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Singular solutions of the Briot-Bouquet type partial differential equations

By Hiroshi YAMAZAWA

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Abstract. In 1990, Gérard-Tahara [**2**] introduced the Briot-Bouquet type partial differential equation $t\partial_t u = F(t, x, u, \partial_x u)$, and they determined the structure of singular solutions provided that the characteristic exponent $\rho(x)$ satisfies $\rho(0) \notin \{1, 2, \dots\}$. In this paper the author determines the structure of singular solutions in the case $\rho(0) \in \{1, 2, \dots\}$.

1. Introduction.

In this paper, we will study the following type of nonlinear singular first order partial differential equations:

$$t\partial_t u = F(t, x, u, \partial_x u) \tag{1.1}$$

where $(t, x) = (t, x_1, \dots, x_n) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n$, $\partial_x u = (\partial_1 u, \dots, \partial_n u)$, $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$ for $i = 1, \dots, n$, and $F(t, x, u, v)$ with $v = (v_1, \dots, v_n)$ is a function defined in a polydisk $\triangle$ centered at the origin of $\boldsymbol{C}_t \times \boldsymbol{C}_x^n \times \boldsymbol{C}_u \times \boldsymbol{C}_v^n$. Let us denote $\triangle_0 = \triangle \cap \{t = 0, u = 0, v = 0\}$.

The assumptions are as follows:

(A1) $F(t, x, u, v)$ is holomorphic in $\triangle$,

(A2) $F(0, x, 0, 0) = 0$ in $\triangle_0$,

(A3) $\dfrac{\partial F}{\partial v_i}(0, x, 0, 0) = 0$ in $\triangle_0$ for $i = 1, \dots, n$.

DEFINITION 1.1 ([**2**], [**3**]). If the equation (1.1) satisfies (A1), (A2) and (A3) we say that the equation (1.1) is of Briot-Bouquet type with respect to t.

DEFINITION 1.2 ([**2**], [**3**]). Let us define

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.1).

Let us denote by
1. $\mathscr{R}(\boldsymbol{C} \setminus \{0\})$ the universal covering space of $\boldsymbol{C} \setminus \{0\}$,
2. $S_\theta = \{t \in \mathscr{R}(\boldsymbol{C} \setminus \{0\}); |\arg t| < \theta\}$,

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3. $S(\varepsilon(s)) = \{t \in \mathscr{R}(\boldsymbol{C}\backslash\{0\}); 0 < |t| < \varepsilon(\arg t)\}$ for some positive-valued function $\varepsilon(s)$ defined and continuous on $\boldsymbol{R}$,
4. $D_R = \{x \in \boldsymbol{C}^n; |x_i| < R \text{ for } i = 1, \dots, n\}$,
5. $\boldsymbol{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\boldsymbol{C}^n$.

DEFINITION 1.3. We define the set $\tilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions;

1. $u(t,x)$ is holomorphic in $S(\varepsilon(s)) \times D_R$ for some $\varepsilon(s)$ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R

$$\max_{x \in K} |u(t,x)| = O(|t|^a) \quad \text{as } t \to 0 \text{ in } S_\theta.$$

We know some results on the equation (1.1) of Briot-Bouquet type with respect to t. We concern the following result. R. Gérard and H. Tahara studied in [**2**] the structure of holomorphic and singular solutions of the equation (1.1) and proved the following result;

THEOREM 1.4 (R. Gérard and H. Tahara). *If the equation* (1.1) *is of Briot-Bouquet type and* $\rho(0) \notin \boldsymbol{N}^* = \{1, 2, 3, \dots\}$ *then we have*;

(1) (*Holomorphic solutions*) *The equation* (1.1) *has a unique solution* $u_0(t,x)$ *holomorphic near the origin of* $\boldsymbol{C} \times \boldsymbol{C}^n$ *satisfying* $u_0(0,x) \equiv 0$.

(2) (*Singular solutions*) *Denote by* S_+ *the set of all* $\tilde{\mathcal{O}}_+$*-solutions of* (1.1).

$$S_+ = \begin{cases} \{u_0(t,x)\} & \textit{when } \operatorname{Re}\rho(0) \le 0, \\ \{u_0(t,x)\} \cup \{U(\varphi); 0 \ne \varphi(x) \in \boldsymbol{C}\{x\}\} & \textit{when } \operatorname{Re}\rho(0) > 0, \end{cases}$$

where $U(\varphi)$ *is an* $\tilde{\mathcal{O}}_+$*-solution of* (1.1) *having an expansion of the following form*:

$$U(\varphi) = \sum_{i \ge 1} u_i(x) t^i + \sum_{i+2j \ge k+2,\, j \ge 1} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

In the case $\rho(0) \in \boldsymbol{N}^*$, Yamane [**7**] showed that the equation (1.1) has a holomolphic solution in a region $\{(t,x) \in \boldsymbol{C} \times \boldsymbol{C}^n; |x| < c|t|^d \ll 1\}$ for some $c > 0$ and $d > 0$, but the solution is not in S_+.

The purpose of this paper is to determine S_+ in the case $\rho(0) \in \boldsymbol{N}^*$.

The main result of this paper is;

THEOREM 1.5. *If the equation* (1.1) *is of Briot-Bouquet type and if* $\rho(0) = N \in \boldsymbol{N}^*$ *and* $\rho(x) \not\equiv \rho(0)$, *then*

$$S_+ = \{U(\varphi); \varphi(x) \in \boldsymbol{C}\{x\}\},$$

where $U(\varphi)$ *is an* $\tilde{\mathcal{O}}_+$*-solution of* (1.1) *having an expansion of the following form*:

$$U(\varphi) = u_1^0(x) t + u_0^{e_0}(x) \phi_N(t,x) + \sum_{\substack{i+|\beta| \ge 2,\, |\beta| < \infty \\ [\beta] \le i+|\beta|-2}} u_i^\beta(x) t^i \Phi_N^\beta$$

$$+ w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{\substack{i+j+|\beta| \ge 2,\, |\beta| < \infty \\ j \ge 1,\, [\beta] \le i+j+|\beta|-2}} \sum_{\substack{k \le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_N^\beta,$$

where $u_N^0(x) \equiv 0$, $w_{0,1,0}^0(x) = \varphi(x)$ *is an arbitrary holomorphic function and the other coefficients* $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ *are holomorphic functions determined by* $w_{0,1,0}^0(x)$ *and defined in a common disk, and*

$$l = (l_1, \dots, l_n) \in \boldsymbol{N}^n, \quad |l| = l_1 + \cdots + l_n, \quad \beta = (\beta_l \in \boldsymbol{N}; l \in \boldsymbol{N}^n),$$

$$|\beta| = \sum_{|l| \ge 0} \beta_l, \quad |\beta|_p = \sum_{|l| = p} \beta_l \text{ for } p \ge 0, \quad [\beta] = \sum_{|l| \ge 2} (|l| - 1)\beta_l,$$

$$\Phi_N^\beta = \prod_{|l| \ge 0} \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l}, \quad \partial_x^l = \partial_1^{l_1} \cdots \partial_n^{l_n}, \quad \phi_N(t, x) = \frac{t^{\rho(x)} - t^N}{\rho(x) - N}.$$

The following lemma will play an important role in the proof of Theorem 1.5. At first, we define some notations. We set for $l \in \boldsymbol{N}^n$, $e_l = (\beta_k; k \in \boldsymbol{N}^n)$ with $\beta_l = 1$ and $\beta_k = 0$ for $k \ne l$ and for $p \in \{1, 2, \dots, n\}$, $e(p) = (i_1, \dots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \ne p$, and define $l^1 < l^0$ by $|l^1| < |l^0|$ and $l_i^1 \le l_i^0$ for $i = 1, \dots, n$.

Lemma 1.6. *Let* $\rho(x)$, ϕ_N *and* Φ_N^β *be as in Theorem* 1.5. *Then we have*;

1. $\partial_p \Phi_N^\beta = \sum_{|l| \ge 0} \beta_l (l_p + 1) \Phi_N^{\beta - e_l + e_{l+e(p)}}$ *for* $i = 1, \dots, n$,
2. $t\partial_t \phi_N = \rho(x)\phi_N + t^N$,
3. $t\partial_t \Phi_N^\beta = |\beta|\rho(x)\Phi_N^\beta + \beta_0 t^N \Phi_N^{\beta - e_0} + \sum_{|l^0| \ge 1} \sum_{l^1 < l^0} \beta_{l^0} \dfrac{\partial_x^{l^0 - l^1} \rho(x)}{(l^0 - l^1)!} \Phi_N^{\beta - e_{l^0} + e_{l^1}}$.

Proof.
1. By $\partial_p (\partial_x^l \phi_N / l!)^{\beta_l} = \beta_l (\partial_x^l \phi_N / l!)^{\beta_l - 1} \partial_x^{l+e(p)} \phi_N / l!$, we have the result 1.
2. By $t\partial_t \phi_N = (\rho(x) t^{\rho(x)} - N t^N)/(\rho(x) - N)$, we have the result 2.
3. By 2, we have

$$t\partial_t \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l} = \beta_l \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l - 1} \frac{\partial_x^l (\rho(x)\phi_N + t^N)}{l!}.$$

Therefore we have

$$t\partial_t \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l} = \begin{cases} \beta_0 \rho(x) \phi_N^{\beta_0} + \beta_0 t^N \phi_N^{\beta_0 - 1} & \text{if } l = 0, \\ \beta_l \rho(x) \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l} + \sum_{0 \le l^1 < l} \beta_l \frac{\partial_x^{l - l^1} \rho(x)}{(l - l^1)!} \frac{\partial_x^{l^1} \phi_N}{l^1!} \left(\frac{\partial_x^l \phi_N}{l!} \right)^{\beta_l - 1} & \text{if } |l| > 0. \end{cases}$$

Hence we have the desired result. □

2. Construction of formal solutions in the case $\rho(0) = 1$.

By [**2**] (Gérard-Tahara), if the equation (1.1) is of Briot-Bouquet type with respect to t, then it is enough to consider the following equation:

$$Lu = t\partial_t u - \rho(x)u = a(x)t + G_2(x)(t, u, \partial_x u) \tag{2.1}$$

where $\rho(x)$ and $a(x)$ are holomorphic functions in a neighborhood of the origin, and the function $G_2(x)(t, X_0, X_1, \dots, X_n)$ is a holomorphic function in a neighborhood of the origin in $\boldsymbol{C}_x^n \times \boldsymbol{C}_t \times \boldsymbol{C}_{X_0} \times \boldsymbol{C}_{X_1} \times \cdots \times \boldsymbol{C}_{X_n}$ with the following expansion:

$$G_2(x)(t, X_0, X_1, \dots, X_n) = \sum_{p + |\alpha| \ge 2} a_{p,\alpha}(x) t^p \{X_0\}^{\alpha_0} \{X_1\}^{\alpha_1} \cdots \{X_n\}^{\alpha_n}$$

and we may assume that the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha|\geq 2}$ are holomorphic functions on D_{R_0} for a sufficiently small $R_0>0$. Let $0<R<R_0$. We put $A_{p,\alpha}(R):=\max_{x\in D_R}|a_{p,\alpha}(x)|$ for $p+|\alpha|\geq 2$. Then for $0<r<R$

$$\sum_{p+|\alpha|\geq 2}\frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}}t^p X_0^{\alpha_0}X_1^{\alpha_1}\times\cdots\times X_n^{\alpha_n} \tag{2.2}$$

is convergent in a neighborhood of the origin.

In this section, we assume $\rho(0)=1$ and $\rho(x)\not\equiv 1$ and we will construct formal solutions of the equation (2.1).

PROPOSITION 2.1. *If $\rho(0)=1$ and $\rho(x)\not\equiv 1$, the equation* (2.1) *has a family of formal solutions of the form*:

$$\begin{aligned} u = u_0^{e_0}(x)\phi_1 &+ \sum_{m\geq 2}\sum_{\substack{i+|\beta|=m\\ [\beta]\leq m-2}} u_i^{\beta}(x)t^i\Phi_1^{\beta} \\ &+ w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m\geq 2}\sum_{\substack{i+j+|\beta|=m\\ j\geq 1,[\beta]\leq m-2}}\sum_{\substack{k\leq i+|\beta|_0+|\beta|_1\\ +2(j-1)}} w_{i,j,k}^{\beta}(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^{\beta} \end{aligned} \tag{2.3}$$

where $w_{0,1,0}^0(x)$ is an arbitrary holomorphic function and the other coefficients $u_i^{\beta}(x)$, $w_{i,j,k}^{\beta}(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk.

REMARK 2.2. By the relation $[\beta]\leq m-2$ in summations of the above formal solution, we have $\beta_l=0$ for any $l\in \boldsymbol{N}^n$ with $|l|\geq m$.

We define the following two sets U_m and W_m for $m\geq 1$ to prove Proposition 2.1.

DEFINITION 2.3. We denote by U_m the set of all functions u_m of the following forms:

$$\begin{aligned} u_1 &= u_1^0(x)t + u_0^{e_0}(x)\phi_1, \\ u_m &= \sum_{\substack{i+|\beta|=m\\ [\beta]\leq m-2}} u_i^{\beta}(x)t^i\Phi_1^{\beta} \quad \text{for } m\geq 2, \end{aligned} \tag{2.4}$$

and denote by W_m the set of all functions w_m of the following forms:

$$\begin{aligned} w_1 &= w_{0,1,0}^0(x)t^{\rho(x)}, \\ w_m &= \sum_{\substack{i+j+|\beta|=m\\ j\geq 1,[\beta]\leq m-2}}\sum_{\substack{k\leq i+|\beta|_0+|\beta|_1\\ +2(j-1)}} w_{i,j,k}^{\beta}(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^{\beta} \quad \text{for } m\geq 2 \end{aligned} \tag{2.5}$$

where $u_i^{\beta}(x),\ w_{i,j,k}^{\beta}(x)\in \boldsymbol{C}\{x\}$.

We can rewrite the formal solution (2.3) as follows:

$$u=\sum_{m\geq 1}(u_m+w_m) \quad \text{where } u_m\in U_m, w_m\in W_m.$$

Let us show important relations of u_m and w_m for $m\geq 2$. By Lemma 1.6, we have

$$\partial_p u_m = \sum_{\substack{i+|\beta|=m \\ [\beta]\le m-2}} \left\{ \partial_p u_i^\beta(x) t^i \Phi_1^\beta + \sum_{|l|=0}^{m-1} (l_p+1)\beta_l u_i^\beta(x) t^i \Phi_1^{\beta-e_l+e_{l+e(p)}} \right\},$$

$$\begin{aligned}\partial_p w_m = \sum_{\substack{i+j+|\beta|=m \\ j\ge 1, [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} &\Big\{ \partial_p w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \\ &+ j\partial_p\rho(x) w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^{k+1} \Phi_1^\beta \\ &+ \sum_{|l|=0}^{m-1} (l_p+1)\beta_l w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta-e_l+e_{l+e(p)}} \Big\}\end{aligned} \tag{2.6}$$

for $p = 1, \dots, n$, and we have

$$\begin{aligned}Lu_m = \sum_{\substack{i+|\beta|=m \\ [\beta]\le m-2}} &\Big\{ \{i+(|\beta|-1)\rho(x)\} u_i^\beta(x) t^i \Phi_1^\beta + \beta_0 u_i^\beta(x) t^{i+1} \Phi_1^{\beta-e_0} \\ &+ \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} \beta_{l^0} \frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0-l^1)!} u_i^\beta(x) t^i \Phi_1^{\beta-e_{l^0}+e_{l^1}} \Big\},\end{aligned} \tag{2.7}$$

$$\begin{aligned}Lw_m = \sum_{\substack{i+j+|\beta|=m \\ j\ge 1, [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} &\Big\{ \{i+(j+|\beta|-1)\rho(x)\} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \\ &+ k w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^{k-1} \Phi_1^\beta + \beta_0 w_{i,j,k}^\beta(x) t^{i+j\rho(x)+1} \{\log t\}^k \Phi_1^{\beta-e_0} \\ &+ \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} \beta_{l^0} \frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0-l^1)!} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta-e_{l^0}+e_{l^1}} \Big\}.\end{aligned}$$

We show two lemmas.

LEMMA 2.4. *If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.*

PROOF. We prove $Lu_m \in U_m$. We will see all the exponents of each terms in (2.7). For the second term in (2.7), we have $i+1+|\beta-e_0| = i+|\beta| = m$ and $[\beta-e_0] = [\beta] \le m-2$.

For the third term, we have $i+|\beta-e_{l^0}+e_{l^1}| = i+|\beta| = m$ and $[\beta-e_{l^0}+e_{l^1}] = [\beta]$ (if $|l^0|=1$), $=[\beta]-(|l^0|-1)$ (if $|l^0|>1$ and $|l^1|\le 1$), $=[\beta]-|l^0|+|l^1|$ (if $|l^0|>1$ and $|l^1|>1$). Therefore by $l^1<l^0$, we have $[\beta-e_{l^0}+e_{l^1}] \le [\beta] \le m-2$. Hence we have $Lu_m \in U_m$.

We can prove $Lw_m \in W_m$ in the same way. □

LEMMA 2.5. *If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold for $i, j = 1, \dots, n$,*

1. $a(x)U_m \subset U_m$ *and* $a(x)W_m \subset W_m$ *for any holomorphic function* $a(x)$,
2. $tU_m,\ \phi_1 U_m \subset U_{m+1}$ *and* $t^{\rho(x)}U_m,\ tW_m,\ t^{\rho(x)}W_m,\ \phi_1 W_m \subset W_{m+1}$,
3. $u_m \times u_n,\ \partial_i u_m \times \partial_j u_n,\ \partial_i u_m \times u_n \in U_{m+n}$,
4. $w_m \times w_n,\ \partial_i w_m \times \partial_j w_n,\ \partial_i w_m \times w_n \in W_{m+n}$,
5. $u_m \times w_n,\ \partial_i u_m \times w_n,\ u_m \times \partial_j w_n,\ \partial_i u_m \times \partial_j w_n \in W_{m+n}$.

Proof. This is verified by the relations (2.6). □

Let us show that u_m and w_m are determined inductively on $m \geq 1$. By substituting $\sum_{m\geq 1}(u_m + w_m)$ into (2.1), we have

$$(1-\rho(x))u_1^0(x) + u_0^{e_0}(x) = a(x), \tag{2.8}$$

and for $m \geq 2$

$$Lu_m = \sum_{\substack{p+|\alpha|\geq 2\\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n}\prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}}, \tag{2.9}$$

$$\begin{aligned}Lw_m = &\sum_{\substack{p+|\alpha|\geq 2\\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} (u_{m_{0,h_0}} + w_{m_{0,h_0}}) \prod_{j=1}^{n}\prod_{h_j=1}^{\alpha_j} \partial_j (u_{m_{j,h_j}} + w_{m_{j,h_j}})\\ &- \sum_{\substack{p+|\alpha|\geq 2\\ p+|m_n|=m}} a_{p,\alpha}(x)t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n}\prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}},\end{aligned} \tag{2.10}$$

where $|m_n| = \sum_{i=0}^{n} m_i(\alpha_i)$ and $m_i(\alpha_i) = m_{i,1} + \cdots + m_{i,\alpha_i}$ for $i = 0, 1, \ldots, n$.

We take any holomorphic function $\varphi(x) \in \boldsymbol{C}\{x\}$ and put $w_{0,1,0}^0(x) = \varphi(x)$, and by (2.8), we put $u_1^0(x) \equiv 0$ and $u_0^{e_0}(x) = a(x)$.

For $m \geq 2$, let us show that u_m and w_m are determined by induction. By Lemma 2.5, the right side of (2.9) belongs to U_m and the right side of (2.10) belongs to W_m. Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \ldots, \alpha_j$ and $j = 0, \ldots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi_1^\beta$ and $t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta$ respectively for (2.9) and (2.10), then put

$$\begin{aligned}&\{i + (|\beta|-1)\rho(x)\}u_i^\beta(x)\\ &\qquad + (\beta_0+1)u_{i-1}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1}\sum_{0\leq l^1<l^0} (\beta_{l^0}+1)\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0-l^1)!}u_i^{\beta+e_{l^0}-e_{l^1}}(x)\\ &\quad = f_i^\beta(\{a_{p,\alpha}\}_{2\leq p+|\alpha|\leq m}, \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'|<m})\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\{i + (j+|\beta|-1)\rho(x)\}w_{i,j,k}^\beta(x) + (k+1)w_{i,j,k+1}^\beta(x)\\ &\qquad + (\beta_0+1)w_{i-1,j,k}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1}\sum_{0\leq l^1<l^0} (\beta_{l^0}+1)\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0-l^1)!}w_{i,j,k}^{\beta+e_{l^0}-e_{l^1}}(x)\\ &\quad = g_{i,j,k}^\beta(\{a_{p,\alpha}\}_{2\leq p+|\alpha|\leq m}, \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'|<m}, \{w_{i',j',k'}^{\beta'}(x)\}_{i'+j'+|\beta'|<m}).\end{aligned} \tag{2.12}$$

We define an order for the multi indices (i, β) and (i, j, k, β) to show that $u_i^\beta(x)$ and $w_{i,j,k}^\beta(x)$ are determined by (2.11) and (2.12).

Definition 2.6. The relation $(i', \beta') < (i, \beta)$ is defined by the following orders;
1. $i' + |\beta'| < i + |\beta|$.
2. If $i' + |\beta'| = i + |\beta|$, then $i' < i$.
3. If $i' + |\beta'| = i + |\beta|$ and $i' = i$, then $|\beta'|_0 < |\beta|_0$.
4. If $i' + |\beta'| = i + |\beta|$, $i' = i$, $|\beta'|_0 = |\beta|_0, \dots, |\beta'|_l = |\beta|_l$, then $|\beta'|_{l+1} < |\beta|_{l+1}$.

The relation $(i', j', k', \beta') < (i, j, k, \beta)$ is defined by the following orders;
1. $i' + j' + |\beta'| < i + j + |\beta|$.
2. If $i' + j' + |\beta'| = i + j + |\beta|$, then $i' < i$.
3. If $i' + j' + |\beta'| = i + j + |\beta|$ and $i' = i$, then $j' < j$.
4. If $i' + j' + |\beta'| = i + j + |\beta|$, $i' = i$ and $j' = j$, then $|\beta'|_0 < |\beta|_0$.
5. If $i' + j' + |\beta'| = i + j + |\beta|$, $i' = i$, $j' = j$, $|\beta'|_0 = |\beta|_0, \dots, |\beta'|_l = |\beta|_l$, then $|\beta'|_{l+1} < |\beta|_{l+1}$.
6. If $(i', j', \beta') = (i, j, \beta)$, then $k' > k$.

For $m \geq 2$, we have $i + (|\beta| - 1)\rho(x) \neq 0$ and $i + (j + |\beta| - 1)\rho(x) \neq 0$ by $\rho(0) = 1$. Therefore all the coefficients $u_i^\beta(x)$ and $w_{i,j,k}^\beta(x)$ are determined in the order of Definition 2.6. Hence we obtain Proposition 2.1.

3. Convergence of the formal solutions in the case $\rho(0) = 1$.

In this section, we show that the formal solution (2.3) converges in $\tilde{\mathcal{O}}_+$.

Proposition 3.1. *Let γ satisfy $0 < \gamma < 1$ and let λ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result;*

For any $\theta > 0$ there is an $\varepsilon > 0$ such that the formal solution (2.3) *converges in the following region*:

$$\{(t, x) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n; |\eta(t, \lambda)t| < \varepsilon, |\eta(t, \lambda)^2 t^{\rho(x)}| < \varepsilon, |\eta(t, \lambda)t^\gamma| < \varepsilon, t \in S_\theta \text{ and } x \in D_r\},$$

where $\eta(t, \lambda) = \max\{|(\log t)/\lambda|, 1\}$.

In this section, we put $w_{i,0,0}^\beta(x) = u_i^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (2.3). Then the formal solution (2.3) is as follows:

$$u = w_{0,0,0}^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta. \tag{3.1}$$

Let us define the following set V_m for (3.1).

Definition 3.2. We denote by V_m the set of all the functions v_m of the following forms:

$$v_1 = w_{0,0,0}^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)},$$

$$v_m = \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^\beta \quad \text{for } m \geq 2. \tag{3.2}$$

We define the following estimate for the function v_m.

DEFINITION 3.3. For the function (3.2), we define

$$\|v_1\|_{r,c,\lambda} = \|v_1\|_{r,c} := \frac{\|w^{e_0}_{0,0,0}\|_r}{c} + \|w^0_{0,1,0}\|_r,$$

$$\|v_m\|_{r,c,\lambda} := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+\beta_1 \\ +2(j-1)}} \frac{\|w^{\beta}_{i,j,k}\|_r \lambda^k}{c^{\langle\beta\rangle}} \quad \text{for } m \ge 2, \tag{3.3}$$

for $c > 0$ and $\lambda > 0$, where

$$\|w^{\beta}_{i,j,k}\|_r = \max_{x\in D_r} |w^{\beta}_{i,j,k}(x)| \quad \text{and} \quad \langle\beta\rangle = \sum_{|l|\ge 0} (|l|+1)\beta_l.$$

We will make use of

LEMMA 3.4. *For a holomorphic function $f(x)$ on D_{R_0}, we have*

$$\|\partial_x^{\alpha} f\|_R \le \frac{\alpha!}{(R_0 - R)^{|\alpha|}} \|f\|_{R_0} \quad \textit{for } 0 < R < R_0.$$

PROOF. By Cauchy's integral formula, we have the desired result. □

LEMMA 3.5. *If a holomorphic function $f(x)$ on D_R satisfies*

$$\|f\|_r \le \frac{C}{(R-r)^p} \quad \textit{for } 0 < r < R$$

then we have

$$\|\partial_i f\|_r \le \frac{Ce(p+1)}{(R-r)^{p+1}} \quad \textit{for } 0 < r < R, \quad i = 1, \dots, n.$$

For the proof, see Hörmander ([**5**], lemma 5.1.3).

Let us show the following estimate for the function Lv_m.

LEMMA 3.6. *Let $0 < R < R_0$. Then there exists a positive constant σ such that for $m \ge 2$, if $v_m \in V_m$ we have*

$$\|Lv_m\|_{r,c,\lambda} \ge \frac{\sigma}{2} m \|v_m\|_{r,c,\lambda} \quad \textit{for } 0 < r \le R$$

for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$.

PROOF. Let us give an estimate the second, the third and the fourth term in the right side of the second relation in (2.7) respectively.

For the second term, since $k \le i + |\beta|_0 + |\beta|_1 + 2(j-1) \le 2m$ by $i + j + |\beta| = m$ we have

$$T_2 := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} k \frac{\|w^{\beta}_{i,j,k}\|_r \lambda^{k-1}}{c^{\langle\beta\rangle}} \le \frac{2m}{\lambda} \|v_m\|_{r,c,\lambda}.$$

For the fourth term, we have

$$T_4 := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} \frac{\beta_{l^0}}{(l^0-l^1)!} \frac{\|\partial_x^{l^0-l^1}\rho w_{i,j,k}^{\beta}\|_r \lambda^k}{c^{\langle\beta-e_{l^0}+e_{l^1}\rangle}}$$

$$\le \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} c^{|l^0|-|l^1|}\beta_{l^0} \frac{\|\partial_x^{l^0-l^1}\rho\|_R}{(l^0-l^1)!} \frac{\|w_{i,j,k}^{\beta}\|_r \lambda^k}{c^{\langle\beta\rangle}}. \tag{3.4}$$

By Lemma 3.4, we have

$$\sum_{l^1<l^0} c^{|l^0|-|l^1|} \frac{\|\partial_x^{l^0-l^1}\rho\|_R}{(l^0-l^1)!} \le \sum_{l^1<l^0} \left(\frac{c}{R_0-R}\right)^{|l^0|-|l^1|} \|\rho\|_{R_0} \le \frac{cn\|\rho\|_{R_0}}{R_0-R}\left(\frac{R_0-R}{R_0-R-c}\right)^n \tag{3.5}$$

for sufficiently small $c>0$. Therefore by (3.4) and (3.5), we have

$$T_4 \le \kappa(c) \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \sum_{|l^0|=1}^{m-1} \beta_{l^0} \frac{\|w_{i,j,k}^{\beta}\|_r \lambda^k}{c^{\langle\beta\rangle}}$$

where $\kappa(c) := (cn/(R_0-R))((R_0-R)/(R_0-R-c))^n\|\rho\|_{R_0}$.

For the third term, we have

$$T_3 := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \beta_0 \frac{\|w_{i,j,k}^{\beta}\|_r \lambda^k}{c^{\langle\beta-e_0\rangle}} = \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} c\beta_0 \frac{\|w_{i,j,k}^{\beta}\|_r \lambda^k}{c^{\langle\beta\rangle}}.$$

Therefore, since $c\beta_0 + \kappa(c)\sum_{|l^0|=1}^{m-1}\beta_{l^0} \le (\sigma/3)m$ by the conditions $\kappa(0)=0$ and $i+j+|\beta|=m\ge 2$ for sufficiently small $c>0$ and some $\sigma>0$ we have

$$T_2+T_3+T_4 \le \left(\frac{2m}{\lambda}+\frac{\sigma}{3}m\right)\|v_m\|_{r,c,\lambda}.$$

Further we have $|i+(j+|\beta|-1)\rho(x)| \ge \sigma m$ by the condition $\rho(0)=1$ and $i+j+|\beta|=m\ge 2$. Therefore we have

$$\|Lv_m\|_{r,c,\lambda} \ge \left(\sigma m - \frac{2m}{\lambda} - \frac{\sigma}{3}m\right)\|v_m\|_{r,c,\lambda}.$$

Hence for sufficiently small $c>0$ and sufficiently large $\lambda>0$, we obtain the desired result. □

Let us estimate the function $\partial_i v_m$.

Definition 3.7. For the function $v_m \in V_m$ we define

$$D_p v_m := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \partial_p w_{i,j,k}^{\beta}(x) t^{i+j\rho(x)}\{\log t\}^k \Phi_1^{\beta}$$

for $p=1,\dots,n$.

Lemma 3.8. *If $v_m \in V_m$, then for $i = 1, \dots, n$, we have*

$$\|\partial_i v_m\|_{r,c,\lambda} \le \|D_i v_m\|_{r,c,\lambda} + c_0 \lambda m \|v_m\|_{r,c,\lambda} + \frac{3m-2}{c}\|v_m\|_{r,c,\lambda} \quad \text{for } 0 < r \le R. \tag{3.6}$$

Proof. We have

$$\sum_{|l|\ge 0} (l_p + 1)\beta_l \le \sum_{|l|=0}^{m-1} (|l|+1)\beta_l = 2|\beta| + [\beta] \le 3m-2. \tag{3.7}$$

We put $c_0 = \max_{i=1,\dots,n}\{\|\partial_i \rho\|_R\}$, and by the relations (2.6), (3.7) and $j \le m$ we obtain the desired estimate. □

Therefore by the relations (2.9), (2.10) and Lemma 3.8, we have the following lemma.

Lemma 3.9. *If $u = \sum_{m\ge 1} v_m$ is a formal solution of the equation* (2.1) *constructed in Section* 2, *we have the following inequality for v_m $(m \ge 2)$:*

$$\begin{aligned}\|Lv_m\|_{r,c,\lambda} \le & \sum_{\substack{p+|\alpha|\ge 2\\ p+|m_n|=m}} \|a_{p,\alpha}\|_r \prod_{h_0=1}^{\alpha_0} \|v_{m_{0,h_0}}\|_{r,c,\lambda} \\ & \times \prod_{i=1}^{n}\prod_{h_i=1}^{\alpha_i}\left\{\|D_i v_{m_{i,h_i}}\|_{r,c,\lambda} + c_0\lambda m_{i,h_i}\|v_{m_{i,h_i}}\|_{r,c,\lambda} + \frac{3m_{i,h_i}-2}{c}\|v_{m_{i,h_i}}\|_{r,c,\lambda}\right\}.\end{aligned}$$

Let us define a majorant equation to show that the formal solution (3.1) converges. We take A_1 so that

$$\frac{\|w_{0,0,0}^{e_0}\|_R}{c} + \|w_{0,1,0}^{0}\|_R \le A_1,$$

$$\frac{\|\partial_i w_{0,0,0}^{e_0}\|_R}{c} + \|\partial_i w_{0,1,0}^{0}\|_R \le A_1$$

for $i = 1, \dots, n$.

Then we consider the following equation:

$$\frac{\sigma}{2} Y = \frac{\sigma}{2} A_1 t_1 + \frac{1}{R-r}\sum_{p+|\alpha|\ge 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t_1^p Y^{\alpha_0} \prod_{i=1}^{n}\left(eY + c_0\lambda Y + \frac{3}{c}Y\right)^{\alpha_i}. \tag{3.8}$$

The equation (3.8) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0, 0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:

$$Y = \sum_{m\ge 1} Y_m t_1^m \quad \text{with } Y_m = \frac{C_m}{(R-r)^{m-1}}$$

where $Y_1 = C_1 = A_1$ and $C_m \ge 0$ for $m \ge 1$.

Then we have;

Lemma 3.10. *For $m \geq 1$, we have*

$$m\|v_m\|_{r,c,\lambda} \leq Y_m \quad \textit{for } 0 < r < R \tag{3.9}$$

$$\|D_i v_m\|_{r,c,\lambda} \leq eY_m \quad \textit{for } 0 < r < R, \tag{3.10}$$

for $i = 1, \ldots, n$.

Proof. By $A_1 = Y_1$ and the definition of A_1, (3.9) and (3.10) hold for $m = 1$. By induction on m, let us show that (3.9) and (3.10) hold for $m \geq 2$. By substituting the solution $Y = \sum_{m \geq 1} Y_m t_1^m$ into the equation (3.8), we have the following relation:

$$\frac{\sigma}{2} Y_m = \frac{1}{R-r} \sum_{\substack{p+|\alpha| \geq 2 \\ p+|m_n|=m}} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} \prod_{h_0=1}^{\alpha_0} Y_{m_{0,h_0}}$$

$$\times \prod_{i=1}^{n} \prod_{h_i=1}^{\alpha_i} \left\{ eY_{m_{i,h_i}} + c_0 \lambda Y_{m_{i,h_i}} + \frac{3}{c} Y_{m_{i,h_i}} \right\} \tag{3.11}$$

for $m \geq 2$. Therefore if we assume that (3.9) and (3.10) hold for $m_{i,h_i} < m$, by (3.11), Lemma 3.6 and Lemma 3.9 we obtain

$$\frac{\sigma}{2} m\|v_m\|_{r,c,\lambda} \leq (R-r)\frac{\sigma}{2} Y_m.$$

Therefore we have

$$m\|v_m\|_{r,c,\lambda} \leq (R-r)Y_m \leq Y_m. \tag{3.12}$$

The relation (3.12) is rewrited as follows:

$$m \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{\|w_{i,j,k}^{\beta}\|_r \lambda^k}{c^{\langle \beta \rangle}} \leq \frac{C_m}{(R-r)^{m-2}}. \tag{3.13}$$

By (3.13) and Lemma 3.5, we have

$$m\|D_i v_m\|_{r,c,\lambda} \leq \frac{(m-1)eC_m}{(R-r)^{m-1}}$$

for $i = 1, \ldots, n$ and $0 < r < R < 1$. Therefore we have

$$\|D_i v_m\|_{r,c,\lambda} \leq \frac{eC_m}{(R-r)^{m-1}} = eY_m.$$

Hence (3.9) and (3.10) hold for $m \geq 2$. □

Let us show that the formal solution (3.1) converges by using (3.9) in Lemma 3.10. We rewrite (3.1) as follows:

$$u = u_0^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)}$$

$$+ \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{w_{i,j,k}^{\beta}(x)\lambda^k}{c^{\langle \beta \rangle}} t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi_1^{\beta},$$

where

$$\Psi_1^\beta = \prod_{|l|\geq 0}\left(c^{|l|+1}\frac{\partial_x^l\phi_1}{l!}\right)^{\beta_l}. \tag{3.14}$$

Firstly let us estimate (3.14). For $\|\phi_1\|_R$, we have the following lemma.

LEMMA 3.11. *For any γ with $0<\gamma<1$, there is an $R>0$ such that*

$$\|\phi_1\|_R = O(|t|^\gamma) \quad \textit{as } t\to 0 \textit{ in } S_\theta$$

holds for any $\theta>0$.

PROOF. We put

$$\phi_1 = t^\gamma\frac{t^{\rho_0(x)+\alpha}-t^\alpha}{\rho_0(x)}$$

with $\alpha+\gamma=1$ and $\rho_0(x)=\rho(x)-1$. Then we can take $R>0$ with

$$\|\rho_0\|_R<\alpha$$

by $\rho_0(0)=0$. Therefore we have

$$\left\|\frac{t^{\rho_0(x)+\alpha}-t^\alpha}{\rho_0(x)}\right\|_R \leq |\log t|\,|t|^{\alpha-\|\rho_0\|_R}\to 0 \quad \text{as } t\to 0 \text{ in } S_\theta$$

for any $\theta>0$. Hence we have the desired result. □

By Lemma 3.11, there exists a positive constant c_1 such that

$$\|\phi_1\|_R\leq c_1|t|^\gamma \quad \text{in } S_\theta. \tag{3.15}$$

By Lemma 3.4 and (3.15), for $|l|\geq 0$ we have

$$\|\partial_x^l\phi_1\|_r\leq\frac{l!}{(R-r)^{|l|}}\|\phi_1\|_R\leq\frac{l!c_1}{(R-r)^{|l|}}|t|^\gamma \quad \text{for } 0<r<R. \tag{3.16}$$

Therefore, we have

$$\|\Psi_1^\beta\|_r\leq\prod_{|l|\geq 0}\left(c^{|l|+1}\frac{c_1}{(R-r)^{|l|}}|t|^\gamma\right)^{\beta_l} = \left(\frac{c}{R-r}\right)^{\langle\beta\rangle}(c_1(R-r)|t|^\gamma)^{|\beta|} \tag{3.17}$$

for $0<R<R_0$ in S_θ.

Let us estimate $t^{i+j\rho(x)}((\log t)/\lambda)^k\Psi_1^\beta$.

We put $\eta(t,\lambda)=\max\{|(\log t)/\lambda|,1\}$, $c_2=\max\{c/(R-r),1\}$ and $c_3=c_1(R-r)$. Since we have $[\beta]\leq m-2<m=i+j+|\beta|$,

$$\langle\beta\rangle\leq 2|\beta|+[\beta]\leq i+j+3|\beta|$$

and

$$k\leq i+|\beta|_0+|\beta|_1+2(j-1)\leq i+|\beta|+2j,$$

we obtain

$$\left\| t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi_1^\beta \right\|_r \le \{|c_2\eta(t,\lambda)t|\}^i \{\|c_2\eta(t,\lambda)^2 t^{\rho(x)}\|_r\}^j \{|(c_2)^3 c_3 \eta(t,\lambda)t^\gamma|\}^{|\beta|}$$

in S_θ. For any sufficiently small $\varepsilon > 0$, there exists a sufficiently small $\delta > 0$ such that for any $t \in S_\theta$ with $0 < |t| < \delta$ we have

$$|c_2\eta(t,\lambda)t| < \varepsilon, \quad \|c_2\eta(t,\lambda)^2 t^{\rho(x)}\|_r < \varepsilon, \quad |(c_2)^3 c_3\eta(t,\lambda)t^\gamma| < \varepsilon,$$

and we obtain

$$\left\| t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right) \Psi_1^\beta \right\|_r \le \varepsilon^m.$$

Then by Lemma 3.10, we have

$$\|u\|_r \le \sum_{m\ge 1} Y_m \varepsilon^m \tag{3.18}$$

for sufficiently small $|t|$ in S_θ. Hence the formal solution (3.1) converges for $x \in D_r$ and sufficiently small $|t|$ in S_θ. □

4. Completion of the proof of Theorem 1.5 in the case $\rho(0) = 1$.

In this section, let us complete the proof of Theorem 1.5 in the case $\rho(0) = 1$. We know the following theorem.

Theorem 4.1. *If $u_i(t,x) \in \tilde{\mathcal{O}}_+$ $(i = 1, 2)$ are solutions of (2.1), we have;*
1. *For any $a < \rho(0) = 1$, we have $t^{-a}(u_1 - u_2) \in \tilde{\mathcal{O}}_+$.*
2. *If $t^{-b}(u_1 - u_2) \in \tilde{\mathcal{O}}_+$ for some $b \ge \rho(0) = 1$, we have $u_1(t,x) = u_2(t,x)$ in $\tilde{\mathcal{O}}_+$.*

For the proof, see Gérard and Tahara ([**2**], Theorem 3).

By the discussions in sections 2, 3 and 4, we already know the following results:

(C1) If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, for any $\varphi(x) \in \boldsymbol{C}\{x\}$, the equation (1.1) has an $\tilde{\mathcal{O}}_+$-solution $U(\varphi)(t,x)$ having an expansion of the form

$$\begin{aligned} U(\varphi) = {} & w_{0,0,0}^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m\ge 2}\sum_{\substack{i+|\beta|=m\\ [\beta]\le m-2}} u_i^\beta(x)t^i\Phi_1^\beta \\ & + \sum_{m\ge 2}\sum_{\substack{i+j+|\beta|=m\\ j\ge 1,[\beta]\le m-2}}\sum_{\substack{k\le i+|\beta|_0+|\beta|_1\\ +2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^\beta \end{aligned} \tag{4.1}$$

with $w_{0,1,0}^0(x) = \varphi(x)$, where all the coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic in a common disk centered at the origin of $\boldsymbol{C}_x^n$. If we take $\varphi(x) = 0$, then the solution $U(0)(t,x)$ has the expansion

$$U(0)(t,x) = u_0^{e_0}(x)\phi_1 + \sum_{m\ge 2}\sum_{\substack{i+|\beta|=m\\ [\beta]\le m-2}} u_i^\beta(x)t^i\Phi_1^\beta. \tag{4.2}$$

(C2) If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, and if a solution $u(t,x) \in \tilde{\mathcal{O}}_+$ of the equation (1.1) is expressed in the form

$$t^{-1}(u(t,x) - u_0^{e_0}(x)\phi_1(t,x) - \varphi(x)t^{\rho(x)}) \in \tilde{\mathcal{O}}_+,$$

then the coefficient $u_0^{e_0}(x)$ is uniquely determined by the equation (1.1), and they are independent of $\varphi(x)$.

Moreover, by (C2) and Theorem 4.1 we can easily see that $U(\varphi)$ in (4.1) is uniquely determined by $\varphi(x)$. If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, by (C1) we have

$$S_+ \supset \{U(\varphi); \varphi(x) \in \boldsymbol{C}\{x\}\}. \tag{4.3}$$

Hence it is sufficient to prove the following proposition to complete the proof of the main theorem.

Proposition 4.2. *Assume* (A1), (A2) *and* (A3). *If* $\rho(0) = 1$ *and* $\rho(x) \not\equiv 1$, *and if* $u(t,x) \in S_+$, *then we can find a* $\varphi(x) \in \boldsymbol{C}\{x\}$ *such that* $u(t,x) \equiv U(\varphi)(t,x)$ *holds in* $\tilde{\mathcal{O}}_+$.

The proof of this proposition is almost the same as that of Proposition 2 in Gérard and Tahara [**1**]; so we may omit the details.

By (4.3) and Proposition 4.2 we obtain the main theorem 1.5 in the case $\rho(0) = 1$ and $\rho(x) \not\equiv 1$.

5. Proof of Theorem 1.5 in the case $\rho(0) = N$.

In Section 2, 3 and 4, we have proved Theorem 1.5 in the case $\rho(0) = 1$. In this section, we will prove Theorem 1.5 in the case $\rho(0) = N \geq 2$ and $\rho(x) \not\equiv N$.

We set

$$u(t,x) = \sum_{i=1}^{N-1} u_i(x)t^i + t^{N-1}w(t,x), \tag{5.1}$$

where $u_i(x) \in \boldsymbol{C}\{x\}$ $(1 \leq i \leq N-1)$ and $w(t,x) \in \tilde{\mathcal{O}}_+$.

Then by an easy calculation we see

Lemma 5.1. *If the function* (5.1) *is a solution of the equation* (2.1), *the functions* $u_1(x), \dots, u_{N-1}(x)$ *are uniquely determined and* $w(t,x)$ *satisfies an equation of the following form*:

$$\begin{aligned}(t\partial_t - \rho(x) + N - 1)w = ta(t,x) + tA_0(t,x)w + t\sum_{i=1}^{n} A_i(t,x)\partial_i w \\ + \sum_{|\alpha|\geq 2} t^{(N-1)(|\alpha|-1)}A_\alpha(t,x)w^{\alpha_0}\prod_{i=1}^{n}(\partial_i w)^{\alpha_i},\end{aligned} \tag{5.2}$$

where

$$a(t,x) = \frac{1}{t^N}(G_2(x)(t, w_0, \partial_x w_0) + ta(x) - (t\partial_t - \rho(x))w_0)$$

with $w_0 = \sum_{i=1}^{N-1} u_i(x)t^i$ *and*

$$A_i(t,x) = \frac{1}{t}\frac{\partial G_2}{\partial X_i}(x)(t, w_0, \partial_x w_0), \quad i = 0, 1, \dots, n,$$

$$A_\alpha(t,x) = \frac{1}{\alpha!}\frac{\partial^{|\alpha|} G_2}{\partial X^\alpha}(x)(t, w_0, \partial_x w_0), \quad |\alpha| \geq 2.$$

Since the equation (5.2) satisfies the conditions (A1), (A2), (A3) and the characteristic exponent $\rho^N(x) = \rho(x) - N + 1$ satisfies $\rho^N(0) = 1$, we can apply the results in sections 2, 3 and 4.

Further, by the form of all the nonlinear parts of the equation (5.2), we see that the formal solution constructed in Section 2 has the following form:

$$\begin{aligned} w &= u_0^{N,e_0}(x)\phi_{N,1} + w_{0,1,0}^{N,0}(x)t^{\rho^N(x)} \\ &\quad + \sum_{i \geq 2} u_i^N(x)t^i + \sum_{m \geq 2} \sum_{\substack{i+|\beta|=m \\ [\beta] \leq m-2, |\beta| \geq 1}} u_i^{N,\beta}(x)t^{i+(N-1)(|\beta|-1)}\Phi_{N,1}^\beta \\ &\quad + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^{N,\beta}(x)t^{i+(N-1)(j+|\beta|-1)+j\rho^N(x)}\{\log t\}^k \Phi_{N,1}^\beta \end{aligned} \tag{5.3}$$

where $\Phi_{N,1}^\beta = \prod_{|l| \geq 0}\left(\frac{\partial_x^l \phi_{N,1}}{l!}\right)^{\beta_l}$ and $\phi_{N,1} = \frac{t^{\rho^N(x)} - t}{\rho^N(x) - 1}$. Therefore we have

$$\begin{aligned} u &= \sum_{i=1}^{N-1} u_i(x)t^i + u_0^{N,e_0}(x)\phi_N + w_{0,1,0}^{N,0}(x)t^{\rho(x)} \\ &\quad + \sum_{i \geq 2} u_i^N(x)t^{i+N-1} + \sum_{m \geq 2} \sum_{\substack{i+|\beta|=m \\ [\beta] \leq m-2, |\beta| \geq 1}} u_i^{N,\beta}(x)t^i\Phi_N^\beta \\ &\quad + \sum_{m \geq 2} \sum_{\substack{i+j+|\beta|=m \\ j \geq 1, [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^{N,\beta}(x)t^{i+j\rho(x)}\{\log t\}^k \Phi_N^\beta. \end{aligned} \tag{5.4}$$

We put

$$u_i^N(x) \mapsto u_{i+N-1}(x) \quad \text{for } i \geq 2, \quad u_i^{N,\beta}(x) \mapsto u_i^\beta(x) \quad \text{for } |\beta| \geq 1,$$

$$w_{i,j,k}^{N,\beta}(x) \mapsto w_{i,j,k}^\beta(x) \quad \text{for any } (i,j,k,\beta),$$

and we have $u_N^0(x) \equiv 0$ by the form of the solution (5.3) and the above relations. Hence this completes the proof of Theorem 1.5. □

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Hiroshi YAMAZAWA
Department of Language and Culture
Caritas Junior College
Azamino, Aoba-ku
Yokohama, 225-0011
Japan