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Some sums involving Farey fractions II

To the memory of Professor Pan Cheng Dong

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Abstract. Let F_x denote the Farey series of order $[x]$, i.e. the increasing sequence of irreducible fractions $\rho_\nu \in (0,1]$ whose denominators do not exceed x. We shall obtain precise asymptotic formulae for the sum $\sum_{\nu=1}^{\Phi(x)} \rho_\nu^z$ for complex z and related sums, $\Phi(x) = \sharp F_x$ coinciding the summatory function of Euler's function. In particular, we shall prove an asymptotic formula for $\sum \rho_\nu^{-1}$ with as good an estimate as for the prime number theorem by extracting an intermediate error term occurring in the asymptotic formula for $\Phi(x)$.

1. Introduction and statement of results.

For a variable $x \geq 1$, let $F_x = F_{[x]}$ denote the Farey series of order $[x]$ ($[x]$ = integral part of x), i.e. the sequence of irreducible fractions $\in (0,1]$ whose denominators $\leq x$, arranged in increasing order of magnitude:

$$F_x = \left\{ \rho_\nu = \frac{b_\nu}{c_\nu} \,\middle|\, (b_\nu, c_\nu) = 1, 0 < b_\nu \leq c_\nu \leq x \right\}.$$

Supplementarily, we put $\rho_0 = b_0/c_0 = 0/1$.

The number of ρ_ν's in F_x with $c_\nu = n$ is $\sum_{k=1,(k,n)=1}^{n} = \phi(n)$, Euler's function. The total number of ρ_ν's in F_x is therefore $\Phi(x) = \sum_{n \leq x} \phi(n)$.

Let $Q_x = Q_{[x]}$ denote all pairs $(c_\nu, c_{\nu+1})$ of denominators of consecutive Farey fractions $\rho_\nu = b_\nu/c_\nu$ and $\rho_{\nu+1} = b_{\nu+1}/c_{\nu+1}$ (with $c_0 = 1$):

$$Q_x = \{(c_\nu, c_{\nu+1}) \,|\, 0 \leq \nu \leq \Phi(x) - 1\} \tag{1}$$

($\sharp Q_x = \sharp F_x = \Phi(x)$). E.g. $Q_3 = \{(1,3),(3,2),(2,3),(3,1)\}$.

In Part I [**15**] we considered the m-th power moments $s_m(x)$ for $m = 2, 3$ (Theorems 1 and 2) and revealed the connection between these and the (associated) error terms of Euler's function, where

$$s_m(x) = \sum_{(c_\nu, c_{\nu+1}) \in Q_x} (c_\nu c_{\nu+1})^{-m} \quad (m \in \boldsymbol{N}).$$

In view of the basic relation

$$b_{\nu+1} c_\nu - b_\nu c_{\nu+1} = 1,$$

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we have $\rho_{v+1} - \rho_v = (c_v c_{v+1})^{-1}$, so that

$$s_m(x) = \sum_{v=0}^{\Phi(x)-1} (\rho_{v+1} - \rho_v)^m$$

is indeed the m-th power moment of the difference of consecutive Farey fractions. In particular, $s_1(x) = \rho_{\Phi(x)} - \rho_0 = 1 - 0 = 1$. Rather surprisingly, our Theorems [**15**] as well as those of Hall-Tenenbaum [**9**] caught attention of physicists in connection with circle maps (Cvitanović [**4**]).

In this Part II our objective is twofold. The first-half of the paper is a direct continuation as well as an improved generalization of results in Part I, and the second-half is the presentation of results in the unpublished MS of the first author's former paper (jointly with R. Sita Rama Chandra Rao and A. Siva Rama Sarma) which was once submitted to these volumes, but withdrawn for better organization, years ago; although preparation of the new MS has been suspended because of untimely death of Sita Rama Chandra Rao, the first author's friend and collaborator, that withdrawn MS is now completely rewritten, incorporating new developments.

Namely, Theorem 1 contains not only improvements of Theorems 1 and 2 of Part I by a factor of $(\log\log x)^{4/3}$ but also rather precise asymptotic formulas for $s_m(x)$ in the cases $m = 4$ and $m \geq 5$. The proof goes on the similar lines as those of Theorems 1 and 2; however, we shall prove all the results *at a stretch*, i.e. new asymptotic formulas for $s_m(x)$ for $m \geq 4$ as well as improvements, by modifying proofs of Part I and introducing new lemmas, and also as byproducts evaluate certain infinite series.

Then we proceed to give refinements of the former paper (referred to above), namely improvements over Mikolás' results [**22**] concerning the asymptotic behavior of sums $\sum_{v=1}^{\Phi(x)} \rho_v^{-a}$ for $a > 0$ (for positive power sums, see [**16**]) and precise asymptotic formulas for unsymmetric form of $s_m(x)$, i.e.

$$\sum_{(c_v, c_{v+1}) \in Q_x} \frac{1}{c_v^a c_{v+1}},$$

the special case $a = 2$ of which constitutes an improvement of Hans and Dumir's results [**10**].

All these results rest on Lehner-Newman's first sum formula [**19**]:

LEMMA (Lehner-Newman's first sum formula [**19**]). *For any complex-valued function $f(u, v)$ defined at least for positive integral arguments u, v, we have*

$$s_f(x) := \sum_{v=0}^{\Phi(x)-1} f(c_v, c_{v+1}) \tag{2}$$

$$= f(1,1) + \sum_{2 \leq r \leq x} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \{f(k,r) + f(r,k) - f(k, r-k)\}.$$

On the other hand, our Theorems 6 and 7 are based on Lehner-Newman's second sum formula:

Lemma (Lehner-Newman's second sum formula).

$$\sum_{\nu=0}^{\Phi(x)-1} f(c_\nu, c_{\nu+1}) = \sum_{r\le x} \sum_{\substack{x-r<k\le x\\(k,r)=1}} f(r,k). \tag{3}$$

By these formulas we can transform the sum over Farey points into one over primitive lattice points, for which we can apply ordinary apparatus of number theory. Successful applications were made by Lehner and Newman [**19**], Hall [**7**], subsequently by the first author [**14**] and then in Part I [**15**].

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Notation and known results. The associated error term $E(x)$ is defined by

$$\Phi(x) = \sum_{n\le x} \phi(n) = \frac{1}{2\zeta(2)} x^2 + E(x), \tag{4}$$

and it is known that

$$\begin{cases} E(x) = O(x\log x) & \text{(Mertens)},\\ E(x) = O(x\log^{2/3} x(\log\log x)^{4/3}) & \text{(Walfisz [28])}. \end{cases} \tag{5}$$

The closely related error term $H(x)$ is defined by

$$\sum_{n\le x} \frac{\phi(n)}{n} = \frac{1}{\zeta(2)} x + H(x). \tag{6}$$

It is known that

$$E(x) = xH(x) + O(x\delta(x)), \tag{7}$$

where

$$\delta(x) = \exp(-A(\log x)^{0.6}(\log\log x)^{-0.2}) \tag{8}$$

$A > 0$ being an absolute constant, is the reducing factor which appears in the prime number theorem

$$M(x) := \sum_{n\le x} \mu(n) = O(x\delta(x)), \tag{9}$$

where $\mu(n)$ denotes the Möbius function. Let $B_k(t)$ denote the k-th Bernoulli polynomial and let $\bar{B}_k(t) := B_k(t-[t])$ denote the k-th periodic Bernoulli polynomial. Define

$$U(x) = \sum_{n\le x} \frac{\mu(n)}{n} \bar{B}_1\left(\frac{x}{n}\right). \tag{10}$$

$U(x)$ is the main term of $E(x)$, i.e.

$$E(x) = -xU(x) + O(x\delta(x)), \tag{11}$$

and, as a basis of the second estimate in (5),

$$U(x) = O(\log^{2/3} x(\log\log x)^{4/3}) \tag{12}$$

is the hitherto best known estimate. The estimate of Saltykov used in Part I is erroneous and Pétermann [**24**] has confirmed, after amending some errors in Saltykov's paper, that Saltykov's method yields only (12).

Let δ_ν denote the distance from the ν-th division point $\nu/(\Phi(x))$ of the unit interval to the ν-th Farey point ρ_ν:

$$\delta_\nu = \rho_\nu - \frac{\nu}{\Phi(x)}, \quad \nu = 1, 2, \ldots, \Phi(x).$$

The celebrated Riemann hypothesis (RH) is equivalent in the first place to one of the forms of the prime number theorem:

$$\mathrm{RH} \Leftrightarrow M(x) = O(x^{1/2+\varepsilon}), \tag{13}$$

for every $\varepsilon > 0$ (hereafter we always use ε in this context). Franel's theorem [**5**], [**18**] asserts that the RH is equivalent to the mean square of δ_ν:

$$\mathrm{RH} \Leftrightarrow \sum_{\nu=1}^{\Phi(x)} \delta_\nu^2 = O(x^{-1+\varepsilon}). \tag{14}$$

In what follows we always denote by s the complex variable with $\Re s = \sigma$, and by m a fixed integer ≥ 2.

Define

$$S_s(x) = \sum_{n \leq x} M\left(\frac{x}{n}\right) n^{-s}. \tag{15}$$

Then it is well known that

$$S_{-1}(x) = \Phi(x), \quad S_0(x) = 1, \tag{16}$$

and that

$$\mathrm{RH} \quad \Leftrightarrow \quad S_s(x) = O(x^{1/2+\varepsilon}), \quad \text{for some } s \text{ with } \sigma \geq \frac{1}{2}. \tag{17}$$

We are now in a position to state main results of the paper.

Theorem 1. *As $x \to \infty$, we have for $s_m(x) = \sum_{\nu=0}^{\Phi(x)-1} (c_\nu c_{\nu+1})^{-m}$*

(i) $$s_2(x) = \frac{2}{\zeta(2)x^2}\left\{\log x + \gamma + \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\} + \frac{4U(x)\log x}{x^3} + O\left(\frac{\log x}{x^3}\right)$$

(ii) $$s_3(x) = \frac{2\zeta(2)}{\zeta(3)x^3} + \frac{3}{\zeta(2)x^4}\left\{\log x + \gamma - \frac{1}{4} - \frac{\zeta'(2)}{\zeta(2)} + 2\zeta^2(2)c_2^{(1)}(x)\right\} + \frac{12U(x)\log x}{x^5} + O\left(\frac{\log x}{x^5}\right)$$

(iii) $$s_4(x) = \frac{2\zeta(3)}{\zeta(4)x^4} + \frac{1}{x^5}\left\{\frac{4\zeta(2)}{\zeta(3)} + 8\zeta(3)c_3^{(1)}(x)\right\} + \frac{20}{3\zeta(2)x^6}\left\{\log x + \gamma - \frac{13}{30} - \frac{\zeta'(2)}{\zeta(2)} + 3\zeta^2(2)c_2^{(1)}(x) + 3\zeta(2)\zeta(3)c_2^{(2)}(x)\right\} + \frac{40U(x)\log x}{x^7} + O\left(\frac{\log x}{x^7}\right),$$

and for $m \geq 5$

(iv) $$s_m(x) = \frac{2\zeta(m-1)}{\zeta(m)x^m} + \frac{m}{\zeta(m-1)x^{m+1}}\{\zeta(m-2) + 2\zeta^2(m-1)c_{m-1}^{(1)}(x)\} + \frac{m(m+1)}{x^{m+2}}\left\{\frac{\zeta(m-3)}{3\zeta(m-2)} + \zeta(m-2)c_{m-2}^{(1)}(x) + \zeta(m-1)c_{m-2}^{(2)}(x)\right\} + \frac{35\theta'\log x}{2\zeta(2)x^{m+3}} + O\left(\frac{1}{x^{m+3}}\right),$$

where $U(x)$ *is defined by* (10), γ *is Euler's constant,* θ' *equals* 0 *or* 1 *according as* $m = 5$ *or* $m \geq 6$, *and for* $\sigma > 1$ *and* $r \in \boldsymbol{N}$,

$$c_s^{(r)}(x) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\bar{B}_r\left(\frac{x}{n}\right), \tag{18}$$

which satisfies

$$\liminf_{x\to\infty} c_\sigma^{(r)}(x) \leq \frac{B_r}{\zeta(\sigma)}, \tag{19}$$

so that $c_\sigma^{(1)}(x) \not\equiv 0(1)$, $x \to \infty$, *where* B_r *denotes the r-th Bernoulli number.*

In the course of proof we shall obtain the identity

$$\sum_{j=0}^{m-2}\sum_{r=2}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r} \frac{2\binom{m+j}{j+1}}{r^{m+j+1}k^{m-j-1}} = 1,$$

which, combined with Lemmas 9, 11, yields a vast amount of identities.

Corollary 1. *We have the following evaluations*

(i) $$2\sum_{j=0}^{m-2}\binom{m+j}{j+1}\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}} = \zeta(2m)\left\{1 + 2\sum_{j=0}^{m-2}\binom{m+j}{j+1}\right\}$$

(i)* $$2\sum_{j=0}^{m-2}\binom{m+j}{j+1}\sum_{r=1}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}} = 1+2\sum_{j=0}^{m-2}\binom{m+j}{j+1}$$

(i)$^+$ $$2\sum_{j=0}^{m-3}\binom{m+j}{j+1}\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}}$$
$$= \zeta(2m)\left\{1+2\sum_{j=0}^{m-2}\binom{m+j}{j+1} - (2m+1)\binom{2m-2}{m-1}\right\}$$
$$+\binom{2m-2}{m-1}\sum_{i=1}^{2m-3}\zeta(2m-i-1)\zeta(i+1)$$

(ii) $$2\sum_{j=0}^{m-3}\binom{m+j}{j+1}\sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^{m-j-1}k^{m+j+1}}$$
$$= 2\sum_{j=0}^{m-3}\binom{m+j}{j+1}\zeta(m+j+1)\zeta(m-j-1)$$
$$+2\left\{(2m-1)\binom{2m-2}{m-1}-1\right\}\zeta(2m)$$
$$-\binom{2m-2}{m-1}\sum_{i=1}^{2m-3}\zeta(2m-i-1)\zeta(i+1),$$

where empty sums are understood to be 0.

REMARK 1. (i) is a new essential formula, of which (i)* is the relatively prime version and (i)$^+$ the reciprocal version, and so is Formula (ii), which has relatively prime and reciprocal version. Each of these has its relatively prime and reciprocal versions. E.g. the reciprocal version of (i)*, (i)$^{*+}$, states that

(i)$^{*+}$ $$2\sum_{j=0}^{m-3}\binom{m+j}{j+1}\sum_{r=1}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}}$$
$$= 1+2\sum_{j=0}^{m-2}\binom{m+j}{j+1} - (2m+1)\binom{2m-2}{m-1}$$
$$+\binom{2m-2}{m-1}\sum_{i=1}^{2m-3}\zeta(2m-i-1)\zeta(i+1).$$

EXAMPLE. Let H and H^* denote functions in Lemma 10. Then

(i) $$H(3,1) = \sum_{r=1}^{\infty}\sum_{k=1}^{r}\frac{1}{r^3k} = \frac{5}{4}\zeta(4),$$

(ii) $H(4,2) = \sum_{r=1}^{\infty} \sum_{k=1}^{r} \frac{1}{r^4 k^2} = \zeta^2(3) - \frac{\zeta(6)}{3} = \zeta^2(3) - \frac{\pi^6}{2835}$,

(iii) $H^*(2,4) = \sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{1}{r^2 k^4} = \frac{37}{12} - \frac{\zeta^2(3)}{\zeta(6)} = \frac{37}{12} - \frac{945\zeta^2(3)}{\pi^6}$,

etc.

PROOF. By taking $m = 2$ in (i) of Corollary 1, $m = 3$ in $(\mathrm{i})^+$ of Corollary 1 and in $(\mathrm{i})^{*+}$ of Remark 1 we obtain the above identities. □

The identities including (i) can be traced back to Nielsen's book on gamma function and are also later given by Williams (see [**24**] and references given there).

Each of these has its relatively prime versions. E.g. $H^*(3,1) = 5/4$ due to the first author [**14**]; for (i) itself, see Matsuoka [**21**]. These evaluations do not seem possible to deduce from Apostol and Vu's consideration [**2**].

REMARK 2. Those identities in Lemma 11 as well as those in Corollary 1 may be of particular interest in the light of recent investigations of Zagier [**31**].

For $r_1, \ldots, r_l \in \boldsymbol{N}$ (l is called the depth) define the multiple zeta-values

$$\zeta(r_1, \ldots, r_l) = \sum_{n_1 > \cdots > n_l > 0} \frac{1}{n_1^{r_1} \cdots n_l^{r_l}}.$$

Question. Consider the numbers $Z_r := \zeta(r)$ $(r \in \boldsymbol{N})$ as known. Then to what extent can one write all $\zeta(r_1, \ldots, r_l)$ as polynomials in $Z_1, Z_2, \ldots$?

A partial answer has been given by himself.

THEOREM (Zagier [**31**]). *Let $\mathfrak{A}_k^{\le 2}$ be the $\boldsymbol{Z}$-span of all multiple zeta-values of weight $k := r_1 + r_2$, depth $l \le 2$. Then*

$$\mathfrak{A}_k^{\le 2} = \sum_{n=0}^{(k-3)/2} \boldsymbol{Q}\zeta(2n)\zeta(k-2n) \quad \left(\dim = \frac{k-1}{2}\right)$$

if $2 \nmid k$, while if $2 | k$, then

$$\boldsymbol{Q}\mathfrak{A}_k^{\le 2} = \boldsymbol{Q}Z_k + \sum_{\substack{3 \le n \le k/2 \\ n \text{ odd}}} \boldsymbol{Q}Z_n Z_{k-n} + \left(\textit{space of } \dim\left[\frac{k-2}{6}\right]\right).$$

Lemma 11, (i), Example (i), (ii) are in conformity with this theorem with $l = 2$, and $k = a + 1$, $k = 4$, and $k = 6$, respectively.

For related topics, see Zagier [**32**] and Arakawa and Kaneko [**1**].

COROLLARY 2 (Maier [**20**]). *For integers g, $h \ge 0$ define*

$$s_{g,h} = s_{g,h}(x) := \sum_{\nu=1}^{\Phi(x)-1} (\delta_\nu)^g (\delta_{\nu+1})^h.$$

Then

$$s_{2,0} \ (= s_{0,2}) = \sum_{\nu=1}^{\Phi(x)-1} \delta_\nu^2$$
$$= s_{1,1} + \frac{1}{\zeta(2)x^2}\left(\log x + \gamma - \frac{\zeta'}{\zeta}(2)\right) + O\left(\frac{\log x}{x^3}\right),$$

and

$$s_{2,1} = \frac{\zeta(2)}{3\zeta(3)x^3} + O(x^{-4}).$$

Franel's sum $s_{2,0}$ has been considered by several authors, the initiator being Franel, whose theorem (see Formula (14) above) is based on the Franel identity, which in turn has its genesis in the 3-term relation for the 2-dimensional Dedekind sums (for this and references, see [**16**]).

Regarding power moments, we have

THEOREM 2. *For* $2 \le g \in \boldsymbol{N}$,

$$\sum_{\nu=1}^{\Phi(x)} |\delta_\nu|^g = \Omega_+(x^{1-g}),$$

and on the RH, for $1 \le g \in \boldsymbol{R}$,

$$\sum_{\nu=1}^{\Phi(x)} |\delta_\nu|^g = O(x^{2-(3/2)g+\varepsilon}) + O(x^{1-g+\varepsilon}).$$

In particular, for Maier's sums with $g \in \boldsymbol{N}$,

$$s_{2g,0} = \begin{cases} O(x^{1-2g+\varepsilon}) & \text{on the RH} \\ \Omega_+(x^{1-2g}), o(x^{2-2g}) & \text{unconditionally.} \end{cases}$$

THEOREM 3 (Refinements of Mikolás' results). *For* $\Re a > 0$, $a \ne 1$

$$\text{(i)} \quad \sum_{\nu=1}^{\Phi(x)} \rho_\nu^{-a} = \frac{\zeta(a)x^{a+1}}{(a+1)\zeta(a+1)} - \frac{1}{a-1}\Phi(x)$$
$$- \begin{cases} \zeta(a)c_a^{(1)}(x)x^a, & \Re a > 1 \\ 0, & 0 < \Re a \le 1, \ a \ne 1 \end{cases}$$
$$+ O_a(x^{\max\{\Re a - 1, 1\}}(\log x)^\theta),$$

where $\theta = \theta(a) = 1$ *or* 0 *according as* $\Re a = 2$ *or* $\Re a \ne 2$, *and*

$$\text{(ii)} \quad \sum_{\nu=1}^{\Phi(x)} \frac{1}{\rho_\nu} = \Phi(x)\left\{\log x + \gamma - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right\} + O(x\log^2 x)$$
$$= \frac{x^2}{2\zeta(2)}\left(\log x + \gamma - \frac{1}{2} - \frac{\zeta'}{\zeta}(2)\right) - xD(x) - \gamma x U(x) + O(x\delta(x)),$$

where

$$\begin{aligned}
D(x) &:= \sum_{n \le x} \frac{\mu(n)}{n} \bar{B}_1\left(\frac{x}{n}\right) \log \frac{x}{n} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \bar{B}_1\left(\frac{x}{n}\right) \log \frac{x}{n} + O(\delta(x)) \\
&= O(\log^2 x) \quad (\textit{trivially}) \\
&= U(x) \log x + O((\log x)^{4/3} (\log\log x)^{8/3}) \\
&= O((\log x)^{5/3} (\log\log x)^{4/3}).
\end{aligned}$$

Although only the formula (ii) in Theorem 3 expresses the relation between the distribution of Farey points and the error term of the prime number theorem, it is possible to refine the formula in (i) in a similar way, with some additional effects, to be carried out in detail elsewhere.

On the other hand, if we extract the main term, we again get equivalent conditions to the RH in terms of negative powers of Farey fractions.

Theorem 4. *For*

$$1 < \lambda \le \lambda_0 := \sqrt{\frac{60\zeta(3)}{\zeta(3/2)\zeta(7/2)} + \frac{1}{4}} - \frac{3}{2} = 3.4752\ldots,$$

let

$$f(t) = f_\lambda(t) = \frac{1}{t^\lambda} - \zeta_\lambda(t) \ \ (= -\zeta_\lambda(t+1)),$$

where

$$\zeta_\lambda(t) = \sum_{n=0}^{\infty} \frac{1}{(n+t)^\lambda}$$

denotes the Hurwitz zeta-function, and let

$$E_f(x) = \sum_{v=1}^{\Phi(x)} f(\rho_v) - \Phi(x) \int_0^1 f(t)\, dt.$$

Then

$$\mathrm{RH} \Leftrightarrow E_f(x) = O(x^{(1/2)+\varepsilon}).$$

This is a direct consequence of the modified Theorem 4 of Yoshimoto [**29**]:

Theorem 5. *Let* $f \in \mathscr{C}^6[0,1]$ *satisfy (at least) one of the conditions that* $f^{(4)}$ *is of constant sign, in which case* $c = 15/8$; $f^{(4)}$ *and* $f^{(6)}$ *are of the same constant sign,* $c = 1$; $f^{(4)}$ *and* $f^{(6)}$ *are of opposite constant sign,* $c = 7/8$. *For thus defined* c, *suppose* f

satisfies the inequality

$$\frac{|f'(1)-f'(0)|}{|f^{(3)}(1)-f^{(3)}(0)|} \geq \frac{c}{60}\cdot\frac{\zeta(3/2)\zeta(7/2)}{\zeta(3)}.$$

Then statement (E) *of* [**29**] *holds, i.e.*

$$\mathrm{RH} \Leftrightarrow E_f(x) = O(x^{(1/2)+\varepsilon}).$$

Theorem 6 (Improved generalization of Hans and Dumir [**10**]).

(i) $$\sum_{v=1}^{\Phi(x)-1}\frac{1}{c_v^2 c_{v+1}}$$
$$= \frac{1}{\zeta(2)x}\left(\log x+\gamma+1-\frac{\zeta'}{\zeta}(2)\right)+\frac{U(x)\log x}{x^2}+O\left(\frac{\log x}{x^2}\right),$$

(ii) $$\sum_{r=1}^{\Phi(x)-1}\frac{1}{c_v^3 c_{v+1}}$$
$$= \frac{\zeta(2)}{\zeta(3)x}+\frac{1}{2\zeta(2)x^2}\left(\log x+\gamma-\frac{1}{2}-\frac{\zeta'}{\zeta}(2)+2\zeta^2(2)c_2^{(1)}(x)\right)$$
$$+\frac{U(x)\log x}{x^3}+O\left(\frac{\log x}{x^3}\right),$$

(iii) $$\sum_{r=1}^{\Phi(x)-1}\frac{1}{c_v^4 c_{v+1}}$$
$$= \frac{\zeta(3)}{\zeta(4)x}+\frac{1}{2\zeta(2)x^2}(\zeta(2)+2\zeta^2(3)c_3^{(1)}(x))$$
$$+\frac{1}{3\zeta(2)x^3}\left(\log x+\gamma-\frac{7}{6}-\frac{\zeta'}{\zeta}(2)+3\zeta^2(2)c_2^{(1)}(x)+3\zeta(2)\zeta(3)c_2^{(2)}(x)\right)$$
$$+\frac{U(x)\log x}{x^4}+O\left(\frac{\log x}{x^4}\right),$$

and for every integer $a \geq 5$

(iv) $$\sum_{v=1}^{\Phi(x)-1}\frac{1}{c_v^a c_{v+1}}$$
$$= \frac{\zeta(a-1)}{\zeta(a)x}+\frac{1}{2\zeta(a-1)x^2}(\zeta(a-2)+2\zeta^2(a-1)c_{a-1}^{(1)}(x))$$
$$+\frac{1}{3\zeta(a-2)x^3}(\zeta(a-3)+3\zeta^2(a-2)c_{a-2}^{(1)}(x)$$

$$+ 3\zeta(a-2)\zeta(a-1)c_{a-2}^{(2)}(x))$$

$$+ \frac{\theta' \log x}{(a-1)\zeta(2)x^{a-1}} + O\left(\frac{1}{x^4}\right),$$

where θ' *is* 1 *or* 0 *according as* $a = 5$ *or* $a \geq 6$.

In the course of proof we prove Lemma 11, (i)*, and hence (i) also.

Theorem 7 (Refinements over results of [**14**], which in turn are refinements of Lehner and Newman's results [**19**]).

(i) *For non-negative reals* a, b,

$$\sum_{\nu=0}^{\Phi(x)-1} c_\nu^a c_{\nu+1}^b = c_{a,b} x^{a+b+2} - \frac{\pi^2}{6}(a+b+2)c_{a,b} U(x) x^{a+b+1}$$

$$+ O(x^{a+b+1}(\log\log x)^{\theta''}),$$

where $\theta'' = 1$ *if* $0 < b \leq 1/2$ *and* 0 *if either* $b = 0$ *or* $b > 1/2$, *and*

$$c_{a,b} = \frac{6}{\pi^2}\left\{\frac{1}{(1+a)(1+b)} - \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(3+a+b)}\right\}.$$

(ii) $$\sum_{\nu=0}^{\Phi(x)-1} \frac{1}{c_\nu c_{\nu+1}(c_\nu + c_{\nu+1})} = \frac{12\log 2}{\pi^2 x} + \frac{2\log 2}{x^2} U(x) + O\left(\frac{1}{x^2}\right),$$

$$\sum_{\nu=0}^{\Phi(x)-1} \frac{1}{c_\nu + c_{\nu+1}} = \frac{6}{\pi^2}(2\log 2 - 1)x + (1 - 2\log 2)U(x) + O(1),$$

and

$$\sum_{\nu=0}^{\Phi(x)-1} \frac{c_\nu c_{\nu+1}}{c_\nu + c_{\nu+1}} = \frac{11 - 12\log 2}{3\pi^2} x^3 + (11 - 12\log 2)\frac{xE(x)}{6} + O(x^2).$$

As we deduce Corollary 1 in the course of proof of Theorem 1, we reprove Gupta's identity

$$\sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{1}{r^2(r+k)} = \frac{3}{4},$$

which follows also from Apostol and Vu's consideration [**2**]. Indeed, defining

$$T'(s,z) = \sum_{r=1}^{\infty} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \frac{1}{r^s k^z (r+k)},$$

one has

$$T'(s,z) = \frac{1}{\zeta(s+z+1)} T(s,z),$$

where $T(s,z)$ is defined by (19) of [**2**]. Their Eq. (22) gives, for $a = 1$, Gupta's identity above.

Regarding Gupta's identity and generalizations thereof, we refer to an extensive work of Hata [**11**] which enables us to evaluate a large class of infinite series in closed form, using Farey fractions, or rather the intervals between them.

In regard to Theorem 1 we note the recent papers [**6**], [**7**] of Hall who considers the moments of length of Farey arcs. From the point of view of circle maps as well as the analogy with the difference between zeros of the Riemann zeta-function on the critical line these may be of particular interest and may shed some new light.

2. Preliminaries.

We collect basic lemmas some of which are of interest in their own right.

LEMMA 1 (Ishibashi-Kanemitsu [**12**], Lemmas 3 and 8). *For any* $u \in \boldsymbol{C}$ *define*

$$L_u(x) = \sum_{n \le x} n^u \quad \text{and} \quad M_u(x) = \sum_{n \le x} n^u \log n,$$

where for $n^u = e^{u \log n}$ *with* $\log n \in \boldsymbol{R}$. *Then for any* $l \in \boldsymbol{N}$ *satisfying* $l > \Re u + 1$ (*if* $u \in \boldsymbol{N} \cup \{0\}$, *we take* $l = u + 1$) *we have*

$$\text{(i)} \quad L_u(x) = \begin{cases} \dfrac{1}{u+1} x^{u+1} + \zeta(-u), & u \ne -1 \\ \log x + \gamma, & u = -1 \end{cases}$$

$$+ \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} \bar{B}_r(x) x^{u+1-r}$$

$$+ \begin{cases} (-1)^l \dbinom{u}{l} \displaystyle\int_x^\infty \bar{B}_l(t) t^{u-l}\, dt, & u \notin \boldsymbol{N} \cup \{0\}, \\ 0, & u \in \boldsymbol{N} \cup \{0\}, \end{cases}$$

where the error term can also be estimated as $O(x^{\Re u - l})$, *and*

$$\text{(ii)} \quad M_u(x)$$

$$= \begin{cases} \dfrac{1}{u+1} x^{u+1} \log x - \dfrac{1}{(u+1)^2} x^{u+1} - \zeta'(-u), & u \ne -1 \\ \dfrac{1}{2} \log^2 x + \gamma_1, & u = -1 \end{cases}$$

$$+\begin{cases} \displaystyle\sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} \bar{B}_r(x)(\log x + j_{r-2}(u)) x^{u+1-r}, & u \notin \boldsymbol{N} \\ \displaystyle\sum_{r=1}^{u+1} \frac{(-1)^r}{r} \binom{u}{r-1} \bar{B}_r(x)(\log x + j_{r-2}(u)) x^{u+1-r} & \\ \qquad \displaystyle +(-1)^u \sum_{r=u+2}^{l} \frac{u!(r-u-2)!}{r!} \bar{B}_r(x) x^{u+1-r}, & u \in \boldsymbol{N} \end{cases}$$

$$+\begin{cases} \displaystyle(-1)^l \binom{u}{l} \int_x^\infty \bar{B}_l(t) t^{u-l}(\log t + j_{l-1}(u))\, dt, & u \notin \boldsymbol{N} \cup \{0\} \\ \displaystyle(-1)^u \frac{u!(l-u-1)!}{l!} \int_x^\infty \bar{B}_l(t) t^{u-l}\, dt, & u \in \boldsymbol{N} \cup \{0\}, \end{cases}$$

where γ_1 is the first generalized Euler constant defined by

$$\gamma_1 = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{\log n}{n} - \frac{1}{2} \log^2 x \right),$$

and

$$j_{r-2}(u) = \begin{cases} \displaystyle\sum_{h=0}^{r-2} \frac{1}{u-h}, & 2 \le r \in \boldsymbol{N} \\ 0, & r = 1 \\ \displaystyle -\frac{1}{u+1}, & r = 0, \end{cases}$$

and the error term can be estimated as $O(x^{\Re u - l} \log x)$, respectively.

Corollary. *Let $k \in \boldsymbol{N}$ be given. Then for any $u \in \boldsymbol{C}$ and any $l \in \boldsymbol{N}$ with $l > \Re u + 1$, we have*

$$\begin{aligned} L_u^*(x) = L_{u,k}^*(x) &= \sum_{n \le x,\, (n,k)=1} n^u \\ &= \begin{cases} \displaystyle\frac{1}{u+1} \frac{\phi(k)}{k} x^{u+1} + \zeta(-u) \sum_{d|k} \mu(d) d^u, & u \ne -1 \\ \displaystyle\frac{\phi(k)}{k} (\log x + \alpha(k) + \gamma), & u = -1 \end{cases} \\ &\quad + \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{u}{r-1} x^{u+1-r} \sum_{d|k} \mu(d) \bar{B}_r\left(\frac{x}{d}\right) d^{r-1} + O(x^{\Re u - l} \sigma_l(k)), \end{aligned}$$

where

$$\sigma_l(n) = \sum_{d|n} d^l,$$

and

$$\alpha(k) = -\frac{k}{\phi(k)}\sum_{d|k}\frac{\mu(d)}{d}\log d = \sum_{p|n}\frac{\log p}{p-1} = \frac{k}{\phi(k)}t(k) - \log k,$$

with $\alpha(k)$ denoting Davenport's function and $t(k)$ denoting the function in Lemma 5 *below.*

LEMMA 2 (Kanemitsu-Yoshimoto [**16**]). *Let $0 < \xi \le 1$ and let $f(t)$ be defined at the points m/n $((m,n) = 1,\ 1 \le m \le n \le x)$. Then*

$$h(\xi, f) := \sum_{\rho_v \le \xi} f(\rho_v) = \sum_{n \le x}(\mu * V_\xi)(n) = \sum_{n \le x} M\left(\frac{x}{n}\right)V_\xi(n),$$

where

$$V_\xi(n) := \sum_{k \le n\xi} f\left(\frac{k}{n}\right)$$

and $*$ denotes the Dirichlet convolution.

LEMMA 3 (Lemma 6 of [**15**]). *For complex s we have*

$$\sum_{n \le x}\frac{\phi(n)\log n}{n^{s+1}}$$
$$= \begin{cases} \dfrac{1}{(s-1)^2\zeta(2)}\left(1 - \dfrac{(s-1)\log x}{x^{s-1}} - \dfrac{1}{x^{s-1}}\right), & s \ne 1 \\[2ex] \dfrac{1}{2\zeta(2)}\log^2 x, & s = 1 \end{cases}$$
$$+ \begin{cases} \dfrac{\zeta(s)\zeta'(s+1)}{\zeta^2(s+1)} - \dfrac{\zeta'(s)}{\zeta(s+1)} - \dfrac{1}{(s-1)^2\zeta(2)}, & s \ne 1, \sigma > 0 \\[2ex] \dfrac{1}{\zeta(2)}\left(-\dfrac{\zeta(2)}{8}\left(\dfrac{1}{\zeta(s)}\right)''\Big|_{s=2} + \gamma\dfrac{\zeta'}{\zeta}(2) + \gamma_1\right), & s = 1 \\[2ex] 0, & \textit{otherwise} \end{cases}$$
$$+ x^{-s}H(x)\log x + O\left(\frac{\log x}{x^\sigma}\right).$$

COROLLARY.

$$s\int_1^x \frac{H(t)\log t}{t^{s+1}}\,dt$$

$$= \begin{cases} \dfrac{\zeta(s)\zeta'(s+1)}{\zeta^2(s+1)} - \dfrac{\zeta'(s)}{\zeta(s+1)} + \dfrac{\zeta(s)}{s\zeta(s+1)} - \dfrac{s}{(s-1)^2\zeta(2)}, & \sigma > 0, s \neq 1 \\ \dfrac{1}{\zeta(2)}\left(\dfrac{\zeta''(2)\zeta(2) - 2\zeta'(2)^2}{\zeta(2)^2} + \gamma\dfrac{\zeta'}{\zeta}(2) + \gamma_1\right) & \\ \qquad + \dfrac{1}{\zeta(2)}\left(\gamma - 1 - \dfrac{\zeta'}{\zeta}(2)\right), & s = 1 \\ \displaystyle\int_1^x \frac{H(t)}{t^{s+1}}\,dt, & \sigma = 0 \\ 0, & \sigma < 0 \end{cases}$$

$$+ O(x^{-\sigma}\log x).$$

In particular,

$$\int_1^x \frac{H(t)\log t}{t}\,dt = O(\log x). \tag{20}$$

LEMMA 4. *For complex s,*

$$\sum_{n\le x}\frac{\phi(n)}{n^{s+1}} = \begin{cases} \dfrac{\zeta(s)}{\zeta(s+1)} - \dfrac{1}{(s-1)\zeta(2)x^{s-1}}, & s \neq 1 \\ \dfrac{1}{\zeta(2)}\left(\log x + \gamma - \dfrac{\zeta'}{\zeta}(2)\right), & s = 1 \end{cases}$$

$$+ x^{-s}H(x) + O(x^{-\sigma}),$$

and

$$s\int_1^\infty \frac{H(t)}{t^{s+1}}\,dt = \frac{\zeta(s)}{\zeta(s+1)} - \frac{s}{(s-1)\zeta(2)} \quad (s \neq 1), \tag{21}$$

$$\int_1^\infty \frac{H(t)}{t^2}\,dt = \frac{1}{\zeta(2)}\left(\gamma - 1 - \frac{\zeta'}{\zeta}(2)\right), \tag{22}$$

and

$$\int_x^\infty \frac{H(t)}{t^{s+1}}\,dt = O(x^{-\sigma}), \quad \sigma > 0$$

$$\int_1^x \frac{H(t)}{t^{s+1}}\,dt = O(x^{-\sigma}), \quad \sigma \le 0.$$

LEMMA 5. *Define the arithmetic function $t(k)$ by $t(k) := \sum_{d|k}((\mu(d))/d)(\log(k)/d)$, and let $T(x) = \sum_{k\le x} t(k)$. Then we have as $x \to \infty$*

$$T(x) = \frac{x}{\zeta(2)}\left\{\log x - 1 - \frac{\zeta'(2)}{\zeta(2)}\right\} - D(x) + \frac{1}{2} + O(\delta(x))$$
$$= \frac{x}{\zeta(2)}\left\{\log x - 1 - \frac{\zeta'(2)}{\zeta(2)}\right\} - D(x) + O(1),$$

where $D(x)$ *is defined in Theorem* 3.

PROOF. Rewriting the 2-dimensional sum into a double sum and noting that $\phi(n) = \sum_{d|n} \mu(d)(n/d)$, we may write

$$T(x) = S_1 - S_2, \tag{23}$$

where

$$S_1 = \sum_{n \le x} \frac{\phi(n)}{n} \log n, \quad S_2 = \sum_{d\delta \le x} \frac{\mu(d)}{d} \log d.$$

To treat S_1, we apply the refinement of Pan Cheng Dong's (Pan Cheng Tung's) result [**23**]

$$\sum_{n \le x} \frac{\phi(n)}{n} \log \frac{x}{n} = \frac{x}{\zeta(2)} + O(\delta(x)).$$

The left-hand side is

$$(\log x) \sum_{n \le x} \frac{\phi(n)}{n} - \sum_{n \le x} \frac{\phi(n)}{n} \log n$$
$$= \log x \left(\frac{x}{\zeta(2)} + H(x)\right) - S_1,$$

whence

$$S_1 = \frac{x}{\zeta(2)}(\log x - 1) + H(x) \log x + O(\delta(x)).$$

Comparing (7) and (11), we have

$$H(x) = -U(x) + O(\delta(x)), \tag{24}$$

whence

$$S_1 = \frac{x}{\zeta(2)}(\log x - 1) - U(x) \log x + O(\delta(x)). \tag{25}$$

On the other hand,

$$S_2 = \sum_{d \le x} \frac{\mu(d)}{d} \left[\frac{x}{d}\right] \log d = \sum_{d \le x} \frac{\mu(d)}{d} \left(\frac{x}{d} - \frac{1}{2} - \bar{B}_1\left(\frac{x}{d}\right)\right) \log d \tag{26}$$
$$= x \sum_{d \le x} \frac{\mu(d)}{d^2} \log d - \frac{1}{2} \sum_{d \le x} \frac{\mu(d)}{d} \log d - \sum_{d \le x} \frac{\mu(d)}{d} \bar{B}_1\left(\frac{x}{d}\right) \log d.$$

Applying

$$\sum_{d \le x} \frac{\mu(d)}{d} \log d = -1 + O(\delta(x)), \tag{27}$$

$$\sum_{d \le x} \frac{\mu(d)}{d^2} \log d = \frac{\zeta'(2)}{\zeta(2)^2} + O\left(\frac{\delta(x)}{x}\right), \tag{28}$$

we conclude that

$$S_2 = \frac{\zeta'(2)}{\zeta(2)^2} x - \sum_{d \le x} \frac{\mu(d)}{d} \bar{B}_1\left(\frac{x}{d}\right) \log d + \frac{1}{2} + O(\delta(x)). \tag{29}$$

Putting together, we conclude the assertion.

For the sake of completeness, we give a proof of (24). Similarly to (27), (28), we have a form of the prime number theorem

$$\sum_{d \le x} \frac{\mu(d)}{d} = O(\delta(x)), \tag{30}$$

$$\sum_{d \le x} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O\left(\frac{\delta(x)}{x}\right). \tag{31}$$

Hence

$$\begin{aligned}
\sum_{n \le x} \frac{\phi(n)}{n} &= \sum_{d\delta \le x} \frac{\mu(d)}{d} = \sum_{d \le x} \frac{\mu(d)}{d} \sum_{\delta \le x/d} 1 = \sum_{d \le x} \frac{\mu(d)}{d} \left[\frac{x}{d}\right] \\
&= -\sum_{d \le x} \frac{\mu(d)}{d} \left(\bar{B}_1\left(\frac{x}{d}\right) - \frac{x}{d} + \frac{1}{2}\right) \\
&= -U(x) + x \sum_{d \le x} \frac{\mu(d)}{d^2} - \frac{1}{2} \sum_{d \le x} \frac{\mu(d)}{d} \\
&= \frac{x}{\zeta(2)} - U(x) + O(\delta(x)),
\end{aligned}$$

and (24) follows. □

As a well-known generalization of Euler's function ($\phi(n) = J_1(n)$), we introduce Jordan's totient function

$$J_k(n) = \sum_{d \mid n} \mu(d) \left(\frac{n}{d}\right)^k$$

($= n^k \prod_{p \mid n} (1 - p^{-k})$ = the number of ordered sets of k positive integers such that the g.c.d. of them is relatively prime to n).

Lemma 6. *For complex* s *and* $k \in \boldsymbol{R}$ *satisfying either* (i) $2 \le k < \sigma - 1$ *or* (ii) $\sigma - 1 \le k \le 0$ *it holds that*

(i) $$\sum_{n \le x} n^{-s} J_k(n) = \frac{x^{k-s+1}}{(k-s+1)\zeta(k+1)} + \frac{\zeta(s-k)}{\zeta(s)} + c_k^{(1)}(x) x^{k-s}$$

$$+ \begin{cases} O(x^{1-\sigma} \log x), & k = 2 \\ \dfrac{k-s}{2} c_{k-1}^{(2)}(x) x^{k-s-1} + O(x^{1-\sigma} \log^\theta x), & 2 < k \le 3 \\ \dfrac{k-s}{2} c_{k-1}^{(2)}(x) x^{k-s-1} + O(x^{k-1-\sigma}), & 3 < k, \end{cases}$$

where θ is 1 or 0 according as $k = 3$ or not.

In particular, for $k \ge 2$

$$\sum_{n \le x} J_k(n) = \frac{x^{k+1}}{(k+1)\zeta(k+1)} + c_k^{(1)}(x) x^k$$

$$+ \begin{cases} O(x \log x), & k = 2 \\ \dfrac{k}{2} c_{k-1}^{(2)}(x) x^{k-1} + O(x \log^\theta x), & 2 < k \le 3 \\ \dfrac{k}{2} c_{k-1}^{(2)}(x) x^{k-1} + O(x^{k-1}), & 3 < k. \end{cases}$$

(ii) $$\sum_{n \le x} n^{-s} J_k(n) = O(x^{1-\sigma}).$$

The special case of (i) gives an improvement of Jarden's formula [**13**]. Of course, in other cases, we can obtain asymptotic formulas. The case of $s = 1$ reduces to Lemma 4. If e.g. $0 < \sigma - 1 \le k < 1$, then

$$\sum_{n \le x} n^{-s} J_k(n) = \begin{cases} \dfrac{x^{k-s+1}}{(k-s+1)\zeta(k+1)} + \dfrac{\zeta(s-k)}{\zeta(s)}, & k - s \ne -1 \\ \dfrac{1}{\zeta(s)} (\log x + \gamma) - \dfrac{\zeta'(s)}{\zeta^2(s)}, & k - s = -1 \end{cases}$$

$$+ O(x^{1-\sigma}).$$

Lemma 7. *For a, $b \in \boldsymbol{R}$, let*

$$S^{(a,b)}(x) = \sum_{r > x}^{\infty} r^{-a} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} k^{-b}.$$

Then for $a > 1$,

$$S^{(a,1)}(x) = \sum_{r>x}^{\infty} r^{-a} \sum_{\substack{k=1\\(k,r)=1}}^{r} \frac{1}{k} \tag{32}$$

$$= \frac{1}{(a-1)\zeta(2)x^{a-1}}\left(\log x + \gamma + \frac{1}{a-1} - \frac{\zeta'}{\zeta}(2)\right)$$

$$+ \frac{U(x)}{x^a}\log x + O\left(\frac{\log x}{x^a}\right),$$

and for $a > 1,\ b \ge 2$

$$S^{(a,b)}(x) = \frac{\zeta(b)}{(a-1)\zeta(b+1)x^{a-1}} + \frac{1}{x^a}\left(\zeta(b)c_b^{(1)}(x) - \frac{\alpha}{a\zeta(2)}\right) \tag{33}$$

$$+ \frac{1-\alpha}{2x^{a+1}}\left(a\zeta(b)c_{b-1}^{(2)}(x) - \frac{(b-2)\beta}{(a+1)\zeta(2)}\right) + O\left(\frac{\log^\beta x}{x^{a+2-\alpha}}\right),$$

where $\alpha = [2/a]$ *and* $\beta = [3/b]$.

Proof. Since

$$S^{(a,b)}(x) = \sum_{r>x} r^{-a} L^*_{-b}(r)$$

we substitute from Corollary to Lemma 1 to get

$$S^{(a,b)}(x) = \begin{cases} \displaystyle\sum_{r>x}\frac{t(r)}{r^a} + \gamma\sum_{r>x}\frac{\phi(r)}{r^{a+1}}, & b = 1 \\[2ex] \displaystyle\zeta(b)\sum_{r>x} r^{-a-b}J_b(r) - \frac{1}{b-1}\sum_{r>x}\frac{\phi(r)}{r^{a+b}}, & b \ne 1 \end{cases}$$

$$+ O\left(\sum_{r>x}\frac{\sigma_1(r)}{r^{a+b}}\right).$$

Applying Lemmas 4–6 as well as Corollary to Lemma 3 to respective sums and the trivial estimate $\sum_{n\le x}\sigma_1(n) = O(x^2)$ completes the proof. $\square$

Lemma 8. *For* $0 \le j \in \boldsymbol{Z}$ *let*

$$A_j(x) = 2\binom{m+j}{j+1}\sum_{r=2}^{[x]}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}}.$$

Then for $m \ge 2$

(i) $\displaystyle s_m(x) = 1 - \sum_{j=0}^{m-2} A_j(x),$

and

$$\text{(ii)} \quad \lim_{x\to\infty} A_j(x) = 2\binom{m+j}{j+1}\sum_{r=2}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{r^{m+j+1}k^{m-j-1}}.$$

PROOF. We apply Formula (2) with $f(x,y) = (xy)^{-m}$ to get

$$s_m(x) = 1 - {\sum}'\sum_{j=1}^{m-1}\binom{m}{j}\frac{1}{r^m k^{m-j}(r-k)^j} = 1 - s_m'(x),$$

say, where $\sum'$ stands for the double sum extended over $2 \le r \le x$, $1 \le k \le r$ with $(k,r) = 1$.

Now, putting for $0 \le i \le n-2$,

$$C_i(x) = {\sum}'\sum_{j=1}^{m-i-1}\frac{\binom{m+i}{j+i}}{r^{m+i}k^{m-j-i}(r-k)^j},$$

we see that

$$C_i(x) = A_i(x) + C_{i+1}(x).$$

Since $s_m'(x) = C_0(x)$ and $C_{m-2}(x) = A_{m-2}(x)$, it follows from this that

$$s_m'(x) = \sum_{j=0}^{m-2} A_j(x),$$

thereby proving our assertion. □

LEMMA 9 (Relatively prime version). *Suppose* $f(tx,ty) = t^\alpha f(x,y)$ *for* $\alpha < -1$. *Then*

$$\sum_{r=1}^{\infty}\sum_{k=1}^{r} f(k,r) = \zeta(-\alpha)\sum_{r=1}^{\infty}\sum_{\substack{k=1\\(k,r)=1}}^{r} f(k,r),$$

the series converging simultaneously.

LEMMA 10 (Reciprocity relation). *For* $\sigma > \max\{1, 2-\Re z\}$, *let*

$$H(s,z) = \sum_{r=1}^{\infty} r^{-s}\sum_{k=1}^{r} k^{-z}.$$

Then

(i) $H(s,z) + H(z,s) = \zeta(s)\zeta(z) + \zeta(s+z)$.

Similarly, defining

$$H^*(s,z) = \sum_{r=1}^{\infty} r^{-s}\sum_{\substack{k=1\\(k,r)=1}}^{r} k^{-z} = \frac{1}{\zeta(s+z)}H(s,z)$$

H^* *satisfies* (i)* *with the right-hand side divided by* $\zeta(s+z)$.

Lemma 11. *Let $a \geq 2$ be an integer. Then*

(i) $$2\sum_{r=1}^{\infty}\frac{1}{r^a}\sum_{k=1}^{r}\frac{1}{k} = (a+2)\zeta(a+1) - \sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1),$$

or equivalently

(i)* $$2\sum_{r=1}^{\infty}\frac{1}{r^a}\sum_{\substack{k=1\\(k,r)=1}}^{r}\frac{1}{k} = a+2-\frac{1}{\zeta(a+1)}\sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1).$$

For Lemmas 9, 10, 11 see Sita Rama Chandra Rao and Siva Rama Sarma [**26**], [**27**] and references thereof.

The following Lemmas 12–14 are needed solely for the proof of the estimate for $D(x)$ in Theorem 3, (ii).

Lemma 12 (Lemma 2, [**15**]). *Let $\{a_n\}_{n=1}^{\infty}$ be a complex sequence such that $a_n = O((\log n)^K)$, $K > 0$ and suppose that*

$$\sum_{n \leq x} a_n = O(x\delta(x)).$$

Then

$$\sum_{n \leq x} a_n \left\{\frac{x}{n}\right\}^2 = O(x\delta(x))$$

uniformly in x.

Lemma 13. *Let*

$$X = \left[\frac{1}{80000}(\log x)^{1/3}(\log\log x)^{-4/3}\right].$$

Then for any Q, Q' satisfying

$$Q \leq Q' \leq 2Q, \quad x^{6/X} \leq Q \leq Q' \leq x\exp(-(\log x)^{1/2}),$$

we have

$$\sum_{m=Q}^{Q'} \mu(m)\psi\left(\frac{x}{m}\right) = O(Q(\log x)^{-1}).$$

This follows immediately from Walfisz [**28**], Lemma 4.5.4.

Lemma 14. *For any $a > 0$ and an arithmetical function $f(n)$ we have*

(34) $$\sum_{a<n\leq x} \mu(n)f(n)\left\{\frac{x}{n}\right\} = x\sum_{a<n\leq x}\mu(n)\frac{f(n)}{n} - \sum_{m \leq x/a} M\left(\frac{x}{m}\right)f\left(\frac{x}{m}\right) + \int_a^x \left[\frac{x}{u}\right]M(u)f'(u)\,du + \left[\frac{x}{a}\right]M(a)f(a).$$

In particular,

$$\sum_{n\le x}\mu(n)f(n)\left\{\frac{x}{n}\right\} = x\sum_{n\le x}\mu(n)\frac{f(n)}{n} - \sum_{m\le x} M\left(\frac{x}{m}\right)f\left(\frac{x}{m}\right) + \int_1^x \left[\frac{x}{u}\right] M(u)f'(u)\,du. \tag{35}$$

Also,

$$\begin{aligned}\sum_{a<n\le x}\mu(n)f(n)\bar{B}_1\left(\frac{x}{n}\right) &= x\int_a^x M(u)\frac{f(u)}{u^2}\,du - \int_a^x \bar{B}_1\left(\frac{x}{u}\right)M(u)f'(u)\,du \\ &\quad - \sum_{m\le x/a} M\left(\frac{x}{m}\right)f\left(\frac{x}{m}\right) - \bar{B}_1\left(\frac{x}{a}\right)M(a)f(a) + \frac{1}{2}M(x)f(x).\end{aligned} \tag{36}$$

PROOF. It suffices to get an asymptotic formula for the sum $G = G(x) = \sum_{a<n\le x}\mu(n)f(n)[x/n]$. As in Walfisz [**28**], we transform G in the following manner by noting that $[x/n] = m$ if $x/(m+1) < n \le x/m$:

$$\begin{aligned}G(x) &= \sum_{x/[x/a]<n\le x} + \sum_{a<n\le x/[x/a]} \\ &= \sum_{m\le x/a-1} m \sum_{x/(m+1)<n\le x/m}\mu(n)f(n) + \left[\frac{x}{a}\right]\sum_{a<n\le x/[x/a]}\mu(n)f(n).\end{aligned}$$

Then by partial summation

$$\begin{aligned}G(x) &= \sum_{m\le x/a-1} m\left(M\left(\frac{x}{m}\right)f\left(\frac{x}{m}\right) - M\left(\frac{x}{m+1}\right)f\left(\frac{x}{m+1}\right) - \int_{x/(m+1)}^{x/m} M(u)f'(u)\,du\right) \\ &\quad + \left[\frac{x}{a}\right]\left(M\left(\frac{x}{[x/a]}\right)f\left(\frac{x}{[x/a]}\right) - M(a)f(a) - \int_a^{x/[x/a]} M(u)f'(u)\,du\right) \\ &= \sum_{m\le x/a-1} M\left(\frac{x}{m}\right)f\left(\frac{x}{m}\right)(m-(m-1)) - \left[\frac{x}{a}\right]M(a)f(a) \\ &\quad - \sum_{m\le x/a-1}\int_{x/(m+1)}^{x/m}\left[\frac{x}{u}\right]M(u)f'(u)\,du - \int_a^{x/[x/a]}\left[\frac{x}{u}\right]M(u)f'(u)\,du,\end{aligned}$$

whence we infer that

$$G(x) = \sum_{m\le x/a} M\left(\frac{x}{m}\right)f\left(\frac{x}{m}\right) - \left[\frac{x}{a}\right]M(a)f(a) - \int_a^x\left[\frac{x}{u}\right]M(u)f'(u)\,du. \tag{37}$$

Now Formula (34) follows from Formula (37) on noting that $\{x/n\} = x/n - [x/n]$, and Formula (35) follows from Formula (34) on taking a between 0 and 1.

Formula (36) follows from Formula (34) by expressing the LHS as

$$\sum_{a<n\le x} \mu(n) f(n) \left\{\frac{x}{n}\right\} - \frac{1}{2} \sum_{a<n\le x} \mu(n) f(n)$$

and substituting partial summation formulas for $\sum_{a<n\le x} \mu(n) f(n)/n$ and $\sum_{a<n\le x} \mu(n) \cdot f(n)$. □

3. Proofs of theorems.

PROOF OF LEHNER-NEWMAN'S FIRST AND SECOND SUM FORMULAS. Lehner and Newman's original proof of their sum formulas (2) and (3) depend on geometrical consideration. For the sake of completeness, we give here a proof of (2) and (3) based on the following (algebraic) expression for Q_x, thus avoiding any appeal to geometry:

$$Q_x = \{(a,b) | 1 \le a, b \le x; (a,b) = 1; a+b > x\}. \tag{38}$$

(cf. also the recent paper of Kargaev and Zhigljavsky [**17**])

Thus for the integral variable $r \ge 1$

$$s_f(r) = \sum_{\substack{1\le a,b\le r\\(a,b)=1\\a+b\ge r+1}} f(a,b).$$

Extracting from $s_f(r)$ the extremal terms with a or b equal to r, we are left with the sum over $1 \le a, b \le r-1$, which is the same as $s_f(r-1)$ with those terms with $a+b=r$ subtracted. Hence $s_f(r) - s_f(r-1)$ gives exactly the inner sum on the RHS of (2). Adding these resulting equalities for $r = 2, 3, \dots, [x]$ and noting that $s_f(1) = f(1,1)$ and $s_f(0) = 0$, we get (2).

Formula (3) is a simple transformation of the 2-dimensional sum over the region (38) into a double sum. □

PROOF OF THEOREM 1. Since, for $m \ge 2$, $s_m(x) = o(1)$, it follows from Lemma 8 that

$$s_m(x) = \sum_{j=0}^{m-2} \sum_{r>x} \sum_{\substack{k<r\\(k,r)=1}} \frac{2\binom{m+j}{j+1}}{r^{m+j+1} k^{m-j-1}} \tag{39}$$

(note that we have proved altogether Formula (18)). Now (39) can be rewritten as

$$s_m(x) = 2\binom{2m-2}{m-1} S^{(2m-1,1)}(x) + \sum_{j=0}^{m-3} \left\{ 2\binom{2m+j}{j+1} S^{(m+j+1,m-j-1)}(x) \right\}$$

and so formulas (i)–(iv) can be readily read off from this and Lemma 7. □

PROOF OF THEOREM 2. For $0 < \xi \le 1$ define

$$E(\xi; x) = \sum_{\rho_\nu \le \xi} 1 - \xi \Phi(x).$$

Then we see easily that for $k \in \boldsymbol{N}$

$$\begin{aligned}\int_0^1 |E(\xi; x)|^{2k} \, d\xi &= \frac{1}{(2k+1)\Phi(x)} \sum_{g=0}^{k} \binom{2k+1}{2g} s_{2g,0} \Phi(x)^{2g} \\ &= \Phi(x)^{2k-1}(s_{2k,0} + O(x^{-2k})).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^1 |E(\xi; x)|^{2k} \, d\xi &\ge \int_0^{1/2x} \\ &= \int_0^{1/2x} (-\xi\Phi(x))^{2k} \, d\xi \\ &= \frac{1}{(2k+1)2^{2k+1}} \frac{\Phi(x)^{2k}}{x^{2k+1}} \\ &= \Omega_+(x^{2k-1}),\end{aligned}$$

whence

$$s_{2k,0} = \Omega_+(x^{1-2k}). \qquad \square$$

PROOF OF THEOREM 3. By Lemma 2,

$$\sum_{\nu=1}^{\Phi(x)} \rho_\nu^{-a} = \sum_{n \le x} M\left(\frac{x}{n}\right) n^a L_{-a}(n).$$

Hence, substituting from Lemma 1, (i) with $l = 2$, we have

$$\begin{aligned}\sum_{\nu=1}^{\Phi(x)} \rho_\nu^{-a} &= \begin{cases} \zeta(a) S_{-a}(x) - \dfrac{1}{a-1} S_{-1}(x), & a \ne 1 \\ -S'_{-1}(x) + \gamma S_{-1}(x), & a = 1 \end{cases} + \frac{1}{2} S_0(x) \\ &\quad - \frac{a}{12} S_1(x) + \frac{a(a+1)}{2} \sum_{n \le x} M\left(\frac{x}{n}\right) \frac{1}{n} \int_1^\infty \bar{B}_2(nt) t^{-a-2} \, dt \\ &= \begin{cases} \zeta(a) S_{-a}(x) - \dfrac{1}{a-1} \Phi(x), & a \ne 1 \\ -S'_{-1}(x) + \gamma \Phi(x), & a = 1 \end{cases} + O(x\delta(x)),\end{aligned}$$

on using (16) and estimating the sum $\sum_{n \le z} M(x/n)(1/n)$ by dint of (9) (first we need to divide it into subsums), where $S_{-a}(x)$ is defined in (15) and

$$S'_{-1}(x) = -\sum_{n \le x} M\left(\frac{x}{n}\right) n \log n.$$

(i) $S_{-a}(x)$ is easy to treat. Indeed, using Lemma 2 and Lemma 1, (i) with $l = 1$ or $l > [\Re a] + 1$ as the case may be, we deduce

$$\begin{aligned}
S_{-a}(x) &= \sum_{n \le x} \mu(n) L_a\left(\frac{x}{n}\right) \\
&= \frac{x^{a+1}}{a+1} \sum_{n \le x} \frac{\mu(n)}{n^{a+1}} - x^a \sum_{n \le x} \frac{\mu(n)}{n^a} \bar{B}_1\left(\frac{x}{n}\right) \\
&\quad + \begin{cases} O(x^{\Re a - 1}), & \Re a > 2 \\ O(x \log x), & \Re a = 2 \\ O(x), & 0 < \Re a < 2, \end{cases}
\end{aligned}$$

whence, transforming the finite sums into series, (i) follows.

(ii) Treating $S'_{-1}(x)$ similarly, we rewrite

$$-S'_{-1}(x) = \sum_{n \le x} \mu(n) M_1\left(\frac{x}{n}\right)$$

and apply Lemma 1, (ii) with $l = 3$ with error estimate to get

$$-S'_{-1}(x) = \frac{x^2}{2\zeta(2)}\left(\log x - \frac{\zeta'}{\zeta}(2) - \frac{1}{2}\right) - xD(x) + O(x),$$

after transforming finite sums into infinite series.

This only gives the trivial error term $O(x)$ and not the final form of assertion (ii), although it covers Mikolás' Theorem 4 by using the trivial estimate $D(x) = O(\log^2 x)$.

To prove the final form of the theorem we use Lemma 2 with $V_\xi(n) = n \log n$ to get

$$\begin{aligned}
-S'_{-1}(x) &= \sum_{n \le x} \sum_{d \mid n} \mu(d) \frac{n}{d} \log \frac{n}{d} \\
&= \sum_{n \le x} n t(n) \\
&= x \sum_{n \le x} t(n) - \sum_{n \le x} (x - n) t(n) \\
&= xT(x) - \int_1^x T(u)\, du.
\end{aligned} \tag{40}$$

Form now on we use the decomposition (23) and calculate the integrals $\int S_1\, du$ and $\int S_2\, du$ separately.

First we treat

$$\int S_1\, du = \sum_{n \le x} \frac{\phi(n) \log n}{n} (x - n).$$

This can be written as

$$\int S_1\,du = xS_1 + \Sigma_1 - \Phi(x)\log x, \tag{41}$$

where

$$\Sigma_1 := \sum_{n\le x}\phi(n)\log\frac{x}{n} = \int_1^x \frac{\Phi(u)}{u}\,du.$$

In a similar way as we prove Formula (24), we can prove that

$$\Phi(x) = \frac{x^2}{2}\sum_{n\le x}\frac{\mu(n)}{n^2} - x\sum_{n\le x}\frac{\mu(n)}{n}\bar{B}_1\left(\frac{x}{n}\right) + \frac{1}{2}\sum_{n\le x}\mu(n)\left(\bar{B}_2\left(\frac{x}{n}\right) - \frac{1}{6}\right),$$

whence follows in the first place that

$$\Phi(x) = \frac{1}{2\zeta(2)}x^2 - xU(x) + O(x\delta(x)), \tag{42}$$

or Formula (11) on using the estimate (to be proved later)

$$\sum_{n\le x}\mu(n)\bar{B}_2\left(\frac{x}{n}\right) = O(x\delta(x)), \tag{43}$$

and in the second place that

$$\Sigma_1 = \frac{1}{2}\sum_{n\le x}\frac{\mu(n)}{n^2}\left(\frac{x^2}{2} - \frac{n^2}{2}\right) - \sum_{n\le x}\mu(n)\int_n^x \bar{B}_1\left(\frac{u}{n}\right)\frac{du}{n} + \frac{1}{2}\sum_{n\le x}\mu(n)\int_n^x\left(\bar{B}_2\left(\frac{u}{n}\right) - \frac{1}{6}\right)\frac{du}{u}. \tag{44}$$

The first term on the RHS of (44) can be rewritten as

$$\frac{x^2}{4\zeta(2)} - \frac{1}{4}M(x) + O(x\delta(x)),$$

while the second becomes

$$-\frac{1}{2}\sum_{n\le x}\mu(n)\bar{B}_2\left(\frac{x}{n}\right) + \frac{1}{12}M(x),$$

which is $O(x\delta(x))$ on account of (43).

Changing the order of summation and integration, the third term on the RHS of (44) becomes

$$\frac{1}{2}\int_1^x M(u)\left(\bar{B}_2\left(\frac{x}{u}\right) - \frac{1}{6}\right)\frac{du}{u},$$

to which we apply the estimate

$$\int_1^x \frac{|M(u)|}{u}\,du = O(x\delta(x)) \tag{45}$$

to obtain again the same order of error term $O(x\delta(x))$.

Estimate (45) follows immediately on dividing the range of integration into $[1, \sqrt{x}]$ and $[\sqrt{x}, x]$ and applying (9) to the latter.

Thus, (44) simplifies to

$$\Sigma_1 = \frac{x^2}{4\zeta(2)} + O(x\delta(x)). \tag{46}$$

Substituting (46) into (41), we deduce that

$$\int_1^x S_1\, du = xS_1 - \frac{x^2}{2\zeta(2)}(\log x - 1) + x \log x U(x) + O(x\delta(x)). \tag{47}$$

It remains to prove (43). Writing

$$\sum_{n \le x} \mu(n) \bar{B}_2\left(\frac{x}{n}\right) = \sum_{n \le x} \mu(n) \left\{\frac{x}{n}\right\}^2 - \sum_{n \le x} \mu(n) \left\{\frac{x}{n}\right\} + \frac{1}{6} M(x),$$

we see that the first and the third term can be estimated by Lemma 12 and (9), respectively. By Lemma 14, (35) with $f(n) = 1$ and $\sum_{m \le x} M(x/m) = S_0(x) = 1$ (by (16)), we see that the second sum is

$$\sum_{n \le x} \mu(n) \left\{\frac{x}{n}\right\} = x \sum_{n \le x} \frac{\mu(n)}{n} - 1 = O(x\delta(x))$$

by (30). This proves (43).

Similarly, we integrate both sides of (26) to get

$$\int_1^x S_2\, du = \frac{x^2}{2} \sum_{n \le x} \frac{\mu(n)}{n^2} \log n - \frac{x}{2} \sum_{n \le x} \frac{\mu(n)}{n} \log n - \frac{1}{2}\Sigma_2 + \frac{1}{12}\Sigma_3, \tag{48}$$

where

$$\Sigma_2 = \sum_{n \le x} \mu(n) \log n \bar{B}_2\left(\frac{x}{n}\right), \quad \Sigma_3 = \sum_{n \le x} \mu(n) \log n.$$

First, by partial summation and (9) we have

$$\Sigma_3 = O(x\delta(x)). \tag{49}$$

Secondly, we apply the similar argument that we applied to prove (43). Namely, we apply Lemma 12 to the first and (49) to the third term, respectively, of the expression

$$\Sigma_2 = \sum_{n \le x} \mu(n) \log n \left\{\frac{x}{n}\right\}^2 - \sum_{n \le x} \mu(n) \log n \left\{\frac{x}{n}\right\} + \frac{1}{6} \sum_{n \le x} \mu(n) \log n$$

to get

$$\Sigma_2 = -\Sigma_4 + O(x\delta(x)), \tag{50}$$

where Σ_4 denotes the 2nd term.

Since

$$-\sum_{m\le x} M\left(\frac{x}{m}\right)\log\frac{x}{m} = -\log x + \sum_{m\le x} M\left(\frac{x}{m}\right)\log m \tag{51}$$

$$= -\log x + \sum_{n\le x}\Lambda(n),$$

we deduce from Lemma 14, (35) with $f(n) = \log n$ that

$$\Sigma_4 = x\sum_{n\le x}\frac{\mu(n)}{n}\log n - \log x + \sum_{n\le x}\Lambda(n) + \int_1^x \left[\frac{x}{u}\right]\frac{M(u)}{u}\,du. \tag{52}$$

We express the last integral as

$$x\int_1^x \frac{M(u)}{u^2}\,du - \int_1^x \left\{\frac{x}{u}\right\}\frac{M(u)}{u}\,du. \tag{53}$$

Since

$$\int_1^x \frac{M(u)}{u^2}\,du = x\sum_{n\le x}\frac{\mu(n)}{n} - M(x) = O(x\delta(x))$$

by (30), and the second integral in (53) is $O(x\delta(x))$ by (45), it follows that the integral in (52) is $O(x\delta(x))$. Hence, putting all together, we conclude from (27) and a form of the prime number theorem for the von Mangoldt function $\Lambda(n)$ that

$$\Sigma_4 = -x + x + O(x\delta(x)) = O(x\delta(x)),$$

so that by (50), $\Sigma_2 = O(x\delta(x))$.

Substituting this and (49) into (48), and using (27) and (28), we infer that

$$\int_1^x S_2\,du = \frac{\zeta'(2)}{2\zeta^2(2)}x^2 + \frac{x}{2} + O(x\delta(x)). \tag{54}$$

Hence, substituting (25), (29), (47) and (54) into

$$-S'_{-1}(x) = xS_1 - xS_2 - \int_1^x S_1\,du + \int_1^x S_2\,du,$$

we conclude that

$$-S'_{-1}(x) = \frac{x^2}{2\zeta(2)}\left(\log x - \frac{\zeta'}{\zeta}(2) - \frac{1}{2}\right) - xD(x) + O(x\delta(x)). \tag{55}$$

Finally, we substitute (42) and (55) into

$$\sum_{\nu=1}^{\Phi(x)}\rho_\nu^{-1} = -S'_1(x) + \gamma\Phi(x) + O(x\delta(x)),$$

to complete the proof.

It remains to establish the estimate of $D(x)$. We shall do this, following Walfisz's argument [**28**].

To make the dependence of our proof on Walfisz's result explicit, we use the notation $\psi(u)$ for the present defined by

$$\psi(u) = \begin{cases} \bar{B}_1(u) & \text{if } u \in \boldsymbol{R} \backslash \boldsymbol{Z} \\ 0 & \text{if } u \in \boldsymbol{Z}. \end{cases}$$

Then $D(x)$ differs from $D_1(x) = \sum_{n \le x} \psi(x/n)(\mu(n)/n) \log n$ only when x is an integer, in which case, however, the difference is in absolute value not greater than

$$\sum_{\substack{p|x \\ p\,\text{prime}}} \frac{\log p}{p-1} \le \sum_{p|x} \log p \le \log x.$$

This difference being negligible, we may as well consider $D_1(x)$ for $D(x)$. For simplicity we write $D(x)$ for $D_1(x)$:

$$D(x) = \sum_{n \le x} \psi\left(\frac{x}{n}\right) \frac{\mu(n)}{n} \log n. \tag{56}$$

We divide the sum over $1 \le n \le x$ into three parts $1 \le n \le Q_0$, $Q_0 < n \le R$, $R < n \le x$, where

$$Q_0 = Q_0(x) = x^{6/X}, \quad R = R(x) = x \exp(-\sqrt{\log x}),$$

X being defined in Lemma 13.

Then the first sum is in absolute value

$$\begin{aligned} O\left(\sum_{n \le Q_0} \frac{\log n}{n}\right) &= O((\log Q_0)^2) \\ &= O((\log x)^{4/3} (\log\log x)^{8/3}), \end{aligned}$$

which gives the error term stated in the theorem.

We are thus left with two sums $D_2(x)$ and $D_3(x)$ to estimate, where

$$D_2(x) = \sum_{Q_0 < n \le R} \psi\left(\frac{x}{n}\right) \frac{\mu(n)}{n} \log n$$

$$D_3(x) = \sum_{R < n \le x} \psi\left(\frac{x}{n}\right) \frac{\mu(n)}{n} \log n.$$

We apply to $D_2(x)$ the standard technique of expressing the sum as the union of subsums of length $2Q_0$.

Let κ be the largest integer satisfying

$$2^\kappa Q_0 \le R,$$

i.e.

$$\kappa = [\log_2(Q_0^{-1} R)] = O(\log x).$$

Then we may express $D_2(x)$ as κ subsums of the form $\sum_{2^k Q_0 \le m < 2^{k+1} Q_0}$, $k = 0, \dots, \kappa$, where $2^{\kappa+1} Q_0$ is to be replaced by R.

Each of three subsums being of the form

$$\sum_{Q \le m < Q'} \mu(m) \psi\left(\frac{x}{m}\right) \frac{\log m}{m}, \quad Q \le Q' \le 2Q,$$

we apply the estimate

$$\sum_{Q \le m < Q'} \mu(m) \psi\left(\frac{x}{m}\right) \frac{\log m}{m} = O\left(\frac{\log Q}{\log x}\right),$$

which follows from Lemma 13 by partial summation.

Then

$$\begin{aligned} D_2(x) &= O\left((\log x)^{-1} \sum_{k=0}^{\kappa-1} (\log 2^k Q_0) + (\log x)^{-1} \log 2^\kappa Q_0\right) \\ &= O((\log x)^{-1} \kappa \log Q_0 + (\log x)^{-1} \kappa^2) \\ &= O(\log x). \end{aligned}$$

Now we use Lemma 14 to supersede the trivial bound $O((\log x)^{3/2})$ for $D_3(x)$.

Formula (36) of Lemma 14 with $a = R = x \exp(-\sqrt{\log x})$ and $f(t) = (\log t)/t$ states that

$$\begin{aligned} &\sum_{R < n \le x} \mu(n) \bar{B}_1\left(\frac{x}{n}\right) \frac{\log n}{n} \\ &\quad = x \int_R^x M(u) \frac{\log u}{u^3}\, du - \int_R^x \bar{B}_1\left(\frac{x}{u}\right) M(u) \frac{1 - \log u}{u^2}\, du \\ &\qquad - \frac{1}{x} \sum_{m \le x/R} M\left(\frac{x}{m}\right) m \log \frac{x}{m} - \bar{B}_1\left(\frac{x}{R}\right) M(R) \frac{\log R}{R} \\ &\qquad + \frac{1}{2} M(x) \frac{\log x}{x}. \end{aligned}$$

Applying Estimate (9) to each term, we see that

$$D_3(x) = O\left(\frac{x}{R} \delta(R) \log R\right) + O(\delta(R) \log^2 x).$$

By $x/R = \exp(\sqrt{\log x})$, $\log R = O(\log x)$, we conclude that the first term is

$$O(\exp((\log x)^{1/2} - c(\log x)^{0.6} (\log \log x)^{-0.2})) = O(\delta(x)),$$

and so is the second term, completing the proof. □

PROOF OF THEOREM 5. We take a closer look at the error term $R_2 = R_2(x)$ of the Euler-Maclaurin formula

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) = n\int_0^1 f(t)\,dt + \frac{1}{2}(f(1)+f(0)) + \frac{B_2}{2}\frac{1}{n}(f'(1)-f'(0)) + R_2,$$

where

$$R_2 = R_2(x) = -\frac{1}{4!n^3}\int_0^1 \phi_4(nt)f^{(4)}(t)\,dt,$$

with

$$\phi_4(t) = \bar{B}_4(t) - B_4.$$

We can express R_2 corresponding to three cases stated in our Theorem 5:

(57) $$R_2 = c\frac{B_4}{4!}\frac{\theta}{n^3}(f^{(3)}(1) - f^{(3)}(0)), \quad 0 \le {}^{\exists}\theta \le 1.$$

Then, slightly modifying the argument of proof of Theorem 4 [**29**] we have, with $b(n) = (\mu * b')(n)$

$$\begin{aligned} F(s) &= \zeta(s+1)\left(\frac{f'(1)-f'(0)}{12} - \sum_{n=1}^{\infty}\frac{b(n)}{n^{s+1}}\right) \\ &= \frac{f'(1)-f'(0)}{12}\zeta(s+1) - \sum_{n=1}^{\infty}\frac{b'(n)}{n^{s+1}}, \end{aligned}$$

where

$$b'(n) = \frac{1}{4!n^2}\int_0^1 \phi_4(nt)f^{(4)}(t)\,dt.$$

Using (57) for the estimate of $|b'(n)|$, we obtain

$$\begin{aligned} |F(s)| &\ge \frac{|f'(1)-f'(0)|}{12}|\zeta(s+1)| - \frac{c}{720}|f^{(3)}(1) - f^{(3)}(0)|\zeta(\sigma+3) \\ &> \frac{1}{12}\frac{\zeta(3)}{\zeta(3/2)}|f'(1)-f'(0)| - \frac{c}{720}|f^{(3)}(1) - f^{(3)}(0)|\zeta(7/2) \\ &\ge 0. \end{aligned}$$

Solving the last inequality, we conclude the assertion. □

Deduction of Theorem 4 from Theorem 5. Theorem 4 follows from Theorem 5 with $f(t) = f_\lambda(t) = -\zeta_\lambda(t+1)$. The constant λ_0 is the maximum of λ for which the imposed inequality holds.

Proof of Theorem 6. By Lehner-Newman's first sum formula with $f(x,y) = x^{-a}y^{-1}$, we have

$$\begin{aligned} \sum_{v=1}^{\Phi(x)-1} c_v^a c_{v+1} &= s_f(x) \\ &= 1 + \sum_{2\le r\le x}\sum_{\substack{k=1 \\ (k,r)=1}}^{r}\left(\frac{1}{k^a r} + \frac{1}{r^a(r-k)} - \frac{1}{k^a(r-k)}\right), \end{aligned}$$

whence, rewriting the summand as $-\sum_{i=1}^{a-1} r^{-i-1}k^{i-a}$, we deduce that

$$\sum_{\nu=0}^{\Phi(x)-1} c_\nu^a c_{\nu+1} = a - \sum_{i=1}^{a-1}\sum_{r\le x}\frac{1}{r^{i+1}}L_{i-a}^*(r),$$

where

$$L_{i-a}^*(r) = \sum_{\substack{k=1\\(k,r)=1}}^{r} k^{i-a}$$

is the sum in Corollary to Lemma 1. Using (38), we see that

$$\begin{aligned} s_f(x) &\le \sum_{\substack{1\le k, r\le x\\ k+r>x}} k^{-2}r^{-1} \\ &= O\left(\sum_{r\le x}\frac{1}{r([x]+1-r)}\right) \\ &= O\left(\frac{\log x}{x}\right), \end{aligned}$$

$$\sum_{\nu=0}^{\Phi(x)-1} c_\nu^a c_{\nu+1} = O\left(\frac{\log x}{x}\right),$$

and hence that

$$\sum_{\nu=0}^{\Phi(x)-1} c_\nu^a c_{\nu+1} = \sum_{i=1}^{a-1}\sum_{r>x}\frac{1}{r^{i+1}}L_{i-a}^*(r).$$

Substituting from Lemma 7 completes the proof. □

The proof of Theorem 7 rests on Lehner-Newman's second sum formula (only the first formula in (ii) rests on the first sum formula) and similar reasonings in this paper with frequent use of Corollary to Lemma 1.

We shall not, however, give a proof of Theorem 7 as it requires another series of lemmas. We shall publish it as well as detailed proofs of (generalizations of) some lemmas in §2 elsewhere.

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