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The homotopy groups of the L_2-localized mod 3 Moore spectrum

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Abstract. At each prime number p, the homotopy groups $\pi_*(L_2S^0)$ of the $v_2^{-1}BP$-localized sphere spectrum play an crucial role to understand the category of $v_2^{-1}BP$-local spectra. For $p > 3$, they are determined by using the Adams-Novikov spectral sequence (ANSS), which collapses in this case.

At the prime 3, $\pi_*(L_2V(1))$ is also determined by using the ANSS, in which $E_\infty = E_{10}$ in this case. Here $V(1)$ denotes the Toda-Smith 4-cells spectrum. In this paper, we determine the homotopy groups $\pi_*(L_2V(0))$ of the mod 3 Moore spectrum from $\pi_*(L_2V(1))$ by the Bockstein spectral sequence (BSS). Actually, we first compute the E_2-term of the ANSS by the BSS and then study the Adams-Novikov differentials, and obtain $E_\infty = E_{10}$ as well.

§1. Introduction.

Let $\mathscr{S}_p$ denote the category of p-local spectra for each prime number p and BP the Brown-Peterson spectrum at p. Then we have the Bousfield localization functor $L_n : \mathscr{S}_p \to \mathscr{S}_p$ with respect to $v_n^{-1}BP$ for the generator v_n of $BP_* = \boldsymbol{Z}_{(p)}[v_i : i > 0]$. The category $L_n\mathscr{S}_p$ is easier to be understood than $\mathscr{S}_p$ itself and reflects some properties of it. $L_n\mathscr{S}_p$ is, in a sense, generated by the L_n-localized sphere spectrum L_nS^0, because $L_nX = X \wedge L_nS^0$ for any spectrum X by the smash product theorem [**10**, Th. 7.5.6]. Besides, we have the chromatic convergence theorem due to Hopkins and Ravenel [**10**, Th. 7.5.7], which says that $\operatorname{holim}_{\overleftarrow{n}} L_nX = X$ for a finite spectrum X. Therefore it is very important to compute the homotopy groups $\pi_*(L_nS^0)$. So far we know the homotopy groups $\pi_*(L_nS^0)$ for $n < 2$ given in [**8**] and for $n = 2$ and $p > 3$ in [**12**]. The next place to study is the case where $n = 2$ and $p = 3$. They are computed by using the Bockstein spectral sequences $\pi_*(L_nV(k)) \Rightarrow \pi_*(L_nV(k-1))$, where $V(n)$ denotes the Toda-Smith spectrum, and is known to exist if $n < 4$ and $p > 2n$ (*cf.* [**9**]). (For $L_nV(k)$, we have some other existence theorems in [**13**] and [**14**].) Note that $V(-1) = S^0$ and $V(0)$ is the mod p Moore spectrum. On the other hand, $\pi_*(L_nV(n-1))$ is computed by Ravenel (*cf.* [**9**]) in case of $n < 4$ and $n < p-1$, by Mahowald [**5**] in case of $n = p-1 = 1$, and by the author [**11**] and Henn and Mahowald [**4**] in case of $n = p-1 = 2$. In this paper we study the Bockstein spectral sequence $\pi_*(L_2V(1)) \Rightarrow \pi_*(L_2V(0))$ and determine $\pi_*(L_2V(0))$ at the prime number 3. Our main tool is the Adams-Novikov spectral

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sequence. In [**6**] Miller, Ravenel and Wilson introduced the chromatic spectral sequence converging to the E_2-term of the Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(V(n))$. We use here the modified chromatic spectral sequence which converges to the E_2-term of the Adams-Novikov spectral sequence $E_2^{*,*}(L_2V(0)) \Rightarrow \pi_*(L_2V(0))$ based on $E(2)$ with E_1-terms $H^*M_1^1$ and $H^*M_1^0$. Here $E(2)$ denotes the Johnson-Wilson spectrum with coefficient $E(2)_* = \boldsymbol{Z}_{(3)}[v_1, v_2^{\pm 1}]$, $M_1^0 = v_1^{-1}E(2)_*/(3)$ and $M_1^1 = E(2)_*/(3, v_1^\infty)$ are the $E(2)_*E(2)$-comodules and $H^*M = \mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$. $H^*M_1^0$ was determined by Ravenel [**7**], and so it suffices to determine $H^*M_1^1$ for the E_2-term $E_2^{*,*}(L_2V(0))$. In the first half of this paper, we actually determine $H^*M_1^1$ by using the Bockstein spectral sequence $H^*K(2)_* \Rightarrow H^*M_1^1$, where $K(2)_* = M_2^0 = E(2)_*/(3, v_1)$ and $H^*K(2)_*$ is determined by Ravenel [**7**]. The structure of $H^*M_1^1$ is stated in Theorem 2.5, and obtain the E_2-term $E_2^{*,*}(L_2V(0))$ in Theorem 2.6. In [**6**] $H^0M_1^1$ is determined, and we studied $H^1M_1^1$ in [**1**]. Unfortunately Theorem 4.4 of [**1**] is incorrect, and so are Proposition 5.2 and Theorem 1.1 consequently. Here we replace it by Lemma 4.2, which is proved in §7, and obtain $H^1M_1^1$. In the second half of this paper, we determine the Adams-Novikov differentials d_r on $E_r^{*,*}(L_2W)$ with $E_2^{*,*}(L_2W) = H^*M_1^1$, and then the homotopy groups $\pi_*(L_2W)$ which are described in Theorem 2.8. Here W denotes a cofiber of the localization map $V(0) \to \underrightarrow{\mathrm{holim}}_{\alpha}\, V(0)$ for the Adams map $\alpha : \Sigma^4V(0) \to V(0)$. The homotopy groups $\pi_*(L_2V(0))$, which is our main result, are obtained in Theorem 2.11 as a corollary of Theorem 2.8. The results would have applications. Here we treat the β-family of the homotopy groups of $\pi_*(L_2S^0)$ at the prime 3. We note that though the result of [**2**] depends on a result of [**1**], it remains correct since the proof does not require the incorrect part.

This paper is organized as follows: In the next section, we state our results. Then we prove Theorem 2.5 in §3 assuming the behavior of the connecting homomorphisms $\delta_s : H^sM_1^1 \to H^{s+1}K(2)_*$ which will be studied in the following sections. In §4, assuming the behavior of the differential of the cobar complex $\Omega^*E(2)_*$ which will be studied in §§6 and 7, we prove Proposition 3.4 which determines the differentials of the Bockstein spectral sequence and is the key lemma to determine the E_2-term $E_2^{*,*}(L_2V(0))$. In order to study the differential of the cobar complex $\Omega^*E(2)_*$, we need some relations in $E(2)_*E(2)$, which is given in §5. In §8, we compute the differentials of the Adams-Novikov spectral sequence, and prove Theorem 2.8. The last section is devoted to applications for β-elements.

§2. Statement of results.

Throughout this paper everything is localized at the prime 3. Let $V(0)$ denote the mod 3 Moore spectrum and W be the cofiber of the localization map $V(0) \to L_1V(0)$. Since $L_1V(0) = \underrightarrow{\mathrm{holim}}_{\alpha}\, V(0)$ for the Adams map $\alpha : \Sigma^4V(0) \to V(0)$, we can define W as follows: Let $V(1)_j$ denote a cofiber of $\alpha^j : \Sigma^{4j}V(0) \to V(0)$. In particular, $V(1)_1 = V(1)$, the Toda-Smith spectrum. Then we have the canonical maps $\pi_j : V(1)_j \to V(1)_{j-1}$ and $\iota_j : V(1)_j \to V(1)_{j+1}$. We define $W = \underrightarrow{\mathrm{holim}}_{\iota_j}\, V(1)_j$. By definition, we have cofiber sequences $V(1) \xrightarrow{v_1^{j-1}} V(1)_j \xrightarrow{\pi_j} V(1)_{j-1}$, whose homotopy colimit yields another one

(2.1) $$V(1) \xrightarrow{i} W \xrightarrow{v_1} W.$$

Apply the Johnson-Wilson homology $E(2)_*(-)$ with coefficient $E(2)_* = \boldsymbol{Z}_{(3)}[v_1, v_2^{\pm 1}]$ to the cofiber sequence (2.1), and we have a short exact sequence

(2.2) $$0 \longrightarrow K(2)_* \xrightarrow{i_*} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0,$$

where $K(2)_* = (\boldsymbol{Z}/3)[v_2^{\pm 1}], M_1^1 = E(2)_*/(3, v_1^\infty)$ and $i_*(x) = x/v_1$. Note that $K(2)_*$ is the coefficient ring of the second Morava K-theory $K(2)_*(-)$, and M_1^1 consists of elements of the form x/v_1^j for $x \in K(2)_*$ and $j > 0$, with $k(1)_* = (\boldsymbol{Z}/3)[v_1]$-action given by the relation: $v_1^l(x/v_1^j) = x/v_1^{j-l}$ if $j > l$, and $= 0$ otherwise. Apply the functor $H^*(-) = \mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, -)$ to the exact sequence (2.2), and we obtain the Bockstein spectral sequence

$$E_2^{*,*}(L_2V(1)) = H^*K(2)_* \quad \Rightarrow \quad H^*M_1^1 = E_2^{*,*}(L_2W).$$

Here $E_2^{*,*}(X)$ denotes the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$.

In [**3**], Henn computed the E_2-term $H^*K(2)_*$ as follows:

Theorem 2.3 (*cf.* [**11**, Th. 5.8, Prop. 5.9].). *The E_2-term $E_2^{*,*}(L_2V(1)) = H^*K(2)_*$ is isomorphic to the $K(2)_*[b_{10}]$-module*

$$F \otimes K(2)_*[b_{10}] \otimes \Lambda(\zeta_2).$$

Here $F = (\boldsymbol{Z}/3)\{1, h_{10}, h_{11}, b_{11}, \xi, \psi_0, \psi_1, b_{11}\xi\}$. Besides, we have relations:

$$h_{10}h_{11} = 0, \quad h_{10}\xi = 0, \quad h_{11}\xi = 0,$$

$$v_2^2h_{10}b_{10} = h_{11}b_{11}, \quad v_2h_{11}b_{10} = -h_{10}b_{11},$$

$$b_{11}\xi = v_2h_{10}\psi_1 = v_2h_{11}\psi_0, \quad b_{10}\xi = -h_{10}\psi_0 = v_2^{-1}h_{11}\psi_1,$$

$$v_2^3b_{10}^2 = -b_{11}^2, \quad b_{10}\psi_1 = -v_2^{-1}b_{11}\psi_0, \quad \textit{and} \quad b_{10}\psi_0 = v_2^{-2}b_{11}\psi_1.$$

The bidegrees of these generators are as follows:

$$\|v_2\| = (0,16), \quad \|h_{10}\| = (1,4), \quad \|h_{11}\| = (1,12), \quad \|b_{10}\| = (2,12),$$

$$\|b_{11}\| = (2,36), \quad \|\xi\| = (2,8), \quad \|\psi_0\| = (3,16), \quad \|\psi_1\| = (3,24).$$

In this paper, we first deduce the Adams-Novikov E_2-term $E_2^{*,*}(L_2W)$ from Theorem 2.3 by using the Bockstein spectral sequence. In order to state the E_2-term, consider $k(1)_*$-modules

(2.4)
$$\begin{aligned}
F &= E(2,1)_*\{v_2^{\pm 1}/v_1, v_2h_{10}/v_1^2, v_2^2h_{11}/v_1^2, v_2^{\pm 1}b_{11}/v_1\}\\
F^* &= E(2,1)_*\{\xi/v_1^2, v_2^{2\pm 1}\psi_0/v_1, v_2^{\pm 1}\psi_1/v_1, b_{11}\xi/v_1^2\}\\
F_n &= E(2,n+2)_*\{v_2^{\pm 3^{n+1}}/v_1^{4\cdot 3^n-1}, v_2^{3^{n+1}}h_{10}/v_1^{6\cdot 3^n+1},\\
&\qquad v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}, v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}\}.
\end{aligned}$$

Here $\|v_1\| = (0,4)$ and $E(2,n)_* = (\boldsymbol{Z}/3)[v_1, v_2^{\pm 3^n}]$. We also use the notation $K(1)_* = v_1^{-1}k(1)_* = (\boldsymbol{Z}/3)[v_1^{\pm 1}]$. Then

THEOREM 2.5. *The E_2-term $E_2^{*,*}(L_2W) = H^*M_1^1$ of the Adams-Novikov spectral sequence converging to $\pi_*(L_2W)$ is isomorphic to the direct sum of $k(1)_*$-modules $(K(1)_*/k(1)_*) \otimes \Lambda(h_{10}, \zeta_2)$,*

$$\sum_{n\geq 0} F_n \otimes \Lambda(\zeta_2), \quad \textit{and} \quad (F \oplus F^*) \otimes (\boldsymbol{Z}/3)[b_{10}] \otimes \Lambda(\zeta_2).$$

The short exact sequence associated to the cofiber sequence $V(0) \to L_1V(0) \to W$ yields the long exact sequence

$$H^*E(2)_*/(3) \to H^*M_1^0 \to H^*M_1^1 \xrightarrow{\partial} H^{*+1}E(2)_*/(3),$$

in which $M_1^0 = (\boldsymbol{Z}/3)[v_1^{\pm 1}, v_2^{\pm 1}]$ and the structures of $H^*M_1^0$ is determined to be $K(1)_* \otimes \Lambda(h_{10})$ by Ravenel [**7**]. Observing the exact sequence, we obtain the E_2-term from Theorem 2.5:

THEOREM 2.6. *The E_2-term $E_2^{*,*}(L_2V(0)) = H^*E(2)_*/(3)$ is isomorphic to the direct sum of the $k(1)_*$-modules $K(1)_*/k(1)_*\{\partial(\zeta_2)\} \otimes \Lambda(h_{10})$,*

$$k(1)_* \otimes \Lambda(h_{10}) \quad \textit{and} \quad \Lambda(\zeta_2) \otimes \partial\left(\sum_{n\geq 0} F_n \oplus (F \oplus F^*) \otimes (\boldsymbol{Z}/3)[b_{10}]\right).$$

In order to state the homotopy groups $\pi_*(L_2V(0))$, we introduce more notations:

$$\tilde{F}_0 = E(2,2)_*\{v_2^3/v_1^2, v_2^{-3}/v_1^3, v_2^3h_{10}/v_1^6, v_2^8h_{10}/v_1^{10}, v_2^8\xi/v_1^4, v_2^2\xi/v_1^3\}$$

$$\tilde{F}_1 = E(2,3)_*\{v_2^{\pm 9}/v_1^{11}, v_2^9h_{10}/v_1^{18}, v_2^{24}h_{10}/v_1^{31}, v_2^{16\pm 9}\xi/v_1^{11}\}$$

$$\begin{aligned}\tilde{F}_n = E(2,n+2)_*\{&v_2^{\pm 3^{n+1}}/v_1^{4\cdot 3^n-1}, v_2^{3^{n+1}}h_{10}/v_1^{2\cdot 3^{n+1}},\\ &v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n}, v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n-1}\} \quad (n \geq 2)\end{aligned}$$

$$\begin{aligned}\tilde{F} = &B_5(2,2)_*\{v_2/v_1, v_2h_{10}/v_1^2\}\\ &\oplus B_4(2,2)_*\{v_2^5h_{11}/v_1^2, v_2^4b_{11}/v_1\}\\ &\oplus B_3(2,2)_*\{v_2^2/v_1, v_2^5/v_1, v_2^7h_{10}/v_1\}\\ &\oplus B_2(2,2)_*\{v_2h_{10}/v_1, v_2^2h_{11}/v_1, v_2^5h_{11}/v_1, v_2^5b_{11}/v_1, v_2^{-1}b_{11}/v_1\}\\ \tilde{F}^* = &B_5(2,2)_*\{v_2^7\psi_1/v_1\}\\ &\oplus B_4(2,2)_*\{v_2^3\xi/v_1^2, v_2^3\psi_0/v_1, v_2^6b_{11}\xi/v_1^2\}\\ &\oplus B_3(2,2)_*\{v_2^{-1}\psi_1/v_1\}\\ &\oplus B_2(2,2)_*\{\xi/v_1, v_2^2\psi_1/v_1, v_2^4\psi_0/v_1, v_2^3b_{11}\xi/v_1, v_2^7\psi_0/v_1, v_2^6b_{11}\xi/v_1\}\\ &\oplus \sum_{n\geq 1}(B_3(2,n+2)_*\{v_2^{9u+3}\xi/v_1 \mid u \in \boldsymbol{Z} - I(n)\}\\ &\qquad \oplus B_2(2,n+2)_*\{v_2^{9u+3}\xi/v_1 \mid u \in I(n)\}),\end{aligned} \tag{2.7}$$

where $B_k(2,n)_* = (\boldsymbol{Z}/3)[v_1, v_2^{\pm 3^n}, b_{10}]/(b_{10}^k)$, and

$$I(n) = \{x \in \boldsymbol{Z} | x = (3^{n-1}-1)/2 \text{ or } x = 5\cdot 3^{n-2} + (3^{n-2}-1)/2\}.$$

Studying the Adams-Novikov differentials d_5 and d_9 by results of [**4**] and [**11**], we obtain the following

Theorem 2.8. *The homotopy groups $\pi_*(L_2W)$ are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the direct sum of $k(1)_*$-modules $(K(1)_*/k(1)_*) \otimes \Lambda(h_{10})$, $\sum_{n\geq 0}\tilde{F}_n$ and $\tilde{F} \oplus \tilde{F}^*$.*

For describing more homotopy groups, we further introduce the $k(1)_*$-modules:

$$\bar{F}_0 = E(2,2)_*\{v_1v_2^2h_{10}, v_2^{-4}h_{10}, v_1v_2\xi, v_1v_2^5\xi, v_2^6\psi_1, v_1\psi_1\}$$

$$\bar{F}_1 = E(2,3)_*\{v_2^{\pm 9-3}h_{10}, v_1v_2^4\xi, v_2^{16}\xi, v_1v_2^{12\pm 9}\psi_1\}$$

$$\bar{F}_n = E(2,n+2)_*\{v_2^{\pm 3^{n+1}-3^n}h_{10}, v_1v_2^{(3^{n+1}-1)/2}\xi,$$
$$v_1v_2^{5\cdot 3^n+(3^n-1)/2}\xi, v_1v_2^{3^{n+1}(1\pm 1)+3(3^n-1)/2}\psi_1\} \quad (n\geq 2)$$

$$\begin{aligned}
\bar{F} &= B_5(2,2)_*\{h_{11}, b_{10}\} \oplus B_4(2,2)_*\{v_2^3b_{11}, v_2^3h_{11}b_{11}\} \\
&\quad \oplus B_3(2,2)_*\{v_2h_{11}, v_2^4h_{11}, v_1v_2^6b_{10}\} \\
&\quad \oplus B_2(2,2)_*\{v_1b_{10}, v_1b_{11}, v_1v_2^3b_{11}, v_2^4h_{11}b_{11}, v_2^{-2}h_{11}b_{11}\} \\
\overline{F^*} &= B_5(2,2)_*\{v_2^7\xi b_{10}\} \oplus B_4(2,2)_*\{v_2^2\psi_0, v_2b_{11}\xi, v_2^6\psi_1b_{10}\} \\
&\quad \oplus B_3(2,2)_*\{v_2^{-1}\xi b_{10}\} \\
&\quad \oplus B_2(2,2)_*\{v_1v_2^{-1}\psi_0, v_2^2\xi b_{10}, v_2^2b_{11}\xi, v_1v_2^3\psi_1b_{10}, v_2^5b_{11}\xi, v_1v_2^6\psi_1b_{10}\} \\
&\quad \oplus \sum_{n\geq 1}(B_3(2,n+2)_*\{v_1v_2^{9u+2}\psi_0 \mid u \in \boldsymbol{Z} - I(n)\} \\
&\quad \oplus B_2(2,n+2)_*\{v_1v_2^{9u+2}\psi_0 \mid u \in I(n)\}),
\end{aligned} \tag{2.9}$$

where $\overline{M}$ is isomorphic to $\tilde{M}$ for $M = F_n, F, F^*$ as $k(1)_*$-modules while there is one dimension shift. Furthermore, put $k(1)_*^\wedge = \varprojlim_j k(1)_*/(v_1^j)$. Since $\operatorname{holim}_{\leftarrow j} L_2V(1)_j = L_{K(2)}V(0)$ by observing $K(2)_*$ homology, the above theorem implies

Theorem 2.10. *The homotopy groups $\pi_*(L_{K(2)}V(0))$ are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the direct sum of $k(1)_*$-modules $(k(1)_*^\wedge) \otimes \Lambda(h_{10}), \sum_{n\geq 0}\bar{F}_n$ and $\bar{F} \oplus \overline{F^*}$.*

Observing the cofibration $L_2V(0) \to L_1V(0) \to L_2W$ and the homotopy groups $\pi_*(L_1V(0)) = K(1)_* \otimes \Lambda(h_{10})$, we have

Theorem 2.11. *The homotopy groups $\pi_*(L_2V(0))$ are isomorphic to the direct sum of $k(1)_* \otimes \Lambda(h_{10}), (K(1)_*/k(1)_*)\partial\zeta_2 \otimes \Lambda(h_{10}), \sum_{n\geq 0}\bar{F}_n \otimes \Lambda(\zeta_2)$ and $(\bar{F} \oplus \overline{F^*}) \otimes \Lambda(\zeta_2)$.*

Here recall the conjecture due to Ravenel on the β-elements: $\beta_s \in \pi_*(S^0)$ if and only if $s \equiv 0, 1, 2, 3, 5, 6 \mod 9$. (See §9 for the definition of β-elements.) 'Only if' part is shown in [**11**], in which we also show that $\beta_s \in \pi_*(L_2 S^0)$ if $s \equiv 0, 1, 5 \mod 9$. On this conjecture, we have a supporting evidence:

THEOREM 2.12. *$\beta_s \in \pi_*(L_2 S^0)$ if and only if $s \equiv 0, 1, 2, 3, 5, 6 \mod 9$.*

In the E_2-term, the β-elements of the form $\beta_{a/b}$ are defined [**6**] for integers $a, b > 0$ such that $b \le 3^{v(a)}$ if $v(a) \le 1$ and $b < 4 \cdot 3^{v(a)-1}$ otherwise, where the integer $v(a)$ denotes the maximal power of 3 that divides a. Then we have homotopy β-elements:

THEOREM 2.13. *In the Adams-Novikov spectral sequence $E_2^*(L_2 S^0) \Rightarrow \pi_*(L_2 S^0)$, we have the following*:

(a) *The element β_a with $v(a) = 0$ is permanent if $a \equiv 1, 2, 5 \mod 9$.*
(b) *The element $\beta_{a/b}$ with $v(a) = 1$ is permanent if $v(a-3) \ge 2$ and $b < 3$, or if $v(a+3) \ge 2$ and $b \le 3$.*
(c) *Every element $\beta_{a/b}$ with $v(a) \ge 2$ is permanent.*

§3. Proof of Theorem 2.5.

The proof of Theorem 2.5 is based on the following lemma due to [**6**, Remark 3.11]. To state the lemma, we set up notations: Let K denote a $\boldsymbol{Z}/3$-basis of the submodule $F \otimes K(2)_*[b_{10}]$ of $H^*K(2)_*$ given in Theorem 2.3, and $\bar{x}/v_1^j$ denote an element of $H^*M_1^1$ such that $v_1^{j-1}(\bar{x}/v_1^j) = x/v_1$ for an element $x \in K$ and an integer $j > 0$. Consider the maps i_* and δ_s in the long exact sequence associated to the short one (2.2)

$$(3.1) \qquad \cdots \longrightarrow H^s K(2)_* \xrightarrow{i_*} H^s M_1^1 \xrightarrow{v_1} H^s M_1^1 \xrightarrow{\delta_s} H^{s+1} K(2)_* \longrightarrow \cdots.$$

Note that $i_*(x) = x/v_1$. For each base $x \in K \subset H^s K(2)_*$, define an integer $j(x)$ by $j(x) = j$ if $\delta_s(\bar{x}/v_1^j) \ne 0$, and $j(x) = \infty$ otherwise. Define a $k(1)_*$-submodule B of $H^*M_1^1$ by

$$B = k(1)_*\{\bar{x}/v_1^{j(x)} \mid x \in K \text{ and } i_*(x) \ne 0 \in H^*M_1^1\}.$$

Here note that an element of the form $\bar{x}/v_1^\infty$ generates a $k(1)_*$-module isomorphic to $K(1)_*/k(1)_*$ whose $(\boldsymbol{Z}/3)$-basis is $\{\bar{x}/v_1^j \mid j > 0\}$.

LEMMA 3.2. *For the submodule B defined above, $H^*M_1^1 = B \otimes \Lambda(\zeta_2)$ if B satisfies the condition that the set $\{\delta(\bar{x}/v_1^{j(x)}) | x \in K, i_*(x) \ne 0, j(x) < \infty\}$ is linearly independent.*

Therefore, we will study the connecting homomorphism $\delta_s : H^s M_1^1 \to H^{s+1} K(2)_*$ to find $j(x)$ for each $x \in K$. Note that if $x \notin \operatorname{Im} \delta_s$, then $i_*(x) \ne 0$ in $H^{s+1} M_1^1$. Thus a computation of δ_s shows us all information that we need. We will not distinguish x and $\bar{x}$ in the sequel. The following is our key lemma:

LEMMA 3.3. *The connecting homomorphism $\delta_s : H^s M_1^1 \to H^{s+1} K(2)_*$ acts as follows*:

1. *On* F_n $(n \geq 0)$,

$$\delta_0(v_2^{3^{n+1}}/v_1^{4\cdot 3^n-1}) = \lambda v_2^{2\cdot 3^n}h_{10} \qquad (\lambda = 1 \text{ if } n = 0, \lambda = -1 \text{ if } n > 0)$$

$$\delta_1(v_2^{3^{n+1}}h_{10}/v_1^{6\cdot 3^n+1}) = (-1)^n v_2^{(3^{n+1}-1)/2}\xi$$

$$\delta_1(v_2^{8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}) = -v_2^{5\cdot 3^n+(3^n-1)/2}\xi$$

$$\delta_2(v_2^{3^n(5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}) = \pm v_2^{3^{n+1}(1\pm 1)+3(3^n-1)/2}\psi_1 \mp v_2^{3^{n+1}(1\pm 1)+(3^{n+1}-1)/2}\xi\zeta_2.$$

2. *On* F,

$$\delta_0(v_2/v_1) = h_{11}$$

$$\delta_1(v_2h_{10}/v_1^2) = b_{10}$$

$$\delta_1(v_2^2h_{11}/v_1^2) = b_{11}$$

$$\delta_2(v_2b_{11}/v_1) = v_2^2h_{10}b_{10}$$

3. *On* F^*,

$$\delta_2(\xi/v_1^2) = -v_2^{-1}\psi_0$$

$$\delta_3(v_2(v_2^{-1}\psi_0)/v_1) = v_2(v_2^{-3}b_{11}\xi)$$

$$\delta_3(v_2\psi_1/v_1) = v_2\xi b_{10}$$

$$\delta_4(b_{11}\xi/v_1^2) = \psi_1 b_{10}.$$

This gives rise to all the differentials of the Bockstein spectral sequence. In fact, suppose that $\delta_s(x/v_1^j) = y$ in the above lemma. Then for an element $a \in H^tE(2)_*/(3, v_1^j)$, we have $\delta_{s+t}(ax/v_1^j) = ay$. Take a to be an element of $E(2,n)_*/(v_1) = (\boldsymbol{Z}/3)[v_2^{\pm 3^n}] \subset H^0E(2)_*/(3, v_1^j)$ with $j < 4\cdot 3^{n-1}$, or $b_{10}^t\zeta_2^\varepsilon \in H^{2t+\varepsilon}E(2)_*/(2, v_1^j)$ for $t \geq 0$ and $\varepsilon = 0, 1$. Then we have

$$\delta_{s+2t+\varepsilon}(v_2^{3^nu}xb_{10}^t\zeta_2^\varepsilon/v_1^j) = v_2^{3^nu}yb_{10}^t\zeta_2^\varepsilon,$$

for an integer u. Therefore, we see that

Proposition 3.4. *The connecting homomorphism* $\delta_s : H^sM_1^1 \to H^{s+1}K(2)_*$ *acts as follows*:

$$\delta_0(v_2^{3t\pm 1}/v_1) = \pm v_2^{3t-1\pm 1}h_{11}$$

$$\delta_0(v_2^{3(3t\pm 1)}/v_1^3) = \pm v_2^{9t-1\pm 3}h_{10}$$

$$\delta_0(v_2^{3^n(3t\pm 1)}/v_1^{4\cdot 3^{n-1}-1}) = \mp v_2^{3^{n-1}(9t-1\pm 3)}h_{10} \quad (n > 1)$$

$$\delta_1(v_2^{3t+1}h_{10}/v_1^2) = v_2^{3t}b_{10}$$

$$\delta_1(v_2^{3^n(3t+1)}h_{10}/v_1^{2\cdot 3^n+1}) = (-1)^{n-1}v_2^{3^{n+1}t+(3^n-1)/2}\xi \quad (n > 0)$$

$$\delta_1(v_2^{3^n(9t+8)}h_{10}/v_1^{10\cdot 3^n+1}) = -v_2^{3^{n+2}t+5\cdot 3^n+(3^n-1)/2}\xi \quad (n \geq 0)$$

$$\delta_1(v_2^{3t+2}h_{11}/v_1^2) = v_2^{3t}b_{11};$$

$$\delta_2(v_2^{3t\pm 1}b_{10}/v_1) = \pm v_2^{3t-1\pm 1}h_{11}b_{10}$$

$$\delta_2(v_2^{3t\pm 1}b_{11}/v_1) = \pm v_2^{3t+1\pm 1}h_{10}b_{10}$$

$$\delta_2(v_2^{3t}\xi/v_1^2) = -v_2^{3t-1}\psi_0$$

$$\delta_2(v_2^{3^n(9t+5\pm 3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}) = \pm v_2^{3^{n+1}(3t+1\pm 1)+3(3^n-1)/2}\psi_1 \mp v_2^{3^{n+1}(3t+1\pm 1)+(3^{n+1}-1)/2}\xi\zeta_2 \quad (n \geq 0);$$

$$\delta_{2s+3}(v_2^{3t+2}h_{11}b_{10}^{s+1}/v_1^2) = v_2^{3t}b_{11}b_{10}^{s+1}$$

$$\delta_{2s+3}(v_2^{3t+1}h_{10}b_{10}^{s+1}/v_1^2) = v_2^{3t}b_{10}^{s+2}$$

$$\delta_{2s+3}(v_2^{3t\pm 1}(v_2^{-1}\psi_0)b_{10}^s/v_1) = \pm v_2^{3t\pm 1}(v_2^{-3}b_{11}\xi)b_{10}^s$$

$$\delta_{2s+3}(v_2^{3t\pm 1}\psi_1 b_{10}^s/v_1) = \pm v_2^{3t\pm 1}\xi b_{10}^{s+1};$$

$$\delta_{2s+4}(v_2^{3t\pm 1}b_{10}^{s+2}/v_1) = \pm v_2^{3t-1\pm 1}h_{11}b_{10}^{s+2}$$

$$\delta_{2s+4}(v_2^{3t\pm 1}b_{11}b_{10}^{s+1}/v_1) = \pm v_2^{3t+1\pm 1}h_{10}b_{10}^{s+2}$$

$$\delta_{2s+4}(v_2^{3t}\xi b_{10}^{s+1}/v_1^2) = -v_2^{3t-1}\psi_0 b_{10}^{s+1}$$

$$\delta_{2s+4}(v_2^{3t}b_{11}\xi b_{10}^s/v_1^2) = v_2^{3t}\psi_1 b_{10}^{s+1}$$

for $n, s \geq 0$ *and* $t \in \boldsymbol{Z}$.

Corollary 3.5. *The map* $i_*: H^sK(2)_* \to H^sM_1^1$ *sends each of the following elements in* K *to a non-zero element*:

$$h_{10}, \quad v_2^{3^k s}h_{10}, \quad v_2^{3t-1}h_{11};$$

for $s, t \in \boldsymbol{Z}$ *with* $s \equiv 1$ (3) *or* $s \equiv 8$ (9).

$$v_2^{3t\pm 1}b_{10}, \quad v_2^{3t\pm 1}b_{11}, \quad v_2^u\xi,$$

for $t, u \in \boldsymbol{Z}$ *with* $u \in 3\boldsymbol{Z}$ *or* $u = 3^n(9t+5\pm 3)+(3^n-1)/2$.

$$v_2^{3t-1}h_{11}b_{10}^{s+1}, \quad v_2^{3t+1}h_{10}b_{10}^{s+1}, \quad v_2^{3t-1\pm 1}\psi_0 b_{10}^s, \quad v_2^{3t\pm 1}\psi_1 b_{10}^s$$

for $s, t \in \boldsymbol{Z}$ *with* $s \geq 0$.

These elements in Corollary 3.5 form the set B, and Lemma 3.2 shows Theorem 2.5.

§4. Computation of the connecting homomorphism.

In this section, we will prove Lemma 3.3 by assuming some results on the cobar complex $\Omega^*E(2)_*$ which will be shown in the following sections.

Let $(E(2)_*, E(2)_*E(2))$ denote the Hopf algebroid associated to the Johnson-Wilson spectrum $E(2)$. For an $E(2)_*E(2)$-comodule M with coaction $\psi: M \to$

$M \otimes_{E(2)_*} E(2)_*E(2), H^*M = \mathrm{Ext}^*_{E(2)_*E(2)}(E(2)_*, M)$ is given as the cohomology of the cobar complex Ω^*M with $\Omega^sM = M \otimes_{E(2)_*} E(2)_*E(2)^{\otimes s}$ and the differential $d_s : \Omega^sM \to \Omega^{s+1}M$ defined by $d_s(x \otimes y) = \psi(x) \otimes y + \sum_{i=1}^{s}(-1)^i x \otimes y_1 \otimes \cdots \otimes \Delta(y_i) \otimes \cdots \otimes y_s + (-1)^{s+1} x \otimes y \otimes 1$ for $x \in M$ and $y = y_1 \otimes \cdots \otimes y_s \in E(2)_*E(2)^{\otimes s}$. Consider the connecting homomorphism $\delta_s : H^sM_1^1 \to H^{s+1}K(2)_*$ associated to (2.2). By definition, we see that

(4.1) $\quad$ if $d_s(x) \equiv v_1^j y \bmod (3, v_1^{j+1})$ in $\Omega^{s+1}E(2)_*$, then $\delta_s([x/v_1^j]) = [y]$.

Here $[x]$ denotes a cohomology class represented by a cocycle x.

Now we state several lemmas:

Lemma 4.2.*) *There exists a cochain* $x(8 \cdot 3^n) \in \Omega^1E(2)_*$ *for each* $n \geq 0$ *such that* $x(8 \cdot 3^n) \equiv v_2^{8 \cdot 3^n} t_1 \bmod (3, v_1)$ *and*

$$d_1(x(8 \cdot 3^n)) \equiv -v_1^{10 \cdot 3^n + 1} v_2^{5 \cdot 3^n + (3^n - 1)/2} X \mod (3, v_1^{10 \cdot 3^n + 2}).$$

Here X denotes a cocycle that represents ξ.

Lemma 4.3. *In the cobar complex $\Omega^3E(2)_*$, we have cochains f_i $(i = 0, 1)$ such that f_i represents ψ_i in $E_2^*(L_2V(1))$ and*

$$d_3(f_0) \equiv v_1 v_2^{-2} b_{11} \otimes X \mod (3, v_1^2),$$

$$d_3(f_1) \equiv 0 \mod (3, v_1^2).$$

Lemma 4.4. *There exist cochains* $X(0), \mathrm{X}(2)$ *and* $X(8) \in \Omega^2E(2)_*$ *such that* $X(n) \equiv v_2^n X \bmod (3, v_1)$ *and*

(a) $d_2(X(0)) \equiv -v_1^2 v_2^{-1} f_0 \mod (3, v_1^3)$,

(b) $d_2(X(2)) \equiv v_1^4 z^3 \otimes X^3 - v_1^4 v_2^{-3} f_1^3 \mod (3, v_1^5)$ and

$$d_2(X(8)) \equiv -v_1^4 v_2^3 z^3 \otimes X^9 + v_1^4 v_2^{-6} f_1^9 \mod (3, v_1^5),$$

for the elements f_i $(i = 0, 1)$ of Lemma 4.3. Here z represents ζ_2.

Assuming these lemmas we will prove Lemma 3.3 by which we obtain Proposition 3.4.

Proof of Lemma 3.3. In [**6**, Prop. 5.4], it is shown that $d_0(v_2) \equiv v_1 t_1^3 \bmod (3, v_1^3)$, $d_0(v_2^3) \equiv v_1^3 v_2^2 t_1 \bmod (3, v_1^4)$ and $d_0(v_2^{3^n}) \equiv -v_1^{4 \cdot 3^{n-1} - 1} v_2^{2 \cdot 3^{n-1}} t_1 \bmod (3, v_1^{4 \cdot 3^{n-1}})$ for $n \geq 2$, which implies the first equations in the parts 1 and 2 of Lemma 3.3. In fact, $h_{1i} = [t_1^{3^i}]$. Besides, we see that $d_2(v_2 b_{11}) \equiv v_1 t_1^3 \otimes b_{11} \bmod (3, v_1^3)$, since $d_2(b_{11}) \equiv 0$ and $d_0(v_2) \equiv v_1 t_1^3$. Therefore the fourth one in the part 2 follows from the relation $h_{11} b_{11} = v_2^2 h_{10} b_{10}$ of Theorem 2.3.

In [**1**, Prop.s 5.2, 5.3], it is shown that

$$d_1(x(1)) \equiv v_1^2 b_{10} \mod (3, v_1^3), \quad d_1(y(2)) \equiv v_1^2 b_{11} \mod (3, v_1^3),$$

$$d_1(x(3^n)) \equiv -(-1)^n v_1^{6 \cdot 3^{n-1} + 1} v_2^{(3^n - 1)/2} X \mod (3, v_1^{6 \cdot 3^{n-1} + 2}) \quad (n > 0),$$

*) This is the correction of the last congruence in [**1**, Prop. 5.2].

where $x(n)$ and $y(n)$ are elements such that

$$x(n) \equiv v_2^n t_1 \quad \text{and} \quad y(n) \equiv v_2^n t_1^3 \mod (3, v_1).$$

These show the second equation in the part 1 and the second and the third ones in the part 2. The third one in the part 1 follows from Lemma 4.2. Since $f_1^{3^n} \equiv v_2^{3(3^n-1)/2} f_1$ and $X^{3^n} \equiv v_2^{(3^n-1)/2} X \bmod (3, v_1)$ up to homology by Theorem 2.3, we see the fourth one of the part 1 from Lemma 4.4 (b).

Now turn to the part 3. By Lemma 4.4 (a) we obtain the first one. The fourth one also follows from it, since $b_{10}\psi_1 = -v_2^{-1} b_{11}\psi_0$ by Theorem 2.3. The second one follows immediately from Lemma 4.3. Since $d_0(v_2) \equiv v_1 t_1^3$ and $d_3(f_1) \equiv 0 \bmod (3, v_1^2)$ by Lemma 4.3, we see that $d_3(v_2 f_1) \equiv v_1 t_1^3 \otimes f_1 \bmod (3, v_1^2)$, which is homologous to $v_1 v_2 X \otimes b_{10}$ by the relation $h_{11}\psi_1 = v_2 b_{10}\xi = v_2 \xi b_{10}$ of Theorem 2.3. This shows the third equation. □

§5. Some relations in $E(2)_* E(2)$.

Note that in $E(2)_* E(2)$ we have the following relations (*cf.* [**1**, (3.8), (3.9)]):

(5.1)
$$\begin{aligned} t_1^9 &= v_2^{-1} t_1 \eta_R(v_2^3) - v_1 v_2^{-1} t_2^3 - v_1^2 v_2^{-1} V + v_1^9 v_2^{-1} t_2 \\ &\equiv v_2^2 t_1 - v_1 v_2^{-1} t_2^3 + v_1^2 v_2 t_1^3 + v_1^3 (v_2^{-1} t_1^{10} + t_1^6) - v_1^4 v_2 t_1 + v_1^5 t_1 \mod (3, v_1^5) \\ t_n^9 &\equiv v_2^{3^n - 1} t_n - v_1 v_2^{-1} t_{n+1}^3 \mod (3, v_1^2), \end{aligned}$$

in which $\eta_R(v_2^3) = v_2^3 + v_1^3 t_1^9 - v_1^9 t_1^3$ and

(5.2)
$$V = -v_2^2 t_1^3 - v_1 v_2 t_1^6 + v_1^2 v_2^2 t_1 - v_1^3 v_2 t_1^4 + v_1^4 t_1^7 - v_1^5 v_2 t_1^2 - v_1^6 t_1^5.$$

Therefore we see the following relation in the cobar complex $\Omega^* E(2)_*$:

(5.3)
$$c^9 \equiv v_2^{2t} c \mod (3, v_1),$$

for a cochain $c \in \Omega^{*,4t} E(2)_*$.

Note that $d_0 : E(2)_* \to E(2)_* E(2)$ is computed by $d_0(x) = \eta_R(x) - x$. Since $\eta_R(v_2) \equiv v_2 + v_1 t_1^3 - v_1^3 t_1 \mod (3)$ by Landweber's formula, we compute the following mod $(3, v_1^8)$,

(5.4)
$$\begin{aligned} d_0(v_2^2) &\equiv -v_1 v_2 t_1^3 + v_1^2 t_1^6 + v_1^3 v_2 t_1 + v_1^4 t_1^4 + v_1^6 t_1^2 \\ d_0(v_2^4) &\equiv v_1 v_2^3 t_1^3 + v_1^4 \tau^3 + v_1^5 v_2^2 t_1^3 + v_1^6 v_2 t_1^6 - v_1^7 v_2^2 t_1 \\ d_0(v_2^5) &\equiv -v_1 v_2^4 t_1^3 + v_1^2 v_2^3 t_1^6 - v_1^3 v_2^2 t_1^9 + v_1^4 v_2 t_2^3 \\ &\quad + v_1^5(-v_2^3 t_1^3 + t_1^3 t_2^3 + t_1^{15}) + v_1^6 (v_2^2 t_1^6 + v_2^3 t_1^2) + v_1^7 t_1^{13} \\ d_0(v_2^7) &\equiv v_1 v_2^6 t_1^3 + v_1^3 v_2^4 t_1^9 - v_1^4 v_2^3 (t_2^3 + t_1^{12}) + v_1^5 v_2^5 t_1^3 + v_1^6 v_2^4 t_1^6 + v_1^7 t_1^{21} \\ d_0(v_2^8) &\equiv -v_1 v_2^7 t_1^3 + v_1^2 v_1^6 t_1^6 \mod (3, v_1^4), \end{aligned}$$

where $\tau = t_1^4 - t_2$.

Moreover, we have

(5.5) (*cf.* [**9**]) For $d_1 : E(2)_*E(2) \to E(2)_*E(2)^{\otimes 2}$,

$$d_1(t_1) = 0$$
$$d_1(t_2) = -t_1 \otimes t_1^3 - v_1 b_{10},$$

where $b_{10} = -t_1 \otimes t_1^2 - t_1^2 \otimes t_1$.

Lemma 5.6. *There exist cochains T', T, and $\overline{T}$ in $\Omega^1 E(2)_*$ such that*

$$d_1(T') \equiv -t_1^9 \otimes z^9 - b_{11} - v_2^{-9} g_0^9 \mod (3, v_1^3),$$
$$d_1(T) \equiv -v_2 t_1^9 \otimes z^9 - v_2 b_{11} - v_2^{-8} g_0^9 + v_1 v_2^{-27} t_1^3 \otimes (t_3^9 - t_1^{81} t_2^9) \mod (3, v_1^3),$$
$$d_1(\overline{T}) \equiv -v_2 z^9 \otimes t_1^9 - v_2 b_{11} - v_2^{-26} g_0'^9 + v_1 v_2^{-27} t_1^3 \otimes (t_3^9 - t_1^9 t_2^{27}) - v_1^3 v_2^{-27} t_1 \otimes (t_3^9 - t_1^9 t_2^{27}) \mod (3, v_1^6).$$

Here

$$z = v_2^{-1} t_2 + v_2^{-3} t_2^3 - v_2^{-3} t_1^{12}, \quad b_{11} = -t_1^3 \otimes t_1^6 - t_1^6 \otimes t_1^3,$$
$$g_0 = t_1 \otimes t_2 - t_1^2 \otimes t_1^3 \quad \textit{and} \quad g_0' = t_2^3 \otimes t_1 - t_1^3 \otimes t_1^{10}.$$

Proof. First consider the cochains $\hat{t}_3 = t_3 - t_1^9 t_2$ and $\tilde{t}_3 = t_3 - t_1 t_2^3$. Then we see

$$d_1(\hat{t}_3) \equiv -v_2^3 t_1 \otimes z - v_2 b_{11} - v_2^2 g_0 \mod (3, v_1) \quad \text{and}$$
$$d_1(\tilde{t}_3) \equiv -v_2^3 z \otimes t_1 - v_2 b_{11} - g_0' \mod (3, v_1),$$

by computing

$$\begin{aligned} d_1(t_3) &\equiv -t_1 \otimes t_2^3 - t_2 \otimes t_1^9 - v_2 b_{11} \\ &\equiv -t_1 \otimes t_2^3 - v_2^2 t_2 \otimes t_1 - v_2 b_{11}, \\ d_1(-t_1^9 t_2) &\equiv t_1^{10} \otimes t_1^3 + t_1 \otimes t_1^{12} + t_1^9 \otimes t_2 + t_2 \otimes t_1^9 \\ &\equiv -v_2^2 g_0 + t_1 \otimes t_1^{12} - v_2^2 t_1 \otimes t_2 + t_2 \otimes t_1^9, \\ d_1(-t_1 t_2^3) &\equiv t_1^4 \otimes t_1^9 + v_1^3 \otimes t_1^{10} + t_1 \otimes t_2^3 + t_2^3 \otimes t_1 \\ &\equiv t_1^{12} \otimes t_1 + t_1 \otimes t_2^3 - g_0' - t_2^3 \otimes t_1 \end{aligned}$$

$\mod (3, v_1)$.

Now put $T' = v_2^{-27} \hat{t}_3^9$, and 9-th power of $d_1(\hat{t}_3)$ yields $d_1(T')$. In fact, by (5.3) we see that $b_{11}^9 \equiv v_2^{18} b_{11} \mod (3, v_1^6)$.

We define $T = v_2 T'$ and compute $d_1(T) \equiv d_1(v_2 T') \equiv v_1 t_1^3 \otimes T' - v_2^{-26}(v_2^{27} t_1^9 \otimes z^9 + v_2^9 b_{11}^9 + v_2^{18} g_0^9) \mod (3, v_1^3)$.

In the same manner we verify that the element $\overline{T} = v_2 \overline{T'}$ for $\overline{T'} = v_2^{-27} \tilde{t}_3^9$ satisfies the last congruence. □

§6. Proofs of Lemmas 4.3 and 4.4.

Let x denote a cochain that represents ξ in $H^2K(2)_*$, and define $X \in \Omega^2E(2)_*$ by

$$X = v_2^{-4}x^9.$$

Lemma 6.1. *The element X is a cocycle of $\Omega^*K(2)_*$ that represents ξ in $H^2K(2)_*$, and it satisfies the following in the cobar complex $\Omega^3E(2)_*$:*

$$d_2X \equiv -v_1v_2^{-1}t_1^3 \otimes X - v_1^4v_2^{-4}\tau^3 \otimes X - v_1^5v_2^2t_1^3 \otimes X \mod (3, v_1^6).$$

Proof. By definition with (5.4) we see that

$$\begin{aligned} 0 &\equiv d_2(x^9) \equiv d_2(v_2^4X) \\ &\equiv v_1v_2^3t_1^3 \otimes X + v_1^4\tau^3 \otimes X + v_1^5v_2^2t_1^3 \otimes X \\ &\quad + v_1^6v_2t_1^6 \otimes X + v_2^4d_2(X) \mod (3, v_1^7). \end{aligned}$$ □

As we noted in [**1**, (5.1)], we have an element w such that $d_1(w) = x^3 + v_2x$ in $\Omega^*K(2)_*$. Since $x^9 = v_2^4x$ by (5.3), we obtain that $d_1(w^3 - v_2^3w) = 0$, that is $w^3 - v_2^3w$ is a cocycle. Theorem 2.3 shows that $w^3 - v_2^3w$ is bounded, but nothing bounds it by degree reason. Therefore, w satisfies

$$d_1(w) = x^3 + v_2x \quad \text{and} \quad w^3 = v_2^3w \quad \text{in } \Omega^*K(2)_*. \tag{6.2}$$

By this, we have $d_1(w^9) \equiv x^{27} + v_2^9x^9 \equiv v_2^{12}X^3 + v_2^{13}X \mod (3, v_1^9)$, and so by (5.4),

$$d_1(v_2^{-12}w^9) \equiv -v_1^3v_2^{-15}t_1^9 \otimes w^9 + v_1^6v_2^{-18}t_1^{18} \otimes w^9 + X^3 + v_2X \tag{6.3}$$

$\mod (3, v_1^9)$ since $d_0(v_2^{-12}) \equiv v_2^{-18}(d_0(v_2^2))^3$.

Since $h_{10}\xi = 0$ in $H^3K(2)_*$ by Theorem 2.3, we have cocycles y_0 such that $d_2(y_0) = t_1 \otimes x$ in $\Omega^3K(2)_*$. Define $y_1 = -v_2^{-1}(y_0^3 + t_1^3 \otimes w)$. Then $d_2(y_1) = -v_2^{-1}(t_1^3 \otimes x^3 - t_1^3 \otimes (x^3 + v_2x)) = t_1^3 \otimes x$.

Put $Y_0 = v_2^{-6}y_0^9$ and $Y_1 = v_2^{-10}y_1^9$. Then we have

Lemma 6.4. $Y_i \equiv y_i \mod (3, v_1)$ $(i = 0, 1)$ *and* $Y_0^3 \equiv -v_2Y_1 - v_2^{-18}t_1^{27} \otimes w^9 \mod (3, v_1^9)$. *Besides,*

$$d_2(Y_0) \equiv t_1 \otimes X + v_1v_2^{-3}\tau^3 \otimes X + v_1^2v_2^{-1}t_1^3 \otimes X \mod (3, v_1^3), \quad \textit{and}$$

$$d_2(Y_1) \equiv t_1^3 \otimes X - v_1v_2^{-1}(t_1^3 \otimes Y_1 - t_1^6 \otimes X) + v_1^3(v_2^{-3}t_1^9 \otimes Y_1 - v_2^{-6}t_2^9 \otimes X) \mod (3, v_1^4).$$

Proof. The first one follows from (5.3). By definition, $Y_0^3 = v_2^{-18}y_0^{27} \equiv -v_2^{-9}y_1^9 - v_2^{-18}t_1^{27} \otimes w^9 = -v_2Y_1 - v_2^{-18}t_1^{27} \otimes w^9 \mod (3, v_1^9)$.

A direct computation with (5.1) shows the following:

$$d_2(v_2^{-6}y_0^9) \equiv v_2^{-6}t_1^9 \otimes x^9 \equiv v_2^{-4}(t_1 - v_1v_2^{-3}t_2^3 + v_1^2v_2^{-1}t_1^3) \otimes v_2^4X$$
$$\equiv v_2^{-4}\eta_R(v_2^4)(t_1 - v_1v_2^{-3}t_2^3 + v_1^2v_2^{-1}t_1^3) \otimes X \mod (3, v_1^3),$$

and $\eta_R(v_2^4) = v_2^4 + d_0(v_2^4)$ is read off from (5.4).

For Y_1, noticing that $d_0(v_2^{-10}) \equiv v_2^{-18}d_0(v_2^8) \mod (3, v_1^4)$, we compute with (5.4) the following:

$$\begin{aligned} d_2(v_2^{-10}y_1^9) &\equiv -v_1v_2^{-11}t_1^3 \otimes y_1^9 + v_1^2v_2^{-12}t_1^6 \otimes y_1^9 + v_2^{-10}t_1^{27} \otimes x^9 \\ &\equiv -v_1v_2^{-11}t_1^3 \otimes v_2^{10}Y_1 + v_1^2v_2^{-12}t_1^6 \otimes v_2^{10}Y_1 \\ &\quad + v_2^{-4}t_1^3 \otimes v_2^4X - v_1^3v_2^{-13}t_2^9 \otimes v_2^4X \\ &\equiv t_1^3 \otimes X - v_1v_2^{-1}(t_1^3 \otimes Y_1 - t_1^6 \otimes X) \\ &\quad + v_1^3v_2^{-3}t_1^9 \otimes Y_1 - v_1^3v_2^{-9}t_2^9 \otimes X \mod (3, v_1^4). \end{aligned}$$ □

Proof of Lemma 4.3. Define first $f_0' = t_1^2 \otimes X + t_1 \otimes Y_0$ and $f_1' = t_1 \otimes Y_1 - t_2 \otimes X$. Then Lemma 6.4 shows that f_i' represents $\psi_i = \langle h_{10}, h_{1i}, \xi \rangle$ for each $i = 0, 1$.

By Lemmas 5.6, 6.1 and 6.4 with (5.3) we compute

$$\begin{aligned} d_3(f_0') &\equiv -v_1(v_2^{-3}t_1 \otimes \tau^3 \otimes X - v_2^{-1}t_1^2 \otimes t_1^3 \otimes X) \\ &\equiv v_1(t_1 \otimes z \otimes X - v_2^{-1}g_0 \otimes X) \\ d_3(-v_1v_2^{-3}T \otimes X) &\equiv v_1v_2^{-3}(v_2^3t_1 \otimes z + v_2b_{11} + v_2^2g_0) \otimes X \\ d_3(v_1z \otimes Y_0) &\equiv -v_1z \otimes t_1 \otimes X \\ d_3(-v_1zt_1 \otimes X) &\equiv v_1z \otimes t_1 \otimes X + v_1t_1 \otimes z \otimes X \end{aligned}$$

$\mod (3, v_1^2)$. Then we have the first one by putting $f_0 = f_0' - v_1v_2^{-3}T \otimes X + v_1z \otimes Y_0 - v_1zt_1 \otimes X$.

Similarly, we compute

$$\begin{aligned} d_3(f_1') &\equiv -t_1 \otimes t_1^3 \otimes X + v_1v_2^{-1}t_1 \otimes (t_1^3 \otimes Y_1 - t_1^6 \otimes X) \\ &\quad + t_1 \otimes t_1^3 \otimes X + v_1b_{10} \otimes X - v_1v_2^{-1}t_2 \otimes t_1^3 \otimes X \\ d_3(v_1v_2^{-1}t_2 \otimes Y_1) &\equiv -v_1v_2^{-1}t_1 \otimes t_1^3 \otimes Y_1 - v_1v_2^{-1}t_2 \otimes t_1^3 \otimes X \\ d_3(v_1v_2^{-9}\overline{T}^3 \otimes X) &\equiv v_1v_2^{-9}(-v_2^9z \otimes t_1^3 - v_2^9b_{10} - g_0'^3) \otimes X \\ d_3(-v_1z \otimes Y_1) &\equiv v_1z \otimes t_1^3 \otimes X \end{aligned}$$

$\mod (3, v_1^2)$. Notice that $v_1v_2^{-1}(t_2 \otimes t_1^3 - t_1 \otimes t_1^6) = v_1v_2^{-9}g_0'^3$, and we obtain the second one by setting $f_1 = f_1' + v_1v_2^{-1}t_2 \otimes Y_1 + v_1v_2^{-9}\overline{T}^3 \otimes X - v_1z \otimes Y_1$. □

Proof of Lemma 4.4. (a) Set $\bar{X} = X + v_1v_2^{-1}Y_1$ and $\bar{f}_0 = v_2^{-1}(t_1^3 \otimes Y_1 + t_1^6 \otimes X)$, and we obtain

$$d_2(\bar{X}) \equiv v_1^2v_2^{-1}\bar{f}_0 \quad \mathrm{mod}\,(3, v_1^3)$$

by the computation:

$$d_2(X) \equiv -v_1v_2^{-1}t_1^3 \otimes X$$

$$d_2(v_1v_2^{-1}Y_1) \equiv -v_1^2v_2^{-2}t_1^3 \otimes Y_1 + v_1v_2^{-1}t_1^3 \otimes X - v_1^2v_2^{-2}(t_1^3 \otimes Y_1 - t_1^6 \otimes X)$$

$\mathrm{mod}\,(3, v_1^3)$. Indeed, these are seen by using Lemmas 6.1 and 6.4 and the congruence $d_0(v_2^{-1}) \equiv -v_1v_2^{-2}t_1^3 \ \mathrm{mod}\,(3, v_1^2)$ seen by (5.4).

For a while, we argue in the E_2-term $E_2^{*,*}(L_2V(1))$. Notice that $\bar{f}_0$ represents an element in $v_2^{-1}\langle h_{11}, h_{11}, \xi\rangle$, and ψ_0 is an element of $\langle h_{10}, h_{10}, \xi\rangle$. By a relation of the Massey products, we see that $h_{11}\langle h_{11}, h_{11}, \xi\rangle = \langle h_{11}, h_{11}, h_{11}\rangle\xi$ and $\langle h_{11}, h_{11}, h_{11}\rangle = -b_{11}$. Therefore, $h_{11}[\bar{f}_0] = -v_2^{-1}b_{11}\xi = -h_{11}\psi_0$ by a relation in Theorem 2.3, and so $[\bar{f}_0] = -\psi_0$ mod Ker h_{11}. Theorem 2.3 also shows us that Ker $h_{11} \subset E_2^{3,16}(L_2V(1))$ is generated by $h_{10}b_{10}$. Therefore we have an integer $k \in \boldsymbol{Z}/3$ such that $[\bar{f}_0] = -\psi_0 + kh_{10}b_{10}$ and a cochain e_0 such that $d_2(e_0) = \bar{f}_0 + f_0 - kt_1 \otimes b_{10}$ for f_0 given in the proof of Lemma 4.3. Furthermore the relation $v_2^2h_{10}b_{10} = h_{11}b_{11}$ certifies the existence of a cochain B such that $d_1(B) = v_2^2t_1 \otimes b_{10} - t_1^3 \otimes b_{11}$.

Putting $X(0) = \bar{X} + kv_1v_2^{-1}b_{11} - v_1^2(v_2^{-1}e_0 + kv_2^{-3}B)$ leads us to the desired congruence.

(b) Put $X(2) = v_2X^3 - v_1Y_0^3 - v_1^3v_2^{-2}Y_1^3$ and note that $f_1 \equiv t_1 \otimes Y_1 - t_2 \otimes X$ $\mathrm{mod}\,(3, v_1)$ for f_1 in the proof of Lemma 4.3. We then obtain $d_2(X(2))$ from computation

$$\begin{aligned} d_2(v_2X^3) &\equiv v_1t_1^3 \otimes X^3 - v_1^3t_1 \otimes X^3 - v_1^3v_2^{-2}t_1^9 \otimes X^3 \\ &\equiv v_1t_1^3 \otimes X^3 + v_1^3(t_1 + v_1v_2^{-3}t_2^3) \otimes X^3 \\ d_2(-v_1Y_0^3) &\equiv -v_1t_1^3 \otimes X^3 - v_1^4v_2^{-9}\tau^9 \otimes X^3 \\ d_2(-v_1^3v_2^{-2}Y_1^3) &\equiv -v_1^4v_2^{-3}t_1^3 \otimes Y_1^3 - v_1^3v_2^{-2}t_1^9 \otimes X^3 \\ &\equiv -v_1^4v_2^{-3}t_1^3 \otimes Y_1^3 - v_1^3(t_1 - v_1v_2^{-3}t_2^3) \otimes X^3 \end{aligned}$$

$\mathrm{mod}\,(3, v_1^5)$ by Lemmas 6.1 and 6.4.

By defining $X(8) = v_2^4X^9 - v_1v_2^{-3}Y_1^9$, we compute that

$$d(v_2^4X^9) \equiv v_1v_2^3t_1^3 \otimes X^9 + v_1^4\tau^3 \otimes X^9$$

$$d(-v_1v_2^{-3}Y_1^9) \equiv v_1^4v_2^{-6}t_1^9 \otimes Y_1^9 - v_1(v_2^3t_1^3 - v_1^3v_2^{-6}t_2^9) \otimes X^9$$

$\mathrm{mod}\,(3, v_1^5)$. Since $\tau^3 + v_2^{-6}t_2^9 \equiv -v_2^3z^3 - v_2^{-6}t_2^9$, we have the result. □

§7. Proof of Lemma 4.2.

In this section, we correct [**1**, Th. 4.4], whose X should be replaced by our $x(8)$.

LEMMA 7.1. *There exists an element* $x(7)$ *such that* $x(7) \equiv v_2^7 t_1 \mod (3, v_1)$, *and*

$$d_1(x(7)) \equiv v_1^2 b_{11}^3 - v_1^7 v_2^5 X \mod (3, v_1^8),$$

for a cocycle X *that represents* ξ.

PROOF. Put $x(7)' = v_2^5 t_1^9 - v_1 v_2^4 t_2^3 + v_1^3 v_2^2 t_1^{18} - v_1^2 T'^3 - v_1 v_2^7 z^9 - v_1^4 v_2^4 t_1^9 z^9 + v_1^4 v_2^3 \overline{T} - v_1^5 v_2^3 t_2^3 - v_1^6 v_2^{-4} T^3$ for T', T and $\overline{T}$ given in Lemma 5.6. Using (5.3), (5.4), (5.5) and Lemma 5.6, we compute the following $\mod (3, v_1^8)$:

$$\begin{aligned}
d_1(v_2^5 t_1^9) \equiv{}& (-\underline{v_1 v_2^4 t_1^3}_1 + \underline{v_1^2 v_2^3 t_1^6}_{(a1)} - \underline{v_1^3 v_2^2 t_1^9}_2 + \underline{v_1^4 v_2 t_2^3}_c \\
&+ v_1^5(-\underline{v_2^3 t_1^3}_6 + t_1^3 t_2^3 + t_1^{15}) + v_1^6(\underline{v_2^2 t_1^6}_e + \underline{v_2^3 t_1^2}_d) + v_1^7 t_1^{13})) \otimes t_1^9 \\
d_1(-v_1 v_2^4 t_2^3) \equiv{}& -v_1(\underline{v_1 v_2^3 t_1^3}_{(a1)} + v_1^4 \tau^3 + \underline{v_1^5 v_2^2 t_1^3}_e + v_1^6 v_2 t_1^6) \otimes t_2^3 \\
&+ \underline{v_1 v_2^4 t_1^3 \otimes t_1^9}_1 + \underline{v_1^4 v_2^4 b_{11}}_3 \\
d_1(v_1^3 v_2^2 t_1^{18}) \equiv{}& -\underline{v_1^4 v_2 t_1^3 \otimes t_1^{18}}_c + v_1^5 t_1^6 \otimes t_1^{18} + \underline{v_1^6 v_2 t_1 \otimes t_1^{18}}_d \\
&+ v_1^7 t_1^4 \otimes t_1^{18} + \underline{v_1^3 v_2^2 t_1^9 \otimes t_1^9}_2 \\
d_1(-v_1^2 T'^3) \equiv{}& \underline{v_1^2 t_1^{27} \otimes z^9}_b + v_1^2 b_{11}^3 + \underline{v_1^2 v_2^{-27} g_0^{27}}_a \\
d_1(-v_1 v_2^7 z^9) \equiv{}& -\underline{v_1^2 v_2^6 t_1^3 \otimes z^9}_b - \underline{v_1^4 v_2^4 t_1^9 \otimes z^9}_4 + v_1^5 v_2^3 t_2^3 \otimes z^9 \\
&+ v_1^5 v_2^3 t_1^{12} \otimes z^9 - \underline{v_1^6 v_2^5 t_1^3 \otimes z^9}_7 - v_1^7 v_2^4 t_1^6 \otimes z^9 \\
d_1(-v_1^4 v_2^4 t_1^9 z^9) \equiv{}& -v_1^5 v_2^3 t_1^3 \otimes t_1^9 z^9 + \underline{v_1^4 v_2^4 t_1^9 \otimes z^9}_4 + \underline{v_1^4 v_2^4 z^9 \otimes t_1^9}_5 \\
d_1(v_1^4 v_2^3 \overline{T}) \equiv{}& v_1^7 t_1^9 \otimes \overline{T} + v_1^4 v_2^3(-v_2 z^9 \otimes t_1^9 - v_2 b_{11} - v_2^{-26} g_0'^9 \\
&+ v_1 v_2^{-27} t_1^3 \otimes (t_3^9 - t_1^9 t_2^{27}) - v_1^3 v_2^{-27} t_1 \otimes (t_3^9 - t_1^9 t_2^{27})) \\
\equiv{}& v_1^4 v_2^3(-\underline{v_2 z^9 \otimes t_1^9}_5 - \underline{v_2 b_{11}}_3 - \underline{v_2^{-26} g_0'^9}_c \\
&+ v_1 v_2^{-27} t_1^3 \otimes (t_3^9 - t_1^9 t_2^{27})) \\
d_1(-v_1^5 v_2^3 t_2^3) \equiv{}& \underline{v_1^5 v_2^3 t_1^3 \otimes t_1^9}_6 \\
d_1(-v_1^6 v_2^{-4} T^3) \equiv{}& \underline{v_1^6 v_2^5 t_1^3 \otimes z^9}_7 + \underline{v_1^6 v_2^5 b_{10}}_d + \underline{v_1^6 v_2^2 g_0^3}_e + v_1^7 v_2^{-5} t_1^3 \otimes T^3.
\end{aligned}$$

The elements underlined with the same number are cancelled each other. Since the sum of the elements underlined with $(a1)$ is $-v_1^2 v_2^3 g_0^3$ and $g_0^9 \equiv v_2^{10} g_0 - v_1(v_2^9 t_1^4 \otimes t_2 + v_2 t_1 \otimes t_3^3 + v_2^7(t_2^3 \otimes t_2 + t_1 t_2^3 \otimes t_1^3))$, the sum of the elements underlined with $(a1)$ and a is:

$$v_1^2 v_2^{-27} g_0^{27} - v_1^2 v_2^3 g_0^3 \equiv -v_1^5(t_1^{12} \otimes t_2^3 + v_2^{-24} t_1^3 \otimes t_3^9 + v_2^{-6}(t_2^9 \otimes t_2^3 + t_1^3 t_2^9 \otimes t_1^9)).$$

Besides, the underlined parts with b, c and d are computed as follows:

$$v_1^2t_1^{27}\otimes z^9 - v_1^2v_2^6t_1^3\otimes z^9 \equiv -v_1^5v_2^{-3}t_2^9\otimes z^9$$

$$v_1^4v_2t_2^3\otimes t_1^9 - v_1^4v_2t_1^3\otimes t_1^{18} - v_1^4v_2^{-23}g_0'^9 \equiv v_1^7(v_2^2t_3\otimes t_1 - v_2^4t_2\otimes t_1^2),\quad \text{and}$$

$$v_1^6(v_2^3t_1^2\otimes t_1^9 + v_2t_1\otimes t_1^{18}) + v_1^6v_2^5b_{10} \equiv -v_1^7v_2^2(t_1^2\otimes t_2^3 - t_1\otimes t_1t_2^3)$$

using (5.1). Now we obtain

$$d_1(x(7)') \equiv v_1^2b_{11}^3 + v_1^5Z + v_1^7x'$$

for

$$\begin{aligned} Z = {} & \underline{(t_1^3t_2^3+t_1^{15})\otimes t_1^9}_A - \underline{\tau^3\otimes t_2^3}_C + t_1^6\otimes t_1^{18} + \underline{v_2^3t_2^3\otimes z^9}_D \\ & + \underline{v_2^3t_1^{12}\otimes z^9}_D - \underline{v_2^3t_1^3\otimes t_1^9z^9}_B + v_2^{-24}t_1^3\otimes(\underline{t_3^9}_1 - \underline{t_1^9t_2^{27}}_B) \\ & - (\underline{t_1^{12}\otimes t_2^3}_C + \underline{v_2^{-24}t_1^3\otimes t_3^9}_1 + v_2^{-6}(\underline{t_2^9\otimes t_2^3}_C + \underline{t_1^3t_2^9\otimes t_1^9}_A)) - \underline{v_2^{-3}t_2^9\otimes z^9}_D \\ x' = {} & t_1^{13}\otimes t_1^9 + t_1^4\otimes t_1^{18} - v_2^4t_1^6\otimes z^9 + v_2^{-5}t_1^3\otimes T^3 \\ & + v_2^2t_3\otimes t_1 - v_2^4t_2\otimes t_1^2 - v_2^2(t_1^2\otimes t_2^3 - t_1\otimes t_1t_2^3). \end{aligned}$$

We introduce an element $w = -z - v_2^{-1}t_2 = v_2^{-1}t_1^4 + v_2^{-1}t_2 - v_2^{-3}t_2^3$. Notice that $z^9 \equiv z^3 \mod (3, v_1^3)$. Then the parts underlined with A, B, C and D amount to $v_2^3t_1^3\otimes t_1^9(w\otimes 1)$, $v_2^3t_1^3\otimes t_1^9(1\otimes w)$, $v_2^3w^3\otimes t_2^3$ and $v_2^6w^3\otimes z^3 \mod (3, v_1^3)$, respectively, and so we have

$$Z \equiv v_2^3t_1^3\otimes t_1^9(w^3\otimes 1 + 1\otimes w^3) + t_1^6\otimes t_1^{18} - v_2^6w^3\otimes w^3.$$

Since we have $d_1(v_2w) \equiv t_1\otimes t_1^3 \mod (3, v_1)$ by (5.5), we obtain

$$d_1(v_2^6w^6) \equiv -(v_2^3t_1^3\otimes t_1^9(w^3\otimes 1 + 1\otimes w^3) + t_1^6\otimes t_1^{18} - v_2^6w^3\otimes w^3)$$

$\mod (3, v_1^3)$ by (5.3). Therefore the cochain $x(7) = x(7)' + v_1^5v_2^6w^6$ satisfies the desired congruence by putting $x' = -v_2^5X$. In fact, ξ is represented by a cocycle whose leading term is $v_2^{-3}t_1\otimes t_3 + v_2^{-10}t_3^3\otimes t_1^3$, and moreover T is congruent to $t_3 - t_1^9t_2 \mod (3, v_1)$. □

PROOF OF LEMMA 4.2. Put $x(8) = -V^3 + v_1^4x(7)$ for V in (5.2). Lemma 7.1 implies the lemma for $n = 1$, since $V^3 \equiv -v_2^8t_1 \mod (3, v_1)$ by (5.1) and (5.2), and $d_1(V) \equiv v_1^2b_{11} \mod (3, v_1^8)$ by [**1**, (3.7)]. For a large n, use (6.3) to obtain the lemma. □

§8. The Adams-Novikov differentials on $E_2^{*,*}(L_2W)$.

In this section, we compute the Adams-Novikov differential $d_r : E_r^{s,t}(L_2W) \to E_r^{s+r,t+r-1}(L_2W)$ for $r \geq 2$. Note that $E_2^*(L_2W)$ is given in Theorem 2.5 and that $d_r = 0$ unless $r \equiv 1 \bmod 4$ by degree reason.

PROPOSITION 8.1. *For all $r \geq 2$, $d_r = 0$ on $K(1)_*/k(1)_* \otimes \Lambda(h_{10}, \zeta_2)$.*

PROOF. Suppose that there are elements $x \in \Lambda(h_{10}, \zeta_2)$ and $y \in E_r^{s+r}(L_2W)$ with filt $x = s$ for integers $0 \leq s \leq 2$, $r > 4$ and $j > 0$, such that $d_u(x/v_1^j) = 0$ for $u < r$ and

$$d_r(x/v_1^j) = y \neq 0.$$

Then $d_{r'}(x/v_1^{j+1}) = y' \neq 0$ for some $r' \leq r$ and $y' \in E_{r'}^{s+r'}(L_2W)$. Since r is finite, we may assume that for each $k \geq 0$, $d_u(x/v_1^{j+k}) = 0$ for $u < r$ and

$$d_r(x/v_1^{j+k}) = y/v_1^k \neq 0 \in E_r^{s+r}(L_2W)$$

from the beginning. Thus y generates a module isomorphic to $K(1)_*/k(1)_*$ in $E_r^{s+r}(L_2W)$. On the other hand, Theorem 2.5 shows that $E_2^{s+r}(L_2W) = H^{s+r}M_1^1$ does not contain such a module since $s + r \geq r > 4$. This is a contradiction. □

In the following, an equation $d_r(x) = y$ means not only the indicated one but also $d_s(x) = 0$ for $s < r$.

Lemma 8.2. *Suppose that $d_r(x) = y$ on an element x of F_n or $F \oplus F^*$, then we get*

$$d_r(xb_{10}^s\zeta_2^\varepsilon) = yb_{10}^s\zeta_2^\varepsilon$$

for $s \geq 0$, $\varepsilon = 0, 1$ and $xb_{10}^s\zeta_2^\varepsilon \in E_2^{*,*}(L_2W)$.

Proof. Since b_{10} represents the homotopy element β_1, the relation $d_r(x) = y$ implies $d_r(xb_{10}^s) = yb_{10}^s$.

The same proof that shows $d_r(x) = y$ in the spectral sequence for $\pi_*(L_2W)$ works to show $d_r(x\zeta_2) = y\zeta_2$ in it, since the proof depends on the result of the differentials of the spectral sequence for $\pi_*(L_2V(1))$ in which it is shown in [**11**] that $d_r(x) = y$ if and only if $d_r(x\zeta_2) = y\zeta_2$. □

For the differentials on $F \oplus F^*$ and F_n, we study the exact sequence

$$\cdots \longrightarrow E_2^{s,*}(L_2V(1)) \xrightarrow{i_*} E_2^{s,*}(L_2W) \xrightarrow{v_1} E_2^{s,*}(L_2W) \xrightarrow{\delta_s} E_2^{s+1,*}(L_2V(1)) \longrightarrow \cdots$$

associated to the cofiber sequence (2.1) in order to use the results on $E_2^{*,*}(L_2V(1))$:

(8.3) ([**11**, Prop.s 8.4, 9.13], [**4**]) *The differential d_5 of the spectral sequence for $\pi_*(L_2V(1))$ acts as follows*:

$$d_5(v_2^{3t+1}) = -tv_2^{3t-1}h_{11}b_{10}^2, \qquad d_5(v_2^{3t+1}b_{11}) = -(t-1)v_2^{3t+1}h_{10}b_{10}^3,$$

$$d_5(v_2^{3t+3}\psi_0) = -tv_2^{3t}b_{11}\xi b_{10}^2, \qquad d_5(v_2^{3t+1}\psi_1) = -(t+1)v_2^{3t}\xi b_{10}^3;$$

$$d_5(v_2^{3t-1}) = -(t+1)v_2^{3t-3}h_{11}b_{10}^2, \qquad d_5(v_2^{3t-1}b_{11}) = -tv_2^{3t-1}h_{10}b_{10}^3,$$

$$d_5(v_2^{3t+1}\psi_0) = -(t+1)v_2^{3t-2}b_{11}\xi b_{10}^2 \qquad d_5(v_2^{3t-1}\psi_1) = -(t-1)v_2^{3t-2}\xi b_{10}^3;$$

$$d_5(v_2^{3t}b_{10}) = -tv_2^{3t-2}h_{11}b_{10}^3, \qquad d_5(v_2^{3t}b_{11}) = -(t-1)v_2^{3t}h_{10}b_{10}^3,$$

$$d_5(v_2^{3t-1}\psi_0) = -(t-1)v_2^{3t-4}b_{11}\xi b_{10}^2, \quad d_5(v_2^{3t}\psi_1 b_{10}) = -(t+1)v_2^{3t-1}\xi b_{10}^3.$$

Here we take 1 for λ in [**11**], for short, and the undetermined integer k of [**11**] is shown to be 1 by [**4**]. Thus the results of (8.3) follow.

Lemma 8.4. *Let x be an element of F_n or $F \oplus F^*$. Then we see the following*:
(1) *If $x = i_*(\bar{x})$ and $d_r(\bar{x}) = \bar{y}$ in $E_r^{*,*}(L_2V(1))$, then*

$$d_r(x) = i_*(\bar{y}) \quad \text{in } E_r^{*,*}(L_2(V(1)).$$

(2) *If* $d_r(\delta_s(x)) = \delta_{s+r}(y)$, *then*

$$d_r(x) = y + \cdots$$

Here $\cdots$ *denotes an element of* J *given by*

$$J = E(2,1)_*\{v_2h_{10}/v_1, v_2^2h_{11}/v_1, \xi/v_1, b_{11}\xi/v_1\} \otimes (\boldsymbol{Z}/3)[b_{10}] \otimes \Lambda(\zeta_2).$$

Furthermore the generators of J *have the bidegrees*:

$$\|v_2^{3s+1}h_{10}/v_1\| = (1, 16(3s+1)), \quad \|v_2^{3s+2}h_{11}/v_1\| = (1, 16(3s+2)+8),$$

$$\|v_2^{3s}\xi/v_1\| = (2, 48s+4), \quad \|v_2^{3s}b_{11}\xi/v_1\| = (4, 48s+40),$$

$$\|b_{10}\| = (2, 12) \quad \text{and} \quad \|\zeta_2\| = (1, 0).$$

PROOF. Part (1) follows from the naturality of the differential d_r.

The hypothesis $d_r(\delta_s(x)) = \delta_{s+r}(y)$ of (2) implies $d_r(x) \equiv y \bmod \operatorname{Ker}\delta_{s+r} = \operatorname{Im} v_1$ by naturality. Besides, $d_r(x) \in E_r^{s,*}$ for $s \geq 5$ and $\bigoplus_{s\geq 5} E_r^{s,*} \subset G$, where $G = (F \oplus F^*) \otimes (\boldsymbol{Z}/3)[b_{10}] \otimes \Lambda(\zeta_2)$. Define $J = \operatorname{Im}(v_1 : G \to G)$, and we see $d_r(x) \equiv y \bmod J$. The structure of J follows from Theorem 2.5. □

PROPOSITION 8.5. *The differential* d_5 *on* $(F \oplus F^*) \otimes (\boldsymbol{Z}/3)[b_{10}] \otimes \Lambda(\zeta_2)$ *is read off from the following results on* $F \oplus F^*$ (*by Lemma* 8.2):

(a) $d_5(v_2^{3t+1}/v_1) = -tv_2^{3t-1}h_{11}b_{10}^2/v_1$

(a′) $d_5(v_2^{3t-1}/v_1) = 0$

(b) $d_5(v_2^{3t+1}h_{10}/v_1^2) = tv_2^{3t-1}b_{10}^3/v_1$

(c) $d_5(v_2^{3t+2}h_{11}/v_1^2) = (t-1)v_2^{3t-1}b_{11}b_{10}^2/v_1 + kv_2^{3t+1}h_{10}b_{10}^2\zeta_2/v_1$ *for some* $k \in \boldsymbol{Z}/3$

(d) $d_5(v_2^{3t+1}b_{11}/v_1) = -(t-1)v_2^{3t+1}h_{10}b_{10}^3/v_1$

(d′) $d_5(v_2^{3t-1}b_{11}/v_1) = 0$

(a)* $d_5(v_2^{3t+3}\psi_0/v_1) = -tv_2^{3t}b_{11}\xi b_{10}^2/v_1$

(a′)* $d_5(v_2^{3t+1}\psi_0/v_1) = 0$

(b)* $d_5(v_2^{3t}\xi/v_1^2) = -(t-1)v_2^{3t-2}\psi_0 b_{10}^2/v_1$

(c)* $d_5(v_2^{3t}b_{11}\xi/v_1^2) = (t+1)v_2^{3t-1}\psi_1 b_{10}^3/v_1$

(d)* $d_5(v_2^{3t+1}\psi_1/v_1) = -(t+1)v_2^{3t}\xi b_{10}^3/v_1$

(d′)* $d_5(v_2^{3t-1}\psi_0/v_1) = 0.$

PROOF. The first four equations of (8.3) give rise to (a), (d), (a)* and (d)* by Lemma 8.4 (1).

Proposition 3.4 shows the following:

$$v_2^{3t-3}h_{11}b_{10}^2 = \delta_4(v_2^{3t-2}b_{10}^2/v_1), \qquad v_2^{3t-1}h_{10}b_{10}^3 = \delta_6(v_2^{3t-2}b_{11}b_{10}^2/v_1),$$

$$v_2^{3t-2}b_{11}\xi b_{10}^2 = \delta_7(v_2^{3t}\psi_0 b_{10}^2/v_1), \quad v_2^{3t-2}\xi b_{10}^3 = \delta_7(v_2^{3t-2}\psi_1 b_{10}^2/v_1).$$

Since $i_*\delta_s(y)=0$, Lemma 8.4 (1) implies that the second four equations of (8.3) yield (a′), (d′), (a′)*, and (d′)*.

Furthermore, Proposition 3.4 shows the following:

$$\begin{array}{ll} \delta_1(v_2^{3t+1}h_{10}/v_1^2)=v_2^{3t}b_{10}, & \delta_1(v_2^{3t+2}h_{11}/v_1^2)=v_2^{3t}b_{11}, \\ \delta_2(v_2^{3t}\xi/v_1^2)=-v_2^{3t-1}\psi_0, & \delta_4(v_2^{3t}b_{11}\xi/v_1^2)=v_2^{3t}\psi_1b_{10}; \quad \text{and} \\ v_2^{3t-2}h_{11}b_{10}^3=-\delta_6(v_2^{3t-1}b_{10}^3/v_1), & v_2^{3t}h_{10}b_{10}^3=-\delta_6(v_2^{3t-1}b_{11}b_{10}^2/v_1), \\ v_2^{3t-4}b_{11}\xi b_{10}^2=-\delta_7(v_2^{3t-2}\psi_0b_{10}^2/v_1), & v_2^{3t-1}\xi b_{10}^3=-\delta_9(v_2^{3t-1}\psi_1b_{10}^3/v_1). \end{array}$$

Therefore, we apply Lemma 8.4 (2) to show

$$\begin{aligned} d_5(v_2^{3t+1}h_{10}/v_1^2)&=tv_2^{3t-1}b_{10}^3/v_1+\cdots \\ d_5(v_2^{3t+2}h_{11}/v_1^2)&=(t-1)v_2^{3t-1}b_{11}b_{10}^2/v_1+\cdots \\ d_5(v_2^{3t}\xi/v_1^2)&=-(t-1)v_2^{3t-2}\psi_0b_{10}^2/v_1+\cdots \\ d_5(v_2^{3t}b_{11}\xi/v_1^2)&=(t+1)v_2^{3t-1}\psi_1b_{10}^3/v_1+\cdots \end{aligned}$$

by using the last four equations of (8.3). Let $(J)^{s,t}$ denotes the submodule of J with bidegree (s,t). Then we see that $(J)^{s,u}=0$ if $(s,u)=(6,16(3t-1)+32)$, $(7,16(3t-2)+36)$ or $(9,16(3t-1)+56)$, and $=(\boldsymbol{Z}/3)\{v_2^{3t+1}h_{10}b_{10}^2\zeta_2/v_1\}$ if $(s,u)=(6,16(3t-1)+56)$. Therefore we obtain (b), (c), (b)* and (c)*. □

We here recall the folklore lemma which will be used later:

Lemma 8.6. *Let* $X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{h}\Sigma X$ *be a cofiber sequence with* $E(2)_*(h)=0$. *Then in the exact sequence* $E_2^*(L_2X)\xrightarrow{f_*}E_2^*(L_2Y)\xrightarrow{g_*}E_2^*(L_2Z)\xrightarrow{\delta}E_2^{*+1}(X)$, *we have the following*:

(1) *If we have a chart*

$$\begin{array}{ccccc} y & \xrightarrow{g_*} & z' & & \\ & & \nwarrow d_5 & & \\ & & & z & \xrightarrow{\delta} x, \end{array}$$

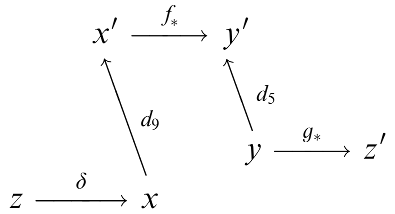
and x *is a permanent cycle, then* $f_*([x])=[y]$, *where* $[\cdot]$ *denotes a homotopy class.*

(2) *If we have a chart*

$$\begin{array}{ccccccc} x' & \xrightarrow{f_*} & y' & & & & \\ & & \nwarrow d_5 & & & & \\ & & & y & \xrightarrow{g_*} & z' & \\ & & & & & \nwarrow d_5 & \\ & & & & & & z \xrightarrow{\delta} x, \end{array}$$

then $d_9(x)=x'$.

(3) *If we have a chart*



then $d_5(z) = z'$.

COROLLARY 8.7. *Consider the cofiber sequence* $V(1) \xrightarrow{i} \Sigma^4 W \xrightarrow{v_1} W \xrightarrow{j} \Sigma V(1)$. *Then the induced map* $i_* : \pi_*(L_2 V(1)) \to \pi_{*-4}(L_2 W)$ *acts as follows*:

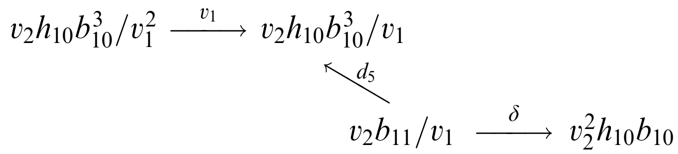
(a) $i_*(v_2^{9t+2}h_{10}) \equiv v_2^{9t+1}h_{10}b_{10}^2/v_1^2 \bmod \operatorname{Ker}\beta_1$

(a′) $i_*(v_2^{9t+2}h_{10}\zeta_2) \equiv v_2^{9t+1}h_{10}b_{10}^2\zeta_2/v_1^2 \bmod \operatorname{Ker}\beta_1$

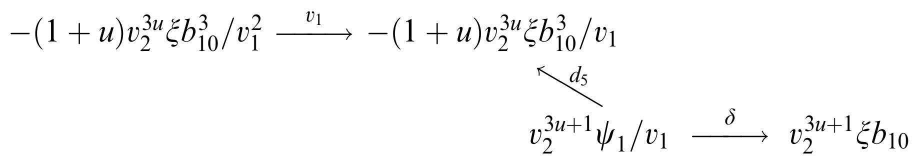
(b) $i_*(v_2^{3u+1}\xi) \equiv -(1+u)v_2^{3u}\xi b_{10}^2/v_1^2 \bmod \operatorname{Ker}\beta_1 \quad (u \not\equiv 0 \ (3))$

(b′) $i_*(v_2^{3u+1}\xi\zeta_2) \equiv -(1+u)v_2^{3u}\xi b_{10}^2\zeta_2/v_1^2 \bmod \operatorname{Ker}\beta_1 \quad (u \not\equiv 0 \ (3))$.

PROOF. We obtain the following charts



and



from Propositions 3.4 and 8.5 (d), (d)*. Since $v_2^{9t+2}h_{10}$ and $v_2^{3u+1}\xi$ with $u \not\equiv 0 \bmod 3$ are permanent cycles by [**11**, Th. A], $i_*(v_2^{9t+2}h_{10}b_{10}) = v_2^{9t+1}h_{10}b_{10}^3/v_1^2$ and $i_*(v_2^{3u+1}\xi b_{10}) = (1+u)v_2^{3u}\xi b_{10}^3/v_1^2$ by Lemma 8.6 (1). Now divide it by b_{10} which represents $\beta_1 \in \pi_*(S^0)$, and we obtain (a) and (b).

In the same way, we obtain (a′) and (b′) by Lemma 8.2. □

PROPOSITION 8.8. *The differential* d_5 *on* $F_n \otimes \Lambda(\zeta_2)$ *is read off from the following relation on* F_n (*by Lemma* 8.2):

(a) $d_5(v_2^{3^{n+1}(3t+1)}/v_1^{4\cdot 3^n-1}) = \begin{cases} v_2^{9t+1}h_{10}b_{10}^2/v_1 & n = 0 \\ 0 & n > 0 \end{cases}$

(a′) $d_5(v_2^{3^{n+1}(3t-1)}/v_1^{4\cdot 3^n-1}) = 0$

(b) $d_5(v_2^{3^{n+2}t+3^{n+1}}h_{10}/v_1^{2\cdot 3^{n+1}+1}) = \begin{cases} \pm v_2^{9t}\xi b_{10}^2/v_1 & n = 0 \\ (-1)^n v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/v_1 & n > 0 \end{cases}$

(c) $d_5(v_2^{3^{n+2}t+8\cdot 3^n}h_{10}/v_1^{10\cdot 3^n+1}) = \begin{cases} 0 & n = 0, 1 \\ -v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/v_1 & n > 1 \end{cases}$

(d) $d_5(v_2^{3^n(9t+5\pm3)+(3^n-1)/2}\xi/v_1^{4\cdot 3^n}) = v_2^{3^{n+1}(3t+1\pm1)+3(3^n-1)/2-1}\psi_1 b_{10}^2/v_1 \quad (n>0)$

(e) $d_5(v_2^{9t+8}\xi/v_1^4) = 0$

(e′) $d_5(v_2^{9t+2}\xi/v_1^4) = -v_2^{9t-1}\psi_1 b_{10}^2/v_1.$

Here the symbol $\pm$ *on the right hand sides denotes an undetermined sign.*

Proof. Put $b(0)=9t+2$ and $b(n)=3^n(9t+5\pm3)+(3^n-1)/2$ for $n>0$. Then we compute $d_5(\delta_2(v_2^{b(n)}\xi/v_1^{4.3^n})) = d_5(v_2^{b(n)-3^n-1}\psi_1) = k_n v_2^{b(n)-3^n-2}\xi b_{10}^3 = k_n\delta_7(v_2^{b(n)-3^n-2}\psi_1 b_{10}^2/v_1)$ for $k_0=-1$ and $k_n=1$ $(n>0)$, since $b(n)-3^n-1\equiv 0$ or 3 mod 9. In fact, $d_5(v_2^{3t}\psi_1) = -(1+t)v_2^{3t-1}\xi b_{10}^3$ by [**11**, Prop. 9.13]. Therefore Lemma 8.4 (2) implies (d) and (e′), since $(J)^{s,u}=0$ for $(s,u)=(7,16(b(n)-3^n-2)+44)$ $(n\ge 0)$.

Take $x/v_1^a\in F_n$. For the cases (a), (a′), (b), (c) and (e), $d_5(\delta_*(x/v_1^a))=0$ by the relation $d_5(v_2^th_{10})=0=d_5(v_2^t\xi)$ shown in [**11**, Prop.s 8.4, 9.13]. Therefore, $d_5(x/v_1^a)\in J$ by Lemma 8.4 (2). Comparing degrees, we have (a) for $n>0$ and (a′), and (c) for $n=0$. Besides,

$$d_5(v_2^{9t+3}/v_1^3) = k_1v_2^{9t+1}h_{10}b_{10}^2/v_1 + k_2v_2^{9t}b_{11}\xi\zeta_2/v_1$$

$$d_5(v_2^{9t-3}/v_1^3) = k_3v_2^{9t-5}h_{10}b_{10}^2/v_1 + k_4v_2^{9t-6}b_{11}\xi\zeta_2/v_1$$

$$d_5(v_2^{3^{n+2}t+3^{n+1}}h_{10}/v_1^{2\cdot3^{n+1}+1}) = k_5v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/v_1 \quad (n\ge 0)$$

$$d_5(v_2^{3^{n+2}t+8\cdot3^n}h_{10}/v_1^{10\cdot3^n+1}) = k_6v_2^{3^{n+2}t+5\cdot3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/v_1 \quad (n\ge 1)$$

$$d_5(v_2^{9t+8}\xi/v_1^4) = k_7v_2^{9t+6}\xi b_{10}^2\zeta_2/v_1 + k_8v_2^{9t+5}h_{11}b_{10}^3/v_1$$

for some $k_i\in \boldsymbol{Z}/3$ $(1\le i\le 8)$. Since $v_2^{9t+5}h_{11}b_{10}^3/v_1$ is hit by d_5 of $v_2^{9t+7}b_{10}/v_1$ by Proposition 8.5, we take $k_8=0$ by replacing $v_2^{9t+8}\xi/v_1^4$ by $v_2^{9t+8}\xi/v_1^4 - k_8v_2^{9t+7}b_{10}/v_1$, that is,

$$d_5(v_2^{9t+8}\xi/v_1^4) = k_7v_2^{9t+6}\xi b_{10}^2\zeta_2/v_1.$$

Now we determine the numbers k_i for $1\le i\le 7$. Consider the following charts:

$$\begin{array}{ccccc}
k_2v_2^{-1}\psi_1b_{10}^3\zeta_2/v_1 & & & & \\
\nwarrow d_5 & & & & \\
k_1v_2h_{10}b_{10}^2/v_1^2 + k_2b_{11}\xi\zeta_2/v_1^2 & \xrightarrow{v_1} & k_1v_2h_{10}b_{10}^2/v_1 + k_2b_{11}\xi\zeta_2/v_1 & & \\
 & & \nwarrow d_5 & & \\
 & & v_2^3/v_1^3 & \xrightarrow{\delta_0} & v_2^2h_{10}
\end{array}$$

and

$$\begin{array}{ccccc}
k_3v_2^{-7}b_{10}^5/v_1 - k_4v_2^{-7}\psi_1b_{10}^3\zeta_2/v_1 & & & & \\
\nwarrow d_5 & & & & \\
k_3v_2^{-5}h_{10}b_{10}^2/v_1^2 + k_4v_2^{-6}b_{11}\xi\zeta_2/v_1^2 & \xrightarrow{v_1} & k_3v_2^{-5}h_{10}b_{10}^2/v_1 + k_4v_2^{-6}b_{11}\xi\zeta_2/v_1 & & \\
 & & \nwarrow d_5 & & \\
 & & v_2^{-3}/v_1^3 & \xrightarrow{\delta_0} & -v_2^{-4}h_{10}
\end{array}$$

obtained from Propositions 3.4 and 8.5 (b), (c)*. The numbers k_2, k_3 and k_4 are seen to be zero by Lemma 8.6 (1), since $v_2^2h_{10}$ and $v_2^{-4}h_{10}$ are permanent cycles by [**11**, Cor. 10.7]. Besides Corollary 8.7 shows that $k_1 = 1$. By similar charts, the third and the fourth equations imply

$$i_*(v_2^{3^{n+2}t+(3^{n+1}-1)/2}\xi) = (-1)^n k_5 v_2^{3^{n+2}t+3(3^n-1)/2}\xi b_{10}^2/v_1^2 \quad (n>0)$$

$$i_*(v_2^{3^{n+2}t+5\cdot 3^n+(3^n-1)/2}\xi) = -k_6 v_2^{3^{n+2}t+5\cdot 3^n+3(3^{n-1}-1)/2}\xi b_{10}^2/v_1^2 \quad (n\ge 1)$$

using Propositions 3.4 and 8.5. In fact, for $n>0$, $v_2^{3^{n+2}t+(3^{n+1}-1)/2}\xi$ and $v_2^{3^{n+2}t+5\cdot 3^n+(3^n-1)/2}\xi$ are permanent cycles by [**11**, Cor. 10.7]. Now compare with Corollary 8.7, and we obtain

$$k_5 = (-1)^n \quad \text{if } n>0, \quad \text{and} \quad k_6 = \begin{cases} 0 & n=1 \\ -1 & n>1. \end{cases}$$

If $n=0$, then $k_5 = \pm 1$ by applying Lemma 8.6 (3) to the following chart (up to sign):

$$\begin{array}{ccccccc}
-v_2^{-2}\psi_0 b_{10}^4 & \xrightarrow{i_*} & -v_2^{-2}\psi_0 b_{10}^4/v_1 & & & & \\
 & \nwarrow^{d_9} & & \nwarrow^{d_5} & & & \\
\pm v_2^3 h_{10}/v_1^7 & \xrightarrow{\delta_1} & \pm v_2\xi & & \xi b_{10}^2/v_1^2 & \xrightarrow{v_1} & \xi b_{10}^2/v_1.
\end{array}$$

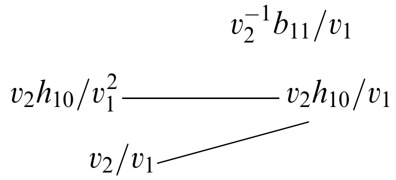
Consider again the chart

$$\begin{array}{ccccc}
k_7 v_2^4\psi_0 b_{10}^5\zeta_2/v_1 & & & & \\
\nwarrow^{d_5} & & & & \\
k_7 v_2^6\xi b_{10}^3\zeta_2/v_1^2 & \xrightarrow{v_1} & k_7 v_2^6\xi b_{10}^6\zeta_2/v_1 & & \\
 & & \nwarrow^{d_5} & & \\
 & & v_2^8\xi/v_1^4 & \xrightarrow{\delta_2} & \delta_2(v_2^8\xi/v_1^4)
\end{array}$$

obtained from Propositions 3.4 and 8.5. Since $\delta_2(v_2^8\xi/v_1^4) = v_2^6\psi_1 - v_2^7\xi\zeta_2 - k_8 v_2^{9t+6}h_{11}b_{10}$ is a permanent cycle by [**11**, Cor. 10.7], we obtain $k_7 = 0$. □

These propositions give us the following charts of E_2-term with d_5, in which holizontal lines of length 4 denote multiplication by v_1, lines of slope $1/3$, multiplication by h_{10} and lines of slope $1/11$, multiplication by h_{11}. The differential d_5 is expressed by arrows of slope -5. Besides, the same pattern of period $(10,2)$ denotes elements obtained by multiplication by b_{10}.

The following chart is the one on $F\otimes(\boldsymbol{Z}/3)[b_{10}]$, where $\dot{\nearrow}$ starting from dimension 12 is multiples by b_{10} with

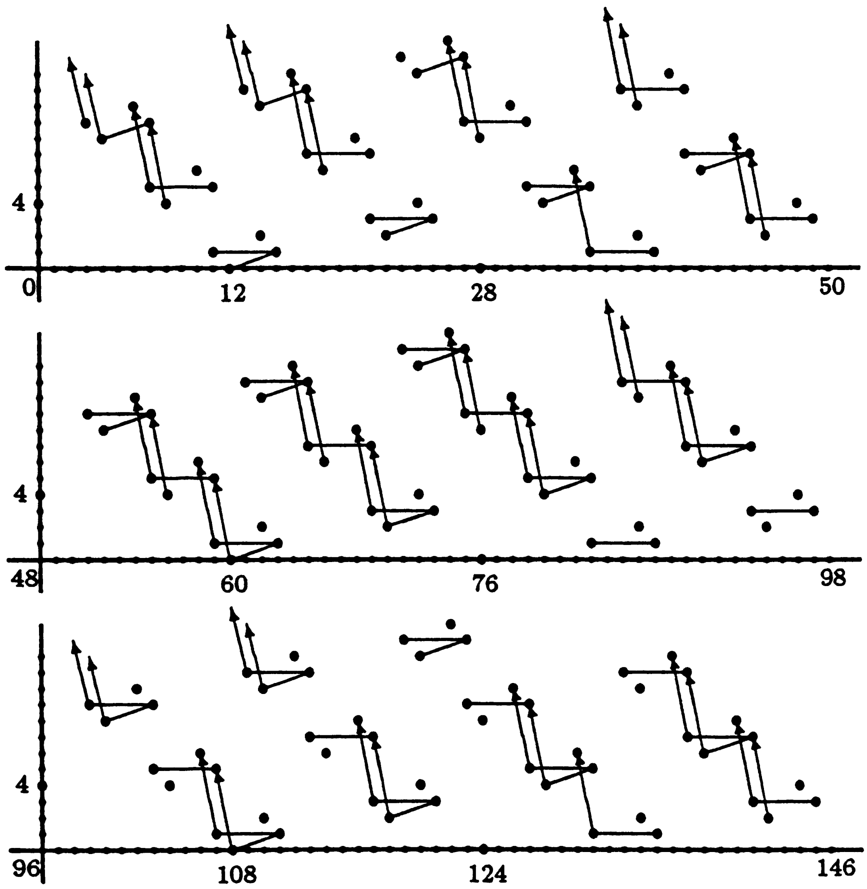


The other one . ⟵ · is generated by

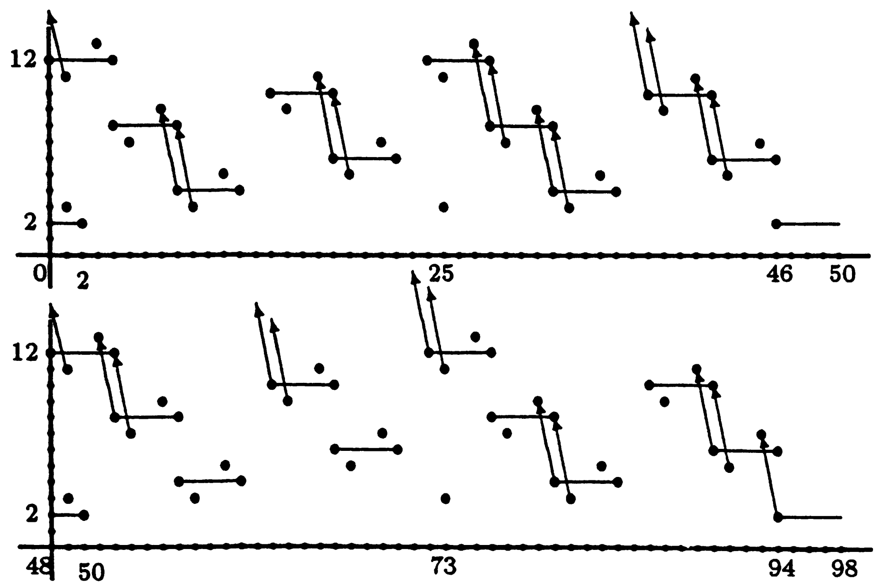
$v_2^2b_{10}/v_1$

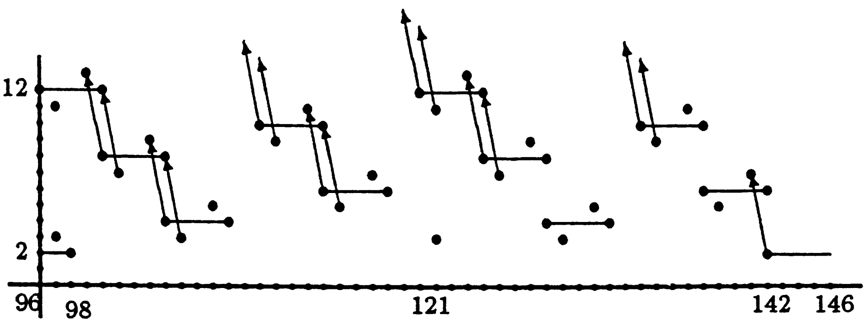
$v_2^2h_{11}/v_1^2$—$v_2^2h_{11}/v_1$

v_2^2/v_1



The next one is the E_2-term with d_5 on $F^* \otimes (\boldsymbol{Z}/3)[b_{10}]$. Each dot can be read off from the degree of the generator.





For the part with ζ_2, just shift the dimension by $(1,0)$.

Now turn to the differential $d_9 : E_9^s(L_2W) \to E_9^{s+9}(L_2W)$. In the following, everything is computed up to sign as in [**11**, §10].

Proposition 8.9. *On the elements of* $E_9^*(L_2W)$ *originating* $(F \oplus F^*) \otimes (\boldsymbol{Z}/3)[b_{10}] \otimes \Lambda(\zeta_2)$, d_9 *is read off from the following* (*by Lemma* 8.2):

(a) $d_9(v_2^{9t+4}h_{10}/v_1) = v_2^{9t+1}b_{10}^5/v_1$

(b) $d_9(v_2^{9t+8}h_{11}/v_1) = v_2^{9t+4}b_{11}b_{10}^4/v_1$

(c) $d_9(v_2^{9t+2}b_{11}/v_1) = v_2^{9t+1}h_{10}b_{10}^5/v_1^2$

(d) $d_9(v_2^{9t+8}/v_1) = v_2^{9t+5}h_{11}b_{10}^4/v_1^2$

(a)* $d_9(v_2^{9t+6}\xi/v_1) = v_2^{9t+3}\psi_0 b_{10}^4/v_1$

(b)* $d_9(v_2^{9t}b_{11}\xi/v_1) = v_2^{9t-2}\psi_1 b_{10}^5/v_1$

(c)* $d_9(v_2^{9t+5}\psi_1/v_1) = v_2^{9t+3}\xi b_{10}^5/v_1^2$

(d)* $d_9(v_2^{9t+1}\psi_0/v_1) = v_2^{9t-3}b_{11}\xi b_{10}^4/v_1^2.$

Proof. It is shown in [**11**, Prop. 10.5]

$$d_9(v_2^{9t+4}h_{10}) = v_2^{9t+1}b_{10}^5, \qquad d_9(v_2^{9t+8}h_{11}) = v_2^{9t+4}b_{11}b_{10}^4,$$
$$d_9(v_2^{9t+6}\xi) = v_2^{9t+3}\psi_0 b_{10}^4, \quad d_9(v_2^{9t}b_{11}\xi) = v_2^{9t-2}\psi_1 b_{10}^5.$$

Then Lemma 8.4 (1) implies immediately (a), (b), (a)* and (b)*.

Put $A = v_2^2 b_{11}$ (resp. $v_2^5\psi_1$, $v_2\psi_0$, $v_2^8 b_{10}$), $B = v_2^2 h_{10}b_{10}^3$ (resp. $v_2^4\xi b_{10}^3$, $v_2^{-2}b_{11}\xi b_{10}^2$, $v_2^6 h_{11}b_{10}^3$), $C = v_2 b_{11}b_{10}^2/v_1$ (resp. $v_2^4\psi_1 b_{10}^2/v_1$, $\psi_0 b_{10}^3/v_1$, $v_2^7 b_{10}^3/v_1$) and $D = v_2 h_{10}b_{10}^5$ (resp. $v_2^3\xi b_{10}^5$, $v_2^{-3}b_{11}\xi b_{10}^4$, $v_2^5 h_{11}b_{10}^5$). Then $d_5(A) = -B$ by (8.3), $\delta_*(C) = B$ by Proposition 3.4, and $d_5(C) = D$ by Proposition 8.5. Now Lemma 8.6 (2) implies $d_9(A/v_1) = D$, which shows (c), (c)*, (d)* and

$$d_9(v_2^{9t+8}b_{10}/v_1) = v_2^{9t+5}h_{11}b_{10}^5/v_1^2.$$

The equality (d) is obtained by dividing this by b_{10}. □

The following completes the computation of the differentials.

PROPOSITION 8.10. *On the elements of* $E_6^*(W)$ *originating* $F_n \otimes \Lambda(\zeta_2)$, d_9 *is given by the following* (*by Lemma* 8.2):

(a) $d_9(v_2^{3^{n+1}(3t+1)}/v_1^{4\cdot 3^n-1}) = 0 \ \ (n > 0)$

(a′) $d_9(v_2^{3^{n+1}(3t-1)}/v_1^{4\cdot 3^n-1}) = 0$

(c) $d_9(v_2^{9t+8}h_{10}/v_1^{11}) = v_2^{9t+3}\xi b_{10}^4/v_1^2$

$d_9(v_2^{3(9t+8)}h_{10}/v_1^{31}) = 0$

(e) $d_9(v_2^{9t+8}\xi/v_1^4) = 0$

PROOF. Consider the total degree

$$|v_2^{3^{n+1}(3t+1)}/v_1^{4\cdot 3^n-1}| \equiv \begin{cases} 100 \mod 144 & n=1 \\ 4 \mod 144 & n \geq 2, \end{cases}$$

$$|v_2^{3^{n+1}(3t-1)}/v_1^{4\cdot 3^n-1}| \equiv \begin{cases} 84 \mod 144 & n=0 \\ 100 \mod 144 & n=1 \\ 4 \mod 144 & n > 0, \end{cases}$$

$$|v_2^{3(9t+8)}h_{10}/v_1^{31}| \equiv 119 \mod 144$$

$$|v_2^{9t+8}\xi/v_1^4| \equiv 118 \mod 144.$$

Then the chart shows that nothing can be hit by d_9 of these elements. Thus we obtain (a), (a′), the second one of (c), and (e). For the first one of (c), we compute

$$\delta_{11}(d_9(v_2^8h_{10}/v_1^{11})) = d_9(v_2^5\xi) = v_2^2\psi_0 b_{10}^4 = \delta_{11}(v_2^3\xi b_{10}^4/v_1^2)$$

by [**11**, Prop. 10.5], and see the equation by Lemma 8.4 (2). □

THEOREM 8.11. *The* E_{10}*-term* $E_{10}(W)$ *is isomorphic to the direct sum of* $k(1)_*$*-modules* $(K(1)_*/k(1)_*) \otimes \Lambda(h_{10}, \zeta_2)$, $\sum_{n\geq 0} \tilde{F}_n \otimes \Lambda(\zeta_2)$ *and* $(\tilde{F} \oplus \tilde{F}^*) \otimes \Lambda(\zeta_2)$ *for* $k(1)_*$*-modules in* (2.7).

PROOF OF THEOREM 2.8. Since $E_{10}^*(W)$ has a holizontal vanishing line by Theorem 8.11, we have $E_{10}^{*,*}(W) = E_\infty^{*,*}(W)$. Furthermore, there arises no extension problem in the spectral sequence, since $\pi_*(L_2W)$ is a $\pi_*(V(0))$-module and so $(\boldsymbol{Z}/3)$-vector space. Therefore we obtain the homotopy groups $\pi_*(L_2W) = E_{10}^*(W)$. □

§9. β-elements.

The β-elements in the E_2-term for $\pi_*(S^0)$ are defined in [**6**]. Here we modifies it in the E_2-term $H^*E(2)$ for $\pi_*(L_2S^0)$ as follows: Let $0 \to E(2)_* \xrightarrow{3} E(2)_* \to E(2)_*/(3) \to 0$ and $0 \to E(2)_*/(3) \to v_1^{-1}E(2)_*/(3) \to M_1^1 \to 0$ be short exact sequences, and $\delta : H^*E(2)_*/(3) \to H^{*+1}E(2)_*$ and $\delta' : H^*M_1^1 \to H^{*+1}E(2)_*/(3)$ the connecting homomorphisms associated to the short exact sequences, respectively. Then for an element of the form v_2^a/v_1^b in $H^0M_1^1$, we define

$$\beta_{a/b} = \delta\delta'(v_2^a/v_1^b) \in H^*E(2)_*$$

and $\beta_a = \beta_{a/1}$, which is essential in the E_2-term $H^*E(2)_*$ for $\pi_*(L_2S^0)$.

Consider the cofiber sequences defining the spectra $V(0)$ and $W : S^0 \xrightarrow{3} S^0 \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S^0$ and $V(0) \xrightarrow{\lambda} L_1V(0) \xrightarrow{\iota} W \xrightarrow{\pi} \Sigma V(0)$, respectively. If an element v_2^a/v_1^b is a permanent cycle, then so is $\beta_{a/b}$ by Geometric Boundary Theorem (*cf.* [**9**]).

PROOF OF THEOREM 2.12. By Theorem 2.8, we see that the elements v_2^j/v_1 for $j \equiv 0, 1, 2, 3, 5, 6 \bmod 9$ are all permanent cycles. Thus 'if' part is shown. 'Only if' part is shown in [**11**]. □

PROOF OF THEOREM 2.13. The element v_2^a/v_1^b with $9|a$ is in $\tilde{F}_1$ or $\tilde{F}_n$ of (2.7), and so Theorem 2.8 shows (c). For the case $9 \nmid a$, the part (a) follows from Theorem 2.12. $v_2^{9t\pm3}/v_1^b$ comes from $\tilde{F}_0$ of (2.7), and we obtain the part (b). □

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