)\|P_j u_i(\lambda,x,t_1)\|_{L^p(\boldsymbol{R}^d\times I)} \le C\int H_i\left(\frac{|t_2|}{2\lambda}\right) G_I(t_2)\frac{dt_2}{|t_2|} \tag{8.25}$$

holds for $i = 1,2,3$, where $H_1(z) = H_2(z) = H(z)^2/z$, $H_3(z) = \chi(z)$. (Note that $2\int_0^\infty \chi(r/z)\chi(r)r\,dr = H(z)^2$, $\int_0^\infty \chi(z/r)\chi(r)\,dr \le \chi(z)$.)

In view of the fact that $\tilde{F}_i(\lambda,x,t)|_{\boldsymbol{R}^d\times I} = F_i(\lambda,x,t)$, these inequalities imply that

$$\begin{aligned}\int_0^a \|F_i(\lambda,x,t)\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d\times I)}\frac{d\lambda}{\lambda} &\le C\int G_I(t_2)\frac{dt_2}{|t_2|}\int_0^a(\lambda^{-b}+1)H_i\left(\frac{|t_2|}{2\lambda}\right)\frac{d\lambda}{\lambda}\\ &\le C'\int G_I(t_2)|t_2|^{-b}\frac{dt_2}{|t_2|}, \quad \text{for } i=1,2,3.\end{aligned} \tag{8.26}$$

Finally, by Hölder's inequality we have

$$\|f^{[0]}(x)\|_{B_{p,1}^{\rho}(\boldsymbol{R}^d)} \le \int \frac{1}{a}\omega\left(\frac{-t_2}{a}\right)\|f(x,t_2)\|_{B_{p,1}^{\rho}(\boldsymbol{R}^d)}\,dt_2 \le Ca^{-1/p}\|f(x,t_2)\|_{L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))}.$$

Hence, by Theorem 7.1 we have

$$\|1_I(t)f^{[0]}(x)\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d\times I)} \le \|\psi(t)f^{[0]}\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^{d+1})} \le Ca^{-1/p}\|f(x,t_2)\|_{L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))},$$

where $1_I(t) = 1$ for any $t\in I$. □

Theorem 8.3. *Let $I = (-a,a)$, $0 < a \le 1$, $1 < p < \infty$, $1/p \le b < 1$, and let ρ be a weight on $\boldsymbol{R}_+$. Then, $B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d\times I) \subset BC(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))$ with continuous inclusion, and*

$$\|f\|_{B_{p,1}^{(\rho,b)}(\boldsymbol{R}^d\times I)} \le C\|G_I(t')|t'|^{-b}\|_{L_*^1(\boldsymbol{R})} + C\|f(x,t_0)\|_{B_{p,1}^{\rho}(\boldsymbol{R}^d)} \tag{8.27}$$

holds, where G_I is the function defined by (8.20), *t_0 is any fixed point in I, and C is a constant independent of a. Here, BC denotes the space of bounded continuous functions.*

Proof. By (8.25) we see that

$$\|u_i(\lambda,x,t_1)\|_{L^p(I;B_{p,1}^{\rho}(\boldsymbol{R}^d))} \le \sum_j \rho(2^j)\|P_j u_i(\lambda,x,t)\|_{L^p(\boldsymbol{R}^{d+1})} \le C\int H_i\left(\frac{|t_2|}{2\lambda}\right)G_I(t_2)\frac{dt_2}{|t_2|}.$$

Since the right-hand side is finite, which follows from $G_I(t_2)|t_2|^{-b}\in L_*^1(\boldsymbol{R})$ (see Proof of Theorem 8.2), this implies that $F_i(\lambda,x,t) = \int(1/\lambda)K_i(t/a,(t-t_1)/\lambda)u_i(\lambda,x,t_1)\,dt_1$ is a bounded continuous $B_{p,1}^{\rho}(\boldsymbol{R}^d)$-valued function of $t\in I$ for any $0<\lambda\le a$. Hence, the inequality

$$(8.28)\qquad \int_0^a \|F_i(\lambda,x,t)\|_{B^\rho_{p,1}(\boldsymbol{R}^d)}\frac{d\lambda}{\lambda} \le C\int_0^a \lambda^{-1/p}\|u_i(\lambda,x,t_1)\|_{L^p(I;B^\rho_{p,1}(\boldsymbol{R}^d))}\frac{d\lambda}{\lambda}$$
$$\le C\int_0^a \lambda^{-1/p}\frac{d\lambda}{\lambda}\int H_i\left(\frac{|t_2|}{2\lambda}\right)G_I(t_2)\frac{dt_2}{|t_2|}$$
$$\le C_i a^{b-1/p}\int |t_2|^{-b}G_I(t_2)\frac{dt_2}{|t_2|}$$

implies that $\int_0^a F_i(\lambda,x,t)(d\lambda/\lambda)$ is a $B^\rho_{p,1}(\boldsymbol{R}^d)$-valued bounded continuous function of $t\in I$ and its norm is estimated by $Ca^{b-1/p}\|G_I(t')|t'|^{-b}\|_{L^1_*(\boldsymbol{R})}$. Thus, by the identity (8.13) we see that $f\in BC(I;B^\rho_{p,1}(\boldsymbol{R}^d))$ and its norm is estimated by $C(a)\|f\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d\times I)}$, where $C(a)$ is a constant indepedent of f (but depend on a).

Furthermore, the estimate (8.28) and the identity (8.13) imply that

$$\|\{f^{[0]}(x)-f(x,t_0)\}1_I(t)\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d\times I)} \le \sum_{i=1}^3\left\|\psi(t)\int_0^a F_i(\lambda,x,t_0)\frac{d\lambda}{\lambda}\right\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^{d+1})}$$
$$\le Ca^{b-1/p}\|G_I(t')|t'|^{-b}\|_{L^1_*}$$

holds for any fixed $t_0\in I$. Therefore, by (8.26) and (8.13) we have

$$\|f\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d\times I)}$$
$$=\left\|\left[\sum_{i=1}^3\int_0^a F_i(\lambda,x,t)\frac{d\lambda}{\lambda}+\{f^{[0]}(x)-f(x,t_0)\}1_I(t)+f(x,t_0)1_I(t)\right]\right\|_{B^{(\rho,b)}_{p,1}(\boldsymbol{R}^d\times I)}$$
$$\le C\|G_I(t')|t'|^{-b}\|_{L^1_*}+Ca^{b-1/p}\|G_I(t')|t'|^{-b}\|_{L^1_*}+C\|f(x,t_0)\|_{B^\rho_{p,1}(\boldsymbol{R}^d)}. \qquad \square$$

Lemma 8.4. $\|f\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^d\times I)}=\|W(-t)f\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^d\times I)}$. *Here* $I=(-a,a)$, $a>0$.

Proof. Let $f\in B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^d\times I)$ and let $\tilde f\in B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^{d+1})$ be any extension of f. Then, it follows from the identity (7.5) that $\|W(-t)f\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^d\times I)}\le\|W(-t)\tilde f\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^{d+1})}=\|\tilde f\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^{d+1})}$. Therefore, $\|W(-t)f\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^d\times I)}\le\|f\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^d\times I)}$.

Conversely, let $W(-t)f\in B^{(\rho,b)}_{2,q}(\boldsymbol{R}^d\times I)$, and let $\tilde g\in B^{(\rho,b)}_{2,q}(\boldsymbol{R}^{d+1})$ be any extension of $W(-t)f$. Then, $\|f\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^d\times I)}\le\|W(t)\tilde g\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^{d+1})}=\|\tilde g\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^{d+1})}$. Therefore, $\|f\|_{B^{(\rho,b)}_{2,q,P}(\boldsymbol{R}^d\times I)}\le\|W(-t)f\|_{B^{(\rho,b)}_{2,q}(\boldsymbol{R}^d\times I)}$. $\square$

Now we are ready *to prove Theorem* 2.5. Let $f\in B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d\times I)$ with $f(x,0)=0$, $1/2\le b<1$, and put $g(x,t)=W(-t)f(x,t)$. We may assume that $a\le 1$. Then, $g\in B^{(\rho,b)}_{2,1}(\boldsymbol{R}^d\times I)$ and $g(x,0)=0$. Therefore, by Theorem 8.2 we see that $G_I(t')|t'|^{-b}\in L^1_*(\boldsymbol{R})$, where $G_I(t')$ is defined by (8.20) with f replaced by g. Since $g(x,0)=0$, Lemma 8.4 and Theorem 8.3 imply that $\|f\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d\times(-\delta,\delta))}=\|g\|_{B^{(\rho,b)}_{2,1}(\boldsymbol{R}^d\times(-\delta,\delta))}\le C\|G_{(-\delta,\delta)}(t')|t'|^{-b}\|_{L^1_*(\boldsymbol{R})}$ for any $0<\delta\le a$, which with the aid of the fact that $G_{(-\delta,\delta)}(t')\le G_I(t')$, and that $G_{(-\delta,\delta)}(t')=0$ if $|t'|\ge 2\delta$ gives

$$\|f\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d\times(-\delta,\delta))}\le C\int_{-2\delta}^{2\delta}\frac{G_{(-\delta,\delta)}(t')}{|t'|^b}\frac{dt'}{t'}\le C\int_{-2\delta}^{2\delta}\frac{G_I(t')}{|t'|^b}\frac{dt'}{t'}\to 0\quad\text{as }\delta\to+0. \qquad \square$$

9. Proof of Main Theorem.

To remove the smallness assumption of the initial data we need one more theorem:

Theorem 9.1. *Let* $\delta = 2^{-p}$, *p a positive integer, and let* $s \le 0$.
(a) *Then we have*

$$\|f(\delta x)\|_{B^s_{2,1}(\boldsymbol{R}^d)} \le \delta^{s-d/2}\|f\|_{B^s_{2,1}(\boldsymbol{R}^d)}, \tag{9.1}$$

$$\|f(\delta x)\|_{B^{s,\#}_{2,1}(\boldsymbol{R}^d)} \le \delta^{s-d/2}\|f\|_{B^{s,\#}_{2,1}(\boldsymbol{R}^d)}. \tag{9.2}$$

(b) *Let* $P(\xi) = \pm|\xi|^2$, $b > 0$, *and let* $I = (-a, a)$. *Assume that a weight* ρ *on* $\boldsymbol{R}_+$ *satisfies the condition*: $\rho(z_2) \le z_1^{-s}\rho(z_1 z_2)$ *for any* $z_1, z_2 \ge 1$. *Then we have*

$$\|f(\delta x, \delta^2 t)\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d\times\delta^{-2}I)} \le \delta^{s-d/2-1}\|f\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d\times I)}, \tag{9.3}$$

$$\|f(\delta x, \delta^2 t)\|_{B^{(s,b),\#}_{2,1,P}(\boldsymbol{R}^d\times\delta^{-2}I)} \le \delta^{s-d/2-1}\|f\|_{B^{(s,b),\#}_{2,1,P}(\boldsymbol{R}^d\times I)}. \tag{9.4}$$

Proof. (a) Let $g(x) := f(\delta x)$. Then the identity

$$\hat{g}(\xi) = c_d\int e^{-ix\xi}f(\delta x)\,dx = c_d\delta^{-d}\int e^{-iy\xi/\delta}f(y)\,dy = \delta^{-d}\hat{f}\left(\frac{\xi}{\delta}\right), \tag{9.5}$$

where $c_d = (2\pi)^{-d/2}$, gives $\|g_j\|_{L^2} = \delta^{-d}\|\varphi_j(|\xi|)\hat{f}(\xi/\delta)\|_{L^2} = \delta^{-d/2}\|\varphi_j(\delta|\eta|)\hat{f}(\eta)\|_{L^2}$. Take $\delta = 2^{-p}$, $\eta := 2^p\xi$ with positive integer p and recall that $\varphi_0(2^{-p}|\eta|) = \sum_{k=0}^p \varphi_k(|\eta|)$ which is a consequence of the identity $\sum_{j=0}^\infty \varphi_j = 1$. Then we obtain the estimates

$$2^{pd/2}\|\varphi_0(2^{-p}|\eta|)\hat{f}(\eta)\|_{L^2} = 2^{pd/2}\left\|\sum_{k=0}^p \varphi_k(|\eta|)\hat{f}(\eta)\right\|_{L^2} \le \delta^{s-d/2}\sum_{k=0}^p 2^{sk}\|f_k\|_{L^2},$$

$$2^{pd/2}2^{sj}\|\varphi_j(2^{-p}|\eta|)\hat{f}(\eta)\|_{L^2} = 2^{pd/2}2^{sj}\|\varphi_1(2^{-j+1-p}|\eta|)\hat{f}(\eta)\|_{L^2} = \delta^{s-d/2}2^{sj+sp}\|f_{j+p}\|_{L^2},$$

which imply that

$$\|g\|_{B^s_{2,1}} \le \delta^{s-d/2}\left\{\sum_{k=0}^p 2^{sk}\|f_k\|_{L^2} + \sum_{j=1}^\infty 2^{sj+sp}\|f_{j+p}\|_{L^2}\right\} = \delta^{s-d/2}\|f\|_{B^s_{2,1}}.$$

Now consider the case $B^{s,\#}_{2,1}$. Let $g(x) := f(\delta x)$, $\delta = 2^{-p}$, $p \in \boldsymbol{N}$, $p > 1$. Then, as above from the identity $\hat{g}(\xi) = \delta^{-d}\hat{f}(\xi/\delta)$ we have $\sum_{j=1}^\infty 2^{sj}\|g_j\|_{L^2} = 2^{pd/2-sp}\sum_{j=p+1}^\infty 2^{sj}\|f_j\|_{L^2}$. On the other hand the identity $\varphi_0(2^{-p}|\eta|) = \sum_{k=0}^p \varphi_k(|\eta|)$ implies that

$$\begin{aligned}
\|g_0^\#\|_{L^2} &= \|(1+|\log|\xi|\,|)^{-2}\varphi_0(\xi)\hat{g}(\xi)\|_{L^2} \\
&= 2^{pd/2}\|\{1+|\log(|2^{-p}\eta|)|\}^{-2}\varphi_0(2^{-p}|\eta|)\hat{f}(\eta)\|_{L^2} \\
&\le 2^{pd/2}\left\|(1+|\log|\eta|\,|)^{-2}\varphi_0(|\eta|)|\hat{f}(\eta)| + \sum_{k=1}^p \varphi_k(|\eta|)|\hat{f}(\eta)|\right\|_{L^2},
\end{aligned}$$

which gives $\|g_0^{\#}\|_{L^2} \le 2^{pd/2}\|f_0^{\#}\|_{L^2} + 2^{pd/2-sp}\sum_{k=1}^{p} 2^{sk}\|f_k\|_{L^2} \le 2^{pd/2-sp}\|f\|_{B^{s,\#}_{2,1}}$. Hence

$$\|g\|_{B^{s,\#}_{2,1}} = \|g_0^{\#}\|_{L^2} + \sum_{j=1}^{\infty} 2^{sj}\|g_j\|_{L^2} \le 2^{pd/2-sp}\left\{\|f_0^{\#}\|_{L^2} + \sum_{j=1}^{\infty} 2^{sj}\|f_j\|_{L^2}\right\} = 2^{pd/2-sp}\|f\|_{B^{s,\#}_{2,1}}.$$

(b) Let $f \in B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})$ and put $g(x,t) := f(\delta x, \delta^2 t)$. Then, $\hat{g}(\xi,\tau) = \delta^{-d-2}\hat{f}(\xi/\delta, \tau/\delta^2)$. Hence, defining $\hat{f}_{j,k,P} := \varphi_j(|\xi|)\varphi_k(\tau - P(\xi))\hat{f}(\xi,\tau)$, $\hat{g}_{j,k,P} := \varphi_j(|\xi|)\varphi_k(\tau - P(\xi))\hat{g}(\xi,\tau)$, it follows that the inequalities

$$\|g_{j,k,P}\|_{L^2} = \delta^{-d/2-1}\|f_{j+p,k+2p,P}\|_{L^2} \quad \text{if } j,k > 0,$$

$$\|g_{j,0,P}\|_{L^2} \le \delta^{-d/2-1}\sum_{k=0}^{2p}\|f_{j+p,k,P}\|_{L^2} \quad \text{if } j > 0,$$

$$\|g_{0,k,P}\|_{L^2} \le \delta^{-d/2-1}\sum_{j=0}^{p}\|f_{j,k+2p,P}\|_{L^2} \quad \text{if } k > 0,$$

$$\|g_{0,0,P}\|_{L^2} \le \delta^{-d/2-1}\sum_{j=0}^{p}\sum_{k=0}^{2p}\|f_{j,k,P}\|_{L^2}$$

hold for the case $\delta = 2^{-p}$, $p \in \boldsymbol{N}$, which gives $\|g\|_{B^{(s,b)}_{2,1,P}(\boldsymbol{R}^{d+1})} \le \delta^{s-d/2-1}\|f\|_{B^{(s,b)}_{2,1,P}(\boldsymbol{R}^{d+1})}$, since $\rho(2^j) \le 2^{-sp}\rho(2^{j+p})$, $2^{bk} < 2^{b(k+2p)}$.

Now, let $\tilde{f} \in B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})$ be an extension of $f \in B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d} \times I)$. Then, $\tilde{g}(x,t) := \tilde{f}(\delta x, \delta^2 t)$ is an extension of $g(x,t) := f(\delta x, \delta^2 t)$. Therefore, it follows from the inequality just proved that $\|g\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^d \times \delta^{-2} I)} \le \|\tilde{g}\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})} \le \delta^{s-d/2-1}\|\tilde{f}\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}^{d+1})}$, which implies the inequality (9.3). We can prove the inequality (9.4) in the same way. □

We only give the proof of Part (a), that is, the case where $N(u,\bar{u}) = c_1 u^2 + c_2 \bar{u}^2$. (Part (b) can be proved in the same way). Let u be a solution to the semilinear Schrödinger equation (1.1) when $|t| < T$ with the initial data $u(x,0) = u_0(x)$. Put $v(x,t) := u(\delta x, \delta^2 t)$. Then v satisfies

$$\partial_t v = i\partial_x^2 v + \lambda N(v,\bar{v}), \quad x \in \boldsymbol{R},\ |t| < a, \tag{9.6}$$

and the initial condition $v(x,0) = v_0(x) := u_0(\delta x)$, where $\lambda = \delta^2$, $a = T\delta^{-2}$.

We take $P(\xi) = -\xi^2$, $a = 1$, and define $\{W(t)f\}(x,t) = \mathscr{F}_x^{-1} e^{itP(\xi)}\mathscr{F}_x f(x,t)$. Then, any solution v to (9.6) with $v(x,0) = v_0(x)$ must satisfy the equation $v = W(t)v_0 + \lambda B(v,v)$, where $B(f,g)$ is defined by (2.6). Put $v = W(t)v_0 + w$. Then the equation to be solved is $w = \Phi(w)$, where $\Phi(w) = \lambda B(W(t)v_0, W(t)v_0) + 2\lambda B(W(t)v_0, w) + \lambda B(w,w)$.

We take $s = -3/4$ and $b \in (1/2, 1)$. Then, it follows from Theorem 7.1 that $W(t)v_0 \in B^{(s,b)}_{2,1,P}(\boldsymbol{R} \times I)$, $\|W(t)v_0\|_{B^{(s,b)}_{2,1,P}(\boldsymbol{R}\times I)} \le C_0\alpha$, where $I = (-1,1)$, $\alpha = \|v_0\|_{B^s_{2,1}}$ and C_0 is a constant independ of v_0. Define

$$X := B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R} \times I), \quad \rho(t) = \log(2+t)t^s. \tag{9.7}$$

Then, Theorem 2.4 implies that

$$\|B(w_1, w_2)\|_X \le C_1\|w_1\|_X\|w_2\|_X, \tag{9.8}$$

$$\|B(W(t)v_0, W(t)v_0)\|_X \le C_1 C_0^2\alpha^2, \tag{9.9}$$

$$\|B(W(t)v_0, w)\|_X \le C_1 C_0\alpha\|w\|_X. \tag{9.10}$$

We assume that $4\lambda C_0 C_1\alpha < 1$, and take β to be the smaller root of the equation: $\lambda C_1 C_0^2\alpha^2 + 2\lambda C_1 C_0\alpha\beta + \lambda C_1\beta^2 = \beta$, that is,

$$2C_1\lambda\beta = 1 - 2C_0C_1\lambda\alpha - \sqrt{1 - 4C_0C_1\lambda\alpha}.$$

Then we see by (9.8), (9.9) and (9.10) that when $\|w\|_X \le \beta$

$$\|\Phi(w)\|_X \le \lambda C_1(C_0^2\alpha^2 + 2C_0\alpha\beta + \beta^2) = \beta,$$

and that when $\|w_1\|_X \le \beta$, $\|w_2\|_X \le \beta$

$$\|\Phi(w_1) - \Phi(w_2)\|_X \le \lambda\|B(2W(t)v_0 + w_1 + w_2, w_1 - w_2)\|_X \le \kappa\|w_1 - w_2\|_X$$

holds, where $\kappa := 2C_1C_0\lambda\alpha + 2C_1\lambda\beta = 1 - \sqrt{1 - 4C_1\lambda\alpha} < 1$. Thus, Φ is a contraction which maps $M = \{w \in X; \|w\|_X \le \beta\}$ into itself. The fixed point theorem says that there exists one and only one fixed point $w \in M$ of Φ. It is easy to show that $v = W(t)v_0(x) + w$ satisfies the equation (9.6) when $t \in (-1, 1)$.

If $\delta = 2^{-p}$, where p is a positive integer, is chosen so that

$$4C_0C_1\|u_0\|_{B^s_{2,1}} < \delta^{-3/4}, \tag{9.11}$$

then Theorem 9.1 implies that $4\lambda C_0C_1\|v_0\|_{B^s_{2,1}} \le 4C_0C_1\delta^{3/4}\|u_0\|_{B^s_{2,1}} < 1$. Therefore, there exists a solution v to the equation (9.6). In conclusion, $u(x,t) = v(\delta^{-1}x, \delta^{-2}t)$ is a solution to the semilinear Schrödinger equation (1.1) in the interval $(-\delta^2, \delta^2)$.

Proof of uniqueness. Assume that u_1, u_2 are solutions to (1.1) such that $u_1(x,0) = u_2(x,0) \in B^s_{2,1}(\boldsymbol{R})$ and

$$u_i(x,t) - W(t)u_i(x,0) \in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R} \times I_T), \quad i = 1, 2, \tag{9.12}$$

where $I_T := (-T, T)$. We take $\delta = 2^{-p}$, $p \in \boldsymbol{N}$ so that $4C_1C_0\delta^{3/4}\|u_0\|_{B^s_{2,1}(\boldsymbol{R})} < 1$, and put $v_i(x,t) := u_i(\delta x, \delta^2 t)$, $i = 1, 2$. Here, $u_0(x) := u_1(x,0) = u_2(x,0)$. Then, v_1, v_2 are solutions to (9.6). Put $v = v_1 - v_2$. Then, v satisfies the equation

$$\partial_t v = i\partial_x^2 v + \lambda c_1 v\{v_1 + v_2\} + \lambda c_2\bar{v}\{\bar{v}_1 + \bar{v}_2\}, \tag{9.13}$$

and $v(x,0) = 0$, where $\lambda = \delta^2$, which implies $v = \lambda B(v, v_1 + v_2)$.

Since $u_1 - u_2 \in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^d \times I_T)$, $u_1 + u_2 \in B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}^d \times I_T)$, by Theorem 9.1 (b) we see that $v_1 - v_2 \in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^d \times I)$, $v_1 + v_2 \in B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}^d \times I)$, where $I := \delta^{-2}I_T$. Put $w(x,t) := v_1(x,t) + v_2(x,t) - 2W(t)v_0(x)$. Then, $w \in B^{(s,1/2)}_{2,1}(\boldsymbol{R} \times I)$ and $w(x,0) = 0$. Hence, it follows from Theorem 2.5 that there exists a positive number $\varepsilon \le \delta^{-2}T$ such that $2C_1\delta^2\|w\|_{B^{(\rho,b)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))} < 1$. Thus, by Theorem 9.1 (a) we have

$$C_1\delta^2\|v_1+v_2\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))} \le C_1\delta^2\{2C_0\|v_0\|_{B^s_{2,1}(\boldsymbol{R})}+\|w\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}\}$$
$$\le 2C_1C_0\delta^{3/4}\|u_0\|_{B^s_{2,1}(\boldsymbol{R})}+C_1\delta^2\|w\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}<1.$$

On the other hand, Theorem 2.4 gives that

$$\|v\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}=\delta^2\|B(v,v_1+v_2)\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}$$
$$\le C_1\delta^2\|v_1+v_2\|_{B^{(s,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}\|v\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}.$$

Therefore, we see that $\|v\|_{B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times(-\varepsilon,\varepsilon))}=0$, which means that $u_1(x,t)=u_2(x,t)$ when $|t|<\varepsilon_1=\varepsilon\delta^2$.

By this result we can prove that $u_1(x,t)=u_2(x,t)$ for any $t\in I_T$ as follows: First, Theorem 8.3 gives that

$$f_i(x,t):=u_i(x,t)-W(t)u_0\in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times I_T)\subset BC(I_T;B^\rho_{2,1}(\boldsymbol{R})),$$

for $i=1,2$, which implies that the set $A:=\{t\in I_T;f_1(\cdot,t)=f_2(\cdot,t)\text{ in }B^\rho_{2,1}(\boldsymbol{R})\}$ is closed.

Secondly, we show that A is open. In fact, let $t_0\in A$, and put $\tilde{u}_i(x,t):=u_i(x,t+t_0)$. Then, $\tilde{u}_1(\cdot,0)=\tilde{u}_2(\cdot,0)\in B^s_{2,1}(\boldsymbol{R})$. By Lemma 8.4 we see that $W(-t-t_0)f_i(x,t+t_0)\in B^{(\rho,1/2)}_{2,1}(\boldsymbol{R}\times(-T_1,T_1))$, where $T_1:=T-|t_0|$. Since $W(t_0)$ maps $B^{(\rho,1/2)}_{2,1}(\boldsymbol{R}\times(-T_1,T_1))$ onto itself, Lemma 8.4 gives that $f_i(x,t+t_0)=W(t)\{W(t_0)W(-t-t_0)f_i(x,t+t_0)\}\in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times(-T_1,T_1))$, which gives, with the help of the fact that $f_i(x,t_0)\in B^\rho_{2,1}(\boldsymbol{R})$,

$$\tilde{f}_i(x,t):=\tilde{u}_i(x,t)-W(t)\tilde{u}_i(x,0)$$
$$=u_i(x,t+t_0)-W(t)\{W(t_0)u_0(x)+f_i(x,t_0)\}$$
$$=f_i(x,t+t_0)-W(t)f_i(x,t_0)\in B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}\times(-T_1,T_1)).$$

Of course, $\tilde{u}_i$, $i=1,2$, are solutions to (1.1). Therefore, the result just proved implies that there exists a positive number ε' such that $\tilde{f}_1(\cdot,t)=\tilde{f}_2(\cdot,t)$ for any $t\in(-\varepsilon',\varepsilon')$, which means that $(t_0-\varepsilon',t_0+\varepsilon')\subset A$, that is, A is an open set. In conclusion, A is an open and closed subset of a connected set I_T, hence $A=I_T$. □

Appendix A. Proof of Theorem 2.1 and Theorem 2.2

We shall prove the basic properties of our Besov type spaces here. First we shall show

LEMMA A.1. *Let $\eta(\xi)\in C_0^\infty(\boldsymbol{R}^d)$, $\psi(\tau)\in C_0^\infty(\boldsymbol{R})$, and let $P(\xi)$ be a real-valued C^∞-function such that $|\partial_\xi^\alpha P(\xi)|\le c_\alpha(1+|\xi|)^{\nu-|\alpha|}$ for any α. Define*

$$\hat{K}_{jk,P}(\xi,\tau):=\eta(2^{-j}\xi)\psi(2^{-k}(\tau-P(\xi))),\quad K_{jk,P}:=\mathscr{F}^{-1}\hat{K}_{jk,P}. \tag{A.1}$$

Then $\|K_{jk,P}*f\|_{L^p}\le C2^{(d+2)\nu j+k}\|f\|_{L^p}$ holds for any $f\in L^p(\boldsymbol{R}^{d+1})$, where C is a constant independent of j,k and f.

PROOF. The estimate

$$|t^n K_{jk,P}(x,t)| \le C_n 2^{dj+k(1-n)} \tag{A.2}$$

follows from the identity

$$t^n K_{jk,P}(x,t) = c_{d+1} i^n 2^{-kn} \iint e^{i(x\xi+t\tau)} \eta(2^{-j}\xi)\psi^{(n)}(2^{-k}(\tau - P(\xi)))\, d\tau d\xi,$$

which is a consequence of integration by parts. The identity

$$x_h^n K_{jk,P}(x,t) = c_{d+1} i^n \iint e^{i(x\xi+t\tau)} \partial_{\xi_h}^n \{\eta(2^{-j}\xi)\psi(2^{-k}(\tau - P(\xi)))\}\, d\tau d\xi$$

is also obtained by integration by parts, and the identity

$$\partial_{\xi_h}^n \{\eta(2^{-j}\xi)\psi(2^{-k}(\tau - P(\xi)))\} = \sum_{m=0}^{n} \binom{n}{m} 2^{-jn+jm} \eta_h^{(n-m)}(2^{-j}\xi) \partial_{\xi_h}^m \{\psi(2^{-k}(\tau - P(\xi)))\},$$

where $\eta_h^{(m)}(\xi) = \partial_{\xi_h}^m \eta(\xi)$, is given by Leibniz' formula. On the other hand, the identity

$$\begin{aligned} &\partial_{\xi_h}^m \{\psi(2^{-k}(\tau - P(\xi)))\} \\ &\quad = \sum_{\alpha_1+2\alpha_2+\cdots+m\alpha_m=m} \prod_{r=1}^{m} \left(\frac{-2^{-k} P_h^{(r)}(\xi)}{r!} \right)^{\alpha_r} \frac{m!\psi^{(\alpha_1+\cdots+\alpha_m)}(2^{-k}(\tau - P(\xi)))}{\alpha_1! \cdots \alpha_m!}, \end{aligned} \tag{A.3}$$

where $P_h^{(r)}(\xi) = \partial_{\xi_h}^r P(\xi)$, which is given by the formula for derivatives of composite functions, implies that

$$|\partial_{\xi_h}^n \{\eta(2^{-j}\xi)\psi(2^{-k}(\tau - P(\xi)))\}| \le \sum_{m=0}^{n} C_{n,m} |\eta_h^{(n-m)}(2^{-j}\xi)| \sum_{\ell=0}^{m} 2^{\nu\ell j - nj - k\ell} |\psi^{(\ell)}(2^{-k}(\tau - P(\xi)))|,$$

since $|P_h^{(r)}(\xi)| \le c_r 2^{\nu j - rj}$ on the support of $\eta_h^{(n-m)}(2^{-j}\xi)$. Therefore it follows that

$$|x_h^n K_{jk,P}(x,t)| \le C_n' 2^{dj - nj + k + n(\nu j - k) \vee 0}. \tag{A.4}$$

This and (A.2) give $(1 + \sum_{h=1}^{d} |x_h|^{d+2} + |t|^{d+2}) |K_{jk,P}(x,t)| \le C 2^{(d+2)\nu j + k}$, which implies

$$\|K_{jk,P}\|_{L^1} \le \iint \frac{C 2^{(d+2)\nu j + k}}{1 + \sum_{h=1}^{d} |x_h|^{d+2} + |t|^{d+2}} dx dt \le C' 2^{(d+2)\nu j + k}. \tag{A.5}$$

This and Lemma 3.1 imply the conclusion of the lemma. □

Next

LEMMA A.2. *$\mathscr{S}(\boldsymbol{R}^{d+1}) \subset B_{p,q,P}^{(\rho,b)}(\boldsymbol{R}^{d+1})$, and the inclusion is continuous.*

PROOF. Let $f \in \mathscr{S}(\boldsymbol{R}^{d+1})$, and define

$$\varphi_{jk,P}(\xi,\tau) := \varphi_j(|\xi|)\varphi_k(\tau - P(\xi)), \tag{A.6}$$

$$\varphi_0^{[\ell]}(z) := \varphi_0(z), \varphi_1^{[\ell]}(z) := z^{-\ell}\varphi_1(z), \varphi_j^{[\ell]}(z) := \varphi_1^{[\ell]}(2^{-j+1}z) \quad \text{for } j \ge 1, \tag{A.7}$$

$$\text{(A.8)} \qquad \varphi_{jk,P}^{[\ell,m]}(\xi,\tau) := \varphi_j^{[\ell]}(|\xi|)\varphi_k^{[m]}(\tau - P(\xi)), \check{\varphi}_{jk,P}^{[\ell,m]} = \mathscr{F}^{-1}\varphi_{jk,P}^{[\ell,m]},$$

$$\text{(A.9)} \qquad f_P^{[2\ell,m]} = \mathscr{F}^{-1}\{|\xi|^{2\ell}\{\tau - P(\xi)\}^m \hat{f}(\xi,\tau)\},$$

$$\text{(A.10)} \qquad \hat{f}_{jk,P}(\xi,\tau) := \varphi_{jk,P}(\xi,\tau)\hat{f}(\xi,\tau).$$

Then it follows that

$$\begin{aligned} f_{jk,P}(x,t) &= 2^{-2\ell j - mk + 2\ell + m}\mathscr{F}^{-1}\{\varphi_{jk,P}^{[2\ell,m]}(\xi,\tau)|\xi|^{2\ell}\{\tau - P(\xi)\}^m \hat{f}(\xi,\tau)\} \\ &= 2^{-2\ell j - mk + 2\ell + m}\check{\varphi}_{jk,P}^{[2\ell,m]} * f_P^{[2\ell,m]}(x,t), \end{aligned}$$

which gives, with the aid of Lemma A.1,

$$\|f_{jk,P}\|_{L^p} = 2^{-2\ell j - mk + 2\ell + m}\|\check{\varphi}_{jk,P}^{[2\ell,m]} * f_P^{[2\ell,m]}\|_{L^p} \le C2^{-2\ell j - mk + 2\ell + m + (d+2)v(j-1) + k - 1}\|f_P^{[2\ell,m]}\|_{L^p}.$$

Thus, assuming that $\rho(z) \le cz^\sigma$ and taking ℓ and m so that $2\ell > \sigma + (d+2)v$, $m > b+1$, we obtain that

$$\|\{\rho(2^j)2^{bk}\|f_{jk,P}\|_{L^p}\}\|_{\ell^q} \le C\|\{2^{\{\sigma - 2\ell + (d+2)v\}j + (b+1-m)k}\}\|_{\ell^q} \cdot \|f_P^{[2\ell,m]}\|_{L^p} \le C'\|f_P^{[2\ell,m]}\|_{L^p}.$$

□

Now we proceed to give Proof of Theorem 2.1. First we shall show that

$$\text{(A.11)} \qquad |\langle f, \bar{\psi}\rangle_{\mathscr{S}' \times \mathscr{S}}| \le C\|f\|_{B_{p,q,P}^{(\rho,b)}} \cdot \|\psi\|_{B_{p',q',P}^{(1/\rho,-b)}}$$

holds for $f \in B_{p,q,P}^{(\rho,b)}$ and $\psi \in \mathscr{S}(\boldsymbol{R}^{d+1})$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

In fact, writing $\tilde{\varphi}_j = \sum_{|k-j|\le 1}\varphi_k$, $\tilde{\varphi}_{jk,P}(\xi,\tau) = \tilde{\varphi}_j(|\xi|)\tilde{\varphi}_k(\tau - P(\xi))$, we have

$$\langle f_{jk,P}, \bar{\psi}\rangle = \langle \hat{f}_{jk,P}, \hat{\bar{\psi}}\rangle = \langle \tilde{\varphi}_{jk,P}\hat{f}_{jk,P}, \hat{\bar{\psi}}\rangle = \langle \hat{f}_{jk,P}, \tilde{\varphi}_{jk,P}\hat{\bar{\psi}}\rangle = \sum_{|\ell-j|\le 1, |k-m|\le 1}\langle \hat{f}_{jk,P}, \hat{\bar{\psi}}_{\ell m,P}\rangle,$$

where $\hat{\psi}_{\ell m,P} = \varphi_{\ell m,P}\hat{\psi}$. Hence, by Hölder's inequality we have

$$\begin{aligned} |\langle f, \bar{\psi}\rangle| &= \left|\left\langle \sum_{j,k} f_{jk,P}, \bar{\psi}\right\rangle\right| \\ &\le \sum_{j,k}\sum_{|\ell-j|\le 1, |k-m|\le 1}\|f_{jk,P}\|_{L^p} \cdot \|\psi_{\ell m,P}\|_{L^{p'}} \\ &\le \|\rho(2^j)2^{bk}\|f_{jk,P}\|_{L^p}\|_{\ell^q} \cdot \left\|\left\{\frac{1}{\rho(2^j)}2^{-bk}\sum_{|\ell-j|\le 1, |k-m|\le 1}\|\psi_{\ell m,P}\|_{L^{p'}}\right\}\right\|_{\ell^{q'}} \\ &\le C\|f\|_{B_{p,q,P}^{(\rho,b)}} \cdot \|\psi\|_{B_{p',q',P}^{(1/\rho,-b)}}. \end{aligned}$$

Next, let $\{f^{(n)}\}$ be a Cauchy sequence in $B_{p,q,P}^{(\rho,b)}$. Then (A.11) shows that $\{\langle f^{(n)}, \psi\rangle_{\mathscr{S}' \times \mathscr{S}}\}$ converges for any $\psi \in \mathscr{S}(\boldsymbol{R}^{d+1})$, which implies that $\{f^{(n)}\}$ converges to a distribution f in $\mathscr{S}'$. We write $\hat{f}_{jk,P}^{(n)} = \varphi_{jk,P}\hat{f}^{(n)}$, $\hat{f}_{jk,P} = \varphi_{jk,P}\hat{f}$. Since the Fourier transform is continuous on $\mathscr{S}'$, we see that $f_{jk,P}^{(n)} \to f_{jk,P}$ in $\mathscr{S}'$, which, with the aid of the fact that $\{f_{jk,P}^{(n)}\}$ is a Cauchy sequence in L^p, gives that $f_{jk,P}^{(n)} \to f_{jk,P}$ in L^p. Thus, for the case $q < \infty$ we have

$$\left[\sum_{j+k\le\ell}\{\rho(2^j)2^{bk}\|f_{jk,P}\|_{L^p}\}^q\right]^{1/q} = \lim_{n\to\infty}\left[\sum_{j+k\le\ell}\{\rho(2^j)2^{bk}\|f_{jk,P}^{(n)}\|_{L^p}\}^q\right]^{1/q} \le \sup_n \|f^{(n)}\|_{B_{p,q,P}^{(\rho,b)}}$$

for any ℓ. This means that $f \in B_{p,q,P}^{(\rho,b)}$, and $\|f\|_{B_{p,q,P}^{(\rho,b)}} \le \sup_n \|f^{(n)}\|_{B_{p,q,P}^{(\rho,b)}}$. The same argument shows that $\|f - f^{(m)}\|_{B_{p,q,P}^{(\rho,b)}} \le \sup_n \|f^{(n)} - f^{(m)}\|_{B_{p,q,P}^{(\rho,b)}}$, which implies that $f^{(n)} \to f$ in $B_{p,q,P}^{(\rho,b)}$. In the same way we can prove the same results for $q = \infty$.

We define here $K_n = \mathscr{F}\varphi_0(2^{-n}|\xi|)\varphi_0(2^{-n}\tau)$, and prove that $K_n * f \to f$ in $B_{p,q,P}^{(\rho,b)}$ as $n \to \infty$ when $f \in B_{p,q,P}^{(\rho,b)}$ and q is finite. Since $K_n(x,t) = 2^{(d+1)n}K_0(2^n x, 2^n t)$, we see that $\|K_n\|_{L^1} = \|K_0\|_{L^1}$. Hence, by making use of Lemma 3.1, we have

$$\|(K_n * f)_{jk,P}\|_{L^p} = \|K_n * (f_{jk,P})\|_{L^p} \le \|K_0\|_{L^1} \cdot \|f_{jk,P}\|_{L^p}.$$

Moreover, $(K_n * f)_{jk,P} = f_{jk,P}$ if $k \le n-2$, $v(j+2) \le n - 1 - \log_2 c_0$, since $|\tau| \le |\tau - P(\xi)| + |P(\xi)| < 2^{k+1} + c_0 2^{(j+2)v} \le 2^n$ on the support of $\hat{f}_{jk,P}(\xi,\tau)$. Hence, putting $I_n := \{(j,k); v(j+2) > n - 1 - \log_2 c_0 \text{ or } k > n-2\}$, we see that

$$\begin{aligned}\|K_n * f - f\|_{B_{p,q,P}^{(\rho,b)}}^q &\le \sum_{(j,k)\in I_n}\{\rho(2^j)2^{bk}\|(f - K_n * f)_{jk,P}\|_{L^p}\}^q \\ &\le (1 + \|K_0\|_{L^1})^q \sum_{(j,k)\in I_n}\{\rho(2^j)2^{bk}\|f_{jk,P}\|_{L^p}\}^q,\end{aligned}$$

which implies that $K_n * f \to f$ in $B_{p,q,P}^{(\rho,b)}$ as $n \to \infty$.

Finally assume that $f \in B_{p,q,P}^{(\rho,b)}$, the support of $\hat{f}$ is contained in the set $\{(\xi,\tau) \in \boldsymbol{R}^{d+1}; |\xi| \le 2^n, |\tau| \le 2^n\}$ for some $n \in \boldsymbol{N}$ and p, q are finite. Then we see that $k \le (n+1)v + |\log_2 c_0| + 1$ if $\hat{f}_{jk,P}(\xi,\tau) \ne 0$, since $2^{k-1} < |\tau - P(\xi)| \le |\tau| + |P(\xi)| \le 2^n + c_0 2^{(n+1)v}$. This implies that $f = \sum_{j=0}^n \sum_{k=0}^{(n+1)v+|\log_2 c_0|+1} f_{jk,P} \in L^p$. For any positive number ε there exists $g \in \mathscr{S}$ such that $\|f - g\|_{L^p} < \varepsilon$, for $\mathscr{S}$ is dense in L^p. The identity $\|K_n\|_{L^1} = \|K_0\|_{L^1}$, Lemma A.1 and Lemma 3.1 imply that

$$\|(f - K_n * g)_{jk,P}\|_{L^p} = \|\{K_n * (f-g)\}_{jk,P}\|_{L^p} \le C_{jk}\|K_n * (f-g)\|_{L^p} \le C_{jk}\|K_0\|_{L^1}\varepsilon.$$

Since the support of $\mathscr{F}\{K_n * (f-g)\}$ is contained in the set $\{(\xi,\tau); |\xi| < 2^{n+1}, |\tau| < 2^{n+1}\}$, by the same argument as above we see that $\{K_n * (f-g)\}_{jk,P} = 0$ when $k > (n+1)v + |\log_2 c_0| + 3$. Thus we have

$$\|f - K_n * g\|_{B_{pq,P}^{\rho,b}}^q = \sum_{j=0}^{n+2}\sum_{k=0}^{(n+1)v+|\log_2 c_0|+3}\rho(2^j)^q 2^{bkq}\|\{K_n * (f-g)\}_{jk,P}\|_{L^p}^q < C_n\varepsilon^q.$$

Since $K_n * g \in \mathscr{S}$, this means that $\mathscr{S}(\boldsymbol{R}^{d+1})$ is a dense subset of $B_{p,q,P}^{(\rho,b)}$ if p and q are finite.

Consider now the space $B_{2,q,P}^{(s,b),\#}$. Let $f, \psi \in \mathscr{S}(\boldsymbol{R}^{d+1})$. Then, from the fact that $(1 + |\log|\xi||)^2\gamma_0(|\xi|) \in L^2$ (γ_0 is the defining function of the interval $[-2,2]$) and the inequality

$$\sup_{\xi,\tau}\frac{\gamma_0(|\xi|)\tilde{\varphi}_k(\tau - P(\xi))}{(1+\tau^2)^m} \le C(m)2^{-2km} \quad \text{for any } m \in \overline{\boldsymbol{N}}, \tag{A.12}$$

where $C(m)$ is a constant independent of k, it follows that the L^2-norm of the function $(1+|\log|\xi||)^2\gamma_0(\xi)\tilde{\varphi}_k(\tau-P(\xi))\hat{\psi}(\xi,\tau)$ is estimated by

$$C2^{-2km}\sup_\xi\left\|\int e^{-ix\xi}(1-\partial_t^2)^m\psi(x,t)\,dx\right\|_{L_t^2}\le C2^{-2km}\|\{\|(1-\partial_t^2)^m\psi(x,t)\|_{L^2(\boldsymbol{R}_t)}\}\|_{L^1(\boldsymbol{R}_x^d)}.$$

Hence, taking m such that $b+2m>0$ we have

$$\begin{aligned}\sum_k|\langle f_{0k,P},\bar{\psi}\rangle| &= \sum_k|\langle\hat{f}^{\#}_{0k,P},(1+|\log|\xi||)^2\gamma_0(\xi)\tilde{\varphi}_k(\tau-P(\xi))\hat{\bar{\psi}}(\xi,\tau)\rangle| \\ &\le c\left\{\sum_k 2^{-2mk}\|f^{\#}_{0k,P}\|_{L^2}\right\}\|\{\|(1-\partial_t^2)^m\psi(x,t)\|_{L^2(\boldsymbol{R}_t)}\}\|_{L^1(\boldsymbol{R}_x^d)} \\ &\le c'\|f\|_{B^{(s,b),\#}_{2,q,P}}\|\{\|(1-\partial_t^2)^m\psi(x,t)\|_{L^2(\boldsymbol{R}_t)}\}\|_{L^1(\boldsymbol{R}_x^d)},\end{aligned}$$

which shows that

$$|\langle f,\bar{\psi}\rangle|\le c\|f\|_{B^{(s,b),\#}_{2,q,P}}[\|\{\|(1-\partial_t^2)^m\psi\|_{L^2(\boldsymbol{R}_t)}\}\|_{L^1(\boldsymbol{R}_x^d)}+\|\psi\|_{B^{(-s,-b)}_{2,q',P}}].$$

Since the second factor of the right-hand side is a continuous norm on $\mathscr{S}$, this means that $B^{(s,b),\#}_{2,q,P}$ is a Banach space continuously imbedded in $\mathscr{S}'(\boldsymbol{R}^{d+1})$. □

Proof of Theorem 2.2. Let $f\in\mathscr{S}$. Then Schwarz's inequality implies

$$\int|\hat{f}_{jk,P}(\xi,\tau)|\,d\tau\le\left\{\int_{|\tau-P(\xi)|\le 2^{k+1}}d\tau\right\}^{1/2}\|\hat{f}_{jk,P}(\xi,\cdot)\|_{L^2(\boldsymbol{R})}\le c2^{k/2}\|\hat{f}_{jk,P}(\xi,\cdot)\|_{L^2(\boldsymbol{R})}.$$

Since $|j-\ell|\le 1$ if $\varphi_\ell(\xi)\hat{f}_{jk,P}(\xi,\tau)=\varphi_\ell(\xi)\varphi_{jk,P}(\xi,\tau)\hat{f}(\xi,\tau)\ne 0$, it follows that

$$\begin{aligned}\|f(x,t)\|_{B^\rho_{2,1}} &\le \sum_\ell\rho(2^\ell)\sum_{|j-\ell|\le 1}\sum_k\left\|\left\{\varphi_\ell(|\xi|)\frac{1}{\sqrt{2\pi}}\int e^{it\tau}\hat{f}_{jk,P}(\xi,\tau)\,d\tau\right\}\right\|_{L^2(\boldsymbol{R}^d)} \\ &\le c\sum_k 2^{k/2}\sum_\ell\rho(2^\ell)\sum_{|j-\ell|\le 1}\|\varphi_\ell(|\xi|)\hat{f}_{jk,P}(\xi,\tau)\|_{L^2(\boldsymbol{R}^{d+1})} \\ &\le c'\sum_k 2^{k/2}\sum_j\rho(2^j)\|f_{jk,P}\|_{L^2(\boldsymbol{R}^{d+1})}=c'\|f\|_{B^{(\rho,1/2)}_{2,1,P}}\end{aligned}$$

holds for any $t\in\boldsymbol{R}$. With the help of the fact that $\mathscr{S}$ is dense in $B^{(\rho,1/2)}_{2,1,P}$, this inequality implies that $B^{(\rho,1/2)}_{2,1,P}(\boldsymbol{R}^{d+1})\subset BC(\boldsymbol{R};B^\rho_{2,1}(\boldsymbol{R}^d))$ and the inclusion is continuous. □

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