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The primitive ideal space of the C^*-algebras of infinite graphs

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Abstract. For any countable directed graph E we describe the primitive ideal space of the corresponding generalized Cuntz-Krieger algebra $C^*(E)$.

0. Introduction.

The primary purpose of this article is to give a description of the primitive ideal space of the C^*-algebra $C^*(E)$ corresponding to an arbitrary countable directed graph E. The two main results of the article, Theorem 2.10 (together with Corollary 2.11) and Theorem 3.4, identify elements of $\mathrm{Prim}(C^*(E))$ and describe the closure operation in the hull-kernel topology. These theorems build on and generalize a long string of previous results on the ideal structure of Cuntz-Krieger type algebras, obtained by a number of researchers over the period of past twenty years. Our present article completes the program of classification of ideals of the generalized Cuntz-Krieger algebras corresponding to arbitrary countable directed graphs.

First fundamental results about the ideal structure of Cuntz-Krieger algebras were obtained by Cuntz, who described all ideals of $\mathcal{O}_A$ for a finite $0-1$ matrix A satisfying Condition (II) [**3**, Theorem 2.5]. Much more recently, an Huef and Raeburn gave a complete description of all gauge-inaviant ideals [**14**, Theorem 3.5] and the primitive ideal space [**14**, Theorem 4.7] for Cuntz-Krieger algebras $\mathcal{O}_A$ corresponding to arbitrary finite $0-1$ matrices. Soon after the Cuntz-Krieger algebras of countably infinite directed graphs (the graph algebras) were introduced and analyzed by Kumjian, Pask, Raeburn and Renault. They described all ideals of $C^*(E)$ for a locally finite graph E satisfying Condition (K) (an analogue of Cuntz's Condition (II)) [**20**, Theorem 6.6]. Since then a number of papers considered the problem of classification of ideals of graph algebras and other generalizations of the classical Cuntz-Krieger algebras. However, to the best of our knowledge, most of those papers dealt only with ideals invariant under the gauge action. This in particular applies to graphs satisfying Condition (K), since for such graphs all closed ideals of $C^*(E)$ are gauge-invariant [**13**, Lemma 2.2]. For row-finite graphs, Bates, Pask, Raeburn and Szymański gave all gauge-invariant ideals of $C^*(E)$ [**2**, Theorem 4.1] and described the primitive ideal space if in addition E satisfies Condition (K) [**2**, Theorem 6.3]. Working with arbitrary countable graphs, Bates, Hong, Raeburn and Szymański described all gauge-invariant ideals [**1**, Theorem 3.6] and identify those of them which are primitive [**1**, Theorem 4.7]. (A brief overview of the results of [**1**] was reported by Hong in [**10**].) About the same time, Drinen and

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Tomforde obtained similar results (through different techniques) for graphs satisfying Condition (K) [**5**, Theorem 3.5 and Theorem 4.10].

Besides graph algebras, there are other interesting generalizations of Cuntz-Krieger algebras. These include Cuntz-Pimsner algebras generated by Hilbert bimodules and Exel-Laca algebras corresponding to infinite $0-1$ matrices. Many, though not all, graph algebras can be viewed as either Cuntz-Pimsner algebras or Exel-Laca algebras (cf. [**9**], [**25**]). Partial results about the ideal structure of Cuntz-Pimsner algebras were obtained by Pinzari [**22**], Kajiwara, Pinzari and Watatani [**15**], and Fowler, Muhly and Raeburn [**8**]. Exel and Laca described ideals of the Cuntz-Krieger algebras $\mathcal{O}_A$ corresponding to infinite $0-1$ matrices, under an extra hypothesis on the matrix A analogous to Condition (K) [**6**, Theorem 15.1].

Obviously, there are many benefits from such a comprehensive description of the ideal structure of a large class of algebras, as presented in [**1**] and the present article. For example, Szymański's proof of a very general criterion of injectivity of homomorphisms of graph algebras [**27**, Theorem 1.2] is based on the analysis of ideals, and so are some arguments from the recent work of Hong and Szymański on non-simple purely infinite graph algebras [**13**]. In fact, we think that graph algebras might play a prominent role in the study and classification of this interesting class of C^*-algebras, whose investigations have been recently initiated by Kirchberg and Rørdam [**18**], [**19**]. Also, graph algebras appearing in the context of some compact quantum manifolds [**11**], [**12**] are not simple, and it is important to know their ideal structure. Furthermore, it does not seem unlikely, that our methods, techniques and results on the ideal structure of graph algebras may help in understanding of other classes of C^*-algebras, Cuntz-Pimsner algebras for example. Indeed, techniques quite similar to those developed for the study of graph algebras have recently been used by Katsura in his analysis of the crossed products of Cuntz algebras by quasi-free actions of locally compact abelian groups [**16**], [**17**]. And certainly, good understanding of the ideal structure of generalized Cuntz-Krieger algebras is a necessary first step towards their classification.

Of course, as a by-product of our analysis of the ideal structure of graph algebras we obtain their simplicity criteria. The problem of simplicity of generalized Cuntz-Krieger algebras was discussed by a number of authors. Partial answers to this question for graph algebras were given by Bates, Pask, Raeburn and Szymański in [**2**, Proposition 5.1], and by Fowler, Laca and Raeburn in [**7**, Theorem 3]. An if and only if criterion was proved by Szymański in [**26**, Theorem 12], and another one somewhat later but independently by Paterson in [**21**, Theorem 4]. A partial result about simplicity of $\mathcal{O}_A$ for an infinite $0-1$ matrix A was given by Exel and Laca in [**6**, Theorem 14.1], and if and only if criterion was supplied by Szymański in [**26**, Theorem 8]. Building on this latter result, Tomforde proved an analogous criterion for the C^*-algebras corresponding to ultragraphs [**28**] (a class of C^*-algebras which contains both graph algebras $C^*(E)$ and Exel-Laca algebras $\mathcal{O}_A$). Related results about simplicity of Cuntz-Pimsner algebras were obtained by Schweizer in [**23**] and [**24**].

The present article is organized as follows. In §1, we review basic facts about graph algebras we need. The main reference to this section is [**1**]. We rely heavily on the results of that paper. In particular, we use the description of quotients of $C^*(E)$ by gauge-invariant ideals [**1**, Proposition 3.4], the description of the intersection of a family of gauge-invariant ideals [**1**, Proposition 3.9 and Corollary 3.10], the concepts of maximal tails and breaking vertices [**1**, §4], the description of all gauge-invariant ideals of $C^*(E)$ [**1**, Theorem 3.6], and the classification of gauge-invariant primitive ideals of $C^*(E)$ [**1**, Theorem 4.7]. In §2, we describe all primitive ideals of $C^*(E)$ which are not invariant under the gauge action (cf. Theorem 2.10). Inside $\mathrm{Prim}(C^*(E))$ they form circles which are in one-to-one correspondence with maximal tails containing a loop without exits (cf. Lemma 2.1). The general plan of our argument is similar to that of [**14**] and is based on sandwiching a non gauge-invariant primitive ideal between two gauge-invariant ideals (cf. Lemmas 2.6 and 2.8). However, the case of arbitrary infinite graphs is technically much more complicated than that of finite graphs. The main result of §2 is Corollary 2.11, which gives a description of all primitive ideals of $C^*(E)$ for an arbitrary countable graph E. In §3, we describe the closure operation in the hull-kernel topology of $\mathrm{Prim}(C^*(E))$ (cf. Theorem 3.4). Since our result covers all possible countable directed graphs, this description is necessarily somewhat involved. It greatly simplifies in the case of row-finite graphs (cf. Corollary 3.5). In §4, we illustrate the main results with a few examples.

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1. Preliminaries on graph algebras.

We recall the definition of the C^*-algebra corresponding to a directed graph [**7**]. Let $E = (E^0, E^1, r, s)$ be a directed graph with countably many vertices E^0 and edges E^1, and range and source functions $r, s : E^1 \to E^0$, respectively. $C^*(E)$ is defined as the universal C^*-algebra generated by families of projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$, subject to the following relations.

(GA1) $p_v p_w = 0$ for $v, w \in E^0$, $v \neq w$.

(GA2) $s_e^* s_f = 0$ for $e, f \in E^1$, $e \neq f$.

(GA3) $s_e^* s_e = p_{r(e)}$ for $e \in E^1$.

(GA4) $s_e s_e^* \leq p_{s(e)}$ for $e \in E^1$.

(GA5) $p_v = \sum_{e \in E^1 : s(e) = v} s_e s_e^*$ for $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

Universality in this definition means that if $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ are families of projections and partial isometries, respectively, satisfying conditions (GA1)–(GA5), then there exists a C^*-algebra homomorphism from $C^*(E)$ to the C^*-algebra generated by $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ such that $p_v \mapsto Q_v$ and $s_e \mapsto T_e$ for $v \in E^0$, $e \in E^1$.

It follows from the universal property that there exists a gauge action $\gamma : \boldsymbol{T} \to \mathrm{Aut}(C^*(E))$ such that $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$ for all $v \in E^0$, $e \in E^1$, $z \in \boldsymbol{T}$. We denote by Γ the corresponding conditional expectation of $C^*(E)$ onto the fixed-point algebra $C^*(E)^\gamma$, such that $\Gamma(x) = \int_{z \in \boldsymbol{T}} \gamma_z(x)\, dz$ for $x \in C^*(E)$. The integral is with respect to the normalized Haar measure on $\boldsymbol{T}$. Note that $\Gamma(p_v) = p_v$ and $\Gamma(s_e) = 0$ for all $v \in E^0$, $e \in E^1$.

If $\alpha_1, \dots, \alpha_n$ are (not necessarily distinct) edges such that $r(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, n-1$, then $\alpha = (\alpha_1, \dots, \alpha_n)$ is a path of length $|\alpha| = n$, with source $s(\alpha) = s(\alpha_1)$ and range $r(\alpha) = r(\alpha_n)$. We also allow paths of length zero, identified with vertices. The set of all paths of length n is denoted by E^n, while the collection of all finite paths in E is denoted by E^*. Given a path $\alpha = (\alpha_1, \dots, \alpha_n)$ we denote $s_\alpha = s_{\alpha_1} \cdots s_{\alpha_n}$, a partial isometry in $C^*(E)$. A loop is a path of positive length whose source and range coincide. A loop α has an exit if there exists an edge $e \in E^1$ and index i such that $s(e) = s(\alpha_i)$ but $e \neq \alpha_i$. If α is a loop all of whose vertices belong to a subset $M \subseteq E^0$ then we say that α has an exit in M if an edge e exists as above with $r(e) \in M$.

By an ideal in a C^*-algebra we always mean a closed two-sided ideal. In order to understand the ideal structure of a graph algebra it is convenient to look at saturated hereditary subsets of the vertex set. As usual, if $v, w \in E^0$ then we write $v \geq w$ when there is a path from v to w, and say that a subset K of E^0 is hereditary if $v \in K$ and $v \geq w$ imply $w \in K$. A subset K of E^0 is saturated if every vertex v which satisfies $0 < |s^{-1}(v)| < \infty$ and $s(e) = v \Rightarrow r(e) \in K$ itself belongs to K. If $X \subseteq E^0$ then $\Sigma(X)$ is the smallest saturated subset of E^0 containing X, and $\Sigma H(X)$ is the smallest saturated hereditary subset of E^0 containing X. If K is hereditary and saturated then I_K denotes the ideal of $C^*(E)$ generated by $\{p_v : v \in K\}$. We have

$$I_K = \overline{\mathrm{span}}\{s_\alpha p_v s_\beta^* : \alpha, \beta \in E^*, v \in K, r(\alpha) = r(\beta) = v\}.$$

As shown in [**1**, Proposition 3.4], the quotient $C^*(E)/I_K$ is naturally isomorphic to $C^*(E/K)$. The quotient graph E/K was defined in [**1**, Section 3]. The vertices of E/K are

$$(E/K)^0 = (E^0 \backslash K) \cup \{\beta(v) : v \in K_\infty^{\mathrm{fin}}\},$$

where

$$K_\infty^{\mathrm{fin}} = \{v \in E^0 \backslash K : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \backslash K)| < \infty\}.$$

The edges of E/K are

$$(E/K)^1 = r^{-1}(E^0 \backslash K) \cup \{\beta(e) : e \in E^1, r(e) \in K_\infty^{\mathrm{fin}}\},$$

with r, s extended by $r(\beta(e)) = \beta(r(e))$ and $s(\beta(e)) = s(e)$. Note that all extra vertices $\beta(K_\infty^{\mathrm{fin}})$ are sinks in E/K. If $K_\infty^{\mathrm{fin}} = \emptyset$ then E/K is simply a subgraph of E, denoted $E \backslash K$. If $v \in K_\infty^{\mathrm{fin}}$ then we write

$$p_{v,K} = \sum_{s(e)=v,\, r(e) \notin K} s_e s_e^*.$$

For $B \subseteq K_\infty^{\mathrm{fin}}$ the ideal of $C^*(E)$ generated by I_K and $\{p_v - p_{v,K} : v \in B\}$ is denoted by $J_{K,B}$. We have

$$J_{K,B} = \overline{\operatorname{span}}\{s_\alpha p_v s_\beta^*, s_\mu(p_w - p_{w,K})s_\nu^* : \alpha, \beta, \mu, \nu \in E^*,$$
$$v \in K, w \in B, r(\alpha) = r(\beta) = v, r(\mu) = r(\nu) = w\}.$$

By [**1**, Corollary 3.5], the quotient $C^*(E)/J_{K,B}$ is naturally isomorphic to $C^*((E/K)\backslash\beta(B))$. As shown in [**1**, Theorem 3.6], all gauge-invariant ideals of $C^*(E)$ are of the form $J_{K,B}$.

A non-empty subset $M \subseteq E^0$ is a maximal tail if it satisfies the following three conditions (cf. [**1**, Lemma 4.1]):

(MT1) If $v \in E^0$, $w \in M$, and $v \geq w$, then $v \in M$.

(MT2) If $v \in M$ and $0 < |s^{-1}(v)| < \infty$, then there exists $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.

(MT3) For every $v, w \in M$ there exists $y \in M$ such that $v \geq y$ and $w \geq y$.

We denote by $\mathscr{M}(E)$ the collection of all maximal tails in E and by $\mathscr{M}_\gamma(E)$ the collection of all maximal tails M in E such that each loop in M has an exit in M. We set $\mathscr{M}_\tau(E) = \mathscr{M}(E)\backslash\mathscr{M}_\gamma(E)$.

If $X \subseteq E^0$ then, as in [**1**], we denote

$$\Omega(X) = \{w \in E^0\backslash X : w \not\geq v \text{ for all } v \in X\}.$$

If X consists of a single vertex $\{v\}$ then we write $\Omega(v)$ instead of $\Omega(\{v\})$. Note that $\Omega(M) = E^0\backslash M$ for every maximal tail M. Moreover, for such an M, $\Omega(M)$ is hereditary by (MT1) and saturated by (MT2).

Along with maximal tails, the set

$$BV(E) = \{v \in E^0 : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v)\backslash r^{-1}(\Omega(v))| < \infty\}$$

plays an important role in the classification of primitive gauge-invariant ideals [**1**]. Its elements are called breaking vertices. Note that if $K \subseteq E^0$ is hereditary and saturated, then $v \in K_\infty^{\text{fin}}$ implies $v \in BV(E)$. If $v \in BV(E)$ then $\Omega(v)$ is hereditary and saturated.

We denote by $\operatorname{Prim}(C^*(E))$ the set of all primitive ideals of $C^*(E)$ and by $\operatorname{Prim}_\gamma(C^*(E))$ the set of all primitive gauge-invariant ideals of $C^*(E)$.

As shown in [**1**, Theorem 4.7], there is a one-to-one correspondence

$$\mathscr{M}_\gamma(E) \cup BV(E) \to \operatorname{Prim}_\gamma(C^*(E))$$

given by

$$\mathscr{M}_\gamma(E) \ni M \mapsto J_{\Omega(M),\Omega(M)_\infty^{\text{fin}}},$$

$$BV(E) \ni v \mapsto J_{\Omega(v),\Omega(v)_\infty^{\text{fin}}\backslash\{v\}}.$$

2. The primitive ideals.

Our goal in this section is to show that any maximal tail in $\mathscr{M}_\tau(E)$ gives rise to a circle of primitive ideals, none of which is gauge-invariant, and that all non gauge-invariant primitive ideals arise in this way. To this end we explicitly construct the corresponding irreducible representations of $C^*(E)$. At first we observe that any maximal tail $M \in \mathscr{M}_\tau(E)$ contains an essentially unique vertex simple loop without exits in

M. A loop $L = (e_1, \dots, e_n)$ is vertex simple if and only if $r(e_i) \neq r(e_j)$ for $i \neq j$. We denote by L^0 the set $\{r(e_i) : i = 1, \dots, n\}$ of the vertices through which L passes, and by L^1 the set $\{e_i : i = 1, \dots, n\}$ of its edges.

LEMMA 2.1. *If M is a maximal tail in E then $M \in \mathcal{M}_\tau(E)$ if and only if there exists a vertex simple loop L with $L^0 \subseteq M$ and such that if $e \in E^1 \backslash L^1$ and $s(e) \in L^0$ then $r(e) \notin M$. Furthermore, such a loop is unique up to a cyclic permutation of the edges comprising it, and $\Omega(M) = \Omega(L^0)$.*

PROOF. The first assertion follows immediately from the definition of $\mathcal{M}_\tau(E)$ as the complement of $\mathcal{M}_\gamma(E)$ in $\mathcal{M}(E)$.

For the uniqueness suppose that there are two loops $L_1 = (e_1, \dots, e_n)$ and $L_2 = (f_1, \dots, f_m)$ with the above properties. By condition (MT3) there is a vertex $v \in M$ and paths α, β such that $s(\alpha) = s(e_1)$, $s(\beta) = s(f_1)$, and $r(\alpha) = r(\beta) = v$. Since L_1, L_2 have no exits in M we must have $v = s(e_k) = s(f_r)$ for some k, r. The absence of exits then implies that $e_k = f_r$, $e_{k+1} = f_{r+1}$, and so on, which proves the claim.

Obviously $\Omega(M) \subseteq \Omega(L^0)$. For the reverse inclusion let $L = (e_1, \dots, e_n)$ and $v \in \Omega(L^0)$. Suppose for a moment that $v \in M$. By (MT3), there exists a vertex $w \in M$ such that $v \leq w$ and $s(e_1) \leq w$. Since L has no exits in M, w must be in L^0. Thus $v \notin \Omega(L^0)$, contrary to the assumption. Hence $v \notin M$. Since M is a maximal tail we have $E^0 \backslash M = \Omega(M)$ and hence $v \in \Omega(M)$, as required. □

From now on, for each maximal tail $M \in \mathcal{M}_\tau(E)$ we choose one vertex simple loop without exits in M as in Lemma 2.1 and call it L_M. For $v \in E^0 \backslash L_M^0$ we denote

$$A_M(v) = \{(\alpha_1, \dots, \alpha_m) \in E^* : s(\alpha_1) = v, r(\alpha_m) \in L_M^0, r(\alpha_i) \notin L_M^0 \text{ if } i \neq m\}$$

and set

$$A_M = \bigcup_{v \in E^0 \backslash L_M^0} A_M(v).$$

DEFINITION 2.2. Let E be a directed graph. Given $M \in \mathcal{M}_\tau(E)$ let $L_M = (e_1, \dots, e_n)$. We denote by $\mathcal{H}_M$ the Hilbert space with an orthonormal basis $\{\xi_\alpha : \alpha \in A_M \cup L_M^0\}$. For $t \in \boldsymbol{T} \subset \boldsymbol{C}$ we define a representation

$$\rho_{M,t} : C^*(E) \to \mathcal{B}(\mathcal{H}_M)$$

so that

$$\rho_{M,t}(p_v)\xi_\alpha = \begin{cases} \xi_\alpha & \text{if } s(\alpha) = v \\ 0 & \text{otherwise,} \end{cases}$$

$$\rho_{M,t}(s_e)\xi_\alpha = \begin{cases} t\xi_{s(e)} & \text{if } e = e_1 \text{ and } \alpha = r(e_1) \\ \xi_{s(e)} & \text{if } r(e) = \alpha \in L_M^0 \text{ and } e_1 \neq e \in L_M^1 \\ \xi_e & \text{if } r(e) = \alpha \in L_M^0 \text{ and } e \notin L_M^1 \\ \xi_{(e, \alpha_1, \dots, \alpha_m)} & \text{if } \alpha, (e, \alpha_1, \dots, \alpha_m) \in A_M \\ 0 & \text{otherwise,} \end{cases}$$

for $v \in E^0$, $e \in E^1$, and $\alpha = (\alpha_1, \dots, \alpha_m) \in A_M \cup L_M^0$.

Remark 2.3. That $\rho_{M,t}$ indeed gives rise to a representation of $C^*(E)$ will be shown in Lemma 2.5. Strictly speaking, this representation depends not only on the maximal tail M but also on the choice of the loop L_M. We slightly abuse the notation by writing $\rho_{M,t}$ instead of more precise $\rho_{L_M,t}$. We will use the latter notation later in Lemma 2.8 for emphasis, when considering L_M and its cyclic permutation L'_M simultaneously.

We will see (cf. Lemma 2.5) that each $\rho_{M,t}$ is an irreducible representation of $C^*(E)$ and thus $\ker \rho_{M,t}$ is a primitive ideal, which turns out to be not invariant under the gauge action. It will be useful to sandwich such an ideal between two gauge-invariant ones (cf. Lemma 2.6), and to this end we need to consider the set

$$K_M := L_M^0 \cup \{v \in E^0 \backslash L_M^0 : |A_M(v)| < \infty\}.$$

It is not difficult to see that each K_M is hereditary, saturated and that $\Sigma H(\Omega(M) \cup L_M^0) \subseteq K_M$. If E is row-finite then $\Sigma H(\Omega(M) \cup L_M^0) = K_M$ but in general K_M may be larger.

Lemma 2.4. *Let E be a directed graph, $M \in \mathscr{M}_\tau(E)$ and $\pi : C^*(E) \to C^*(E)/J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}}$ be the natural surjection. Then for every vertex $v \in K_M \backslash (\Omega(M) \cup L_M^0)$ we have*

$$\pi(p_v) = \sum_{\alpha \in A_M(v)} \pi(s_\alpha s_\alpha^*).$$

Proof. By [**1**, Corollary 3.5], there exists a natural isomorphism between $C^*(E)/J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}}$ and $C^*(E\backslash\Omega(M))$. Since $E\backslash\Omega(M) = (M, r^{-1}(M), r, s)$ we may assume that $M = E^0$ and $\pi = \mathrm{id}$. Now we proceed by induction with respect to the maximum length ℓ of elements of $A_M(v)$. If $\ell = 1$ then the claim follows from condition (GA5). For the inductive step observe that if $v \in K_M \backslash (\Omega(M) \cup L_M^0)$ then the set $\{e \in E^1 : s(e) = v\}$ is non-empty by Lemma 2.1 and finite by the definition of K_M. Thus $p_v = \sum_{e \in E^1, s(e) = v} s_e s_e^*$ by (GA5). Applying the inductive hypothesis to $\{p_w : w = r(e), e \in E^1, s(e) = v\}$ we infer that the desired identity holds. □

If the graph E is row-finite then the smallest gauge-invariant ideal of $C^*(E)$ containing $\ker \rho_{M,t}$ is I_{K_M} (cf. Lemma 2.6). However, if E contains vertices with infinite valencies then such a gauge-invariant ideal must have the form J_{K_M,B_M} for a suitable $B_M \subseteq (K_M)_\infty^{\mathrm{fin}}$. It turns out that

$$B_M := (K_M)_\infty^{\mathrm{fin}} \cap \Omega(M)_\infty^{\mathrm{fin}}$$

does the trick. Note that a vertex $v \in E^0$ belongs to B_M if and only if among $\{e \in E^1 : s(e) = v\}$ there are infinitely many edges e such that $r(e) \in \Omega(M)$, only finitely many e with $r(e) \in E^0 \backslash \Omega(M) = M$, and at least one e such that $r(e) \notin K_M$.

Lemma 2.5. *Let E be a directed graph, $M \in \mathscr{M}_\tau(E)$ and $t \in \boldsymbol{T}$. Then $\rho_{M,t}$ of Definition 2.2 gives rise to an irreducible representation of $C^*(E)$ such that $\rho_{M,t}(J_{K_M,B_M}) = \mathscr{K}(\mathscr{H}_M)$.*

Proof. At first we show that $\{\rho_{M,t}(p_v), \rho_{M,t}(s_e) : v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family. Obviously, conditions (GA1)–(GA4) are satisfied. We verify (GA5). Let v be a vertex such that $0 < |s^{-1}(v)| < \infty$. If $v \in L_M^0$ then there is exactly one edge f with $s(f) = v$ and $r(f) \in L_M^0$. It then follows from the definition of $\rho_{M,t}$ that both $\sum_{e \in E^1, s(e)=v} \rho_{M,t}(s_e)\rho_{M,t}(s_e)^* = \rho_{M,t}(s_f)\rho_{M,t}(s_f)^*$ and $\rho_{M,t}(p_v)$ are projections onto the 1-dimensional subspace of $\mathscr{H}_M$ spanned by ξ_v. If $v \in E^0 \backslash L_M^0$ then $\rho_{M,t}(p_v)$ is a projection onto the subspace of $\mathscr{H}_M$ spanned by $\{\xi_\alpha : \alpha \in A_M(v)\}$. If $e \in E^1$, $s(e) = v$, then $\rho_{M,t}(s_e)^*\xi_\alpha = 0$ for $s(\alpha) \neq v$. If $s(\alpha) = v$ and $\alpha = (f_1, \dots, f_k)$, $f_i \in E^1$, then $\rho_{M,t}(s_e)^*\xi_\alpha = 0$ for $e \neq f_1$ and $\rho_{M,t}(s_e)\rho_{M,t}(s_e)^*\xi_\alpha = \xi_\alpha$ for $e = f_1$. Thus again $\sum_{e \in E^1, s(e)=v} \rho_{M,t}(s_e)\rho_{M,t}(s_e)^* = \rho_{M,t}(p_v)$. Hence, Definition 2.2 gives rise to a representation $\rho_{M,t} : C^*(E) \to \mathscr{B}(\mathscr{H}_M)$.

It follows from Definition 2.2 that all the projections $\rho_{M,t}(p_v)$, $v \in K_M$, and $\rho_{M,t}(p_w - p_{w,K_M})$, $w \in B_M$, are of finite rank. Thus $\rho_{M,t}(J_{K_M,B_M}) \subseteq \mathscr{K}(\mathscr{H}_M)$. On the other hand, the range $\rho_{M,t}(C^*(E))$ contains $\mathscr{K}(\mathscr{H}_M)$. Indeed, if $v \in L_M^0$ then $\{\rho_{M,t}(s_\mu) : \mu \in E^*, r(\mu) = v\}$ are rank-1 partial isometries sending ξ_v to all other elements of the orthonormal basis $\{\xi_\alpha : \alpha \in A_M \cup L_M^0\}$. Consequently $\rho_{M,t}(J_{K_M,B_M}) = \mathscr{K}(\mathscr{H}_M)$. In particular, $\rho_{M,t}$ is irreducible. □

By Lemma 2.5, the representations $\rho_{M,t}$ give rise to primitive ideals $\ker \rho_{M,t}$ of $C^*(E)$. In Lemmas 2.6 and 2.8, below, we show the key property of $C^*(E)$ that for each $M \in \mathscr{M}_\tau(E)$ the union of $\{\ker \rho_{M,t} : t \in \boldsymbol{T}\}$ may be sandwiched between two uniquely determined gauge-invariant ideals whose quotient is Morita equivalent to $C(\boldsymbol{T})$. This result was originally proved by an Huef and Raeburn in [**14**, Lemma 4.5] for the Cuntz-Krieger algebras corresponding to finite matrices. The argument there took advantage of the existence of only finitely many gauge-invariant ideals. In the present article we need a different argument, as algebras corresponding to infinite graphs may have infinitely many gauge-invariant ideals.

Lemma 2.6. *Let E be a directed graph, $M \in \mathscr{M}_\tau(E)$ and $t \in \boldsymbol{T}$. Then the following hold.*

1. *The ideal $J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}}$ is the largest among gauge-invariant ideals of $C^*(E)$ contained in* $\ker \rho_{M,t}$.
2. *The ideal J_{K_M,B_M} is the smallest among gauge-invariant ideals of $C^*(E)$ containing* $\ker \rho_{M,t}$.

Proof. Ad 1. Since $\Omega(M) = \Omega(L_M^0)$ by Lemma 2.1, it is immediate from Definition 2.2 that $\rho_{M,t}(p_v) = 0$ if $v \in \Omega(M)$, and $\rho_{M,t}(p_v - p_{v,\Omega(M)}) = 0$ if $v \in \Omega(M)_\infty^{\mathrm{fin}}$. Hence $J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}} \subseteq \ker \rho_{M,t}$.

Let J_1 be a gauge-invariant ideal of $C^*(E)$ contained in $\ker \rho_{M,t}$. By [**1**, Theorem 3.6] there is a saturated hereditary $K \subseteq E^0$ and a $B \subseteq K_\infty^{\mathrm{fin}}$ such that $J_1 = J_{K,B}$. By [**1**, Corollary 3.10], in order that $J_{K,B} \subseteq J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}}$ we must have $K \subseteq \Omega(M)$ and $B \subseteq \Omega(M) \cup \Omega(M)_\infty^{\mathrm{fin}}$. Let $v \in E^0 \backslash \Omega(M)$. Since $\Omega(M) = \Omega(L_M^0)$ there is a path from v to L_M^0. If α is such a path with the shortest possible length then $\alpha \in A_M \cup L_M^0$ and $\rho_{M,t}(p_v)\xi_\alpha \neq 0$. Thus $\rho_{M,t}(p_v) \neq 0$ and consequently $v \notin K$. This shows that $K \subseteq \Omega(M)$. Now let $v \in B \backslash \Omega(M)$. Then v emits at least one edge into M, since there is a path from v to L_M^0. Furthermore, v emits infinitely many edges into $K \subseteq \Omega(M)$, and only finitely many edges into $E^0 \backslash K$ and hence into $E^0 \backslash \Omega(M)$. Consequently $v \in \Omega(M)^{\mathrm{fin}}_{\infty}$. Thus $B \subseteq \Omega(M) \cup \Omega(M)^{\mathrm{fin}}_{\infty}$.

Ad 2. Let $J = J_{K_M, B_M}$. We denote by $\pi : \mathscr{B}(\mathscr{H}_M) \to \mathscr{B}(\mathscr{H}_M)/\mathscr{K}(\mathscr{H}_M)$ the quotient map of $\mathscr{B}(\mathscr{H}_M)$ onto its Calkin algebra. By Lemma 2.5 we have $\rho_{M,t}(J) = \mathscr{K}(\mathscr{H}_M)$.

At first we show that $\ker \rho_{M,t} \subseteq J$. To this end it suffices to prove injectivity of the homomorphism

$$\phi : C^*(E)/J \to \rho_{M,t}(C^*(E))/\rho_{M,t}(J) = \rho_{M,t}(C^*(E))/\mathscr{K}(\mathscr{H}_M)$$

given by $\phi(x + J) = \pi(\rho_{M,t}(x))$. This follows from the gauge-invariant uniqueness theorem [**1**, Theorem 2.1]. Indeed, by [**1**, Corollary 3.5] $C^*(E)/J$ is naturally isomorphic to $C^*((E/K_M) \backslash \beta(B_M))$, and the gauge action on $C^*(E)/J$ is inherited from the gauge action γ on $C^*(E)$, since J is gauge-invariant. We also need a matching action on $\pi(\rho_{M,t}(C^*(E)))$. For $z \in \boldsymbol{T}$ let U_z be a unitary operator on $\mathscr{H}_M$ such that $U_z(\xi_\alpha) = z^{|\alpha|}\xi_\alpha$ for $\alpha \in A_M \cup L_M^0$. Since $U_z(\rho_{M,t}(p_v))U_z^* = \rho_{M,t}(p_v)$ for $v \in E^0$ and

$$U_z(\rho_{M,t}(s_e))U_z^* = \begin{cases} \rho_{M,t}(s_e) & \text{if } e \in L_M^1 \\ z\rho_{M,t}(s_e) & \text{otherwise} \end{cases}$$

for $e \in E^1$, $\mathrm{Ad}\, U_z$ is an automorphism of $\rho_{M,t}(C^*(E))$. It induces an automorphism of $\pi(\rho_{M,t}(C^*(E)))$. Therefore we may define an action θ of $\boldsymbol{T}$ on $\pi(\rho_{M,t}(C^*(E)))$ by

$$\theta_z(\pi(\rho_{M,t}(x))) = \pi(U_z\rho_{M,t}(x)U_z^*).$$

For all $x \in C^*(E)$ and $z \in \boldsymbol{T}$ we have $\theta_z(\phi(x + J)) = \phi(\gamma_z(x) + J)$, since this identity holds on the generators $\{p_v, s_e\}$ of $C^*(E)$. We must still show that ϕ does not kill any of the generating projections of $C^*(E)/J \cong C^*((E/K_M)\backslash\beta(B_M))$. We set $K' = E^0 \backslash K_M = M \backslash K_M$, $B' = (K_M)^{\mathrm{fin}}_{\infty} \backslash B_M$. Since $((E/K_M)\backslash\beta(B_M))^0 = K' \cup \{\beta(w) : w \in B'\}$ it suffices to show that $\rho_{M,t}(p_v) \notin \mathscr{K}(\mathscr{H}_M)$ for $v \in K'$ and $\rho_{M,t}(p_w - p_{w,K_M}) \notin \mathscr{K}(\mathscr{H}_M)$ for $w \in B'$. That is, we must prove that $\rho_{M,t}(p_v)$ and $\rho_{M,t}(p_w - p_{w,K_M})$ are infinite dimensional for $v \in K'$ and $w \in B'$. For $v \in K'$ this fact is simply contained in the definition of K_M. Since $w \in B'$ there are infinitely many edges $e \in E^1$ such that $s(e) = w$ and $r(e) \in K_M \backslash \Omega(M)$, and consequently $\rho_{M,t}(p_w - p_{w,K_M})$ is infinite dimensional. Thus the hypothesis of the gauge-invariant uniqueness theorem is satisfied, and we may conclude that $\ker \rho_{M,t} \subseteq J$.

Now let J_2 be a gauge-invariant ideal of $C^*(E)$ containing $\ker \rho_{M,t}$. We must show that $J \subseteq J_2$. It follows from part 1 that $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}} \subseteq J_2$. Since $p_{r(L_M)} - (1/t)s_{L_M} \in \ker \rho_{M,t} \subseteq J_2$ and J_2 is gauge-invariant, also $p_{r(L_M)} = \Gamma(p_{r(L_M)} - (1/t)s_{L_M}) \in J_2$ and hence $\{p_v : v \in L_M^0\} \subseteq J_2$. Now if $v \in K_M \backslash (\Omega(M) \cup L_M^0)$ then $p_v \in J_2$ by Lemma 2.4. Consequently $I_{K_M} \subseteq J_2$. If $v \in B_M$ then the finite sum $\sum_{s(e)=v,\, r(e) \in K_M \backslash \Omega(M)} s_e s_e^*$ belongs to I_{K_M}, and $p_v - p_{v,\Omega(M)}$ belongs to $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}}$. Thus

$$p_v - p_{v,K_M} = (p_v - p_{v,\Omega(M)}) + \sum_{s(e)=v,\, r(e) \in K_M \backslash \Omega(M)} s_e s_e^*$$

belongs to $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}} + I_{K_M} \subseteq J_2$, and consequently $J = J_{K_M,B_M} \subseteq J_2$, as required. □

In particular, for each $M \in \mathscr{M}_\tau(E)$ and $t \in \boldsymbol{T}$ we have

$$J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty} \subset \ker \rho_{M,t} \subset J_{K_M,B_M}.$$

From Lemmas 2.5 and 2.6 we deduce the following.

Corollary 2.7. *Let E be a directed graph, $M \in \mathscr{M}_\tau(E)$ and $t \in \boldsymbol{T}$. Then*

$$J_{K_M,B_M} = (\rho_{M,t})^{-1}(\mathscr{K}(\mathscr{H}_M)).$$

In the following Lemma 2.8 we find explicit generators for the ideals $\ker \rho_{M,t}$. The lemma also implies that for each $M \in \mathscr{M}_\tau(E)$ the family $\{\ker \rho_{M,t} : t \in \boldsymbol{T}\}$ imbeds topologically as a circle into the primitive ideal space of $C^*(E)$.

Lemma 2.8. *Let E be a directed graph, $M \in \mathscr{M}_\tau(E)$, $v = r(L_M)$, and $\pi : J_{K_M,B_M} \to J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ be the canonical surjection. Then the following hold.*

1. *The hereditary C^*-subalgebra $\pi(p_v)(J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty})\pi(p_v)$ is a full corner in $J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$, generated by $\pi(s_{L_M})$, and isomorphic to $C(\boldsymbol{T})$. Hence the quotient $J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ is Morita equivalent to $C(\boldsymbol{T})$.*
2. *For $t \in \boldsymbol{T}$ the ideal $\ker \rho_{M,t}$ of $C^*(E)$ is generated by $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ and $s_{L_M} - tp_v$. We have $\ker \rho_{M,t} = \ker \rho_{L'_M,t}$ for any cyclic permutation L'_M of L_M, and $\ker \rho_{M,t} \neq \ker \rho_{M,z}$ if $t \neq z \in \boldsymbol{T}$.*

Proof. Ad 1. Let $J = \overline{\mathrm{span}}\{J_{K_M,B_M} p_v J_{K_M,B_M} + J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}\}$, an ideal of $C^*(E)$ contained in J_{K_M,B_M}. By Lemma 2.4, $\{p_w : w \in K_M \backslash (\Omega(M) \cup L_M^0)\} \subseteq J$, and clearly $\{p_w : w \in \Omega(M) \cup L_M^0\} \subseteq J$. Thus $I_{K_M} \subseteq J$. Also, if $w \in B_M$ then $p_w - p_{w,K_M} \in I_{K_M} + J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ (cf. the argument at the end of the proof of Lemma 2.6), and thus $p_w - p_{w,K_M} \in J$. Consequently $J_{K_M,B_M} = J$ and hence $\pi(p_v)(J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty})\pi(p_v)$ is a full corner in $J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$.

If $\mu, \nu \in E^*$ then $p_v s_\mu s_\nu^* p_v \notin J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ if and only if $s(\mu) = s(\nu) = v$ and $r(\mu) = r(\nu) \in L_M^0$. Thus $\pi(p_v)(J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty})\pi(p_v)$ is generated as a C^*-algebra by $\pi(s_{L_M})$. If $z \in \boldsymbol{T}$ then $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty} \subseteq \ker \rho_{M,z}$ and thus $\rho_{M,z}$ induces a representation $\tilde{\rho}_{M,z}$ of $J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$. Since $\tilde{\rho}_{M,z}(\pi(s_{L_M}))$ equals z-multiple of a rank one projection, the spectrum of the partial unitary $\pi(s_{L_M})$ contains the entire unit circle. Consequently, the corner $\pi(p_v)(J_{K_M,B_M}/J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty})\pi(p_v)$ is isomorphic to $C(\boldsymbol{T})$.

Ad 2. Fix $t \in \boldsymbol{T}$ and let J' be the ideal of $C^*(E)$ generated by $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$ and $s_{L_M} - tp_v$. Since $\rho_{M,t}(s_{L_M} - tp_v) = 0$, Lemma 2.6 implies that

$$J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty} \subseteq J' \subseteq \ker \rho_{M,t} \subseteq J_{K_M,B_M}.$$

We have shown, above, that $\pi(J_{K_M,B_M} p_v J_{K_M,B_M}) = \pi(J_{K_M,B_M})$ and that $\pi(p_v J_{K_M,B_M} p_v) = C^*(\pi(s_{L_M})) \cong C(\boldsymbol{T})$. Hence

$$\pi(p_v J' p_v) = \{g(\pi(s_{L_M})) : g \in C(\boldsymbol{T}), g(t) = 0\} = \pi(p_v(\ker \rho_{M,t})p_v).$$

Thus

$$\begin{aligned}\pi(J') &= \pi((J_{K_M,B_M} p_v J_{K_M,B_M}) J' (J_{K_M,B_M} p_v J_{K_M,B_M})) \\ &= \pi(J_{K_M,B_M} p_v (\ker \rho_{M,t}) p_v J_{K_M,B_M}) \\ &= \pi((J_{K_M,B_M} p_v J_{K_M,B_M})(\ker \rho_{M,t})(J_{K_M,B_M} p_v J_{K_M,B_M})) \\ &= \pi(\ker \rho_{M,t}).\end{aligned}$$

It follows that $J' = \ker \rho_{M,t}$ and consequently the ideal $\ker \rho_{M,t}$ is generated by $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}}$ and $s_{L_M} - tp_v$. This immediately implies that $\ker \rho_{M,t} = \ker \rho_{L'_M,t}$ for any cyclic permutation L'_M of L_M. Finally, if $t \neq z$ then $s_{L_M} - tp_v \in \ker \rho_{M,t} \backslash \ker \rho_{M,z}$ and hence $\ker \rho_{M,t} \neq \ker \rho_{M,z}$. □

For $M \in \mathscr{M}_\tau(E)$ and $t \in \boldsymbol{T}$ we denote by $R_{M,t}$ the closed two-sided ideal of $C^*(E)$ generated by $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}}$ and $s_{L_M} - tp_{r(L_M)}$. By Lemma 2.8

$$R_{M,t} = \ker \rho_{M,t}.$$

Since $\gamma_{\bar{t}}(R_{M,1}) = R_{M,t}$, these ideals are not gauge-invariant. We will show in Theorem 2.10 that each non gauge-invariant primitive ideal of $C^*(E)$ is of the form $R_{M,t}$. To this end we still need the following simple lemma.

Lemma 2.9. *Let E be a directed graph. If $J \neq 0$ is a primitive ideal of $C^*(E)$ such that $p_v \notin J$ for all $v \in E^0$, then $E^0 \in \mathscr{M}_\tau(E)$ and there is a $t \in \boldsymbol{T}$ such that $J = R_{E^0,t}$.*

Proof. Since $p_v \notin J$ for all $v \in E^0$, it follows from [**1**, Lemma 4.1] that E^0 is a maximal tail. If all loops in E had exits then there existed a $v \in E^0$ such that $p_v \in J$, by the Cuntz-Krieger uniqueness theorem [**7**, Theorem 2]. Thus $E^0 \in \mathscr{M}_\tau(E)$. To simplify the notation, in the remaining part of this proof we write $M = E^0$.

Let ρ be an irreducible representation of $C^*(E)$ with kernel J. Since J does not contain any projections p_v, $v \in E^0$, J does not contain the ideal $I_{L^0_M}$. Thus the restriction of ρ to $I_{L^0_M}$ must be irreducible. By Lemma 2.8 there exists a $t \in \boldsymbol{T}$ such that the restrictions of ρ and $\rho_{M,t}$ to $I_{L^0_M}$ coincide. Hence $\rho = \rho_{M,t}$ and consequently $J = \ker \rho = \ker \rho_{M,t} = R_{M,t}$. □

We set $\mathrm{Prim}_\tau(C^*(E)) = \mathrm{Prim}(C^*(E)) \backslash \mathrm{Prim}_\gamma(C^*(E))$, the collection of primitive ideals of $C^*(E)$ which are not invariant under the gauge action γ.

Theorem 2.10. *Let E be a directed graph. The map*

$$\mathscr{M}_\tau(E) \times \boldsymbol{T} \to \mathrm{Prim}_\tau(C^*(E))$$

given by

$$(M,t) \mapsto R_{M,t}$$

is a bijection.

Proof. The map is well-defined by Lemmas 2.5 and 2.8.

Firstly, we show that the map is injective. That is, we must show that the ideals $\{R_{M,t} : M \in \mathscr{M}_\tau(E), t \in \boldsymbol{T}\}$ are distinct. Indeed, if $R_{M,t} = R_{N,z}$ then $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}} =$

$J_{\Omega(N),\Omega(N)^{\rm fin}_\infty}$ by Lemma 2.6. Thus $\Omega(M)=\Omega(N)$ by [**1**, Lemma 3.7], and consequently $M=E^0\backslash\Omega(M)=E^0\backslash\Omega(N)=N$. It then follows from Lemma 2.8 that $t=z$.

Secondly, we show that the map is surjective. Let $J\in{\rm Prim}_\tau(C^*(E))$. We set $K=\{v\in E^0 : p_v\in J\}$ and $B=\{x\in K^{\rm fin}_\infty : p_x-p_{x,K}\in J\}$. Then $J_{K,B}$ is a proper ideal of J and hence $J/J_{K,B}$ is a non-zero primitive ideal of $C^*(E)/J_{K,B}$. By [**1**, Corollary 3.5] we have $C^*(E)/J_{K,B}\cong C^*(F)$, with $F=(E/K)\backslash\beta(B)$. We denote the canonical generating Cuntz-Krieger F-family by $\{q_w,u_f\}$ (cf. [**1**, Proposition 3.4]). By [**1**, Lemma 3.7] the ideal $J/J_{K,B}$ does not contain any projections q_w, $w\in F^0$. Now applying Lemma 2.9 to the ideal $J/J_{K,B}$ of $C^*(F)$ we see that $F^0\in\mathscr{M}_\tau(F)$ and, using also Lemma 2.8, that there is $t\in\boldsymbol{T}$ such that $J/J_{K,B}$ is generated as an ideal of $C^*(F)$ by $u_{L_{F^0}}-tq_{r(L_{F^0})}$.

Let $M=E^0\backslash K$. M is a maximal tail in E by [**1**, Lemma 4.1]. If $L_{F^0}=(f_1,\dots,f_k)$ then all f_i must come from edges in E^1. Clearly the loop L_{F^0} has no exits in M. Thus $M\in\mathscr{M}_\tau(E)$ and L_{F^0} is a cyclic permutation of L_M by Lemma 2.1. Since M is a maximal tail in E we have $K=E^0\backslash M=\Omega(M)$. We also have $B=\Omega(M)^{\rm fin}_\infty$. Indeed, otherwise the graph F would contain a sink $\beta(v)$, $v\in\Omega(M)^{\rm fin}_\infty\backslash B$, contradicting the fact that F^0 belongs to $\mathscr{M}_\tau(F)$. Consequently the ideal J contains $J_{\Omega(M),\Omega(M)^{\rm fin}_\infty}=J_{K,B}$. Since $u_{L_{F^0}}-tq_{r(L_{F^0})}$ belongs to the quotient $J/J_{\Omega(M),\Omega(M)^{\rm fin}_\infty}$, it now follows that $s_{L_M}-tp_{r(L_M)}$ belongs to J. By Lemma 2.8 we have $R_{M,t}\subseteq J$. As both $J/J_{\Omega(M),\Omega(M)^{\rm fin}_\infty}$ and $R_{M,t}/J_{\Omega(M),\Omega(M)^{\rm fin}_\infty}$ are generated by $u_{L_{F^0}}-tq_{r(L_{F^0})}$, it follows that $J=R_{M,t}$. □

Combining Theorem 2.10 with [**1**, Theorem 4.7] we obtain a complete list of primitive ideals of $C^*(E)$ for an arbitrary countable graph E.

Corollary 2.11. *For a countable directed graph E the map*

$$\mathscr{M}_\gamma(E)\cup BV(E)\cup(\mathscr{M}_\tau(E)\times\boldsymbol{T})\to{\rm Prim}(C^*(E))$$

given by

$$\mathscr{M}_\gamma(E)\ni M\mapsto J_{\Omega(M),\Omega(M)^{\rm fin}_\infty}$$

$$BV(E)\ni v\mapsto J_{\Omega(v),\Omega(v)^{\rm fin}_\infty\backslash\{v\}}$$

$$\mathscr{M}_\tau(E)\times\boldsymbol{T}\ni(N,t)\mapsto R_{N,t}$$

is a bijection.

If E is row-finite then $BV(E)=\varnothing$ and $K^{\rm fin}_\infty=\varnothing$ for every saturated hereditary $K\subseteq E^0$. Consequently, for such graphs we have the following simpler description of the primitive ideals.

Corollary 2.12. *For a countable, row-finite directed graph E the map*

$$\mathscr{M}_\gamma(E)\cup(\mathscr{M}_\tau(E)\times\boldsymbol{T})\to{\rm Prim}(C^*(E))$$

given by

$$\mathscr{M}_\gamma(E)\ni M\mapsto I_{\Omega(M)}$$

$$\mathscr{M}_\tau(E)\times\boldsymbol{T}\ni(N,t)\mapsto R_{N,t}$$

is a bijection.

3. The hull-kernel topology.

$\mathrm{Prim}(C^*(E))$ is a topological space with the hull-kernel topology determined by the closure operation

$$\bar{Z} = \{J \in \mathrm{Prim}(C^*(E)) : \cap Z \subseteq J\}.$$

Our goal in this section is to describe this closure operation. Using the bijection of Corollary 2.11 we transport the hull-kernel topology from $\mathrm{Prim}(C^*(E))$ onto $\mathcal{M}_\gamma(E) \cup BV(E) \cup (\mathcal{M}_\tau(E) \times \boldsymbol{T})$. We begin with the following two simple lemmas.

Lemma 3.1. *Let E be a directed graph and $M \neq N \in \mathcal{M}_\tau(E)$. If there exists a path from L_N^0 to L_M^0 then for all $t, z \in \boldsymbol{T}$ we have*

$$R_{M,t} \subset J_{K_M, B_M} \subseteq J_{\Omega(N), \Omega(N)_\infty^{\mathrm{fin}}} \subset R_{N,z}.$$

Proof. By virtue of Lemma 2.6 it suffices to prove that $R_{M,t} \subseteq J_{\Omega(N), \Omega(N)_\infty^{\mathrm{fin}}}$. By Lemma 2.8 this amounts to showing that $J_{\Omega(M), \Omega(M)_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(N), \Omega(N)_\infty^{\mathrm{fin}}}$ and $s_{L_M} - tp_{r(L_M)} \in J_{\Omega(N), \Omega(N)_\infty^{\mathrm{fin}}}$. Indeed, since there is a path from L_N^0 to L_M^0 we have $\Omega(M) \subseteq \Omega(N)$ and $\Omega(M)_\infty^{\mathrm{fin}} \subseteq \Omega(N) \cup \Omega(N)_\infty^{\mathrm{fin}}$. Thus $J_{\Omega(M), \Omega(M)_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(N), \Omega(N)_\infty^{\mathrm{fin}}}$ by [**1**, Corollary 3.10]. Furthermore, Lemma 2.1 implies that there is no path from L_M^0 to L_N^0. Hence both $p_{r(L_M)}$ and $s_{L_M} = s_{L_M} p_{r(L_M)}$ are in $J_{\Omega(N), \Omega(N)_\infty^{\mathrm{fin}}}$. □

Lemma 3.2. *Let E be a directed graph. If $Y \subseteq \mathcal{M}_\tau(E)$ then we have*

$$\bigcap_{U \in Y} J_{\Omega(U), \Omega(U)_\infty^{\mathrm{fin}}} = J_{K, K_\infty^{\mathrm{fin}}} \quad \text{with } K = \bigcap_{U \in Y} \Omega(U).$$

Proof. By [**1**, Proposition 3.9] we have $J = J_{K,B}$ with $K = \bigcap_{U \in Y} \Omega(U)$ and $B = (\bigcap_{U \in Y} \Omega(U) \cup \Omega(U)_\infty^{\mathrm{fin}}) \cap K_\infty^{\mathrm{fin}}$. Fix $U_0 \in Y$ and let $w \in K_\infty^{\mathrm{fin}} \backslash \Omega(U_0)$. Then w emits infinitely many edges into $K = \bigcap_{U \in Y} \Omega(U) \subseteq \Omega(U_0)$ and only finitely many edges outside K, hence also only finitely many edges outside $\Omega(U_0)$. Since there is a path from w to the loop L_{U_0}, w emits at least one edge into $E^0 \backslash \Omega(U_0)$. Consequently $K_\infty^{\mathrm{fin}} \subseteq \bigcap_{U \in Y} \Omega(U) \cup \Omega(U)_\infty^{\mathrm{fin}}$ and hence $B = K_\infty^{\mathrm{fin}}$. □

If K is a hereditary saturated subset of E^0 then the set K_∞^{fin} of vertices which emit infinitely many edges into K and finitely many edges into its complement affects the ideal structure of $C^*(E)$ and hence it affects $\mathrm{Prim}(C^*(E))$. To describe the topology of $\mathrm{Prim}(C^*(E))$ it is also important to consider the set of those vertices which emit infinitely many edges into K and none into its complement. We call this set $K_\infty^\varnothing$. More formally, we define

$$K_\infty^\varnothing := \{v \in E^0 \backslash K : |s^{-1}(v)| = \infty \text{ and } r(e) \in K \text{ for all } e \text{ with } s(e) = v\}.$$

If M is a maximal tail then $\Omega(M)_\infty^\varnothing$ is either empty or consists of exactly one element by (MT3). In the latter case, if $\Omega(M)_\infty^\varnothing = \{w\}$ then M consists of all those vertices $u \in E^0$ for which there exists a path from u to w.

Lemma 3.3. *Let E be a directed graph. Let $M \in \mathcal{M}_\gamma(E)$, $v \in BV(E)$, $N \in \mathcal{M}_\tau(E)$, and $t \in \boldsymbol{T}$. If $Y \subseteq \mathcal{M}_\tau(E)$ and $K = \bigcap_{U \in Y} \Omega(U)$ then the following hold.*

1. $J_{K,K_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}}$ *if and only if* $M \subseteq \bigcup Y$ *and* $s^{-1}(\Omega(M)_\infty^{\varnothing}) \cap r^{-1}(K)$ *is finite*.
2. $J_{K,K_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(v),\Omega(v)_\infty^{\mathrm{fin}}\setminus\{v\}}$ *if and only if* $v \in \bigcup Y$ *and* $s^{-1}(v) \cap r^{-1}(K)$ *is finite*.
3. $J_{K,K_\infty^{\mathrm{fin}}} \subseteq R_{N,t}$ *if and only if* $N \subseteq \bigcup Y$.

PROOF. Ad 1. By [**1**, Corollary 3.10], $J_{K,K_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}}$ if and only if $K \subseteq \Omega(M)$ and $K_\infty^{\mathrm{fin}} \subseteq \Omega(M) \cup \Omega(M)_\infty^{\mathrm{fin}}$. Clearly, $K \subseteq \Omega(M)$ if and only if $M \subseteq \bigcup Y$. Now assuming $M \subseteq \bigcup Y$ we automatically have $K_\infty^{\mathrm{fin}} \setminus \Omega(M)_\infty^{\varnothing} \subseteq \Omega(M) \cup \Omega(M)_\infty^{\mathrm{fin}}$. On the other hand, if $w \in \Omega(M)_\infty^{\varnothing}$ then $w \notin \Omega(M) \cup \Omega(M)_\infty^{\mathrm{fin}}$. Thus we must have $w \notin K_\infty^{\mathrm{fin}}$ and this can only happen if $s^{-1}(\Omega(M)_\infty^{\varnothing}) \cap r^{-1}(K)$ is finite.

Ad 2. By [**1**, Corollary 3.10], $J_{K,K_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(v),\Omega(v)_\infty^{\mathrm{fin}}\setminus\{v\}}$ if and only if $K \subseteq \Omega(v)$ and $K_\infty^{\mathrm{fin}} \subseteq \Omega(v) \cup (\Omega(v)_\infty^{\mathrm{fin}} \setminus \{v\})$. $K \subseteq \Omega(v)$ if and only if $v \in \bigcup Y$. Assuming $v \in \bigcup Y$ we have $(K_\infty^{\mathrm{fin}} \setminus \{v\}) \subseteq \Omega(v) \cup (\Omega(v)_\infty^{\mathrm{fin}} \setminus \{v\})$. Since $v \notin \Omega(v) \cup (\Omega(v)_\infty^{\mathrm{fin}} \setminus \{v\})$ we must have $v \notin K_\infty^{\mathrm{fin}}$, which can only happen if $s^{-1}(v) \cap r^{-1}(K)$ is finite.

Ad 3. Since $J_{K,K_\infty^{\mathrm{fin}}}$ is gauge-invariant it follows from Lemma 2.6 that $J_{K,K_\infty^{\mathrm{fin}}} \subseteq R_{N,t}$ if and only if $J_{K,K_\infty^{\mathrm{fin}}} \subseteq J_{\Omega(N),\Omega(N)_\infty^{\mathrm{fin}}}$. For $K \subseteq \Omega(N)$ we must have $N \subseteq \bigcup Y$, and under this assumption we automatically have $K_\infty^{\mathrm{fin}} \subseteq \Omega(N) \cup \Omega(N)_\infty^{\mathrm{fin}}$, since $\Omega(N)_\infty^{\varnothing} = \varnothing$. □

If $Y \subseteq \mathcal{M}_\tau(E)$ then it is convenient to consider two special subsets of Y, $Y_{\min}$ and Y_∞, defined as follows.

$$Y_{\min} := \{U \in Y : \text{for all } U' \in Y, U' \neq U \text{ there is no path from } L_U^0 \text{ to } L_{U'}^0\},$$

$$Y_\infty := \{U \in Y : \text{for all } V \in Y_{\min} \text{ there is no path from } L_U^0 \text{ to } L_V^0\}.$$

We call $Y_{\min}$ the set of minimal elements of Y. We are now ready to describe the closure operation in $\mathrm{Prim}(C^*(E))$.

THEOREM 3.4. *Let E be a countable directed graph. Let $X \subseteq \mathcal{M}_\gamma(E)$, $W \subseteq BV(E)$, $Y \subseteq \mathcal{M}_\tau(E)$, and let $D(U) \subseteq \mathbf{T}$ for each $U \in Y$. If $M \in \mathcal{M}_\gamma(E)$, $v \in BV(E)$, $N \in \mathcal{M}_\tau(E)$, and $z \in \mathbf{T}$, then the following hold.*

1. $M \in \overline{X}$ *if and only if either*
   (i) $M \in X$, *or*
   (ii) $M \subseteq \bigcup X$ *and* $s^{-1}(\Omega(M)_\infty^{\varnothing}) \cap r^{-1}(\bigcap_{U \in X} \Omega(U))$ *is finite*.
2. $v \in \overline{X}$ *if and only if* $v \in \bigcup X$ *and* $s^{-1}(v) \cap r^{-1}(\bigcap_{U \in X} \Omega(U))$ *is finite*.
3. $(N, z) \in \overline{X}$ *if and only if* $N \subseteq \bigcup X$.
4. $M \in \overline{W}$ *if and only if* $M \subseteq E^0 \setminus \bigcap_{w \in W} \Omega(w)$ *and* $s^{-1}(\Omega(M)_\infty^{\varnothing}) \cap r^{-1}(\bigcap_{w \in W} \Omega(w))$ *is finite*.
5. $v \in \overline{W}$ *if and only if either*
   (i) $v \in W$, *or*
   (ii) $v \in E^0 \setminus \bigcap_{w \in W} \Omega(w)$ *and* $s^{-1}(v) \cap r^{-1}(\bigcap_{w \in W} \Omega(w))$ *is finite*.
6. $(N, z) \in \overline{W}$ *if and only if* $N \subseteq E^0 \setminus \bigcap_{w \in W} \Omega(w)$.
7. *M is in the closure of* $\{(U, t) : U \in Y, t \in D(U)\}$ *if and only if either*
   (i) $M \subseteq \bigcup Y_\infty$ *and* $s^{-1}(\Omega(M)_\infty^{\varnothing}) \cap r^{-1}(\bigcap_{U \in Y_\infty} \Omega(U))$ *is finite or*
   (ii) $M \subseteq \bigcup Y_{\min}$ *and* $s^{-1}(\Omega(M)_\infty^{\varnothing}) \cap r^{-1}(\bigcap_{U \in Y_{\min}} \Omega(U))$ *is finite*.

8. *v is in the closure of $\{(U,t) : U \in Y, t \in D(U)\}$ if and only if either*
   (i) $v \in \bigcup Y_\infty$ *and* $s^{-1}(v) \cap r^{-1}(\bigcap_{U \in Y_\infty} \Omega(U))$ *is finite, or*
   (ii) $v \in \bigcup Y_{\min}$ *and* $s^{-1}(v) \cap r^{-1}(\bigcap_{U \in Y_{\min}} \Omega(U))$ *is finite.*
9. *(N,z) is in the closure of $\{(U,t) : U \in Y, t \in D(U)\}$ if and only if one of the following three conditions holds.*
   (i) $N \subseteq \bigcup Y_\infty$.
   (ii) $N \notin Y_{\min}$ *and* $N \subseteq \bigcup Y_{\min}$.
   (iii) $N \in Y_{\min}$ *and* $z \in \overline{D(N)}$.

Proof. Throughout the proof of cases 1–6 we denote

$$K = \bigcap_{U \in X} \Omega(U), \quad B = \left(\bigcap_{U \in X} \Omega(U) \cup \Omega(U)^{\mathrm{fin}}_\infty \right) \cap K^{\mathrm{fin}}_\infty,$$

$$K' = \bigcap_{w \in W} \Omega(w), \quad B' = \left(\bigcap_{w \in W} \Omega(w) \cup (\Omega(w)^{\mathrm{fin}}_\infty \setminus \{w\}) \right) \cap (K')^{\mathrm{fin}}_\infty.$$

Ad 1. It suffices to consider the case $M \notin X$. By [**1**, Proposition 3.9] we have $\bigcap_{U \in X} J_{\Omega(U),\Omega(U)^{\mathrm{fin}}_\infty} = J_{K,B}$. Thus $M \in \overline{X}$ if and only if $J_{K,B} \subseteq J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_\infty}$. By [**1**, Corollary 3.10] this is equivalent to $K \subseteq \Omega(M)$ and $B \subseteq \Omega(M) \cup \Omega(M)^{\mathrm{fin}}_\infty$. Clearly, $K \subseteq \Omega(M)$ if and only if $M \subseteq \bigcup X$. Assuming $M \subseteq \bigcup X$ we automatically have $(B \setminus \Omega(M)^{\varnothing}_\infty) \subseteq \Omega(M) \cup \Omega(M)^{\mathrm{fin}}_\infty$. On the other hand, if $w \in \Omega(M)^{\varnothing}_\infty$ then $w \notin \Omega(M) \cup \Omega(M)^{\mathrm{fin}}_\infty$. Thus we must have $w \notin B$, which can only happen in one of the following three cases: (i) $\{r(e) : e \in E^1, s(e) = w\} \subseteq \bigcap_{U \in X} \Omega(U)$, (ii) there is a $U \in X$ such that $\{r(e) : e \in E^1, s(e) = w\} \subseteq \Omega(U)$, (iii) $s^{-1}(w) \cap r^{-1}(\bigcap_{U \in X} \Omega(U))$ is finite. Case (i) reduces to case (ii) since we assumed that $M \subseteq \bigcup X$. In case (ii) we have $w \in \Omega(U)^0_\infty$ and hence $M = U \in X$, contrary to the assumption. Therefore only case (iii) remains, and the claim is proved.

Ad 2. Similarly as in case 1 above, $v \in \overline{X}$ if and only if $J_{K,B} \subseteq J_{\Omega(v),\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\}}$, and this is equivalent to $K \subseteq \Omega(v)$ and $B \subseteq \Omega(v) \cup (\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$. Clearly, $K \subseteq \Omega(v)$ if and only if $v \in \bigcup X$. Assuming $v \in \bigcup X$ we automatically have $(B \setminus \{v\}) \subseteq \Omega(v) \cup (\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$. On the other hand, $v \notin \Omega(v) \cup (\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$. Thus we must have $v \notin B$, which can only happen if $s^{-1}(v) \cap r^{-1}(\bigcap_{U \in X} \Omega(U))$ is finite.

Ad 3. Since $\bigcap_{U \in X} J_{\Omega(U),\Omega(U)^{\mathrm{fin}}_\infty}$ is gauge-invariant, by Lemma 2.6 this ideal is contained in $R_{N,z}$ if and only if it is already contained in $J_{\Omega(N),\Omega(N)^{\mathrm{fin}}_\infty}$. Since $N \notin X$ and $\Omega(N)^{\varnothing}_\infty = \varnothing$ the claim is proved similarly to the first part of case 1 above.

Ad 4. We have $\bigcap_{w \in W} J_{\Omega(w),\Omega(w)^{\mathrm{fin}}_\infty \setminus \{w\}} = J_{K',B'}$. Thus, by [**1**, Corollary 3.10], $M \in \overline{W}$ if and only if $K' \subseteq \Omega(M)$ and $B' \subseteq \Omega(M) \cup \Omega(M)^{\mathrm{fin}}_\infty$. $K' \subseteq \Omega(M)$ is equivalent to $M \subseteq E^0 \setminus \bigcap_{w \in W} \Omega(w)$. Assuming this, it suffices to find a condition for $v \notin B'$ (similarly to the argument from case 1 above). It is easy to see that $v \notin B'$ occurs precisely when $s^{-1}(\Omega(M)^{\varnothing}_\infty) \cap r^{-1}(\bigcap_{w \in W} \Omega(w))$ is finite.

Ad 5. Obviously, if $v \in W$ then $v \in \overline{W}$. Thus we may assume that $v \notin W$. Similarly to the above, $v \in \overline{W}$ if and only if $K' \subseteq \Omega(v)$ and $B' \subseteq \Omega(v) \cup (\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$. $K' \subseteq \Omega(v)$ is equivalent to $v \in E^0 \setminus \bigcap_{w \in W} \Omega(w)$. Since $B' \setminus \{v\} \subseteq \Omega(v) \cup (\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$ and $v \notin \Omega(v) \cup (\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$, it suffices to find a condition for $v \notin B'$. But this is equivalent to $s^{-1}(v) \cap r^{-1}(\bigcap_{w \in W} \Omega(w))$ being finite.

Ad 6. The proof is similar to the case 3 above. Indeed, $(N,z) \in \overline{W}$ if and only if $J_{K',B'} \subseteq R_{N,z}$. Since $J_{K',B'}$ is gauge-invariant this is equivalent to $J_{K',B'} \subseteq J_{\Omega(N),\Omega(N)^{\mathrm{fin}}_{\infty}}$ by Lemma 2.6, and by [1, Corollary 3.10] this happens if and only if $K' \subseteq \Omega(N)$ and $B' \subseteq \Omega(N) \cup \Omega(N)^{\mathrm{fin}}_{\infty}$. The latter is automatically satisfied and the former is equivalent to $N \subseteq E^0 \setminus \bigcap_{w \in W} \Omega(w)$.

Ad 7–9. The following observations are used in the proofs of cases 7, 8 and 9.

The closure of $\{(U,t) : U \in Y, t \in D(U)\}$ coincides with the union of the closure of $\{(U,t) : U \in Y_\infty, t \in D(U)\}$ and the closure of $\{(U,t) : U \in Y \setminus Y_\infty, t \in D(U)\}$. Thus it suffices to find these two closures.

In order to determine the closure of $\{(U,t) : U \in Y_\infty, t \in D(U)\}$ we observe that

$$\bigcap_{U \in Y_\infty} \bigcap_{t \in D(U)} R_{U,t} = \bigcap_{U \in Y_\infty} J_{\Omega(U),\Omega(U)^{\mathrm{fin}}_{\infty}}. \tag{1}$$

Indeed, if $U' \in Y_\infty$ then there exists a $U \in Y_\infty$ different from U' such that there is a path from $L^0_{U'}$ to L^0_U. By Lemma 3.1 we have $R_{U',t_1} \cap R_{U,t_2} = J_{\Omega(U'),\Omega(U')^{\mathrm{fin}}_{\infty}} \cap R_{U,t_2}$ and hence in the LHS of (1) we may replace each $R_{U,t}$ by $J_{\Omega(U),\Omega(U)^{\mathrm{fin}}_{\infty}}$. Therefore, a primitive ideal belongs to the closure of $\{(U,t) : U \in Y_\infty, t \in D(U)\}$ if and only if the conditions of Lemma 3.3 are satisfied (with Y_∞ instead of Y).

Similarly, in order to determine the closure of $\{(U,t) : U \in Y \setminus Y_\infty, t \in D(U)\}$ we observe that

$$\bigcap_{U \in Y \setminus Y_\infty} \bigcap_{t \in D(U)} R_{U,t} = \bigcap_{U \in Y_{\min}} \bigcap_{t \in D(U)} R_{U,t} \tag{2}$$

by virtue of Lemma 3.1. By Lemmas 3.2 and 2.6 we get

$$J_{K'',(K'')^{\mathrm{fin}}_{\infty}} = \bigcap_{U \in Y_{\min}} J_{\Omega(U),\Omega(U)^{\mathrm{fin}}_{\infty}} \subseteq \bigcap_{U \in Y_{\min}} \bigcap_{t \in D(U)} R_{U,t},$$

where $K'' = \bigcap_{U \in Y_{\min}} \Omega(U)$. Then a primitive ideal J belongs to the closure of $\{(U,t) : U \in Y \setminus Y_\infty, t \in D(U)\}$ if and only if J contains the RHS of (2), and for this it is necessary that $J \supseteq J_{K'',(K'')^{\mathrm{fin}}_{\infty}}$. Thus it is useful to look at the quotient $C^*(E)/J_{K'',(K'')^{\mathrm{fin}}_{\infty}}$. By [1, Corollary 3.5] we have $C^*(E)/J_{K'',(K'')^{\mathrm{fin}}_{\infty}} \cong C^*(F)$, where $F = (E/K'') \setminus \beta((K'')^{\mathrm{fin}}_{\infty})$ is the subgraph of E such that $F^0 = E^0 \setminus K''$ and $F^1 = \{e \in E^1 : r(e) \notin K''\}$.

Ad 7. $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}}$ contains $J_{K'',(K'')^{\mathrm{fin}}_{\infty}}$ if and only if $M \subseteq \bigcup Y_{\min}$ and $s^{-1}(\Omega(M)^{\varnothing}_{\infty}) \cap r^{-1}(K'')$ is finite, by Lemma 3.3. Assume this holds. Then $J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}}/J_{K'',(K'')^{\mathrm{fin}}_{\infty}}$ is a gauge-invariant primitive ideal of $C^*(F)$ and hence contains all projections corresponding to $\{v \in L^0_U : U \in Y_{\min}\}$, since the loops L_U, $U \in Y_{\min}$ have no exits in F. By Lemmas 2.4 and 2.6 this implies that $R_{U,t} \subseteq J_{K_U,B_U} \subseteq J_{\Omega(M),\Omega(M)^{\mathrm{fin}}_{\infty}}$ for each $U \in Y_{\min}$, $t \in D(U)$, and thus M is in the closure of $\{(U,t) : U \in Y \setminus Y_\infty, t \in D(U)\}$. Consequently, M is in the closure of $\{(U,t) : U \in Y, t \in D(U)\}$ if and only if either (i) $M \subseteq \bigcup Y_\infty$ and $s^{-1}(\Omega(M)^{\varnothing}_{\infty}) \cap r^{-1}(\bigcap_{U \in Y_\infty} \Omega(U))$ is finite, or (ii) $M \subseteq \bigcup Y_{\min}$ and $s^{-1}(\Omega(M)^{\varnothing}_{\infty}) \cap r^{-1}(\bigcap_{U \in Y_{\min}} \Omega(U))$ is finite.

Ad 8. This is proved by an argument very similar to case 7 above.

Ad 9. $R_{N,z}$ contains $J_{K'',(K'')^{\mathrm{fin}}_{\infty}}$ if and only if $N \subseteq \bigcup Y_{\min}$, by Lemma 3.3. Assume this holds. If $N \notin Y_{\min}$ then $R_{N,z}$ contains the RHS of (2) by Lemma 3.1, since there exists a path from L_N^0 to at least one L_U^0, $U \in Y_{\min}$. Suppose $N \in Y_{\min}$. If $z \notin \overline{D(N)}$ then let $g : \boldsymbol{T} \to \boldsymbol{C}$ be a continuous function such that $g|_{\overline{D(N)}} = 0$ and $g(z) \neq 0$. Then $g(s_{L_N})$ is in the RHS of (2) but not in $R_{N,z}$. Thus we must have $z \in \overline{D(N)}$. In this case it follows from Lemma 2.8 that $R_{N,z}$ contains the RHS of (2). Consequently, (N, z) is in the closure of $\{(U, t) : U \in Y, t \in D(U)\}$ if and only if one of the following three conditions holds; (i) $N \subseteq \bigcup Y_\infty$, (ii) $N \notin Y_{\min}$ and $N \subseteq \bigcup Y_{\min}$, (iii) $N \in Y_{\min}$ and $z \in \overline{D(N)}$. □

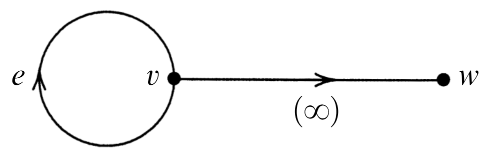
Corollary 3.5. *Let E be a row-finite directed graph. Let $X \subseteq \mathcal{M}_\gamma(E)$, $Y \subseteq \mathcal{M}_\tau(E)$, and let $D(U) \subseteq \boldsymbol{T}$ for each $U \in Y$. If $M \in \mathcal{M}_\gamma(E)$, $N \in \mathcal{M}_\tau(E)$, and $z \in \boldsymbol{T}$, then the following hold.*

1. *$M \in \overline{X}$ if and only if $M \subseteq \bigcup X$.*
2. *$(N, z) \in \overline{X}$ if and only if $N \subseteq \bigcup X$.*
3. *M is in the closure of $\{(U, t) : U \in Y, t \in D(U)\}$ if and only if $M \subseteq \bigcup Y$.*
4. *(N, z) is in the closure of $\{(U, t) : U \in Y, t \in D(U)\}$ if and only if one of the following three conditions holds.*
   (i) *$N \subseteq \bigcup Y_\infty$.*
   (ii) *$N \notin Y_{\min}$ and $N \subseteq \bigcup Y_{\min}$.*
   (iii) *$N \in Y_{\min}$ and $z \in \overline{D(N)}$.*

4. Examples.

We illustrate the main results of this paper with the following three examples. The discussion of gauge-invariant ideals of the algebras corresponding to the last two of them was carried out in [**1**, Section 5]. Now we are in a position to give a complete description of their primitive ideal spaces.

Example 4.1. Let E_1 be the following graph, in which the symbol (∞) indicates that there are infinitely many edges from v to w.



There are two maximal tails in E_1, namely E_1^0 and $M = \{v\}$. E_1^0 is in $\mathcal{M}_\gamma(E_1)$ and M belongs to $\mathcal{M}_\tau(E_1)$. There is a unique breaking vertex v in E_1. The bijection of Corollary 2.11 identifies $\{E_1^0, v\} \cup (M \times \boldsymbol{T})$ with $\mathrm{Prim}(C^*(E_1))$. The topology can be determined by Theorem 3.4. The closure of $\{E_1^0\}$ is the entire space $\mathrm{Prim}(C^*(E_1))$, the closure of $\{v\}$ is $\{v\} \cup (M \times \boldsymbol{T})$, and for every $D \subseteq \boldsymbol{T}$ the closure of $M \times D$ is $M \times \overline{D}$.

The maximal tail E_1^0 corresponds to the primitive ideal $\{0\}$. The breaking vertex v corresponds to the ideal I_w, generated by the projection p_w. This ideal is isomorphic with the compacts and is essential in $C^*(E_1)$ by [**27**, Lemma 1.1]. The quotient $C^*(E_1)/I_w$ is isomorphic to the Toeplitz algebra $\mathcal{T}$ by [**1**, Proposition 3.4]. Thus, there is a short exact sequence

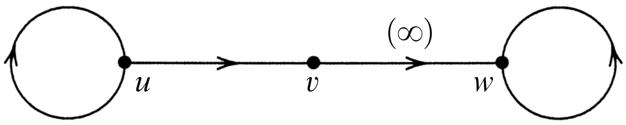
$$0 \to \mathcal{K} \to C^*(E_1) \to \mathcal{T} \to 0.$$

As shown in Lemma 2.6, each non gauge-invariant primitive ideal $R_{M,t}$, corresponding to the maximal tail M and $t \in \boldsymbol{T}$, is sandwiched between two gauge-invariant ideals, namely

$$J_{\{w\},\{v\}} \subset R_{M,t} \subset C^*(E_1).$$

The ideal $J_{\{w\},\{v\}}$ is generated by the projections p_w and $p_v - s_e s_e^*$.

Example 4.2. The following graph E_2, considered in [**8**], is neither row-finite nor does it satisfy Condition (K).



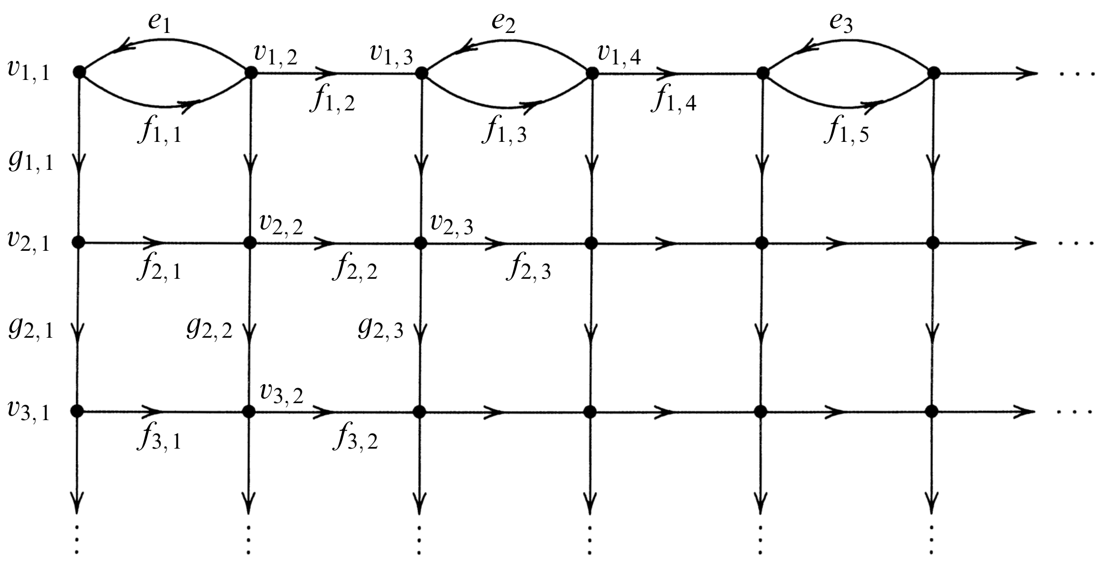
There are no breaking vertices in E_2. There are three maximal tails in E_2: $M_1 = \{u\}$, $M_2 = \{u, v\}$ and $M_3 = \{u, v, w\}$. $M_2 \in \mathcal{M}_\gamma(E_2)$, while M_1 and M_3 belong to $\mathcal{M}_\tau(E_2)$. The bijection of Corollary 2.11 identifies $\{M_2\} \cup (M_1 \times \boldsymbol{T}) \cup (M_3 \times \boldsymbol{T})$ with $\mathrm{Prim}(C^*(E_2))$. The topology is given by Theorem 3.4. The closure of $\{M_2\}$ is $\{M_2\} \cup (M_1 \times \boldsymbol{T})$. For any $D \subseteq \boldsymbol{T}$, the closure of $M_1 \times D$ is $M_1 \times \bar{D}$, and the closure of $M_3 \times D$ is $\{M_2\} \cup (M_1 \times \boldsymbol{T}) \cup (M_3 \times \bar{D})$. As in Lemma 2.6, for any $t \in \boldsymbol{T}$ we have

$$I_{\{v,w\}} \subset R_{M_1,t} \subset C^*(E_2) \quad \text{and} \quad \{0\} \subset R_{M_3,t} \subset I_w.$$

Example 4.3. Let E_3 be the following graph with $E_3^0 = \{v_{i,j} \mid 1 \le i, j < \infty\}$ and $E_3^1 = \{e_i\} \cup \{f_{i,j}\} \cup \{g_{i,j}\}$, where

$$s(e_i) = v_{1,2i}, \quad s(f_{i,j}) = v_{i,j}, \quad s(g_{i,j}) = v_{i,j},$$

$$r(e_i) = v_{1,2i-1}, \quad r(f_{i,j}) = v_{i,j+1}, \quad r(g_{i,j}) = v_{i+1,j}.$$



This is an infinite row-finite graph which does not satisfy Condition (K). Since E_3 is row-finite there are no breaking vertices. There are four families of maximal tails, indexed by the integers $n \ge 1$:

$$M_n = \{v_{i,j} : 1 \le i \le n, 1 \le j < \infty\},$$

$$M^{2n-1} = \{v_{i,j} : 1 \le i < \infty, 1 \le j \le 2n-1\} \cup \{v_{1,2n}\},$$

$$M^{2n} = \{v_{i,j} : 1 \le i < \infty, 1 \le j \le 2n\},$$

$$T_n = \{v_{1,j} : 1 \le j \le 2n\}.$$

In addition, E_3^0 is a maximal tail too. E_3^0 and all M_n and M^n belong to $\mathcal{M}_\gamma(E_3)$. On the other hand, each maximal tail T_n contains a loop without exits and hence $T_n \in \mathcal{M}_\tau(E_3)$. By Corollary 2.11, there is a bijection between $\{E_3^0\} \cup \{M_n : n \ge 1\} \cup \{M^n : n \ge 1\} \cup \bigcup_{n \ge 1}(T_n \times \boldsymbol{T})$ and $\mathrm{Prim}(C^*(E_3))$. The topology of $\mathrm{Prim}(C^*(E_3))$ can be determined with help of Corollary 3.5.

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