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On the local convergence of Newton's method to a multiple root

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Abstract. In the local dynamics of Newton's method of a holomorphic function of two variables, a multiple root of rank 1 has a Cantor family of holomorphic superstable manifolds which consists of quadratically convergent initial values.

1. Introduction.

The aim of this paper is to give a geometric description on the local convergence of Newton's method toward a multiple root, in the case of a holomorphic mapping of two variables.

First recall that the local dynamics of Newton's method is well known in the case of one variable. If $z = z_0 \in \boldsymbol{C}$ is a simple root of the function $f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$, then z_0 is a superattracting fixed point of Newton's method $Nf(z) = z - f(z)/f'(z) = z_0 + a_1^{-1}a_2(z - z_0)^2 + \cdots$. If z_0 is a multiple root of $f(z) = a(z - z_0)^m + \cdots$, $m \geq 2$, then it is an attracting fixed point of $Nf(z) = z_0 + ((m-1)/m)(z - z_0) + \cdots$.

Let $F : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ be a holomorphic map. Newton's method of F is the mapping $NF(z) = z - (DF)_z^{-1}F(z)$ where $z = (x, y) \in \boldsymbol{C}^2$. A multiple root of F is a point z_0 such that $F(z_0) = (0, 0)$ and $\det(DF)_{z_0} = 0$. It can give rise to an indeterminate point. That is, the intersection of the closures $\overline{NF(U \backslash \{z_0\})}$, where $U \subset \boldsymbol{C}^2$ runs through a neighborhood base of z_0, is not a single point. So no definition of the image $NF(z_0)$ makes the mapping NF continuous.

Suppose that the origin $z_0 = (0, 0)$ is a multiple root of F. Since F is a mapping of two variables, $\operatorname{rank}(DF)_{z_0}$ is equal to 1 or 0. In this paper we consider the case $\operatorname{rank}(DF)_{z_0} = 1$. As a general property of Newton's method, it is easy to see that $N(L \circ F) = NF$ if $L : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ is a linear automorphism, and $N(F \circ A) = A^{-1} \circ NF \circ A$ if $A : \boldsymbol{C}^2 \to \boldsymbol{C}^2$ is an affine automorphism. This implies that we can give linear coordinate changes in the domain of definition $\boldsymbol{C}^2$ as well as in the range $\boldsymbol{C}^2$. So we may suppose that

$$(DF)_{z_0} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

without loss of generality. Denote by $F(z) = (x + \cdots, p(x, y) + \cdots)$ where $p(x, y)$ is a

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homogeneous polynomial of degree ≥ 2. In this paper we consider the simplest case of a multiple root, so suppose that $p(x, y)$ is a quadratic homogeneous polynomial with no multiple factor, and that $p(x, y)$ is not divisible by x. By linear coordinate changes we may suppose that $p(x, y) = y^2 - x^2$, and F is written by

$$F(z) = (x + a_2x^2 + a_1xy + a_0y^2 + O(\|z\|^3), y^2 - x^2 + O(\|z\|^3)) \tag{1}$$

as $z = (x, y) \to (0, 0)$, where $\|z\| = \max(|x|, |y|)$ is the box norm. Suppose furthermore that

$$a_2 + a_0 \neq \pm a_1, \tag{2}$$

which gives a transversality condition that will be used later.

The main result of this paper is the following. There exists a neighborhood $U \ni z_0$ that is divided into three subsets

$$U \backslash \{z_0\} = A \cup B \cup C \tag{3}$$

where

- A is called an attracting set. $NF(A) \subset A$. For each $z \in A$, we have $x_n/y_n \to 0$ and $y_{n+1}/y_n \to 1/2$ as $n \to \infty$, where $(NF)^n(z) = (x_n, y_n)$.
- B is called a bursting set. $B = \bigcup_{n=0}^{\infty} B_n$ where $B_0 = U \backslash NF^{-1}(U)$, and $B_{n+1} = U \cap NF^{-1}(B_n)$, $n \geq 0$. Each B_n consists of 2^n components and the image $NF^{n+1}(B_n)$ is unbounded.
- C is called a chaotic set, or a Cantor family of holomorphic superstable manifolds. There exist constants $0 < c_1' < c_2'$ such that $c_1'\|z\|^2 \leq \|NF(z)\| \leq c_2'\|z\|^2$ for each $z \in C$.

By definition, the local stable set $W_{loc}^s(z_0)$ of z_0 is the set of points $z \in U$ such that $NF^n(z_0)$ stays in U for any $n \geq 0$, and $NF^n(z) \to z_0$ as $n \to \infty$. In our case the local stable set of the multiple root $W_{loc}^s(z_0)$ is equal to $A \cup C$.

Under an appropriate local coordinate change, we find a blow-up operation that is defined on a pair of polydiscs and is mapped to an unbounded region transversing the polydiscs. First in Section 2 we study such a dynamics, which is called a 'kebab' (or 'dango') operation that was first given in [**4**]. Later in Section 3 we give the decomposition (3).

By the C^r center manifold theorem (see [**3**]), we see that there exists a C^r invariant manifold of z_0 in the subset A, but its analyticity is not known. In section 4 we consider this problem in a general situation.

A global approach to Newton's method of several variables is given by [**1**], which also includes many references.

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2. Cantor family of superstable manifolds in the kebab operation.

Here we give a model of a local dynamics that gives a Cantor family of holomorphic superstable manifolds for a pair of indeterminate points. Let $i, j = 1, 2$ throughout this section.

Let $\pi(u,v) = (u, uv)$ and $\mathrm{sq}(u,v) = (u^2, v)$ be mappings of $\boldsymbol{C}^2$. Let V_0 be a neighborhood of the origin in $\boldsymbol{C}^2$, and let $V = \pi^{-1}(V_0)$. Consider two points $q_i = (0, \alpha_i)$ and their neighborhoods $V_i \ni q_i$. Let $g_i : V_0 \to V_i$, with $g_i(0,0) = q_i$, be a biholomorphic map $g_i(u,v) = S_i(u,v) + \cdots$ where $S_i(u,v) = (a_i u + b_i v, \alpha_i + c_i u + d_i v)$ is the linear part. Suppose that

$$|a_i + b_i \alpha_j| \neq 0, \quad i, j = 1, 2. \tag{4}$$

We consider the local dynamics

$$f : V_1 \cup V_2 \to V \tag{5}$$

defined by

$$f|_{V_i} = \mathrm{sq} \circ \pi^{-1} \circ g_i^{-1} : V_i \to V.$$

It has two indeterminate points q_i, $i = 1, 2$, since the origin is an indeterminate point of π^{-1}. Denote by $f_i = f|_{V_i}$. The mapping f has two inverse branches

$$f_i^{-1} = g_i \circ \pi \circ \mathrm{sq}^{-1} : V \to V_i$$

which are contracting in the vertical v-direction by the contribution of the blow-down map π, and expanding in the horizontal u-direction by sq^{-1}. (In [**4**], we have studied the dynamics like $\pi^{-1} \circ g_i^{-1} : V_i \to V$ without sq.)

Let $r, r_0, \rho > 0$ be small and $M > 0$ large. Let $\boldsymbol{B}_0 = \bar{\boldsymbol{D}}(0,\rho) \times \bar{\boldsymbol{D}}(0, r_0) \subset \bar{\boldsymbol{D}}(0, \sqrt{\rho}) \times \bar{\boldsymbol{D}}(0, r_0) \subset V_0$ be closed polydiscs centered at the origin. Let $\boldsymbol{B}_i = \bar{\boldsymbol{D}}(0,\rho) \times \bar{\boldsymbol{D}}(\alpha_i, r) \subset V_i$. Let $\boldsymbol{L}_i = \mathrm{Lip}_M(\bar{\boldsymbol{D}}(0,\rho), \bar{\boldsymbol{D}}(\alpha_i, r))$ be the set of Lipschitz functions of $\bar{\boldsymbol{D}}(0,\rho)$ to $\bar{\boldsymbol{D}}(\alpha_i, r)$ with Lipschitz constant $\leq M$. Let $\boldsymbol{H}_i \subset \boldsymbol{L}_i$ be the set of $\tau_i \in \boldsymbol{L}_i$ such that the restriction to the open disk $\tau_i|_{\boldsymbol{D}(0,\rho)}$ is holomorphic. Let $\Sigma(2) = \{1,2\}^{\boldsymbol{N}} \ni w = w_0 w_1 \cdots$ be a Cantor set. Let $s : \Sigma(2) \to \Sigma(2)$, $s(w_0 w_1 w_2 \cdots) = w_1 w_2 \cdots$, be the shift operator.

For each $\tau_j \in \boldsymbol{L}_j$, denote by $\tau_j^* : \bar{\boldsymbol{D}}(0, \sqrt{\rho}) \to \bar{\boldsymbol{D}}(0, \sqrt{\rho}) \times \bar{\boldsymbol{D}}(\alpha_j, r)$ the mapping such that $\mathrm{image}(\tau_j^*) = \mathrm{sq}^{-1}(\mathrm{graph}\, \tau_j)$. It is defined by $\tau_j^*(u) = (u, \tau_j(u^2))$. Let $p_1(u,v) = u$, $p_2(u,v) = v$ be the projections. We are going to define the graph transform

$$\Gamma_{g_i} : \boldsymbol{L}_1 \cup \boldsymbol{L}_2 \to \boldsymbol{L}_1 \cup \boldsymbol{L}_2$$

by

$$\Gamma_{g_i}(\tau_j) = p_2 g_i \pi \tau_j^* [p_1 g_i \pi \tau_j^*]^{-1}|_{\bar{\boldsymbol{D}}(0,\rho)},$$

so that

$$f(\mathrm{graph}(\Gamma_{g_i}(\tau_j))) \subset \mathrm{graph}\, \tau_j, \tag{6}$$

$$\mathrm{graph}(\Gamma_{g_i}(\tau_j)) = \boldsymbol{B}_i \cap f^{-1}(\mathrm{graph}\, \tau_j), \tag{7}$$

and $\Gamma_{g_i}(\boldsymbol{L}_1 \cup \boldsymbol{L}_2) \subset \boldsymbol{L}_i$ hold.

In order to show that Γ_{g_i} is well defined, let $\ell := \mathrm{Lip}(g_i - S_i)$ be the Lipschitz constant as a mapping of $\bar{\boldsymbol{D}}(0, \sqrt{\rho}) \times \bar{\boldsymbol{D}}(0, r_0)$. Note that $\ell \to 0$ as $\rho, r_0 \to 0$. Let $b = \max(|b_1|, |b_2|, |d_1|, |d_2|)$. Choose small $r, r_0, \rho > 0$ and $\delta > 0$ appropriately so that

$$\sqrt{\rho}(|\alpha_i| + r) \leq r_0 \tag{8}$$

and

$$\ell \max(1, |\alpha_j| + r + 2\rho M) + b(r + 2\rho M) < \delta < |a_i + b_i \alpha_j| - \sqrt{\rho}. \tag{9}$$

LEMMA 1. *For each* $\tau_j \in \boldsymbol{L}_j$, $\Gamma_{g_i}(\tau_j) = p_2 g_i \pi \tau_j^* [p_1 g_i \pi \tau_j^*]^{-1}|_{\bar{\boldsymbol{D}}(0,\rho)}$ *is well defined as a mapping of* $\bar{\boldsymbol{D}}(0,\rho)$ *to* $\boldsymbol{C}$. *That is,* $p_1 g_i \pi \tau_j^* : \bar{\boldsymbol{D}}(0, \sqrt{\rho}) \to \boldsymbol{C}$ *is an injective map that overflows* $\bar{\boldsymbol{D}}(0,\rho)$, *i.e.,* $p_1 g_i \pi \tau_j^*(\bar{\boldsymbol{D}}(0, \sqrt{\rho})) \supset \bar{\boldsymbol{D}}(0,\rho)$.

PROOF. By (8) we see that $\pi(\bar{\boldsymbol{D}}(0, \sqrt{\rho}) \times \bar{\boldsymbol{D}}(\alpha, r)) \subset \bar{\boldsymbol{D}}(0, \sqrt{\rho}) \times \bar{\boldsymbol{D}}(0, r_0) \subset V_0$ and the mapping $g_i \pi \tau_j^*$ of $\bar{\boldsymbol{D}}(0, \sqrt{\rho})$ is well defined.

Let $\tau_{j0} \in \boldsymbol{L}_j$ be the constant function $\tau_{j0}(u) \equiv \alpha_j$. Compare $p_1 g_i \pi \tau_j^*$ with the linear mapping $p_1 S_i \pi \tau_{j0}^*(u) = (a_i + b_i \alpha_j) u$ as follows.

$$\begin{aligned}
&\operatorname{Lip}(p_1 g_i \pi \tau_j^* - p_1 S_i \pi \tau_{j0}^*) \\
&\quad \le \operatorname{Lip}(p_1 g_i \pi \tau_j^* - p_1 S_i \pi \tau_j^*) + \operatorname{Lip}(p_1 S_i \pi \tau_j^* - p_1 S_i \pi \tau_{j0}^*) \\
&\quad \le \operatorname{Lip}(p_1) \operatorname{Lip}(g_i - S_i) \operatorname{Lip}(\pi \tau_j^*) + \operatorname{Lip}(p_1 S_i \pi \tau_j^* - p_1 S_i \pi \tau_{j0}^*).
\end{aligned}$$

The second term is the Lipschitz constant of the mapping $u \mapsto (a_i u + b_i u \tau_j(u^2)) - (a_i u + b_i u \alpha_j) = b_i u(\tau_j(u^2) - \alpha_j)$, $u \in \bar{\boldsymbol{D}}(0, \sqrt{\rho})$. So

$$\begin{aligned}
&\operatorname{Lip}(p_1 S_i \pi \tau_j^* - p_1 S_i \pi \tau_{j0}^*) \\
&\quad \le |b_i| \operatorname{Lip}(u) \sup|\tau_j(u^2) - \alpha_j| + |b_i| \sup|u| \operatorname{Lip}(\tau_j(u^2) - \alpha_j) \\
&\quad \le br + b\sqrt{\rho} \cdot 2\sqrt{\rho} M = b(r + 2\rho M).
\end{aligned}$$

By $\pi \tau_j^*(u) = (u, u\tau_j(u^2)) = (u, \alpha_j u + u(\tau_j(u^2) - \alpha_j))$, we have

$$\operatorname{Lip}(\pi \tau_j^*) \le \max(1, |\alpha_j| + r + 2\rho M).$$

Hence

$$\begin{aligned}
\operatorname{Lip}(p_1 g_i \pi \tau_j^* - p_1 S_i \pi \tau_{j0}^*) &\le \ell \max(1, |\alpha_j| + r + 2\rho M) + b(r + 2\rho M) \\
&< \delta.
\end{aligned}$$

Since $|p_1 g_i \pi \tau_j^*(u) - p_1 g_i \pi \tau_j^*(u') - p_1 S_i \pi \tau_j^*(u - u')| \le \delta |u - u'|$, we have

$$|a_i + b_i \alpha_j| - \delta \le \frac{|p_1 g_i \pi \tau_j^*(u) - p_1 g_i \pi \tau_j^*(u')|}{|u - u'|} \le |a_i + b_i \alpha_j| + \delta. \tag{10}$$

By the Lipschitz Inverse Function Theorem (Appendix I of [**3**]), the mapping $p_1 g_i \pi \tau_j^*$ is a homeomorphism of $\bar{\boldsymbol{D}}(0, \sqrt{\rho})$ onto its image, with Lipschitz inverse

$$\operatorname{Lip}([p_1 g_i \pi \tau_j^*]^{-1}) \le (|a_i + b_i \alpha_j| - \delta)^{-1},$$

and the image of $p_1 g_i \pi \tau_j^*$ contains $\bar{\boldsymbol{D}}(0, \sqrt{\rho}(|a_i + b_i \alpha_j| - \delta)) \supset \bar{\boldsymbol{D}}(0, \rho)$. □

Next suppose furthermore that $M > 0$ is so large that

$$M > |a_i + b_i \alpha_j|^{-1} |c_i + d_i \alpha_j|$$

and $\delta, \rho > 0$ are so small that

$$\frac{|c_i + d_i\alpha_j| + \delta}{|a_i + b_i\alpha_j| - \delta} \le M \quad \text{and} \quad \rho M \le r. \tag{11}$$

LEMMA 2. *The graph transform* $\Gamma_{g_i} : \boldsymbol{L}_1 \cup \boldsymbol{L}_2 \to \boldsymbol{L}_i \subset \boldsymbol{L}_1 \cup \boldsymbol{L}_2$ *is well-defined. That is,* $\mathrm{Lip}(\Gamma_{g_i}(\tau_j)) \le M$ *and* $\mathrm{image}(\Gamma_{g_i}(\tau_j)) \subset \bar{\boldsymbol{D}}(\alpha_i, r)$.

PROOF. Compare $\Gamma_{g_i}(\tau_j)$ with the linear function

$$\Gamma_{S_i}(\tau_{j0}) = p_2 S_i \pi \tau_{j0}^* [p_1 S_i \pi \tau_{j0}^*]^{-1} : u \mapsto \alpha_j + \frac{c_i + d_i\alpha_j}{a_i + b_i\alpha_j} u$$

as follows.

$$\begin{aligned}
\mathrm{Lip}(\Gamma_{g_i}(\tau_j)) &\le \mathrm{Lip}(\Gamma_{S_i}(\tau_{j0})) + \mathrm{Lip}(\Gamma_{g_i}(\tau_j) - \Gamma_{S_i}(\tau_{j0})) \\
&\le \mathrm{Lip}(\Gamma_{S_i}(\tau_{j0})) + \mathrm{Lip}(p_2 g_i \pi \tau_j^* - p_2 S_i \pi \tau_{j0}^*)\, \mathrm{Lip}([p_1 g_i \pi \tau_j^*]^{-1}) \\
&\quad + \mathrm{Lip}(p_2 S_i \pi \tau_{j0}^*)\, \mathrm{Lip}([p_1 g_i \pi \tau_j^*]^{-1} - [p_1 S_i \pi \tau_{j0}^*]^{-1}) \\
&\le \left| \frac{c_i + d_i\alpha_j}{a_i + b_i\alpha_j} \right| + \delta \cdot (|a_i + b_i\alpha_j| - \delta)^{-1} \\
&\quad + |c_i + d_i\alpha_j|\, \mathrm{Lip}([p_1 g_i \pi \tau_j^*]^{-1} - [p_1 S_i \pi \tau_{j0}^*]^{-1})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Lip}([p_1 g_i \pi \tau_j^*]^{-1} - [p_1 S_i \pi \tau_{j0}^*]^{-1}) \\
&\quad \le \mathrm{Lip}([p_1 g_i \pi \tau_j^*]^{-1})\, \mathrm{Lip}(p_1 g_i \pi \tau_j^* - p_1 S_i \pi \tau_{j0}^*)\, \mathrm{Lip}([p_1 S_i \pi \tau_{j0}^*]^{-1}) \\
&\quad \le (|a_i + b_i\alpha_j| - \delta)^{-1} \cdot \delta \cdot |a_i + b_i\alpha_j|^{-1}.
\end{aligned}$$

Thus

$$\mathrm{Lip}(\Gamma_{g_i}(\tau_j)) \le \frac{|c_i + d_i\alpha_j| + \delta}{|a_i + b_i\alpha_j| - \delta} \le M.$$

Since $\Gamma_{g_i}(\tau_j)(0) = \alpha_i$ and $\rho M \le r$, we have $\Gamma_{g_i}(\tau_j)(\bar{\boldsymbol{D}}(0, \rho)) \subset \bar{\boldsymbol{D}}(\alpha_i, r)$. □

Note that the restriction to the set of holomorphic functions $\Gamma_{g_i} : \boldsymbol{H}_1 \cup \boldsymbol{H}_2 \to \boldsymbol{H}_i \subset \boldsymbol{H}_1 \cup \boldsymbol{H}_2$ is also well defined because g_i is holomorphic.

By the definition of Γ_{g_i}, it is clear that (6) and (7) hold. This implies that Γ_{g_i} is 'injective' as an operation of germs of functions. That is, if $\Gamma_{g_i}(\tau_j) = \Gamma_{g_i}(\tau_j')$ for $\tau_j, \tau_j' \in \boldsymbol{L}_j$, there exists a small neighborhood $0 \in U' \subset \boldsymbol{D}(0, \rho)$ such that the restrictions to U' coincide: $\tau_j|_{U'} = \tau_j'|_{U'}$. Hence the restriction to the set of holomorphic functions $\Gamma_{g_i}|_{\boldsymbol{H}_1 \cup \boldsymbol{H}_2}$ is injective.

Note also that

$$(\boldsymbol{B}_1 \cup \boldsymbol{B}_2) \cap f^{-1}(\boldsymbol{B}_1 \cup \boldsymbol{B}_2) = \bigcup_{i,j=1}^{2} (f_i|_{\boldsymbol{B}_i})^{-1}(\boldsymbol{B}_j)$$

where $(f_i|_{\boldsymbol{B}_i})^{-1}(\boldsymbol{B}_j) = \boldsymbol{B}_i \cap f_i^{-1}(\boldsymbol{B}_j)$, and furthermore that

$$\bigcap_{k=0}^{n} f^{-k}(\boldsymbol{B}_1 \cup \boldsymbol{B}_2) = \bigcup_{w_0,\dots,w_n=1}^{2} (f_{w_0}|_{\boldsymbol{B}_{w_0}})^{-1} \cdots (f_{w_{n-1}}|_{\boldsymbol{B}_{w_{n-1}}})^{-1}(\boldsymbol{B}_{w_n}).$$

For each $w_0,\dots,w_{n-1} \in \{1,2\}$, there exist open subsets V_1, V_2 of $\boldsymbol{B}_{w_0} \cap \{u \neq 0\}$ such that

$$(f_{w_0}|_{\boldsymbol{B}_{w_0}})^{-1} \cdots (f_{w_{n-1}}|_{\boldsymbol{B}_{w_{n-1}}})^{-1}(\boldsymbol{B}_i)\backslash\{q_{w_0}\} \subset V_i, \quad i = 1,2, \tag{12}$$

and $V_1 \cap V_2 = \emptyset$, since the blow-down operations f_i^{-1} are homeomorphisms when restricted to the outside of the v-axis. This implies that

$$\mathrm{graph}(\Gamma_{w_0\cdots w_{n-1}}(\tau_1)) \cap \mathrm{graph}(\Gamma_{w_0\cdots w_{n-1}}(\tau_2)) = \{q_{w_0}\} \tag{13}$$

where $\Gamma_{w_0\cdots w_{n-1}} := \Gamma_{g_{w_0}} \circ \cdots \circ \Gamma_{g_{w_{n-1}}}$ since

$$\mathrm{graph}(\Gamma_{w_0\cdots w_{n-1}}(\tau_i)) \subset f_{w_0}^{-1} \cdots f_{w_n}^{-1}(\boldsymbol{B}_i), \quad \tau_i \in \boldsymbol{L}_i.$$

As the limit $n \to \infty$, we are going to show that for each $w = w_0 w_1 \cdots \in \Sigma(2)$, there exists a unique function $\sigma(w) \in \boldsymbol{H}_{w_0} \subset \boldsymbol{L}_{w_0}$ such that

$$\bigcap_{n=1}^{\infty} (f_{w_0}|_{\boldsymbol{B}_{w_0}})^{-1} \cdots (f_{w_{n-1}}|_{\boldsymbol{B}_{w_{n-1}}})^{-1}(\boldsymbol{B}_{w_n}) = \mathrm{graph}(\sigma(w)) \subset \boldsymbol{B}_{w_0} \tag{14}$$

is the graph of $\sigma(w)$. Here we suppose $\rho > 0$ is small enough that

$$\lambda := (\ell + b)\sqrt{\rho}(1 + M) < 1. \tag{15}$$

Lemma 3. *The graph transform $\Gamma_{g_i} : \boldsymbol{L}_1 \cup \boldsymbol{L}_2 \to \boldsymbol{L}_i$ is a contraction with respect to the sup norm $\|\cdot\|$ of a function on $\bar{\boldsymbol{D}}(0,\rho)$. That is,*

$$\|\Gamma_{g_i}(\tau_j') - \Gamma_{g_i}(\tau_j)\| \leq \lambda\|\tau_j' - \tau_j\|, \quad \tau_j, \tau_j' \in \boldsymbol{L}_j. \tag{16}$$

Proof. Let $(u,v) \in \bar{\boldsymbol{D}}(0,\sqrt{\rho}) \times \bar{\boldsymbol{D}}(\alpha_j, r)$. Since

$$p_2 g_i \pi(u, \tau_j(u^2)) = \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u, \tau_j(u^2)))$$

we have

$$\begin{aligned} &|p_2 g_i \pi(u,v) - \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u,v))| \\ &\quad \leq |p_2 g_i \pi(u,v) - p_2 g_i \pi(u, \tau_j(u^2))| \\ &\qquad + \mathrm{Lip}(\Gamma_{g_i}(\tau_j))|p_1 g_i \pi(u, \tau_j(u^2)) - p_1 g_i \pi(u,v)| \end{aligned}$$

where

$$\begin{aligned} &|p_k g_i \pi(u,v) - p_k g_i \pi(u, \tau_j(u^2))| \\ &\quad \leq \mathrm{Lip}(p_k)\,\mathrm{Lip}(g_i - S_i)|\pi(u,v) - \pi(u, \tau_j(u^2))| \\ &\qquad + |p_k S_i \pi(u,v) - p_k S_i \pi(u, \tau_j(u^2))| \\ &\quad \leq \ell|u(v - \tau_j(u^2))| + b|u(v - \tau_j(u^2))| \\ &\quad \leq (\ell + b)\sqrt{\rho}|v - \tau_j(u^2)|, \quad k = 1,2. \end{aligned}$$

Thus

$$|p_2 g_i \pi(u,v) - \Gamma_{g_i}(\tau_j)(p_1 g_i \pi(u,v))| \le \lambda |v - \tau_j(u^2)|. \tag{17}$$

Given τ', let $v = \tau_j'(u^2)$ and $u' = p_1 g_i \pi(u, \tau_j'(u^2))$ to obtain

$$|\Gamma_{g_i}(\tau_j')(u') - \Gamma_{g_i}(\tau_j)(u')| \le \lambda |\tau_j'(u^2) - \tau_j(u^2)|.$$

If u^2 runs through in $\bar{\boldsymbol{D}}(0,\rho)$, u' runs through in a region that contains $\bar{\boldsymbol{D}}(0,\rho)$. By taking the supremum over $\bar{\boldsymbol{D}}(0,\rho)$, we obtain the lemma. □

So far we have constructed two contraction mappings $\Gamma_{g_i} : \boldsymbol{L}_1 \cup \boldsymbol{L}_2 \to \boldsymbol{L}_i \subset \boldsymbol{L}_1 \cup \boldsymbol{L}_2$, $i = 1, 2$. Note that the restriction to the set of holomorphic functions $\Gamma_{g_i}|_{\boldsymbol{H}_1 \cup \boldsymbol{H}_2} : \boldsymbol{H}_1 \cup \boldsymbol{H}_2 \to \boldsymbol{H}_i \subset \boldsymbol{H}_1 \cup \boldsymbol{H}_2$ is also a contraction. For each $w = w_0 w_1 \cdots \in \Sigma(2)$, consider the sequence of the mappings $\Gamma_{w_0}, \Gamma_{w_0 w_1}, \ldots, \Gamma_{w_0 \cdots w_{n-1}}, \ldots$ where $\Gamma_{w_0 \cdots w_{n-1}} := \Gamma_{g_{w_0}} \cdots \Gamma_{g_{w_{n-1}}}$. By the contraction mapping principle there exists a unique $\sigma(w) \in \boldsymbol{L}_1 \cup \boldsymbol{L}_2$ such that

$$\{\sigma(w)\} = \bigcap_{n=1}^{\infty} \Gamma_{w_0 \cdots w_{n-1}}(\boldsymbol{L}_{w_n}). \tag{18}$$

Since $\boldsymbol{H}_1 \cup \boldsymbol{H}_2$ is a closed subset of $\boldsymbol{L}_1 \cup \boldsymbol{L}_2$, we have $\sigma(w) \in \boldsymbol{H}_1 \cup \boldsymbol{H}_2$. Note also that

$$\sigma(w) = \Gamma_{g_{w_0}}\left(\bigcap_{n=2}^{\infty} \Gamma_{w_1 \cdots w_{n-1}}(\boldsymbol{L}_{w_n}) \right) = \Gamma_{g_{w_0}}(\sigma(s(w))). \tag{19}$$

Repeated application of (19) implies that

$$\sigma(w) = \Gamma_{w_0 \cdots w_{n-1}}(\sigma(s^n(w))), \quad n > 0. \tag{20}$$

Here let us show (14). By (17), we have

$$|p_2 f_i^{-1}(u',v) - \Gamma_{g_i}(\tau_j)(p_1 f_i^{-1}(u',v))| \le \lambda |v - \tau_j(u')| \tag{21}$$

for any $(u',v) \in \bar{\boldsymbol{D}}(0,\rho) \times \bar{\boldsymbol{D}}(\alpha_j, r)$ and any $\tau_j \in \boldsymbol{L}_j$, where $u = \mathrm{sq}^{-1}(u')$ is in any fixed branch. Given $w = w_0 w_1 \cdots \in \Sigma(2)$, we apply (21) repeatedly to see that

$$\begin{aligned} &|p_2 f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(u',v) - \Gamma_{w_0 \cdots w_{n-1}}(\tau_j)(p_1 f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(u',v))| \\ &\quad \le \lambda^n |v - \tau_j(u')| \end{aligned} \tag{22}$$

for $n > 0$, whenever $p_1 f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(u',v) \in \bar{\boldsymbol{D}}(0,\rho)$. Now let us consider a point $(u'',v'') \in \bigcap_{n=1}^{\infty} f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(\boldsymbol{B}_{w_n})$. Let $(u'',v'') = f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(u',v)$ in (22) and apply (20) to see that

$$|v'' - \sigma(w)(u'')| \le 2r\lambda^n.$$

Taking $n \to \infty$, we have $(u'',v'') \in \mathrm{graph}(\sigma(w))$ because $0 < \lambda < 1$. The other inclusion $\bigcap_{n=1}^{\infty} f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(\boldsymbol{B}_{w_n}) \supset \mathrm{graph}(\sigma(w))$ is obvious from (18), so (14) is proved.

As a consequence we have the following theorem.

THEOREM 4. *Consider the dynamics* (5) *and suppose that* $|a_i + b_i\alpha_j| \neq 0,\ i, j = 1, 2$. *There exist* $r, r_0, \rho > 0$, $M > 0$, *and an embedding* (*homeomorphism onto its image*) $\sigma : \Sigma(2) \to \boldsymbol{H}_1 \cup \boldsymbol{H}_2$ *such that the followings hold.*

1. *For* $w, w' \in \Sigma(2)$ *with* $w \neq w'$, *we have*

$$\mathrm{graph}(\sigma(w)) \cap \mathrm{graph}(\sigma(w')) = \begin{cases} \{q_{w_0}\} & \text{if } w_0 = w'_0 \\ \varnothing & \text{if } w_0 \neq w'_0. \end{cases} \tag{23}$$

The shift operator s *acts on* σ, *the Cantor family of curves. That is,*

$$\sigma(w) = \Gamma_{g_{w_0}}(\sigma(s(w))) \tag{24}$$

and

$$\mathrm{graph}(\sigma(w)) = \boldsymbol{B}_{w_0} \cap f^{-1}(\mathrm{graph}(\sigma(s(w)))) \tag{25}$$

for each $w \in \Sigma(2)$.

2. *The graph* $G(\sigma) := \bigcup_{w \in \Sigma(2)} \mathrm{graph}(\sigma(w))$ *is the maximal local invariant set in* $\boldsymbol{B}_1 \cup \boldsymbol{B}_2$, *that is*

$$G(\sigma) = \bigcap_{n=0}^{\infty} f^{-n}(\boldsymbol{B}_1 \cup \boldsymbol{B}_2). \tag{26}$$

3. *The local stable set of* $\{q_1, q_2\}$, *written by* $W^s_{loc}(\{q_1, q_2\})$, *is equal to* $G(\sigma)$. *That is,* $f^n(z) \to \{q_1, q_2\}$ *as* $n \to \infty$ *for each* $z \in G(\sigma) \backslash \{q_1, q_2\}$.
4. *The local superstable set of* $\{q_1, q_2\}$ *is* $G(\sigma)$. *That is, there exist constants* $0 < c'_1 < c'_2$ *such that*

$$c'_1|u|^2 \leq |p_1 f(u, v)| \leq c'_2|u|^2, \quad (u, v) \in G(\sigma) \backslash \{q_1, q_2\}. \tag{27}$$

PROOF. Choose small $r, r_0, \rho > 0$ and a large $M > 0$ such that (8), (9), (11) and (15) hold. The mapping $\sigma : \Sigma(2) \to \boldsymbol{H}_1 \cup \boldsymbol{H}_2$ is well defined by (18), and is injective because $\Gamma_{g_i}|_{\boldsymbol{H}_1 \cup \boldsymbol{H}_2}$ is injective.

Suppose that $w = w_0 w_1 \cdots$ and $w' = w'_0 w'_1 \cdots \in \Sigma(2)$ are $w \neq w'$. There exists $n \geq 0$ such that $w_0 = w'_0, \ldots, w_{n-1} = w'_{n-1}$ and $w_n \neq w'_n$. From (16) and (20) we see that

$$\begin{aligned} \|\sigma(w) - \sigma(w')\| &= \|\Gamma_{w_0 \cdots w_{n-1}}(\sigma(s^n(w))) - \Gamma_{w_0 \cdots w_{n-1}}(\sigma(s^n(w')))\| \\ &\leq \lambda^n \|\sigma(s^n(w)) - \sigma(s^n(w'))\| \\ &\leq 2r\lambda^n \end{aligned}$$

which implies that σ is continuous since $\lambda < 1$. By (12) we see that σ is a homeomorphism. By (13) and (20), we have (23).

We have already seen (24) and (25) in (19) and (7).

Let $(u, v) \in \bigcap_{n=0}^{\infty} f^{-n}(\boldsymbol{B}_1 \cup \boldsymbol{B}_2)$. For each $n \geq 0$ there exists $w_n \in \{1, 2\}$ such that $f^n(u, v) \in \boldsymbol{B}_{w_n}$. Thus $(u, v) \in \bigcap_{n=1}^{\infty} f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(\boldsymbol{B}_{w_n}) = \mathrm{graph}(\sigma(w))$ by (14), where $w := w_0 w_1 \cdots$. It is obvious that $\mathrm{graph}(\sigma(w)) \subset \bigcap_{n=0}^{\infty} f^{-n}(\boldsymbol{B}_1 \cup \boldsymbol{B}_2)$ and we have (26).

Let $c'_1 = (|a_i + b_i\alpha_j| + \delta)^{-1}$ and $c'_2 = (|a_i + b_i\alpha_j| - \delta)^{-1}$. By (10), we have (27) for $(u, v) \in G(\sigma) \backslash \{q_1, q_2\}$. This also implies that $f^n(u, v) \to \{q_1, q_2\}$ as $n \to \infty$, since $|v - \alpha_i| \leq M|u|$. □

The graph transform Γ_{g_i} determines the power series expansions of holomorphic functions $\sigma(w)$ inductively as follows.

Proposition 5. *Let* $w' = w_0'w_1'\cdots$, $w'' = w_0''w_1''\cdots \in \Sigma(2)$. *Let* $\sigma(w')(u) = \sum_{k=0}^{\infty} \alpha_k' u^k$, $\sigma(w'')(u) = \sum_{k=0}^{\infty} \alpha_k'' u^k$ *be power series expansions. If* $w_k' = w_k''$ *for* $k = 0, \dots, n$, *then we have* $\alpha_k' = \alpha_k''$ *for* $k = 0, \dots, 2^n - 1$.

Proof. Induction on $n \geq 0$. It is easy to see that the case $n = 0$ holds. Suppose that the case n holds, and let $w' = w_0'w_1'\cdots$, $w'' = w_0''w_1''\cdots \in \Sigma(2)$ with $w_k' = w_k''$ for $k = 0, \dots, n+1$. We are going to show that $\alpha_k' = \alpha_k''$ for $k = 0, \dots, 2^{n+1} - 1$.

Let $\sigma(s(w'))(u) = \sum_{k=0}^{\infty} \beta_k' u^k$. By the induction hypothesis, β_k' coincides with the coefficient of u^k in the power series expansion of $\sigma(s(w''))(u)$ for $k = 0, \dots, N$ where $N = 2^n - 1$. The power series expansion of $\pi(u, \sigma(w')(u^2))$ is $(u, \beta_0' u + \beta_1' u^3 + \cdots + \beta_N' u^{2N+1} + \cdots)$. Thus $g_i\pi(u, \sigma(w')(u^2)) = (a_i u + b_i(\sigma(w')(u^2)) + \cdots, \alpha_i + c_i u + d_i(\sigma(w')(u^2)) + \cdots)$ and $g_i\pi(u, \sigma(w'')(u^2))$ have the same power series expansion with respect to the variable u up to higher order terms of degree $>2N+1$. This implies that the coefficients α_k' of the expansion of $\sigma(w') = \Gamma_{g_{w_0'}}(\sigma(s(w')))$ coincides with α_k'' of $\sigma(w'') = \Gamma_{g_{w_0'}}(\sigma(s(w'')))$ for $k = 0, \dots, 2N+1$ where $2N + 1 = 2^{n+1} - 1$. □

3. Local dynamics of Newton's method around a multiple root.

If F is defined as in (1), Newton's method of F is written by

$$NF(z) = \left(\frac{h_1(z)}{2y + h_0(z)}, \frac{y^2 - x^2 + h_2(z)}{2y + h_0(z)}\right)$$

where $|h_0| < c\|z\|^2$, $|h_1| < c\|z\|^3$, and $|h_2| < c\|z\|^3$ in a neighborhood of the origin $z = (0,0) \in \boldsymbol{C}^2$ for some constant $c > 0$.

Suppose that a small $\varepsilon > 0$ is fixed. Let

$$\begin{aligned}
A_0 &= \{(x,y) \in \boldsymbol{C}^2 \mid |x| < \varepsilon|y|\},\\
B_0' &= \{(x,y) \in \boldsymbol{C}^2 \mid |y| < \varepsilon|x|\}, \quad \text{and}\\
C_0 &= C_0^+ \cup C_0^-\\
&= \{(x,y) \in \boldsymbol{C}^2 \mid |y - x| < \varepsilon|x|\} \cup \{(x,y) \in \boldsymbol{C}^2 \mid |y + x| < \varepsilon|x|\}.
\end{aligned}$$

Let $A = \bigcup_{n=0}^{\infty} A_n$ where $A_{n+1} = U \cap NF^{-1}(A_n)$, $n \geq 0$; $B = \bigcup_{n=0}^{\infty} B_n$ where $B_0 = U \backslash NF^{-1}(U)$ and $B_{n+1} = U \cap NF^{-1}(B_n)$, $n \geq 0$; $C = \bigcap_{n=0}^{\infty} C_n$ where $C_{n+1} = U \cap NF^{-1}(C_n)$, $n \geq 0$.

Let $\delta, \rho > 0$ are small. Let

$$U = \{(x,y) \in \boldsymbol{C}^2 \mid |x| < \delta\rho, |y| < \rho\}$$

and

$$B_0'' = \{(x,y) \in U \mid |2y + c_{20}x^2| < \varepsilon|x|^2\}$$

where c_{20} is the coefficient of x^2 in h_0. Suppose that ρ is small enough that $|y^2 + h_2| < (1/2)|x|^2$ and $|-c_{20}x^2 + h_0| < \varepsilon|x|^2$ in B_0''.

Lemma 6. $B_0'' \subset B_0 \subset B_0'$.

Proof. Suppose that $\delta, \rho > 0$ are sufficiently small. If $(x, y) \in U \backslash B_0'$ we have $|y| \geq \varepsilon \max(|x|, |y|) = \varepsilon \|z\|$ and

$$|p_1 NF(x, y)| = \left|\frac{h_1}{2y + h_0}\right| < \frac{c\|z\|}{2\varepsilon - c\|z\|}\|z\| < \frac{c\rho}{2\varepsilon - c\rho}\rho < \delta\rho,$$

$$|p_2 NF(x, y)| = \left|\frac{y^2 - x^2 + h_2}{2y + h_0}\right| < \frac{|y|^2 + |x|^2 + c\|z\|^3}{2|y| - c\|z\|^2}. \tag{28}$$

Denote by $m = |y/x|$. In the case that $\varepsilon|x| \leq |y| \leq |x| = \|z\|$, we have $\varepsilon \leq m \leq 1$ and

$$(28) = \frac{m^2 + 1 + c|x|}{2m - c|x|}|x| \leq \frac{m^2 + 1 + c\delta\rho}{2m - c\delta\rho}\delta\rho \leq \frac{\varepsilon^2 + 1 + c\delta\rho}{2\varepsilon - c\delta\rho}\delta\rho < \rho.$$

If $\delta|y| \leq |x| \leq |y| = \|z\|$, we have $1 \leq m \leq \delta^{-1}$, $|y| = m|x| \leq m\delta\rho \leq \rho$, and

$$(28) = \frac{1 + m^{-2} + c|y|}{2 - c|y|}|y| \leq \frac{1 + m^{-2} + c\rho}{2 - c\rho}m\delta\rho \leq \frac{(m + m^{-1})\delta + c\rho}{2 - c\rho}\rho < \rho.$$

If $|x| \leq \delta|y| \leq |y| = \|z\|$,

$$(28) \leq \frac{1 + \delta^2 + c|y|}{2 - c|y|}|y| \leq \frac{1 + \delta^2 + c\rho}{2 - c\rho}\rho < \rho.$$

Thus $NF(x, y) \in U$. For $(x, y) \in B_0''$, we have

$$|p_2 NF(x, y)| > \frac{|x|^2 - (1/2)|x|^2}{\varepsilon|x|^2 + \varepsilon|x|^2} \geq \rho$$

and $NF(x, y) \notin U$. □

The image $NF(B_0'') \subset NF(B_0)$ is unbounded since the locus of the denominator of NF, $2y + h_0(x, y) = 0$, is a local curve that lies in B_0''.

Under the coordinate systems $(\xi, \eta) = (x, y/x^2)$ and

$$(\Xi, H) = \left(\frac{p_1 NF(x, y)}{p_2 NF(x, y)}, \frac{1}{p_2 NF(x, y)}\right),$$

the point on the η-axis $(\xi, \eta) = (0, \eta)$ is mapped to $(\Xi, H) = (0, -2\eta - c_{20})$. It is a local diffeomorphism around each $(\xi, \eta) = (0, \eta)$ if $a_1 \neq 0$.

Lemma 7. *If* $(x, y) \notin C_0$, *then* $|y^2 - x^2| \geq (\varepsilon/(1 + \varepsilon))\|z\|^2$.

Proof. Let $\zeta = y/x$. By the minimum modulus principle,

$$\min_{(x,y) \notin C_0} |\zeta^2 - 1| = \min_{\zeta = \pm 1 + \varepsilon e^{i\theta}} |\zeta^2 - 1|$$

where $|\zeta^2 - 1| = |2\varepsilon e^{i\theta} + \varepsilon^2 e^{2i\theta}| = \varepsilon|2 + \varepsilon e^{i\theta}| \geq \varepsilon$. Thus $|y^2 - x^2| \geq \varepsilon|x|^2$, and $\varepsilon|x|^2 \geq (\varepsilon/(1 + \varepsilon)) \max(|x|^2, |y|^2)$ if $|y|^2 \leq (1 + \varepsilon)|x|^2$. If $|y|^2 \geq (1 + \varepsilon)|x|^2$, $|y^2 - x^2| \geq |y|^2 - |x|^2 \geq (1 - 1/(1 + \varepsilon))|y|^2 = (\varepsilon/(1 + \varepsilon))\|z\|^2$. □

Lemma 8. $NF(U \backslash C_0) \subset A_0$.

Proof. If $(x, y) \in U \backslash C_0$,

$$\left|\frac{p_1 NF(x, y)}{p_2 NF(x, y)}\right| = \left|\frac{h_1}{y^2 - x^2 + h_2}\right| \leq \frac{c\rho}{(\varepsilon/(1+\varepsilon)) - c\rho} < \varepsilon$$

since $\rho > 0$ is small. □

The lemma above implies that $B_n \subset C_0$ for $n \geq 1$, $A_0 \subset A_1 \subset \cdots$ is an increasing sequence of sets, and that $C_0 \supset C_1 \supset \cdots$ is a decreasing sequence. It is also clear that B_n, $n \geq 0$, are pairwise disjoint and the decomposition (3) holds.

To describe the structure of the set C, let us choose the coordinate system $(u, v) = \phi(x, y) := (x, y/x)$. Let V_0, V_1 and V_2 be neighborhoods of the origin $(u, v) = (0, 0)$, $(u, v) = q_1 = (0, 1)$ and $q_2 = (0, -1)$ respectively. Let $V = \pi^{-1}(V_0)$ be a neighborhood of the v-axis $u = 0$. By the assumption (2), there exist local diffeomorphisms $g_i : V_0 \to V_i$, $i = 1, 2$, such that $(\phi \circ NF \circ \phi^{-1})|_{V_i} = \mathrm{sq} \circ \pi^{-1} \circ g_i^{-1}$, $g_i(0, 0) = q_i$ and

$$(Dg_i)_{(u,v)=(0,0)} = \begin{pmatrix} \sqrt{\pm 2(a_2 + a_0 \pm a_1)^{-1}} & 0 \\ * & \sqrt{\pm 2^{-1}(a_2 + a_0 \pm a_1)} \end{pmatrix}.$$

This gives the local dynamics

$$f : V_1 \cup V_2 \to V, \quad f|_{V_i} = f_i$$

that satisfies the condition (4), under which Theorem 4 can be applied. Thus the set $\phi(C) = \phi(\bigcap_{n=0}^{\infty} C_n)$ is equal to the graph $G(\sigma)$ of the Cantor family of holomorphic curves $\sigma : \Sigma(2) \to \boldsymbol{H}_1 \cup \boldsymbol{H}_2$, by re-choosing sufficiently small neighborhoods if necessary.

Let $B_{11} := \phi(B_1 \cap C_0^+)$, $B_{12} := \phi(B_1 \cap C_0^-)$. It is clear that

$$\phi(B_n) = \bigcup_{w_1, \ldots, w_n = 1}^{2} f_{w_1}^{-1} \cdots f_{w_{n-1}}^{-1}(B_{1w_n})$$

is a disjoint union and each $f_{w_1}^{-1} \cdots f_{w_{n-1}}^{-1}(B_{1w_n})$ is nonempty. Thus B_n consists of 2^n components.

Finally let us consider the dynamics in A_0 under the coordinate system $(u, v) = \varphi(x, y) := (x/y, y)$. Let $p_1(u, v) = u$, $p_2(u, v) = v$ be projections. Both $p_i(\varphi \circ NF \circ \varphi^{-1})(u, v)$, $i = 1, 2$, are divisible by v and

$$D(\varphi \circ NF \circ \varphi^{-1})|_{(u,v)=(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

By an argument similar to Schröder's equation (see [**2**], Theorem 6.2.3 and its Remark),

$$\psi(u, v) := \lim_{n \to \infty} 2^n p_2(\varphi \circ NF^n \circ \varphi^{-1}) = v + \cdots$$

is uniformly convergent in a neighborhood of the origin $(u, v) = (0, 0)$. As a local function around the origin, $\psi = v \cdot \text{unit}$. Thus $p_1(\varphi \circ NF \circ \varphi^{-1})$ is divisible by ψ. By the new coordinate system $(\xi, \eta) = (u, \psi(u, v))$, we obtain the dynamics

$$(\xi, \eta) \mapsto \left(\eta\chi(\xi, \eta), \frac{1}{2}\eta \right) \tag{29}$$

where $\chi = p_1(\varphi \circ NF \circ \varphi^{-1})/\psi$.

By the C^r center manifold theorem (see [**3**], Appendix III), there exists a C^r function $\xi = \mu(\eta) = \mu(\mathrm{Re}(\eta), \mathrm{Im}(\eta))$ around the origin $(\xi, \eta) = (0, 0)$, whose graph is invariant under the dynamics (29). In the next section we will show that it need not be holomorphic.

4. Invariant curve in the attracting set.

Consider the local dynamics

$$(x, y) \mapsto F(x, y) = (yf(x, y), \lambda y),$$

defined in a neighborhood of the origin, where $f(0,0) = 0$ and $0 < |\lambda| < 1$. It is the composition of the mapping $(x, y) \mapsto (\lambda^{-1} f(x, y), \lambda y)$ with the blow-down map $(x, y) \mapsto (xy, y)$. If there exists a local holomorphic curve $x = \mu(y) = \sum_{n=1}^{\infty} c_n y^n$ that passes through the origin and is forward invariant under F, its coefficients c_n are uniquely determined by the functional equation

$$yf(\mu(y), y) = \mu(\lambda y). \tag{30}$$

PROPOSITION 9. *If $f(z) = ax + by$ is a linear function with $ab \neq 0$, there exists no invariant holomorphic curve $x = \mu(y)$ that passes through the origin.*

PROOF. From (30), we obtain $c_1\lambda = 0$, $c_2\lambda^2 = b$ and $c_{n+1}\lambda^{n+1} = ac_n$, $n \geq 2$. Thus $c_n = a^{n-2} b \lambda^{1-n(n+1)/2}$, $n \geq 2$, and the radius of convergence of the power series μ is equal to 0. □

On the other hand, for any holomorphic function $\mu(y) = \sum_{n=2}^{\infty} c_n y^n$ there exists an f such that the curve $x = \mu(y)$ is invariant under F. For instance, $f(x, y) = x - \mu(y) + \mu(\lambda y)/y$.

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