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Quantum ergodicity at a finite energy level

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Abstract. The purpose of this paper is to formulate the notion of *quantum ergodicity at a finite energy level* for certain quantum mechanics, by using the method of Sunada [**Su1**]. Under some assumptions on the corresponding classical mechanics, we obtain a necessary and sufficient condition in terms of *semi-classical* asymptotic behaviour of eigenfunctions of a quantum Hamiltonian so that the classical mechanics is ergodic. We also obtain a result on *quantum weak mixing at a finite energy level* which is a semi-classical analogue of the notion introduced in [**Z4**].

1. Introduction.

The eigenfunctions of a quantum Hamiltonian with *ergodic* classical counterpart have remarkable asymptotic behaviour. For instance, it is well-known that any orthonormal basis of eigenfunctions of the Laplacian on a compact Riemannian manifold with ergodic geodesic flow is, roughly speaking, asymptotically uniformly distributed ([**Sn**], [**Z1**], [**CdV**]).

In a recent paper [**Su1**], Sunada obtained a necessary and sufficient condition in terms of asymptotic behaviour of the eigenfunctions so that the corresponding classical dynamical system is ergodic. He introduced the notion of *quantum ergodicity at infinite energy level* for quantum mechanics, and his result is obtained by studying the relationship between classical ergodicity and quantum ergodicity. That is, he showed that classical ergodicity is equivalent to quantum ergodicity at infinite energy level with an additional condition on the quantum mechanics. This notion introduced by Sunada is a natural quantum analogue of Boltzmann's ergodic hypothesis. In fact, he also noted that a notion of ergodicity *at infinite energy level* can be defined for certain classical systems.

The classical system investigated in the above works is homogeneous Hamilton flow, that is the flow which commutes with $\boldsymbol{R}_+$-action on the cotangent bundle. However, there are natural classical systems which are not homogeneous. For example, the magnetic flow under the uniform magnetic field on a compact Riemann surface with constant negative curvature -1 has different behaviour on different energy surfaces ([**G-U1**], [**Su2**]). This phenomenon arises from the fact that the magnetic flow is not homogeneous. Ergodicity of such dynamical systems affects *semi-classical* asymptotic behaviour of the eigenfunctions of corresponding quantum Hamiltonian ([**H-M-R**], [**S-T**], [**Z2**]).

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Our purposes of this note are to formulate a notion of quantum ergodicity for the quantum mechanics corresponding to such a classical system as the above example using the method of Sunada, and investigate the relationship between classical and quantum ergodicity. We call the notion introduced in this paper quantum ergodicity *at a finite energy level* because we take the dependence of dynamical behaviour on the energy level into consideration.

We will give a brief account of the dynamical system discussed in this paper. The precise formulation of the dynamical system, which is the same as in [**Z2**], is described in the next section.

We note that the magnetic flow is obtained as a *reduction* of the geodesic flow on a compact S^1 bundle with a connection 1-form and with Riemannian metric which is invariant under S^1-action. In this case the magnetic field is represented by the curvature 2-form of the connection form. Therefore, in general, we consider the reduced dynamical system of the Hamilton flow generated by the Hamiltonian which is invariant under group action on the cotangent bundle over a compact principal bundle. The corresponding quantum mechanics is generated by a first order positive elliptic pseudodifferential operator (ψDO for short) which commutes with group action. However, as the case of classical mechanics, we need to consider a *reduced* quantum mechanics. More precisely, we consider the operator restricted to a *ladder subspace* associated with a fixed irreducible representation of the structure group as a reduced quantum Hamiltonian. We will define the notion of quantum ergodicity at a finite energy level for the quantum mechanics generated by the reduced Hamiltonian. To study the relationship between classical and quantum ergodicity, we will use the trace formula due to Guillemin-Uribe [**G-U2**] and Zelditch [**Z2**].

We mention the contents of this paper. Our main theorems will be stated in section 3. In section 4, we will define the notion of quantum ergodicity at a finite energy level, and the main theorems will be proved in section 5 and section 6. Recently, Zelditch [**Z4**] introduced the notion of *quantum weak mixing*. As a semi-classical analogy of this notion, we will introduce the notion of *quantum weak mixing at a finite energy level* in section 7. In section 8, we will mention the example of the magnetic flow on a Riemann surface.

2. Formulation of dynamical systems.

In this section, we will describe the formulation of dynamical systems discussed in this paper, and mention some properties of it.

Let $\pi : P \to M$ be a compact connected principal bundle over a compact Riemannian manifold M with structure group G, a compact connected Lie group. Choosing a biinvariant metric on G and a connection 1-form on P, we have a unique G-invariant metric on P which makes the bundle $\pi : P \to M$ into a Riemannian submertion, with fibers isometric to G. We fix such a metric.

Let $\hat{H}$ be a positive elliptic ψDO of order one on P commuting with G-action and let $H = \sigma_1(\hat{H})$ be its principal symbol. Since $\hat{H}$ commutes with G-action, H is a G-invariant smooth function on the punctured cotangent bundle $T^*P\backslash 0$. The (left) action of G on T^*P is Hamiltonian and its equivariant moment map $\Phi : T^*P \to \mathscr{G}^*$ is given by

$$\langle \Phi(p,\zeta), A\rangle = \zeta(A_p^*), \quad (p,\zeta)\in T^*P, \quad A\in\mathscr{G},$$

where $\mathscr{G}^*$ is the dual space of the Lie algebra $\mathscr{G}$ of G. Let (π_λ, V_λ) be an irreducible representation of G with the highest weight λ in the positive Weyl chamber of a dual Cartan subalgebra, and let $\mathscr{O}_\lambda$ be the coadjoint orbit through λ.

Since the differential map $d\Phi$ of Φ at each point is surjective, $\Phi^{-1}(\mathscr{O}_\lambda)$ is a submanifold in T^*P, and G acts freely on it. The leaves of the null-foliation of the G-invariant closed 2-form $\iota_\lambda^*\Omega_P - \Phi^*\omega_\lambda$ on $\Phi^{-1}(\mathscr{O}_\lambda)$ are just the G-orbits, where Ω_P is the canonical symplectic form on T^*P and ω_λ is the Kostant-Kirillov symplectic form on $\mathscr{O}_\lambda$, and hence it induces the symplectic form Ω_λ on $X_\lambda = \Phi^{-1}(\mathscr{O}_\lambda)/G$. The symplectic manifold $(X_\lambda, \Omega_\lambda)$ is called the *reduced* phase space.

The G-invariant Hamiltonian H descends to the Hamiltonian H_λ on X_λ. Let φ_t^λ denote the restriction of the Hamilton flow of $(H_\lambda, \Omega_\lambda)$ on the energy surface $\Sigma_e^\lambda = H_\lambda^{-1}(e)$, which preserves the Liouville measure ω_e^λ. We thus obtain the classical dynamical system $\mathrm{CD}_e^\lambda = (\Sigma_e^\lambda, \varphi_t^\lambda, \omega_e^\lambda)$. A quantum counterpart of the dynamical system CD_e^λ will be described as follows:

The action of G breaks $L^2(P)$, the Hilbert space of square integrable functions on P, into a direct sum of Hilbert spaces,

$$L^2(P) = \bigoplus_\mu \mathscr{L}_\mu,$$

the sum taken over isotypical subspaces $\mathscr{L}_\mu$ associated to the irreducible representation (π_μ, V_μ) corresponding to the dominant integral weight μ. More precisely, the Hilbert space $\mathscr{L}_\mu$ is the closure of the image of the evaluation map, $\mathrm{Hom}_G(V_\mu, L^2(P)) \otimes V_\mu \to L^2(P)$. The Hilbert space $\mathscr{L}_\mu$ is also obtained by the following way: Since the operator $\hat{H}$ is elliptic and the manifold P is compact, the Hilbert space $L^2(P)$ is the direct sum of finite dimensional eigenspaces of $\hat{H}$. Since the operator $\hat{H}$ commutes with G-action, G acts on each eigenspace, and hence each eigenspace is decomposed into irreducible representations. The Hilbert space $\mathscr{L}_\mu$ is the direct sum of the irreducible representations obtained in this manner which is equivalent to the irreducible representation corresponding to μ. We set

$$\mathscr{V}_\lambda = \bigoplus_{m=1}^{\infty} \mathscr{L}_{m\lambda} (\subset L^2(P)).$$

The subspace $\mathscr{V}_\lambda$ is called the *ladder space* associated to λ ([**G-S2**], [**G-U2**]). Let $e_1(m) \le e_2(m) \le \cdots$ be the eigenvalues of the restriction of the operator $\hat{H}$ to $\mathscr{L}_{m\lambda}$ and let $\{v_j^m\}_{j\in N}$ be the orthonormal basis for $\mathscr{L}_{m\lambda}$ of the eigenfunctions of $\hat{H}$, $\hat{H}v_j^m = e_j(m)v_j^m$.

Now we set up the triple $\mathrm{QD}^\lambda = (\mathscr{V}_\lambda, \hat{H}_\lambda, \Psi_0^\lambda)$ as a quantum dynamical system where $\hat{H}_\lambda$ is the restriction of $\hat{H}$ to $\mathscr{V}_\lambda$, Ψ_0 is the algebra consisted of all ψDO of order zero commuting with G-action and Ψ_0^λ is the algebla of operators which are the restriction of the elements in Ψ_0 to the Hilbert space $\mathscr{V}_\lambda$. We consider the algebra Ψ_0 as the algebra of quantum observables. We will call the dynamical system QD^λ the *reduced* quantum dynamical system.

Before going to discussion of quantum ergodicity, we must mention some properties of the classical dynamical system CD_e^λ.

Let Z_e be the energy surface of H, the principal symbol of $\hat{H}$, at energy e, and let ϕ_t be the Hamilton flow generated by H and the canonical symplectic form Ω_P on T^*P. The flow ϕ_t commutes with G-action, and hence it induces a flow on X_λ. Note that the flow induced by ϕ_t just coincides with the flow φ_t^λ.

We consider the following condition:

(H1) *The Hamiltonian vector field*, X_H, *of* H *is not tangent to the* G-*orbit through any point in* $\tilde{\Sigma}_e^\lambda := Z_e \cap \Phi^{-1}(\mathcal{O}_\lambda)$.

Note that, for example, the Laplacian on P with respect to the fixed metric (see Section 1) satisfies the condition (H1) if $e > |\lambda|$. The assumtion (H1) makes us to obtain the following lemmas.

Lemma 1. *Under the assumption* (H1), *the subset* $\tilde{\Sigma}_e^\lambda$ *is a submanifold in* T^*P, *and thus* $\tilde{\Sigma}_e^\lambda$ *is a principal* G-*bundle over* Σ_e^λ.

Proof. Since the differential map $d\Phi_z$ at $z \in \tilde{\Sigma}_e^\lambda$ is surjective, we have $T_z\Phi^{-1}(\mathcal{O}_\lambda) = d\Phi_z^{-1}(T_{\Phi(z)}\mathcal{O}_\lambda)$. By the equivariance of Φ, we obtain

$$d\Phi_z^{-1}(T_{\Phi(z)}\mathcal{O}_\lambda) = \mathcal{G}(z) + \mathcal{G}(z)^\perp,$$

where $\mathcal{G}(z) = \{A_z^* \in T_zT^*P;\ A \in \mathcal{G}\}$ and "$\perp$" denotes the annihilator of $\mathcal{G}(z)$ with respect to Ω_P. Since H is G-invariant, we have $\mathcal{G}(z) \subset (X_H)^\perp = T_zZ_e$. Therefore we obtain

$$T_zZ_e + T_z\Phi^{-1}(\mathcal{O}_\lambda) = (X_H)^\perp + \mathcal{G}(z)^\perp,$$

and hence

$$(T_zZ_e + T_z\Phi^{-1}(\mathcal{O}_\lambda))^\perp = (X_H) \cap \mathcal{G}(z).$$

By the assumption (H1), the right hand side of the above expression is zero. So the submanifolds Z_e and $\Phi^{-1}(\mathcal{O}_\lambda)$ intersect transversally. Thus we conclude the assertion. □

Lemma 2. *For each smooth function* f *on* Σ_e^λ, *there exists a smooth function* F *on* $T^*P\backslash 0$ *which is* G-*invariant*, *homogeneous of degree zero such that*

$$q_\lambda^* f = F \quad \text{on } \tilde{\Sigma}_e^\lambda,$$

where q_λ *is the projection from* $\tilde{\Sigma}_e^\lambda$ onto Σ_e^λ.

Proof. The function $q_\lambda^* f$ is a G-invariant smooth function on $\tilde{\Sigma}_e^\lambda$, and it can be extended to a smooth function F_0 on Z_e. Averaging F_0 on G and extending to a smooth function on $T^*P\backslash 0$ of degree zero, we obtain a desired function. □

We also note that a G-invariant smooth function a on $T^*P\backslash 0$ descends to a smooth function on X_λ. Then we will continue to denote by a the function on X_λ induced by a.

Next, we will review the ergodicity of the classical dynamical system CD_e^λ.

For each square integrable function $f \in L^2(\Sigma_e^\lambda)$, let f_t be the time average of f up to time t, $f_t = (1/t)\int_0^t f \circ \varphi_\tau^\lambda \, d\tau$. Birkhoff's ergodic theorem says that the (long) time average $\bar{f} = \lim_{t\to\infty} f_t$ exists a.e. The dynamical system CD_e^λ is said to be *ergodic* if the time average $\bar{f}$ identically equals the space average $\langle f \rangle_e^\lambda = \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} f \, d\omega_e^\lambda$, or equivalently

$$\langle |\bar{f}|^2 \rangle_e^\lambda = |\langle f \rangle_e^\lambda|^2 \tag{1}$$

for all smooth function $f \in C^\infty(\Sigma_e^\lambda)$.

LEMMA 3. *For all* $f \in C^\infty(\Sigma_e^\lambda)$, *we have the following*:
(1) $\lim_{t\to\infty} \langle |f_t|^2 \rangle_e^\lambda = \langle |\bar{f}|^2 \rangle_e^\lambda$,
(2) $\langle |\bar{f}|^2 \rangle_e^\lambda \geq |\langle f \rangle_e^\lambda|^2$.

PROOF.
(1) This is a direct consequence of Lebesgue's convergence theorem.
(2) By Birkhoff's ergodic theorem, we have $\langle \bar{f} \rangle_e^\lambda = \langle f \rangle_e^\lambda$. Therefore

$$\langle |\bar{f}|^2 \rangle_e^\lambda - |\langle f \rangle_e^\lambda|^2 = \langle |\bar{f} - \langle f \rangle_e^\lambda|^2 \rangle_e^\lambda \geq 0. \qquad \square$$

3. Statement of main theorems.

To state our main theorems, we prepare some notation. For a fixed constant $c > 0$, let

$$\mathcal{N}_m(e,c) = \{ j \in \boldsymbol{N}; |e_j(m) - me| \leq c \},$$
$$N_m(e,c) = \sharp \mathcal{N}_m(e,c).$$

Then our first theorem can be stated as follows. (See Section 4 for the assumption (H2).)

THEOREM 1. *Assume that the dynamical system* CD_e^λ *satisfies the conditions* (H1), (H2). *Then the dynamical system* CD_e^λ *is ergodic if and only if the following two conditions hold.*

(1) *For every* $A \in \Psi_0$ *and for every orthonormal basis* $\{v_j^m\}_{j,m=1}^\infty$ *of* $\mathcal{V}_\lambda$ *of eigenfunctions of* $\hat{H}$, *we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\substack{j,k \in \mathcal{N}_m(e,c) \\ e_j(m) = e_k(m)}} |\langle A v_j^m, v_k^m \rangle|^2 = \left| \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} \sigma_0(A) \, d\omega_e^\lambda \right|^2. \tag{2}$$

(2) *For every* A, $\{v_j^m\}$ *as above, the following holds*:

$$\lim_{\delta \downarrow 0} \limsup_{m \uparrow \infty} N_m(e,c)^{-1} \sum_{j \in \mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0 < |e_j(m) - e_k(m)| < \delta}} |\langle A v_j^m, v_k^m \rangle|^2 = 0. \tag{3}$$

This theorem is a semi-classical analogy of [**Su1**]. Before proceeding our second theorem, we refer to Zelditch's result [**Z2**].

THEOREM 2 (Zelditch). *Assume that the dynamical system CD_e^λ is ergodic. Then for every orthonormal basis $\{v_j^m\}$ and for every ψDO A of order zero, we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \left| \langle Av_j^m, v_j^m\rangle - \int_{\tilde{S}_e^\lambda} \tilde{\sigma}_0(A)\, d\mu_e^\lambda \right| = 0. \tag{4}$$

We will give a brief explanation for the integral appeared in (4). For details, see [**G-S2**], [**G-U2**], [**Z2**]. Let $(T^*P)_{C(\mathcal{O}_\lambda)}$ be the space of the leaves of the null-foliation on $\Phi^{-1}(C(\mathcal{O}_\lambda))$, where $C(\mathcal{O}_\lambda)$ is the cone through the orbit $\mathcal{O}_\lambda$, $C(\mathcal{O}_\lambda) = \{rf;\, f\in\mathcal{O}_\lambda, r>0\}$. Note that the orbit $\mathcal{O}_\lambda$ is integral, that is, for $f\in\mathcal{O}_\lambda$, there is a character $\chi_f : G_f \to S^1$ (G_f is the stabilizer of f) such that $d\chi_f(A) = 2\pi i\langle f, A\rangle$ for every $A\in\mathcal{G}_f$ ($\mathcal{G}_f$ is the Lie algebra of G_f). Then the leaf of the null-foliation through $z\in\Phi^{-1}(C(\mathcal{O}_\lambda))$ is the orbit through z under the action of the identity component of the kernel, $\ker\chi_f$, of χ_f. The function $\tilde{\sigma}_0(A)$ is the function on $(T^*P)_{C(\mathcal{O}_\lambda)}$ obtained by integration of $\sigma_0(A)$ over the fibers. The natural action of G on the symplectic manifold $(T^*P)_{C(\mathcal{O}_\lambda)}$ is Hamiltonian. Let $\Psi : (T^*P)_{C(\mathcal{O}_\lambda)} \to \mathcal{G}^*$ be the moment map of the above action, and let $p = |\Psi|$. Then the Hamilton flow of p on $(T^*P)_{C(\mathcal{O}_\lambda)}$ is periodic with constant period, and hence it induces an S^1-action on $(T^*P)_{C(\mathcal{O}_\lambda)}$. This S^1-action is obtained by regarding S^1 as $G/\ker\chi_f$. The level surface $p^{-1}(|\lambda|)$ is a S^1-bundle over the Kazhdan-Kostant-Sternberg reduction $X_\lambda^\sharp$ with respect to the orbit $\mathcal{O}_\lambda$. The surface $\tilde{S}_e^\lambda$ appeared in (4) is the intersection $\tilde{S}_e^\lambda = p^{-1}(|\lambda|)\cap \tilde{H}^{-1}(e)$ in $(T^*P)_{C(\mathcal{O}_\lambda)}$, where the function $\tilde{H}$ is the function induced by G-invariant Hamiltonian H on T^*P. The measure μ_e^λ in (4) is the normalized Liouville measure on $\tilde{S}_e^\lambda$.

In the case where the function $\sigma_0(A)$ is invariant under the action of G, the integral in (4) is reduced to the integral over $\Sigma_e^\lambda \subset X_\lambda$ of the function induced by the function $\sigma_0(A)$. We explain this as follows: The level surface $\tilde{S}_e^\lambda$ is a S^1-bundle over the level surface $S_e^\lambda = (H_\lambda^\sharp)^{-1}(e)$ in $X_\lambda^\sharp$, where the function $H_\lambda^\sharp$ is the function on $X_\lambda^\sharp$ induced by H. Note that $X_\lambda^\sharp$ is symplectically diffeomorphic to the product $X_\lambda^\sharp = X_\lambda \times \mathcal{O}_\lambda$ and the action of G on $X_\lambda^\sharp$ is interpreted as the action only on the second component of the product. Since $H_\lambda^\sharp$ is G-invariant, we have $S_e^\lambda = \Sigma_e^\lambda \times \mathcal{O}_\lambda$. Therefore the integral in (4) is reduced to the integral over Σ_e^λ in case where the function $\sigma_0(A)$ is G-invariant.

Before going to state our second theorem, which relates Theorem 1 to the above theorem, we need to prepare some notations.

For every quantum observable $A\in\Psi_0$, we define the (quantum) space average $\langle A\rangle_e^\lambda$ of A by

$$\langle A\rangle_e^\lambda = \lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle Av_j^m, v_j^m\rangle.$$

The existence of the above limit and independence of the choice of the constant c are guaranteed by the semi-classical trace formula due to V. Guillemin-A. Uribe ([**G-U2**]) and S. Zelditch ([**Z2**]) under the assumption (H2). (See Section 4.) We also define the (quantum) time average $\bar{A}$ of $A\in\Psi_0$ by

$$\bar{A} = w\text{-}\lim_{t\to\infty} \frac{1}{t}\int_0^t e^{i\tau\hat{H}} A e^{-i\tau\hat{H}}\, d\tau.$$

Now we can state our second theorem as follows.

Theorem 3. *Assume the condition* (H2). *Then the following three conditions are equivalent*:

(S) *For every* $A \in \Psi_0$,

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \|(\bar{A} - \langle A\rangle_e^\lambda)v_j^m\|^2 = 0, \tag{5}$$

where $\|\cdot\|$ *is the* L^2*-norm.*

(Z) *For every* $A \in \Psi_0$ *and for every orthonormal basis* $\{v_j^m\}_{j,m}$ *for* $\mathscr{V}_\lambda$ *of eigenfunctions of* $\hat{H}$,

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \left| \langle Av_j^m, v_j^m\rangle - \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda \right| = 0. \tag{6}$$

(C) *For every* A, $\{v_j^m\}$ *as in* (Z), *there exists a family* $\{J_m\}_{m\in\mathbf{N}}$ *of subsets of* $\mathcal{N}_m(e,c)$ *satisfying*

$$\lim_{m\to\infty} \frac{\sharp J_m}{N_m(e,c)} = 1 \tag{7}$$

such that

$$\lim_{m\to\infty} \max_{j\in J_m} \left| \langle Av_j^m, v_j^m\rangle - \mathrm{vol}(\Sigma_e^\lambda)^{-1} \int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda \right| = 0. \tag{8}$$

We remark that the conditions (1) in Theorem 1 and (S) in Theorem 3 are equivalent to quantum ergodicity of QD^λ at energy level e defined in Section 4. Note also that the conditions (Z) and (C) in Theorem 3 are equivalent without assuming the condition (H2).

4. Quantum ergodicity at a finite energy level.

This section is devoted to define quantum ergodicity of QD^λ at energy level $e > 0$, following the method in [**Su1**]. Let A be a bounded operator on $L^2(P)$ which commutes with G-action. Then the quantum time average of A is defined by

$$\bar{A} = w\text{-}\lim_{t\to\infty} A_t, \quad A_t = \frac{1}{t}\int_0^t e^{i\tau\hat{H}} A e^{-i\tau\hat{H}}\, d\tau. \tag{9}$$

The above weak limit exists, and the bounded operators A_t, $\bar{A}$ commute with G-action. By the spectral theorem, we have

$$\hat{H} = \sum_\mu \sum_{e(\mu)} e(\mu) P_{e(\mu)}, \quad e^{it\hat{H}} = \sum_\mu \sum_{e(\mu)} e^{ite(\mu)} P_{e(\mu)}, \tag{10}$$

where μ runs over irreducible representations of G, $e(\mu)$ runs over eigenvalues of the restriction of $\hat{H}$ to $\mathscr{L}_\mu$, and $P_{e(\mu)}$ is the projection onto the eigenspace corresponding to the eigenvalue $e(\mu)$. Using the expression (10), we obtain that the time average $\bar{A}$ of A has the form

$$\bar{A} = \sum_\mu \sum_{e(\mu)} P_{e(\mu)} A P_{e(\mu)}. \tag{11}$$

On the other hand, the quantum space average of A is defined by

$$\langle A\rangle_e^\lambda = \lim_{m\to\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle Av_j^m, v_j^m\rangle, \tag{12}$$

if the above limit exists, where $N_m(e,c)$, $\mathcal{N}_m(e,c)$ and v_j^m are as in the previous sections. Note that $\langle A\rangle_e^\lambda = \langle \bar{A}\rangle_e^\lambda$ if the left hand side exists. To guarantee the existence of the space average of A in Ψ_0, we need the following condition.

(H2) *The set of periodic points of* φ_t^λ *on* Σ_e^λ *has Liouville measure zero.*

Under the condition (H2), the semi-classical asymptotic formula due to Guillemin-Uribe [**G-U2**] and Zelditch [**Z2**] holds. That is to say, for every $A \in \Psi_0$ we have the following formula,

$$\sum_{j\in\mathcal{N}_m(e,c)} \langle Av_j^m, v_j^m\rangle = 2c\left(\frac{m}{2\pi}\right)^{n+d-1} \int_{\Sigma_e^\lambda} \sigma_0(A)\, d\omega_e^\lambda + o(m^{n+d-1}). \tag{13}$$

where $n = \dim M$ and $2d = \dim \mathcal{O}_\lambda$. (We refer to [**B-U**], [**G-U2**] and [**Z2**] for the proof of this formula.) The following lemma is the direct consequence of Egorov's theorem [**T**] and the above formula (13).

Lemma 4. (1) *If* $A \in \Psi_0$ *then* $A_t \in \Psi_0$, *and the principal symbol of* A_t *is given by*

$$\sigma_0(A_t) = \frac{1}{t}\int_0^t \sigma_0(A)\circ\phi_\tau\, d\tau.$$

(2) *If the condition* (H2) *is fulfilled, then for every* $A \in \Psi_0$ *we have* $\langle A\rangle_e^\lambda = \langle \sigma_0(A)\rangle_e^\lambda$.

Now we will define quantum ergodicity, which is an analogy of Boltzmann's ergodic hypothesis in a weak sense. (See [**Su1**].)

Definition 1. *The reduced quantum dynamical system* QD^λ *is said to be quantum ergodic at energy level e if for every observable* $A \in \Psi_0$, $\langle \bar{A}^*\bar{A}\rangle_e^\lambda$ *and* $\langle A\rangle_e^\lambda$ *exist and satisfy*

$$\langle \bar{A}^*\bar{A}\rangle_e^\lambda = \langle A\rangle_e^\lambda. \tag{14}$$

The next lemma is the direct consequence of the Definition 1.

Lemma 5. *Assume the condition* (H2). *Then the reduced quantum mechanics* QD^λ *is quantum ergodic at energy level e if and only if the condition* (S) *in the statement of Theorem* 2 *holds.*

Furthermore, we obtain the following proposition, which can be proved by using (11).

Proposition 1. *Assume the condition* (H2). *Then* QD^λ *is quantum ergodic at energy level e if and only if the condition* (1) *in Theorem* 1 *holds.*

In order to prove Theorem 1, we shall prepare the following proposition.

Proposition 2. *Assume the condition* (H2). *Then every quantum observable* $A \in \Psi_0$ *satisfies*

$$\lim_{t\to\infty} \langle A_t^* A_t \rangle_e^\lambda = \langle \bar{A}^* \bar{A} \rangle_e^\lambda \tag{15}$$

if and only if the condition (2) *in Theorem* 1 *holds.*

Proof. This proposition is obtained by essentially the same way as the proof of Lemma 2-2 in [**Su1**]. However, we recall it just to make sure. Note that, by the assumption (H2) and Lemma 3, $\lim_{t\to\infty}\langle A_t^* A_t\rangle_e^\lambda$ exists. A direct computation leads us to

$$A_t v_j^m = \frac{1}{t} \sum_{\substack{k \\ e_k(m)\neq e_j(m)}} \frac{(e^{it(e_k(m)-e_j(m))}-1)}{i(e_k(m)-e_j(m))} \langle A v_j^m, v_k^m \rangle v_k^m + \bar{A} v_j^m, \tag{16}$$

and hence

$$\langle A_t^* A_t v_j^m, v_j^m \rangle = \frac{1}{t^2} \sum_{\substack{k \\ e_k(m)\neq e_j(m)}} \frac{|e^{it(e_k(m)-e_j(m))}-1|^2}{|e_k(m)-e_j(m)|^2} |\langle A v_j^m, v_k^m \rangle|^2 + \langle \bar{A}^* \bar{A} v_j^m, v_j^m \rangle. \tag{17}$$

We set $S(x) = x^{-2}|e^{ix}-1|^2 = 2x^{-2}(1-\cos x)$ and

$$S_t = \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_j(m)\neq e_k(m)}} S(t(e_j(m)-e_k(m)))|\langle A_m v_j^m, v_k^m\rangle|^2.$$

We observe that (15) holds if and only if $\lim_{t\to\infty} S_t = 0$. Note that there exists $\alpha > 0$ such that $S(x) \geq 1/2$ if $|x| < \alpha$. Then we have

$$S_t \geq \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \times \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\leq\alpha/t}} S(t(e_j(m)-e_k(m)))|\langle A v_j^m, v_k^m\rangle|^2$$

$$\geq \frac{1}{2}\limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\leq\alpha/t}} |\langle A v_j^m, v_k^m\rangle|^2. \tag{18}$$

Therefore $\lim_{t\to\infty} S_t = 0$ implies the condition (2) in Theorem 1.

Conversely, we assume the condition (2) in Theorem 1. For any $\varepsilon > 0$, there exists $T > 0$ such that $S(x) < \varepsilon$ if $|x| > T$. Then we have

$$N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ e_j(m)\neq e_k(m)}} S(t(e_j(m)-e_k(m)))|\langle A v_j^m, v_k^m\rangle|^2$$

$$\leq \varepsilon N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ |e_j(m)-e_k(m)|>T/t}} |\langle A v_j^m, v_k^m\rangle|^2$$

$$+ N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\leq T/t}} |\langle A v_j^m, v_k^m\rangle|^2$$

$$\leq \varepsilon\|A\|^2 + N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\leq T/t}} |\langle A v_j^m, v_k^m\rangle|^2. \tag{19}$$

Therefore we have

$$S_t \le \varepsilon\|A\|^2 + \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_j(m)-e_k(m)|\le T/t}} |\langle Av_j^m, v_k^m\rangle|^2. \tag{20}$$

Letting $t\to\infty$, we have $\limsup_{t\uparrow\infty} S_t \le \varepsilon\|A\|^2$. Since $\varepsilon > 0$ is arbitrary, we conclude $\lim_{t\to\infty} S_t = 0$, and hence (15). □

5. Proof of Theorem 1.

In the preceding section, we defined quantum ergodicity of the reduced quantum dynamical system QD^λ at a finite energy level. This notion plays an important role in the proof of Theorem 1 (stated in Section 3). In fact, in view of Propositions 1, 2, we only need to prove the following proposition for the proof of Theorem 1.

Proposition 3. *Assume the conditions* (H1) *and* (H2). *Then the dynamical system* CD_e^λ *is ergodic if and only if the following two conditions hold*:

(1) *The reduced quantum dynamical system* QD^λ *is quantum ergodic at energy level* e.

(2) *For every observable* $A\in\Psi_0$, *we have*

$$\lim_{t\to\infty} \langle A_t^* A_t\rangle_e^\lambda = \langle \bar{A}^*\bar{A}^*\rangle_e^\lambda.$$

Proof. We take an arbitrary $A\in\Psi_0$. Then we have

$$\begin{aligned}
|\langle A\rangle_e^\lambda|^2 &= |\langle \sigma_0(A)\rangle_e^\lambda| \quad \text{(Lemma 4)} \\
&= \langle |\overline{\sigma_0(A)}|^2\rangle_e^\lambda \quad \text{(ergodicity)} \\
&= \lim_{t\to\infty} \langle |\sigma_0(A)_t|^2\rangle_e^\lambda \quad \text{(Lemma 3, (1))} \\
&= \lim_{t\to\infty} \langle \sigma_0(A_t^*A_t)\rangle_e^\lambda \quad \text{(Lemma 4)} \\
&= \lim_{t\to\infty} \langle A_t^*A_t\rangle_e^\lambda \quad \text{(Lemma 4)}.
\end{aligned}$$

By (17), we have $\langle A_t^*A_t v_j^m, v_j^m\rangle \ge \langle \bar{A}^*\bar{A}v_j^m, v_j^m\rangle$, and hence

$$\langle A_t^*A_t\rangle_e^\lambda \ge \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle \bar{A}^*\bar{A}v_j^m, v_j^m\rangle.$$

On the other hand,

$$\begin{aligned}
0 &\le \liminf_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \|(\bar{A} - \langle A\rangle_e^\lambda)v_j^m\|^2 \\
&= \liminf_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle \bar{A}^*\bar{A}v_j^m, v_j^m\rangle - |\langle A\rangle_e^\lambda|^2.
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
|\langle A\rangle_e^\lambda|^2 &= \lim_{t\to\infty} \langle A_t^* A_t\rangle_e^\lambda \\
&\geq \limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle \bar{A}^*\bar{A}v_j^m, v_j^m\rangle \\
&\geq \liminf_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} \langle \bar{A}^*\bar{A}v_j^m, v_j^m\rangle \\
&\geq |\langle A\rangle_e^\lambda|^2.
\end{aligned}
$$

This implies that $\langle \bar{A}^*\bar{A}\rangle_e^\lambda$ exists and

$$
|\langle A\rangle_e^\lambda|^2 = \lim_{t\to\infty} \langle A_t^* A_t\rangle_e^\lambda = \langle \bar{A}^*\bar{A}\rangle_e^\lambda.
$$

We will prove the converse. Let $A \in \Psi_0$. Then we have

$$
\begin{aligned}
|\langle A\rangle_e^\lambda|^2 &= |\langle \sigma_0(A)\rangle_e^\lambda|^2 \quad \text{(Lemma 4)} \\
&\leq \langle |\overline{\sigma_0(A)}|^2\rangle_e^\lambda \quad \text{(Lemma 3, (2))} \\
&= \lim_{t\to\infty} \langle |\sigma_0(A)_t|^2\rangle_e^\lambda \quad \text{(Lemma 3, (1))} \\
&= \lim_{t\to\infty} \langle \sigma_0(A_t^* A_t)\rangle_e^\lambda \quad \text{(Lemma 4)} \\
&= \lim_{t\to\infty} \langle A_t^* A_t\rangle_e^\lambda \quad \text{(Lemma 4)} \\
&= \langle \bar{A}^*\bar{A}\rangle_e^\lambda \quad \text{(Assumption (1))} \\
&= |\langle A\rangle_e^\lambda|^2 \quad \text{(Assumption (2))}.
\end{aligned}
$$

Thus for every $\sigma_0(A)$ of $A \in \Psi_0$, the equation (1) in section 2 holds. Now, by Lemma 2, for every $f \in C^\infty(\Sigma_e^\lambda)$ there exists a smooth function F on $T^*P\backslash 0$ which is G-invariant, homogeneous of degree zero and $q_\lambda^* f = F$ on $\tilde{\Sigma}_e^\lambda$. Let A_0 be the ψDO of order zero whose principal symbol is F. Then the operator $A = \int_G gA_0g^{-1}\,dg$ is in Ψ_0 whose principal symbol is F, and hence $\langle |\overline{\sigma_0(A)}|^2\rangle_e^\lambda = \langle |\bar{f}|\rangle_e^\lambda$. Therefore the dynamical system CD_e^λ is ergodic. □

6. Proof of Theorem 3.

Now we will proceed to the proof of Theorem 3. For this sake, we will define auxiliary notions.

Definition 2. (1) *A family* $\{\mathcal{S}_m; \mathcal{S}_m \subset \sigma_m(e,c)\}$ *of subsets of* $\sigma_m(e,c) = \{\lambda \in \sigma(\hat{H}|_{\mathcal{L}_{m\lambda}}); |\lambda - me| \leq c\}$ *is said to satisfy the condition* (D1) *if it satisfies*

$$
\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\lambda\in\mathcal{S}_m} (\dim V_\lambda) = 1, \tag{21}
$$

where V_λ *is the eigenspace of an eigenvalue* λ *of* $\hat{H}|_{\mathcal{L}_{m\lambda}}$.

(2) *A family* $\{J_m;\, J_m \subset \mathcal{N}_m(e,c)\}$ *of subsets of* $\mathcal{N}_m(e,c) = \{j \in \boldsymbol{N};\, e_j(m) \in \sigma_m(e,c)\}$ *is said to satisfy the condition* (D2) *if we have*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sharp J_m = 1. \tag{22}$$

Let $\boldsymbol{x} = \{x^m\}_{m\in\boldsymbol{Z}}$ be a family of sequences $x^m = \{x_j^m\}_{j\in\mathcal{N}_m(e,c)}$ of nonnegative numbers such that $0 \le x_j^m \le K$ for all m, j, for some constant $K > 0$. For each $\lambda \in \sigma_m(e,c)$, we set

$$x_\lambda^m = (\dim V_\lambda)^{-1} \sum_{\substack{j \\ e_j(m)=\lambda}} x_j^m,$$

so that

$$N_m(e,c)^{-1} \sum_{j\in\mathcal{N}_m(e,c)} x_j^m = N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda) x_\lambda^m.$$

Lemma 6. *The following holds,*

$$\lim_{m\to\infty} N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda) x_\lambda^m = 0, \tag{23}$$

if and only if there exists a family $\{\mathcal{S}_m;\, \mathcal{S}_m \subset \sigma_m(e,c)\}$ *satisfying the condition* (D1) *such that*

$$\lim_{m\to\infty} \max_{\lambda\in\mathcal{S}_m} x_\lambda^m = 0. \tag{24}$$

Proof. Since "if" part is obvious, we will only give a proof of "only if" part. Assume that (23) holds. Then one can find a sequence $\{l_m\}$ of natural numbers which is monotone increasing and goes to infinity as $m \to \infty$ such that

$$N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda) x_\lambda^m < \frac{K}{2^{l_m}}$$

for every $m \in \boldsymbol{N}$. We define

$$\mathcal{S}_m = \left\{\lambda \in \sigma_m(e,c);\, x_\lambda^m < \frac{1}{l_m}\right\}.$$

It is clear that $\mathcal{S}_m$ satisfies (24). Furthermore $\{\mathcal{S}_m\}$ satisfies (D1). Indeed we have

$$\frac{K}{2^{l_m}} > N_m(e,c)^{-1} \sum_{\lambda\in\sigma_m(e,c)} (\dim V_\lambda) x_\lambda^m \ge \{l_m N_m(e,c)\}^{-1} \sum_{\lambda\in\sigma_m(e,c)\setminus\mathcal{S}_m} (\dim V_\lambda),$$

and hence

$$1 - N_m(e,c)^{-1} \sum_{\lambda\in\mathcal{S}_m} (\dim V_\lambda) < K\frac{l_m}{2^{l_m}}.$$

This implies (21). $\square$

Lemma 7. *Let* $\boldsymbol{x} = \{x^m\}_{m \in \boldsymbol{Z}}$ *be a family of sequences* $x^m = \{x_j^m\}_{j \in \mathcal{N}_m(e,c)}$ *of nonnegative numbers as above. Then the following conditions are equivalent*:

(1) *There exists a family* $\{\mathcal{S}_m\}_m$ *satisfying* (D1) *such that* (24) *holds.*

(2) *There exists a family* $\{J_m\}_m$ *satisfying* (D2) *such that*

$$\lim_{m \uparrow \infty} \max_{j \in J_m} x_j^m = 0. \tag{25}$$

Proof. First we assume the condition (1). Then one can find a sequence $\{l_m\}$ of natural numbers which is monotone increasing and goes to infinity as $m \to \infty$ such that for all $\lambda \in \mathcal{S}_m$

$$x_\lambda^m = (\dim V_\lambda)^{-1} \sum_{\substack{j \\ e_j(m) = \lambda}} x_j^m < \frac{K}{2^{l_m}}.$$

We define $J_m \subset \mathcal{N}_m(e,c)$ by

$$J_m = \left\{ j \in \mathcal{N}_m(e,c);\ e_j(m) \in \mathcal{S}_m \text{ and } x_j^m < \frac{1}{l_m} \right\}.$$

This family clearly satisfies (25). Note that we have

$$\begin{aligned} 1 - N_m(e,c)^{-1} \sharp J_m &= N_m(e,c)^{-1} \sum_{\lambda \in \mathcal{S}_m} \sum_{\substack{j \in \mathcal{N}_m(e,c) \setminus J_m \\ e_j(m) = \lambda}} 1 + N_m(e,c)^{-1} \sum_{\lambda \in \sigma_m(e,c) \setminus \mathcal{S}_m} \sum_{\substack{j \\ e_j(m) = \lambda}} 1 \\ &= \mathrm{I} + \mathrm{II} \quad \text{(say)}. \end{aligned}$$

Since $\{\mathcal{S}_m\}$ satisfies the condition (D1), II goes to zero as $m \to \infty$. On the other hand,

$$I = N_m(e,c)^{-1} \sum_{\lambda \in \mathcal{S}_m} (\dim V_\lambda) S_\lambda^m,$$

where we set

$$S_\lambda^m = (\dim V_\lambda)^{-1} \sum_{\substack{j \in \mathcal{N}_m(e,c) \setminus J_m \\ e_j(m) = \lambda}} 1.$$

If $e_j(m) = \lambda \in \mathcal{S}_m$ and $j \notin J_m$, then $x_j^m \geq l_m^{-1}$. Thus, for $\lambda \in \mathcal{S}_m$, we have

$$\frac{K}{2^{l_m}} \geq (\dim V_\lambda)^{-1} \sum_{\substack{j \in \mathcal{N}_m(e,c) \setminus J_m \\ e_j(m) = \lambda}} x_j^m \geq l_m^{-1} S_\lambda^m. \tag{26}$$

This implies that $I < K l_m / 2^{l_m} \to 0$ $(m \to \infty)$. Hence the family $\{J_m\}$ satisfies (D2). The converse is obvious. □

Proposition 4. *The conditions* (Z) *and* (C) *in the statement of Theorem* 2 *are equivalent.*

PROOF. We take an $A \in \Psi_0$ and set

$$x_j^m = |\langle A v_j^m, v_j^m \rangle - \langle \sigma_0(A) \rangle_e^\lambda|$$

for $j \in \mathcal{N}_m(e, c)$. Note that $0 \leq x_j^m \leq \|A\| + \langle \sigma_0(A) \rangle_e^\lambda$. Hence, by Lemmas 6, 7, we conclude the assertion. □

The conditions (Z) and (C) are equivalent without assuming (H2). Next we will prove the equivalence of (S) and (C). For this sake, we prepare the following lemma.

LEMMA 8. *Consider a family* $\{J_m; J_m \subset \mathcal{N}_m(e, c)\}$, *and set* $J_m(\lambda) = \{j \in J_m; e_j(m) = \lambda\}$. *Then* $\{J_m\}_m$ *satisfies* (D2) *if and only if there exists a family* $\{\mathcal{S}_m; \mathcal{S}_m \subset \sigma_m(e, c)\}$ *satisfying* (D1) *such that*

$$\lim_{m \uparrow \infty} \max_{\lambda \in \mathcal{S}_m} (\dim V_\lambda)^{-1} \sharp J_m(\lambda) = 1. \tag{27}$$

PROOF. Set

$$x_j^m = \begin{cases} 0 & \text{if } j \in J_m \\ 1 & \text{if } j \in \mathcal{N}_m(e, c) \backslash J_m. \end{cases}$$

Then for all $\lambda \in \sigma_m(e, c)$, we have $x_\lambda^m = 1 - (\dim V_\lambda)^{-1} \sharp J_m(\lambda)$. Therefore

$$1 - N_m(e, c)^{-1} \sharp J_m = N_m(e, c)^{-1} \sum_{\lambda \in \sigma_m(e, c)} (\dim V_\lambda) x_\lambda^m.$$

Hence by Lemma 7, $\{J_m\}_m$ satisfies (D2) if and only if there exists a family $\{\mathcal{S}_m\}_m$ satisfying (D1) such that $\lim_{m \uparrow \infty} \max_{\lambda \in \mathcal{S}_m} x_\lambda^m = 0$. This implies the assertion. □

Finally we will prove the following proposition, which completes the proof of Theorem 3.

PROPOSITION 5. *Assume that* (H2) *holds. Then the conditions* (S) *and* (C) *in Theorem* 2 *are equivalent.*

PROOF. First, we assume that the condition (S) holds. Note that $\langle A \rangle_e^\lambda = \langle \sigma_0(A) \rangle_e^\lambda$ by Lemma 4. Therefore by setting $x_j^m = \|(\bar{A} - \langle A \rangle_e^\lambda) v_j^m\|$, the condition (C) follows from Lemmas 6, 7 and the inequality

$$|\langle A v_j^m, v_j^m \rangle - \langle \sigma_0(A) \rangle_e^\lambda| \leq \|(\bar{A} - \langle A \rangle_e^\lambda) v_j^m\|.$$

Next we will prove the converse. We may assume, without loss of generality, that $A \in \Psi_0$ is Hermitian. Since the time average $\bar{A}$ commutes with $\hat{H}$ and G-action, we can take an orthonormal basis $\{v_j^m\}$ of $\mathcal{L}_{m\lambda}$ consists of eigenfunctions of $\hat{H}$ such that $\bar{A} v_j^m = \mu_j^m v_j^m$ for some $\mu_j^m \in \boldsymbol{R}$. Note that $\langle \bar{A} v_j^m, v_j^m \rangle = \langle A v_j^m, v_j^m \rangle$. Then we have

$$\|(\bar{A} - \langle A \rangle_e^\lambda) v_j^m\|^2 = |\mu_j^m - \langle A \rangle_e^\lambda|^2 = |\langle A v_j^m, v_j^m \rangle - \langle \sigma_0(A) \rangle_e^\lambda|^2. \tag{28}$$

Let $\{J_m\}_m$ be as in the condition (C). By Lemma 8, one can find a family $\mathcal{S}_m$ satisfying (D1) such that (27) holds. In view of Lemma 6, we only need to prove that this family $\{\mathcal{S}_m\}$ satisfies

$$\lim_{m\uparrow\infty}\max_{\lambda\in\mathscr{S}_m}(\dim V_\lambda)^{-1}\sum_{\substack{j\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)v_j^m\|^2=0. \tag{29}$$

By (27), for arbitrary $\varepsilon>0$ we can find a positive number N_1 such that $m\geq N_1$ implies $(\dim V_\lambda)^{-1}[(\dim V_\lambda)-\sharp J_m(\lambda)]<\varepsilon$ for all $\lambda\in\mathscr{S}_m$. On the other hand, by the condition (C) and (28), one can find $N_2>0$ such that $m\geq N_2$ implies $\|(\bar{A}-\langle A\rangle_e^\lambda)v_j^m\|^2<\varepsilon$ for all $j\in J_m$. Therefore if $m\geq\max\{N_1,N_2\}$ then for all $\lambda\in\mathscr{S}_m$ we have

$$\begin{aligned}
&(\dim V_\lambda)^{-1}\sum_{\substack{j\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)v_j^m\|^2\\
&\quad=(\dim V_\lambda)^{-1}\sum_{\substack{j\in J_m\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)v_j^m\|^2\\
&\qquad+(\dim V_\lambda)^{-1}\sum_{\substack{j\in\mathscr{N}_m(e,c)\setminus J_m\\ e_j(m)=\lambda}}\|(\bar{A}-\langle A\rangle_e^\lambda)v_j^m\|^2\\
&\quad\leq(1+K)\varepsilon,
\end{aligned}$$

where $K=\|A\|+\langle A\rangle_e^\lambda$. Since $\varepsilon>0$ is arbitrary, we obtain (29). □

7. Quantum weak mixing at a finite energy level.

Theorem 1 in this paper says that ergodicity of classical dynamical system is related to the semi-classical asymptotic behaviour of near-diagonal components of quantum observables. It is natural to ask what property of classical mechanics affects the asymptotic behaviour of quantum transition amplitudes. Zelditch [**Z4**] has shown that classical weak mixing is equivalent to the notion of *quantum weak mixing* (see [**Z4**] for the definition) plus an additional condition. In this section, we will discuss quantum weak mixing of QD^λ at a finite energy level.

First, we begin with the review of the notion of classical weak mixing.

For every $\tau\in\boldsymbol{R}$ and every $f\in L^2(\Sigma_e^\lambda)$, we define $f_t(\tau)\in L^2(\Sigma_e^\lambda)$ by

$$f_t(\tau)=\frac{1}{t}\int_0^t e^{-i\tau s}f\circ\varphi_s^\lambda\,ds.$$

By von Neumann's ergodic theorem, the function $f_t(\tau)$ converges to the function $\bar{f}(\tau)\in L^2(\Sigma_e^\lambda)$ satisfying $\bar{f}(\tau)\circ\varphi_t^\lambda=e^{i\tau t}\bar{f}(\tau)$ in L^2-sense as $t\to\infty$. The dynamical system CD_e^λ is said to be *weak mixing* if

$$\bar{f}(\tau)=\langle f\rangle_e^\lambda\delta_{\tau,0},\quad\text{a.e.},$$

(see [**C-F-S**]) or equivalently

$$\langle|\bar{f}(\tau)|^2\rangle_e^\lambda=|\langle\bar{f}(\tau)\rangle_e^\lambda|^2$$

for all $f\in C^\infty(\Sigma_e^\lambda)$.

Next, we will describe a quantum analogue of this notion. For every quantum observable $A \in \Psi_0$ and for every $\tau \in \boldsymbol{R}$, we define the bounded operator $\bar{A}(\tau)$ by

$$\bar{A}(\tau) = w\text{-}\lim_{t\to\infty} A_t(\tau), \quad A_t(\tau) = \frac{1}{t}\int_0^t e^{-i\tau s} e^{is\hat{H}} A e^{-is\hat{H}} ds.$$

The bounded operater $\bar{A}(\tau)$ commutes with the G-action and has the following form

$$\bar{A}(\tau) = \sum_{\mu} \sum_{e(\mu)\in\sigma(\hat{H}|_{\mathscr{L}_\mu})} P_{e(\mu)+\tau} A P_{e(\mu)}.$$

By Egorov's theorem, the operator $A_t(\tau)$ is in Ψ_0 and its principal symbol is given by

$$\sigma_0(A_t(\tau)) = \frac{1}{t}\int_0^t e^{-i\tau s}\sigma_0(A)\circ\phi_s\, ds.$$

DEFINITION 3. *The reduced quantum dynamical system QD^λ is said to be quantum weak mixing at energy level $e > 0$ if for every observable $A \in \Psi_0$ and every $\tau \in \boldsymbol{R}$, $\langle \bar{A}(\tau)^* \bar{A}(\tau)\rangle_e^\lambda$ and $\langle A\rangle_e^\lambda$ exist and satisfy*

$$\langle \bar{A}(\tau)^* \bar{A}(\tau)\rangle_e^\lambda = |\langle A\rangle_e^\lambda|^2 \delta_{\tau,0},$$

or equivalently,

$$\langle \bar{A}(\tau)^* \bar{A}(\tau)\rangle_e^\lambda = |\langle \bar{A}(\tau)\rangle_e^\lambda|^2.$$

The following proposition and theorem can be obtained by the same way as the proofs of Proposition 3 and Theorem 1.

PROPOSITION 6. *Assume the conditions* (H1) *and* (H2). *Then the classical dynamical system CD_e^λ is weak mixing if and only if the following two conditions hold*:

(1) *The reduced quantum dynamical system QD^λ is quantum weak mixing at energy level e.*

(2) *For every observable $A \in \Psi_0$ and for every $\tau \in \boldsymbol{R}$, we have*

$$\lim_{t\to\infty} \langle A_t(\tau)^* A_t(\tau)\rangle_e^\lambda = \langle \bar{A}(\tau)^* \bar{A}(\tau)\rangle_e^\lambda.$$

THEOREM 4. *Assume that the conditions* (H1) *and* (H2) *are fulfilled. Then the classical dynamical system CD_e^λ is weak mixing if and only if the following two conditions hold*:

(1) *For every $A \in \Psi_0$, $\tau \in \boldsymbol{R}$ and orthonormal basis $\{v_j^m\}_{j,m=1}^\infty$ of $\mathscr{V}_\lambda$ of eigenfunctions of $\hat{H}$, we have*

$$\lim_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathscr{N}_m(e,c)} \sum_{\substack{k \\ e_k(m)=e_j(m)+\tau}} |\langle A v_j^m, v_k^m\rangle|^2 = \left|\mathrm{vol}(\Sigma_e^\lambda)^{-1}\int_{\Sigma_e^\lambda}\sigma_0(A)\, d\omega_e^\lambda\right|^2 \delta_{\tau,0}.$$

(2) *For every A, τ and $\{v_j^m\}_{j,m=1}^\infty$ as above, we have*

$$\lim_{\delta\downarrow 0}\limsup_{m\uparrow\infty} N_m(e,c)^{-1} \sum_{j\in\mathscr{N}_m(e,c)} \sum_{\substack{k \\ 0<|e_k(m)-e_j(m)-\tau|<\delta}} |\langle A v_j^m, v_k^m\rangle|^2 = 0.$$

8. Circle bundle case.

In this section, we will apply our main theorems to a circle bundle over a compact Riemannian manifold. We fix a Riemannian metric $\langle\,,\rangle$ and an integral closed 2-form $\boldsymbol{B}$ on a compact Riemannian manifold M. Then one can find a circle bundle $\pi : P \to M$ with connection 1-form α whose curvature 2-form is $\boldsymbol{B}$. We take the irreducible representation $\lambda = 1 \in \boldsymbol{R}$ of the circle S^1, that is the multiplication by elements of S^1. In this case, the corresponding reduced phase space (X_1, Ω_1) is symplectically diffeomorphic to the cotangent bundle T^*M over M with symplectic form $\Omega_M - \pi_M^*\boldsymbol{B}$, where Ω_M is the canonical symplectic form on T^*M and π_M is the projection. Now we will consider the quantum Hamiltonian

$$\hat{H} = \sqrt{\Delta_{hor} + VQ^2},$$

where Δ_{hor} is the horizontal Laplacian, Q is $-\sqrt{-1}$ times the infinitesimal generator of S^1-action on P, and V is the lift of a strictly positive smooth function on M. This operator is a positive elliptic first order ψDO on P. The isotypical subspace $\mathscr{L}_m$ of the character $e^{i\theta} \mapsto e^{im\theta}$ $(m \in \boldsymbol{Z})$ is naturally identified with the Hilbert space $L^2(M, L^{\otimes m})$ of L^2-sections of mth tensor power of the associated line bundle L. Then the restriction of $\hat{H}$ to $\mathscr{L}_m$ is identical with the operator $\hat{H}_m$ given by

$$\hat{H}_m = \sqrt{\nabla_m^* \nabla_m + m^2 V},$$

where ∇_m is the connection on $L^{\otimes m}$ induced by α. The principal symbol of $\hat{H}$ is the Riemannian norm on $T^*P \backslash 0$ with respect to the metric $\pi^*\langle\,,\rangle + V^{-1}\alpha^2$. The corresponding Hamiltonian H_1 on T^*M is given by

$$H_1(x,\xi) = \sqrt{\|\xi\|^2 + V(x)}, \quad (x,\xi) \in T^*M.$$

The flow φ_t generated by (H_1, Ω_1) is called the *electro-magnetic flow* under the magnetic field $\boldsymbol{B}$ and the electric potential V. Furthermore, if we take $V \equiv 1$, then the Liouville measure ω_e on the energy surface $\Sigma_e = H_1^{-1}(e)$ is given by the direct product of the canonical measure on the unit sphere and the volume measure on M up to constant multiple.

Let $f \in C^\infty(M)$ and $A_f \in \Psi_0$ be the multiplication operator by the lift of f. Then we have

$$\langle A_f v_j^m, v_j^m \rangle = \int_M f |v_j^m|^2 \, dV_M, \tag{30}$$

$$\langle \sigma(A_f) \rangle_e^\lambda = \mathrm{vol}(M)^{-1} \int_M f \, dV_M. \tag{31}$$

If M is a Riemann surface with constant negative curvature -1, $\boldsymbol{B}$ is the volume 2-form and $e \geq \sqrt{2}$, then the dynamical system $(\Sigma_e, \varphi_t, \omega_e)$ is ergodic ([**G-U1**], [**Su2**]). Thus we have the following

Corollary. *Let M be a compact Riemann surface with constant negative curvature -1, $\boldsymbol{B}$ the volume 2-form and $e \geq \sqrt{2}$. Then for every orthonormal basis $\{v_j^m\}$ of* *eigenfunctions of $\hat{H}$, there exists a family $\{J_m\}$ satisfying*

$$\lim_{m\to\infty} \frac{\sharp J_m}{N_m(e,c)} = 1 \tag{32}$$

such that for all $f \in C^\infty(M)$ we have

$$\lim_{m\to\infty} \max_{j\in J_m} \left| \int_M f|v_j^m|^2\, dV_M - \mathrm{vol}(M)^{-1} \int_M f\, dV_M \right| = 0. \tag{33}$$

PROOF. In view of Theorem 1 and 3, we only need to prove that a family $\{J_m\}$ can be taken independently of the choice of a smooth function f. For this sake, let $\{\varphi_p\}$ be a orthonormal basis of $L^2(M)$ of eigenfunctions of the Laplacian. For every $l \in \boldsymbol{N}$, let $\{J_m(l)\}$ be a family satisfying (32), (33) for all $f = \varphi_p$ with $p \le l$. We may assume $J_m(l+1) \subset J_m(l)$ for all l. We can find a sequence $\{l_m\}_{m\in\boldsymbol{N}}$ of natural numbers which is monotone increasing and tends to infinity as m goes to infinity such that

$$1 - \frac{1}{2^{l_m}} \le \frac{J_m(l_m)}{N_m(e,c)}.$$

Then the family $\{J_m\}$ defined by $J_m = J_m(l_m)$ is a desired family. □

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