

## Codimension two nonsingular subvarieties of quadrics: scrolls and classification in degree $d \leq 10$

By Mark Andrea A. de CATALDO

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**Abstract.** Let  $X$  be a codimension two nonsingular subvariety of a nonsingular quadric  $\mathcal{Q}^n$  of dimension  $n \geq 5$ . We classify such subvarieties when they are scrolls. We also classify them when the degree  $d \leq 10$ . Both results were known when  $n = 4$ .

### 0. Introduction.

The paper [26] completes the classification of scrolls as codimension two subvarieties of projective space  $P^n$ . Ottaviani's proof consists of three parts. First the sectional genus  $g$  is exhibited as a function of the degree  $d$  of the scroll. The degree  $d$  is then bounded from above by the use of Castelnuovo-type bounds for  $g$ . The final step consists of the construction of varieties with prescribed low invariants which had been accomplished by several authors.

In this paper we classify scrolls as codimension two subvarieties of  $\mathcal{Q}^n$ ; see Theorem 3.1.2. The analysis is quite similar to the one of [26] with the following three differences. The first one is that there are fourfolds scrolls on  $\mathcal{Q}^6$ . The second difficulty is that the method for bounding the degree of scrolls over surfaces on  $\mathcal{Q}^5$  of [26] is not sufficient; we go around the problem using lemmata 3.4.2 and 3.4.3. Lastly, once we obtain a maximal list of invariants we must construct all the scrolls in question. This is essentially the problem of constructing varieties of low degree and codimension two on  $\mathcal{Q}^n$ . We build on the results of [4] and [16] and obtain Theorem 2.1.1, i.e. the complete classification in degree  $d \leq 10$  and  $n \geq 5$ . This result highlights the role that some special vector bundles on quadrics play in the construction of subvarieties of quadrics. As a by-pass result of this classification in low degree we are able to construct all scrolls, except for one case: when the degree  $d = 12$  and the base is a minimal  $K3$  surface. We construct an unirational family of these scrolls; see Theorem 3.4.5. We do not know whether or not this is the only one.

As it is explained in the introduction to [10], the Barth-Larsen Theorem and the double point formulæ put severe constraints on varieties embedded in projective space with small codimension. The same is true for any ambient space, so that it is only

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natural to look at different spaces. In my dissertation I studied codimension two subvarieties of quadrics. This paper is an expanded and completed version of parts of my dissertation [11] the results of which appear in [8], [9] and [10] and in the present paper.

The paper is organized as follows. Section 1 contains preliminary results. Section 2 contains Theorem 2.1.1. Section 3 contains the main result of this paper, Theorem 3.1.2, the proof of which guides the reader through the rest of the section.

NOTATION AND CONVENTIONS. Our basic reference is [18]. We work over any algebraically closed field of characteristic zero. A quadric  $\mathcal{Q}^n$ , here, is a nonsingular hypersurface of degree two in the projective space  $\mathbf{P}^{n+1}$ . Little or no distinction is made between line bundles, associated sheaves of sections and Cartier divisors.  $[t]$  denotes the biggest integer smaller than or equal to  $t$ .  $\sim_n$  denotes the numerical equivalence of divisors on a surface.  $\mathcal{O}_{\mathcal{Q}^n}(1)$  denotes the sheaf  $\mathcal{O}_{\mathbf{P}^{n+1}}(1)|_{\mathcal{Q}^n}$ . If  $F$  is a coherent sheaf on  $\mathcal{Q}^n$  and  $l$  an integer, then  $F(l)$  denotes the sheaf  $F \otimes \mathcal{O}_{\mathcal{Q}^n}(l)$ .

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## 1. Preliminary material.

We now collect the various results that will be necessary in sections 2 and 3. In this section  $X$  is a codimension two, nonsingular subvariety of  $\mathcal{Q}^n$ ,  $d$  is its degree,  $\mathcal{N}_{X, \mathcal{Q}^n}$  is its normal bundle,  $n_i$  is the  $i^{\text{th}}$  Chern class of  $\mathcal{N}_{X, \mathcal{Q}^n}$ ,  $\iota: X \hookrightarrow \mathcal{Q}^n$  is the embedding,  $L$  is the restriction of  $\mathcal{O}_{\mathcal{Q}^n}(1)$  to  $X$ ,  $K_X$  is the canonical dualizing sheaf of  $X$ ,  $S$  ( $C$ , respectively) is a surface (curve, respectively) section of  $X$  obtained by intersecting  $X$  with  $(n-4)$  ( $(n-3)$ , respectively) general hyperplanes of  $\mathbf{P}^{n+1}$  and  $g = g(C)$  is the genus of  $C$ .

### 1.1. Miscellanea.

The cohomology ring of a nonsingular quadric of any dimension is described in [19]. Let  $h$  be the class of any hyperplane section of  $\mathcal{Q}^n$ .

We consider the odd dimensional case first:  $\mathcal{Q}^{2n+1}$ . One can describe  $H^*(\mathcal{Q}^{2n+1}, \mathbf{Z})$  as follows. Let  $A$  be the class of an  $n$ -dimensional linear space in  $\mathcal{Q}^{2n+1}$ . The relevant information is, denoting the cup product by “ $\cdot$ ”:

$$H^{2i+1}(\mathcal{Q}^{2n+1}, \mathbf{Z}) = \{0\}, \forall i; H^{2i}(\mathcal{Q}^{2n+1}, \mathbf{Z}) = \{0\}, \text{ for } i > 2n+1; H^{2i}(\mathcal{Q}^{2n+1}, \mathbf{Z}) = \mathbf{Z}[h^i], \\ i = 0, \dots, n; H^{2(n+j)}(\mathcal{Q}^{2n+1}, \mathbf{Z}) = \mathbf{Z}[A \cdot h^{j-1}], j = 1, \dots, n+1; h^{n+1} = 2A, h^{2n+1} = 2.$$

As to the even dimensional case, we denote by  $A_1, A_2$  the classes of two members of the two rulings of  $\mathcal{Q}^{2n}$  in  $n$ -dimensional linear spaces. One has:

$$H^{2i+1}(\mathcal{Q}^{2n}, \mathbf{Z}) = \{0\}, \forall i; H^{2i}(\mathcal{Q}^{2n}, \mathbf{Z}) = \{0\}, \text{ for } i > 2n; H^{2i}(\mathcal{Q}^{2n}, \mathbf{Z}) = \mathbf{Z}[h^i], i = 0, \dots, \\ n-1; H^{2i}(\mathcal{Q}^{2n}, \mathbf{Z}) = \mathbf{Z}[A_1] \oplus \mathbf{Z}[A_2]; H^{2(n+j)}(\mathcal{Q}^{2n+1}, \mathbf{Z}) = \mathbf{Z}[A_1 \cdot h^{j-1}] = \mathbf{Z}[A_2 \cdot h^{j-1}], j = \\ 1, \dots, n; h^n = A_1 + A_2; h^{2n} = 2; [A_i] \cdot [A_j] = \delta_{ij}, \text{ where } \delta_{ij} \text{ is the Kronecker symbol.}$$

REMARK 1.1.1. The above description of the cohomology ring of  $\mathcal{Q}^n$  implies that, for  $n \geq 5$ ,  $d$  is an even integer.

Mumford’s self intersection formula (cf. [14], page 103) gives, for  $n \geq 5$ :

$$n_2 = \frac{1}{2}dL^2. \tag{1}$$

Consider the twisted ideal sheaves  $\mathcal{I}_{X, \mathcal{Q}^n}(l) := \mathcal{I}_{X, \mathcal{Q}^n} \otimes \mathcal{O}_{\mathcal{Q}^n}(l)$ . We write the total Chern class of these sheaves as  $1 + \sum_{i=1}^n \gamma_i h^i$ . The following is a standard consequence of [14], Theorem and Lemma 15.3.

LEMMA 1.1.2. *Let  $X$  and  $\mathcal{I}_{X, \mathcal{Q}^n}(l)$  be as above, with  $l$  fixed. Assume that  $n \geq 5$ . Then one has the following relations concerning the Chern classes of  $\mathcal{I}_{X, \mathcal{Q}^5}(l)$ :*

$$\gamma_1 = l; \quad \gamma_i = \frac{1}{2}(K_X + (5 - l)L)^{i-2} \cdot L^{n-i}, \quad \forall i = 2, \dots, n.$$

We now make explicit the *Double Point Formula* for the embedding  $\iota$ . The proof is a standard consequence of [14] Theorem 9.3, once we use (1) and the fact that  $n_i = 0$  for  $i \geq 3$ . Denote by  $c_i$  the Chern classes of the tangent bundle of  $X$ .

LEMMA 1.1.3. *Let  $\iota : X \hookrightarrow \mathcal{Q}^n$  be as above with  $n \geq 5$ . Then one has the following relations in the Chow ring of  $X$ :*

$$\frac{1}{2}dL^2 = \frac{1}{2}(n^2 - n + 2)L^2 - nc_1 \cdot L + c_1^2 - c_2; \tag{2}$$

$$c_3 = \frac{1}{6}(n^3 - 3n^2 + 8n - 12)L^3 + \frac{1}{2}(-n^2 + n - 2)c_1L^2 + n(c_1^2 - c_2)L + 2c_1c_2 - c_3^3; \tag{3}$$

$$c_4 = 22L^4 - 24L^3c_1 + 16L^2(c_1^2 - c_2) + 12Lc_1c_2 - 6Lc_1^3 - 6Lc_3 + 2c_1c_3 + (c_1^2 - c_2)^2 - c_1^2c_2. \tag{4}$$

For  $n = 5$  we have:

$$K_X \cdot L^2 = 2(g - 1) - 2d, \tag{5}$$

$$K_X^2 \cdot L = \frac{1}{4}d^2 + \frac{3}{2}d - 8(g - 1) + 6\chi(\mathcal{O}_S), \tag{6}$$

$$K_X^3 = -\frac{9}{2}d^2 + \frac{27}{2}d + gd + 18(g - 1) - 30\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X), \tag{7}$$

$$c_2 \cdot L = -\frac{1}{4}d^2 + \frac{5}{2}d + 2(g - 1) + 6\chi(\mathcal{O}_S) \tag{8}$$

$$c_3 = \frac{1}{4}d^2 - \frac{1}{2}d - 10(g - 1) + gd + 24\chi(\mathcal{O}_S) - 30\chi(\mathcal{O}_X). \tag{9}$$

To prove (5) we use the genus formula. (6) follows from [4], Proposition 2.1, after having realized that  $K_X^2 \cdot L = K_{X|S}^2 = (K_S - L|_S)^2$ . The formula for  $K_X^3$  follows by “cutting” (2) with  $K_X$  and by using the above expressions for  $K_X \cdot L^2$ ,  $K_X^2 \cdot L$ , and the fact that, by Hirzebruch-Riemann-Roch on a threefold,  $c_1c_2 = 24\chi(\mathcal{O}_X)$ . The proof of (8) is similar. (9) is obtained from (3) by first plugging the expression for  $(c_1^2 - c_2)$  that one gets from (2) and then by plugging the above relations into it.

Finally we record the expression for the Hilbert polynomial of a threefold  $X \subseteq \mathcal{Q}^5$ :

$$\chi(\mathcal{O}_X(t)) = \frac{1}{6}dt^3 + \left[ \frac{1}{2} - \frac{1}{2}(g-1) \right]t^2 + \left[ \frac{1}{3}d - \frac{1}{2}(g-1) + \chi(\mathcal{O}_S) \right]t + \chi(\mathcal{O}_X), \tag{10}$$

which is an easy consequence of Hirzebruch-Riemann-Roch on a threefold (cf. [18], page 437) and of the formulæ above.

**FACT 1.1.4** (Unirationality of the Hilbert scheme). Let  $H$  be the connected component of the Hilbert scheme of  $\mathcal{Q}^n$  containing the point corresponding to a fixed  $X \subseteq \mathcal{Q}^n$ . Denote by  $\mathfrak{H}$  the open subscheme of  $H$  corresponding to nonsingular subvarieties. Assume that every subvariety,  $X' \in \mathfrak{H}$ , admits a resolution of its ideal sheaf of the following form:

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}^n}^s \rightarrow E \rightarrow \mathcal{I}_{X'}(c_1(E)) \rightarrow 0,$$

where  $E$  is a fixed, locally free sheaf independent of  $X'$  and  $s$  is a fixed positive integer. Under the above assumptions  $\mathfrak{H}_{red}$  is integral and unirational. In fact it is enough to observe that the natural rational map  $\mathbf{P}(\wedge^s H^0(E)^\vee) \dashrightarrow \mathfrak{H}$  is a dominant one.

In the present context, Lemma 2.3 of [4] gives the following:

**FACT 1.1.5** (Smoothness and dimension of the Hilbert Scheme). If  $h^i(\mathcal{O}_X) = 0$ ,  $i \geq 1$ , then  $h^i(\mathcal{N}_{X, \mathcal{Q}^n}) = 0$ ,  $\forall i \geq 1$ ,  $\mathfrak{H}$  is nonsingular and of dimension  $h^0(\mathcal{N}_{X, \mathcal{Q}^n})$  at  $X$ . Riemann-Roch on a threefold,  $n_1 = K_X + 5L$ ,  $n_2 = (d/2)L^2$ , and formulæ (5), (6), (8) give, for  $n = 5$ :

$$\chi(\mathcal{N}_{X, \mathcal{Q}^n}) = -\frac{5}{4}d^2 + 10d + 10(g-1) + 5\chi(\mathcal{O}_S).$$

**FACT 1.1.6** (The Hilbert scheme of complete intersections). If  $X \subseteq \mathcal{Q}^n$  is a complete intersection of type  $(2, i, j)$  in  $\mathbf{P}^{n+1}$ , then the corresponding Hilbert scheme  $\mathfrak{H}$  is integral, nonsingular and rational.

For  $i < j$ :

$$\begin{aligned} \dim \mathfrak{H} = P(n; i, j) := & [B(n+1+i, n+1) - B(n+1+i-2, n+1) - 1] \\ & + [B(n+j, n) - B(n+j-2, n) - 1], \end{aligned}$$

where  $B(a, b) := a!/[b!(a-b)!]$ , is the usual binomial coefficient.

For  $i = j$ :

$$\dim \mathfrak{H} = Q(n; i) := 2[B(n+1+i, n+1) - B(n+1+i-2, n+1) - 2].$$

The following gives: 1) a method to construct codimension two subvarieties of  $\mathcal{Q}^n$  using vector bundles; 2) a way to reconstruct the ideal sheaf of a codimension two subvariety,  $X$ , given enough sections of twists of its dualizing sheaf  $K_X$ .

**FACT 1.1.7.** The following is a Bertini-type Theorem due to Kleiman; see [22]. Let  $E, F$  two vector bundles on  $\mathcal{Q}^n$  of rank  $m$  and  $m'$  respectively, such that  $E^\vee \otimes F$  is generated by its global sections. Let  $\phi : E \rightarrow F$  be an element of  $H^0(E^\vee \otimes F)$ . Define

$D_k(\phi)$  to be the closed subscheme of  $\mathcal{Q}^n$  defined, locally, by the vanishing of the  $(k + 1) \times (k + 1)$  minors of a matrix representing  $\phi$ . For the general  $\phi$  and for every  $k$ :

a) either  $D_k$  is empty or it has codimension  $(m - k)(m' - k)$  and  $D_k(\phi)_{\text{sing}} \subseteq D_{k-1}(\phi)$ ; in particular, for  $n < (m - k + 1)(m' - k + 1)$ ,  $D_k(\phi)$  is nonsingular;

b) for  $n \geq 5$ , assuming that  $D_k(\phi)$  has codimension two,  $D_k(\phi)$  is connected (see the remarks following Theorem 2.2 of [28]).

The following fact, proved by Vogelaaar, stems from an idea of Serre's; see [24], Theorem I.6.4.2 or [4] §2.3. Let  $X \subseteq \mathcal{Q}^n$  be a local complete intersection of codimension two and  $a$  be an integer such that the twist  $\omega_X(a)$  is generated by  $s$  of its global sections. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}^n}^s \rightarrow F \rightarrow \mathcal{I}_X(n - a) \rightarrow 0,$$

with  $F$  locally free.

**1.2. A lifting criterion and bounds for the genera of curves on  $\mathcal{Q}^3$ .**

The following is well known when  $\mathcal{Q}^n$  is replaced by  $\mathbf{P}^n$ , see [7] for example. The case of  $\mathcal{Q}^4$  is proved in [4], Lemma 6.1. The general case can be proved in the same way. We used it as a tool to prove the finiteness of the number of families of nonsingular codimension two subvarieties of  $\mathcal{Q}^5$  not of general type. See [9], where we prove a more general statement.

**PROPOSITION 1.2.1.** *Let  $X$  be an integral subscheme of degree  $d$  and codimension two on  $\mathcal{Q}^n$ ,  $n \geq 4$ . Assume that for the general hyperplane section  $Y$  of  $X$  we have  $h^0(\mathcal{I}_{Y, \mathcal{Q}^{n-1}}(\sigma)) \neq 0$ , for some positive integer  $\sigma$  such that  $d > 2\sigma^2$ . Then  $h^0(\mathcal{I}_{X, \mathcal{Q}^n}(\sigma)) \neq 0$ .*

1.2.2.  $C$  is an integral curve lying on a smooth three-dimensional quadric  $\mathcal{Q}^3$ ,  $k$  is a positive integer,  $S_k$  is an integral surface in  $|\mathcal{O}_{\mathcal{Q}^3}(k)|$  containing  $C$ ,  $d$  and  $g$  are the degree and the geometric genus of  $C$ , respectively.

**DEFINITION 1.2.3.** Define  $n_0$  and  $\varepsilon$  when  $d > 2k(k - 1)$  and  $\theta_0$  and  $\varepsilon'$  when  $d \leq 2k(k - 1)$  as follows:

$$n_0 := \left\lfloor \frac{d - 1}{2k} \right\rfloor + 1, \quad d \equiv -\varepsilon \pmod{2k}, \quad 0 \leq \varepsilon \leq 2k - 1;$$

$$\theta_0 := \left\lfloor \frac{d - 1}{2k} \right\rfloor + 1, \quad d \equiv -\varepsilon' \pmod{2\theta_0}, \quad 0 \leq \varepsilon' \leq 2\theta_0 - 1.$$

The following class of curves plays a central role in the understanding of the curves whose genus is the maximum possible. Arithmetically Cohen-Macaulay is denoted by a.C.M..

**DEFINITION 1.2.4.** A curve  $C$  as in (1.2.2) is said to be in the class  $\mathfrak{S}(d, k)$ , if it is nonsingular, projectively normal and linked, in a complete intersection on  $\mathcal{Q}^3$  of type  $(k, n_0)$  if  $d > 2k(k - 1)$  (of type  $(\theta_0, k)$  if  $d \leq 2k(k - 1)$ ), to an *a fortiori* a.C.M. curve  $D_\varepsilon$  ( $D_{\varepsilon'}$ , respectively) of degree  $\varepsilon$  ( $\varepsilon'$  respectively) lying on a quadric surface hyperplane section of  $\mathcal{Q}^3$ .

**PROPOSITION 1.2.5** (Cf. [8].) *Notation as in (1.2.2) and Definition 1.2.3. Assume first that  $d > 2k(k - 1)$ . Then*

(a)

$$g - 1 \leq \pi(d, k) - \Xi$$

where

$$\pi(d, k) = \begin{cases} \frac{d^2}{4k} + \frac{1}{2}(k - 3)d - \frac{\varepsilon^2}{4k} - \varepsilon\left(\frac{k - \varepsilon}{2}\right), & \text{if } 0 \leq \varepsilon \leq k, \\ \frac{d^2}{4k} + \frac{1}{2}(k - 3)d - (k - \tilde{\varepsilon})\left(\frac{\tilde{\varepsilon}}{2} - \frac{\tilde{\varepsilon}}{4k} + \frac{1}{4}\right), & \text{if } k + 1 \leq \varepsilon \leq 2k - 1, \tilde{\varepsilon} := \varepsilon - k; \end{cases}$$

and

$$\Xi = \Xi(d, k) = \begin{cases} 0 & \text{if } \varepsilon = 0, 1, 2, 2k - 1, \\ 1 & \text{if otherwise.} \end{cases}$$

(b) *The bound is sharp for  $\varepsilon = 0, 1, 2, 3, 2k - 2, 2k - 1$ . A curve achieves such a maximum possible genus if and only if it is in the class  $\mathfrak{S}(d, k)$ , except, possibly, the cases  $\varepsilon = 3, 2k - 2$ . Assume  $d \leq 2k(k - 1)$ . Then statements a) and b), with  $\pi'(d, k) = \pi(d, \lfloor (d - 1)/2k \rfloor + 1) = \pi(d, \theta_0)$  and with  $\Xi', \varepsilon', (\theta_0, k)$  and  $D_{\varepsilon'}$  replacing  $\Xi, \varepsilon, (k, n_0)$  and  $D_\varepsilon$ , respectively, hold.*

**COROLLARY 1.2.6** (See [4], Proposition 6.4 for the case  $d > 2k(k - 1)$ .) *Notation as above. Then*

$$g - 1 \leq \frac{d^2}{4k} + \frac{1}{2}(k - 3)d.$$

**PROPOSITION 1.2.7** (Cf. [4], Proposition 6.4.) *Let  $C$  be an integral curve in  $\mathcal{Q}^3$ , not contained in any surface of degree strictly less than  $2k$ . Then:*

$$g - 1 \leq \frac{d^2}{2k} + \frac{1}{2}(k - 4)d.$$

**1.3. An inequality.**

In this section we prove an inequality which is an essential tool for our proof of the classification of scrolls over surfaces on  $\mathcal{Q}^5$ .

Let  $X \subseteq \mathcal{Q}^5$  be a three dimensional, nonsingular variety,  $\sigma$  be the smallest integer for which there exists a hypersurface  $V$  in  $|\mathcal{I}_{X, \mathcal{Q}^5}(\sigma)|$  and  $\mathcal{N}$  the normal bundle of  $X$ . By the minimality of  $\sigma$ , the natural section  $\mathcal{O}_X \rightarrow \mathcal{N}(\sigma)$  is not the trivial one. The transposed of this section defines the sheaf of ideals of  $\mathcal{O}_X$  of the singular locus of  $V$  restricted to  $X$ . Let us denote by  $\tilde{\Sigma}$  the associated scheme. We obtain the surjection  $\mathcal{N} \rightarrow \mathcal{I}_{\tilde{\Sigma}}(\sigma)$ .

**DEFINITION 1.3.1.** Let  $D$  be the divisorial component of  $\tilde{\Sigma}$ , i.e. the unique effective Cartier divisor of  $X$  whose sheaf of ideals is the smallest sheaf of principal ideals containing  $\mathcal{I}_{\tilde{\Sigma}}$ ;  $D$  may be empty. Let  $\Sigma$  be the one dimensional component of  $\tilde{\Sigma}$ , i.e. the scheme associated with the sheaf of ideals  $\mathcal{I}_{\tilde{\Sigma}}(D)$ ;  $\Sigma$  is either empty or of pure dimension one.

From the above we get that the following two facts hold.

FACT 1.3.2. The sheaf  $\mathcal{I}_\Sigma$  is either  $\mathcal{O}_X$  or it has homological dimension one.

FACT 1.3.3.  $\mathcal{I}_\Sigma(\sigma L - D) = \mathcal{I}_{\tilde{\Sigma}}(\sigma L)$ ; in particular,  $\mathcal{I}_\Sigma(\sigma L - D)$  is generated by global sections since it is a quotient of  $\mathcal{N}$  which is a quotient sheaf of the globally generated sheaf  $\mathcal{T}_{\mathcal{Q}^5}$ .

PROPOSITION 1.3.4. Let  $s_i$ ,  $i = 1, 2, 3$  be the Segre classes of  $\mathcal{I}_\Sigma(\sigma L - D)$ . Then  $s_1 s_2 \geq s_3 \geq 0$ ,  $s_1$  and  $s_1^2 - s_2$  are represented by effective cycles. Moreover,

$$\chi(\mathcal{O}_S) \geq \frac{1}{6\sigma} \left[ (d - 12\sigma)(g - 1) + \left( \frac{1}{4}\sigma + \frac{3}{2} \right) d^2 - \frac{13}{2}\sigma d \right] - \frac{1}{6\sigma} \left[ \frac{1}{2}dL^2 - (K_X + 5L)^2 \right] D. \quad (11)$$

PROOF. By Fact 1.3.3 there is a surjection  $\mathcal{O}_X^m \rightarrow \mathcal{I}_\Sigma(\sigma L - D)$ , for some  $m$ . By Fact 1.3.2 the kernel,  $F$ , of this surjection is locally free. By the definition of Segre classes,  $s_i = c_i(\tilde{F})$ . The first part of the proposition follows from [7], Lemma 5.1.

As to the proof of the last inequality, first we compute the Chern classes  $C_i$  of  $\mathcal{I}_\Sigma(\sigma L - D)$  using the following exact sequence which is the Koszul resolution of  $\mathcal{I}_\Sigma(\sigma L - D)$ :

$$0 \rightarrow \mathcal{O}_X(K_X - \sigma L + D) \rightarrow \mathcal{N} \rightarrow \mathcal{I}_\Sigma(\sigma L - D) \rightarrow 0;$$

we get

$$C_1 = \sigma L - D,$$

$$C_2 = \frac{1}{2}dL^2 - (K_X + 5L)(\sigma L - D) + (\sigma L - D)^2,$$

$$C_3 = -\frac{1}{2}dL^2(K_X + 5L) + \frac{1}{2}dL^2(\sigma L - D) + (K_X + 5L)^2(\sigma L - D) - 2(K_X + 5L)(\sigma L - D)^2 + (\sigma L - D)^3.$$

The Segre classes of  $\mathcal{I}_\Sigma(\sigma L - D)$  are  $s_1 = C_1$ ,  $s_2 = C_1^2 - C_2$ ,  $s_3 = C_1^3 - 2C_1C_2 + C_3$ . We make explicit these Segre classes using the formulæ for the  $C_i$ . Then we use (6) and (5). We now use the part of the proposition that we have just proved: (11) is  $s_3 \geq 0$ .  $\square$

#### 1.4. Special vector bundles on quadrics.

FACT 1.4.1 (Spinor Bundles). Here we collect some properties of spinor bundles on quadrics. See [1].

Let  $\mathcal{S}$  be the spinor bundle on an odd-dimensional quadric and  $\mathcal{S}'$ ,  $\mathcal{S}''$  be the two spinor bundles on an even dimensional quadric; if  $n$  is the dimension of the quadric, the rank of these bundles is  $2^{\lfloor (n-1)/2 \rfloor}$ .

For  $n = 2m + 1$  (odd) we have an exact sequence:

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_{\mathcal{Q}^n}^{2^{m+1}} \rightarrow \mathcal{S}(1) \rightarrow 0;$$

for  $n = 2m$  (even) we have exact sequences:

$$0 \rightarrow S \rightarrow \mathcal{O}_{\mathcal{Q}^n}^{2^m} \rightarrow S(1) \rightarrow 0,$$

where  $S$  denotes either  $\mathcal{S}'$  or  $\mathcal{S}''$ .

For  $n = 2m + 1$  we have  $\mathcal{S}^\vee \simeq \mathcal{S}(1)$ . For  $n = 4m$  we have  $\mathcal{S}'^\vee \simeq \mathcal{S}'(1)$  and  $\mathcal{S}''^\vee \simeq \mathcal{S}''(1)$ ; for  $n = 4m + 2$  we have  $\mathcal{S}'^\vee \simeq \mathcal{S}''(1)$  and  $\mathcal{S}''^\vee \simeq \mathcal{S}'(1)$ . Let  $i : \mathcal{Q}^{2k-1} \hookrightarrow \mathcal{Q}^{2k}$  be a nonsingular hyperplane section; then  $i^*\mathcal{S}' \simeq i^*\mathcal{S}'' \simeq \mathcal{S}$ . Let  $j : \mathcal{Q}^{2h} \hookrightarrow \mathcal{Q}^{2h+1}$  be a nonsingular hyperplane section; then  $j^*\mathcal{S} \simeq \mathcal{S}' \oplus \mathcal{S}''$ .

An analogue of Horrocks splitting criterion holds on quadrics; recall that spinor bundles carry no intermediate cohomology:

let  $E$  be a vector bundle on  $\mathcal{Q}^n$  then  $h^i(E(t)) = 0, 0 < i < n, \forall t \in \mathbf{Z}$  if and only if  $E$  splits as the direct sum of line bundles and twists of spinor bundles of  $\mathcal{Q}^n$ .

The Chern polynomial of  $\mathcal{S}(l)$  on  $\mathcal{Q}^5$  is:

$$c(\mathcal{S}(l)) = 1 + (4l - 2)h + (6l^2 - 6l + 2)h^2 + (4l^3 - 6l^2 + 4l - 1)h^3 + (l^4 - 2l^3 + 2l^2 - l)h^4.$$

The Chern polynomial of  $\mathcal{S}'(l)$  on  $\mathcal{Q}^6$  is:

$$c(\mathcal{S}'(l)) = 1 + (4l - 2)h + (6l^2 - 6l + 2)h^2 + [(4l^3 - 6l^2 + 4l)h^3 - 2A_1] + [(l^4 - 2l^3 + 2l^2)h^4 - 2lA_1h]$$

Replacing  $A_1$  by  $A_2$  in the formula above, we get  $c(\mathcal{S}''(l))$ .

**FACT 1.4.2 (Cayley bundles).** See [25]. On  $\mathcal{Q}^5$  there is a family of rank two stable vector bundles, called Cayley bundles. Each Cayley bundle  $\mathcal{C}$  has Chern classes  $c_1 = -1, c_2 = 1$  and  $\mathcal{C}(2)$  is generated by global sections. Every stable 2-bundle on  $\mathcal{Q}^5$  with Chern classes  $c_1 = -1, c_2 = 1$  is a Cayley bundle. Cayley bundles are parameterized by a fine moduli space isomorphic to  $\mathbf{P}^7 \setminus \mathcal{Q}^6$ . A Cayley bundle restricts, on a  $\mathcal{Q}^4$ , to a bundle of type  $\dot{E}$  which appears in the description of Type 10) of [4], page 44. The Chern polynomial of a  $\mathcal{C}(l)$  is:  $c(\mathcal{C}(l)) = 1 + (2l - 1)h + (l^2 - l + 1)h^2$ .

**2. Classification for  $d \leq 10$ .**

**2.1. The list.**

In what follows:

- $((a, b, c), \mathcal{O}(1))$  denotes the polarized pair given by a complete intersection of type  $(a, b, c)$  in  $\mathbf{P}^{n+1}$  and the restriction of the hyperplane bundle to it;
- $(X, L)$  denotes the polarized pair given by a variety  $X \subseteq \mathcal{Q}^n$  and  $L := \mathcal{O}_{\mathcal{Q}^n}(1)|_X$ ; if we do not explicitly say the contrary, the embeddings are projectively normal in  $\mathbf{P}^{n+1}$ ; this fact follows from the cohomology of the presentation of the ideal sheaf;
- by a *presentation* of the ideal sheaf  $\mathcal{I}_X$  we mean an injection of locally free coherent sheaves on  $\mathcal{Q}^n, \phi : E \rightarrow F$ , such that  $\text{coker}(\phi) \simeq \mathcal{I}_X(i)$ , where  $i = c_1(F) - c_1(E)$ ; we write the presentations so that the integer  $i$  is the smallest for which the sheaf  $F$  is generated by global sections, so that for that  $i$  so will be the sheaf  $\mathcal{I}_X(i)$ ;
- $\mathfrak{H}$  denotes the Hilbert scheme of  $\mathcal{Q}^n$  of a variety fixed by the context; see Fact 1.1.4;
- $P(n; i, j)$  and  $Q(n; i)$  are defined in Fact 1.1.6;
- a digit “#” refers to the type of the surface section as in [4], page 44 (where  $d \leq 8$ ); type  $Z_{10}^F$  refers to the paper [16];
- $g, q$  and  $p_g$  denote the sectional genus of the embedding line bundle, the irregularity and geometric genus of a surface section, respectively.



**THEOREM 2.1.1.** *Let  $X \subseteq \mathcal{Q}^n$ ,  $n \geq 5$ , a codimension two nonsingular subvariety of degree  $d \leq 10$ . Then the pair  $(X, L)$ , a presentation of the ideal of  $X$  on  $\mathcal{Q}^n$  and the Hilbert scheme,  $\mathfrak{H}$ , of  $X$  on  $\mathcal{Q}^n$  are as follows.*

(•)  $d = 2$

*Type A):  $((1, 1, 2), \mathcal{O}(1)); \mathcal{O}_{\mathcal{Q}^n}(-1) \rightarrow \mathcal{O}_{\mathcal{Q}^n}^2$ ;  $\mathfrak{H}$  is integral, nonsingular, rational, of dimension  $Q(n; 1); 2$ ;  $g = q = p_g = 0$ .*

(•)  $d = 4$

*Type B):  $((1, 2, 2), \mathcal{O}(1)); \mathcal{O}_{\mathcal{Q}^n}(-1) \rightarrow \mathcal{O}_{\mathcal{Q}^n}(1) \oplus \mathcal{O}_{\mathcal{Q}^n}$ ;  $\mathfrak{H}$  is integral, nonsingular, rational and of dimension  $P(n; 1, 2); 6$ ;  $g = 1$ ,  $q = p_g = 0$ .*

*Type C):  $n = 6$ ,  $(\mathbf{P}^1 \times \mathbf{P}^3, \mathcal{O}(1, 1)); \mathcal{O}_{\mathcal{Q}^6}^3 \rightarrow \mathfrak{S}(1)$ , with  $\mathfrak{S} \simeq \mathcal{S}'$ ,  $\mathcal{S}''$ ;  $\mathfrak{H}$  consists of two connected components, which are both nonsingular, integral, unirational and of dimension  $15; 5$ ;  $g = q = p_g = 0$ .*

*Type D):  $n = 5$ ,  $(\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbf{P}^1}(2)), \xi); \mathcal{O}_{\mathcal{Q}^5}^3 \rightarrow \mathcal{S}(1)$ ;  $\mathfrak{H}$  is integral, nonsingular, unirational and of dimension  $15; 5$ ;  $g = q = p_g = 0$ .*

(•)  $d = 6$

*Type E):  $((1, 2, 3), \mathcal{O}(1)); \mathcal{O}_{\mathcal{Q}^n}(-1) \rightarrow \mathcal{O}_{\mathcal{Q}^n}(2) \oplus \mathcal{O}_{\mathcal{Q}^n}$ ;  $\mathfrak{H}$  is integral, nonsingular, rational and of dimension  $P(n; 1, 3); 12$ ;  $g = 4$ ,  $q = 0$ ,  $p_g = 1$ .*

*Type F):  $n = 5$ ,  $(\mathbf{P}(\mathcal{T}_{\mathbf{P}^2}), \xi)$ , embedded using a general codimension one linear system  $\mathcal{L} \subseteq |\xi_{\mathcal{T}_{\mathbf{P}^2}}|$ ;  $\mathcal{O}_{\mathcal{Q}^5} \rightarrow \mathcal{C}(2)$ ;  $\mathfrak{H}$  is integral, nonsingular, unirational and of dimension  $20; 10$ ;  $g = 1$ ,  $q = p_g = 0$ .*

*Type G):  $n = 5$ ,  $f : X \rightarrow \mathbf{P}^1 \times \mathbf{P}^2 =: Y$  a double cover, branched along a divisor of type  $\mathcal{O}_Y(2, 2)$ ,  $L \simeq p^* \mathcal{O}_Y(1, 1)$ ;  $\mathcal{O}_{\mathcal{Q}^5}(-1)^2 \rightarrow \mathcal{O}_{\mathcal{Q}^5}^3$ ;  $\mathfrak{H}$  is integral, nonsingular, unirational and of dimension  $30; 11$ ;  $g = 2$ ,  $q = p_g = 0$ .*

(•)  $d = 8$

*Type H):  $((1, 2, 4), \mathcal{O}(1)); \mathcal{O}_{\mathcal{Q}^n}(-1) \rightarrow \mathcal{O}_{\mathcal{Q}^n}(3) \oplus \mathcal{O}_{\mathcal{Q}^n}$ ;  $\mathfrak{H}$  is integral, nonsingular, rational and of dimension  $P(n; 1, 4); 20$ ;  $g = 9$ ,  $q = 0$ ,  $p_g = 5$ .*

*Type I):  $((2, 2, 2), \mathcal{O}(1)); \mathcal{O}_{\mathcal{Q}^n}(-2) \rightarrow \mathcal{O}_{\mathcal{Q}^n}^2$ ;  $\mathfrak{H}$  is integral, nonsingular, rational and of dimension  $Q(n; 2); 19$ ;  $g = 5$ ,  $q = 0$ ,  $p_g = 1$ .*

*Type L):  $n = 5$ ,  $(\mathbf{P}(E), \xi)$ ,  $E$  a rank two vector bundle on  $\mathcal{Q}^2$  as in [21];  $\mathcal{O}_{\mathcal{Q}^5}^4 \rightarrow \mathcal{S}(1) \oplus \mathcal{O}_{\mathcal{Q}^5}(1)$ ;  $\mathfrak{H}$  is integral, nonsingular, unirational and of dimension  $35; 18$ ;  $g = 4$ ,  $q = p_g = 0$ .*

(•)  $d = 10$

*Type M):  $((1, 2, 5), \mathcal{O}(1)); \mathcal{O}_{\mathcal{Q}^n}(-1) \rightarrow \mathcal{O}_{\mathcal{Q}^n}(4) \oplus \mathcal{O}_{\mathcal{Q}^n}$ ;  $\mathfrak{H}$  is integral, nonsingular, rational and of dimension  $P(n; 1, 5)$ ;  $g = 16$ ,  $q = 0$ ,  $p_g = 14$ .*

*Type N):  $n = 5$ ,  $f_{|K_X+L|} : X \rightarrow \mathbf{P}^1$  is a fibration with general fiber a Del Pezzo surface  $F$ ,  $K_F^2 = 4$ ,  $K_X = -L + f^* \mathcal{O}_{\mathbf{P}^1}(1)$ ;  $\mathcal{O}_{\mathcal{Q}^5}(-1)^2 \rightarrow \mathcal{O}_{\mathcal{Q}^5}(1) \oplus \mathcal{O}_{\mathcal{Q}^5}^2$ ;  $\mathfrak{H}$  is integral, nonsingular, unirational and of dimension  $60$ ; type  $Z_{10}^F$ ;  $g = 8$ ,  $q = 0$ ,  $p_g = 2$ .*

**REMARK 2.1.2.** In this remark, by the symbol  $Q \stackrel{(a,b)}{\sim} R$ , we mean that every variety of Type Q) is linked to a variety of Type R) in a complete intersection of type  $(a, b)$  on  $\mathcal{Q}^5$ . Using Lemma 3.4.7 and the presentations of the ideals of the varieties of the above

theorem we see that:  $A \overset{(1,2)}{\sim} A, A \overset{(1,3)}{\sim} B, A \overset{(1,4)}{\sim} E, A \overset{(2,2)}{\sim} G, A \overset{(1,5)}{\sim} H, A \overset{(1,6)}{\sim} M, A \overset{(2,3)}{\sim} N, B \overset{(2,2)}{\sim} B, B \overset{(2,3)}{\sim} I, G \overset{(2,2)}{\sim} A, G \overset{(2,3)}{\sim} G, G \overset{(2,4)}{\sim} N, I \overset{(2,3)}{\sim} B, I \overset{(2,4)}{\sim} I, I \overset{(3,3)}{\sim} N, N \overset{(3,3)}{\sim} I.$

The simple details are left to the reader. As for Type F), see Proposition 3.4.8.

**2.2. The proof.**

PROOF OF THEOREM 2.1.1. The degree  $d$  is always an even integer by Remark 1.1.1. The statements of the Theorem concerning complete intersections follow from Fact 1.1.6 and [18], III.9.Ex. 9.6. In the sequel, we do not deal with complete intersections.

CLAIM. *The only nonsingular surfaces on  $\mathcal{Q}^4$  which can be a general hyperplane section of a threefold on  $\mathcal{Q}^5$  of degree  $d \leq 10$  are: types 5), 10), 11) and 18) from [4] and type  $Z_F^{10}$  from [16].*

PROOF OF THE CLAIM. Let  $d \leq 8$ . [4] page 44 contains the complete list of nonsingular surfaces on a  $\mathcal{Q}^4$  of degree  $d \leq 8$ . Not all of them can be a general hyperplane section of a threefold on  $\mathcal{Q}^5$ . The complete list of linearly normal, nonsingular subvarieties of projective space of degree  $d \leq 8$  is given in [20] and [21]. We are going to use these results jointly.

Since  $H^2(\mathcal{Q}^5, \mathbf{Z}) \simeq \mathbf{Z}\langle h^2 \rangle$ , the surface section,  $S$ , of a degree  $d$  threefold  $X \subseteq \mathcal{Q}^5$  has cohomology class  $[S] = (d/2)A_1 + (d/2)A_2$ . This implies that the surfaces of type 1), 3), 4), 7), 8), 13), 14), 15) and 16) of [4] page 44 cannot be nonsingular hyperplane sections of any threefold on  $\mathcal{Q}^5$ . Types 2), 6), 12), 19) and 20) are complete intersections.

In what follows, assume that  $S$  is a surface of a given type and that  $X \subseteq \mathcal{Q}^5$  is a threefold with general surface section  $S$ . We now exclude types 9) and 17). Type 9). If  $X$  existed, a comparison with Ionescu’s list [20] would force  $X$  to be a rational scroll over a curve contradicting Proposition 3.3.1. Type 17). This type has sectional genus  $g = 3$  so that, according to Ionescu’s list [21],  $X$  would have to be either a scroll over an elliptic curve, a scroll over  $\mathbf{P}^2$  or  $X$  would have to admit a morphism onto  $\mathbf{P}^1$  with all fibers quadric surfaces. We exclude the first case because type 17) is simply connected, the second one by Proposition 3.4.4. The last one would imply, after having cut (2) with a general fiber  $F \simeq \mathcal{Q}^2$ , the contradiction  $d = 6$ .

It follows that, except for the case of complete intersections, only the following types are admissible as surface sections of codimension two nonsingular subvarieties of quadrics when  $d \leq 8$ : 5), 10), 11), 18).

Let  $d = 10$ . We employ the same technique as above using [16] and [12] instead of [4] and [20], [21]. Looking at the list in [16] we exclude cases  $Z_A^{10}$  and  $Z_B^{10}$  since they do not have a balanced cohomology class. Cases  $Z_D^{10}$  and  $Z_E^{10}$  cannot occur by [12], since they have sectional genus  $g = 7$ .

The case  $C_A^{10}$ , where the sectional genus  $g = 4$  and the irregularity  $q = 1$  is excluded since, by [12], we would have  $q = 0$ . The cases  $C_B^{10}$  and  $C_C^{10}$  are excluded in a similar way.

The case  $Z_C^{10}$ , which is a rational surface with  $g = 6$  is excluded as follows. According to [12] there are only two types of threefolds of degree  $d = 10$  with sectional genus  $g = 6$ ; the first is a Mukai manifold (i.e.  $K_X = -L$ ), the second one a scroll over

$P^2$ . In the former case the surface section would have trivial canonical bundle, contradicting its being rational. The latter case is excluded by Proposition 3.4.4.

The proof of the Claim is complete.

We now show that all the types of the claim occur as nonsingular surface sections of threefolds on  $\mathcal{Q}^5$ , that type 5) is the only one that can occur as a section of a fourfold on  $\mathcal{Q}^6$  and that none of these types can occur as a surface section of any  $(n - 2)$ -fold on  $\mathcal{Q}^n$ , for  $n \geq 7$ .

*The case of Type 5).*

Assume that  $X \subseteq \mathcal{Q}^6$  is a fourfold with surface section of type 5). By Swinnerton-Dyer's classification of varieties of degree  $d = 4$  (see [20] for example),  $(X, L)$  is of Type C); such a type occurs as a subvariety of  $\mathcal{Q}^6$  as pointed out in Proposition 3.3.1.  $K_X = \mathcal{O}_X(-2, -4)$ , so that  $K_X(4)$  is generated by three global sections. Fact 1.1.7 gives us the following exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}^6}^3 \rightarrow F \rightarrow \mathcal{F}_X(2) \rightarrow 0, \tag{12}$$

where  $F$  is locally free. We want to prove that  $F$  is isomorphic to either  $\mathcal{S}'_6(1)$  or to  $\mathcal{S}''_6(1)$ , where the subindices refer to the fact that the bundles are the spinor bundles of  $\mathcal{Q}^6$ . Consider a general threefold section  $T$ , which is of type D), and a general surface section  $S \subset T$ . We have  $K_T \simeq K_X(1) \otimes \mathcal{O}_T$  and  $K_S \simeq K_X(2) \otimes \mathcal{O}_S$ ; there is a canonical identification between  $H^0(K_X(4)), H^0(K_T(3))$  and  $H^0(K_S(2))$  so that the bundles  $\mathcal{F}_T$  and  $\mathcal{F}_S$ , that we obtain repeating for  $T$  and  $S$  the construction we have done for  $X$  using Fact 1.1.7, satisfy  $\mathcal{F}_T \simeq F|_T$  and  $\mathcal{F}_S \simeq F|_S$ .

We know, from [4], that, on  $\mathcal{Q}^4$ ,  $F|_S \simeq \mathcal{S}'(1)_4 \oplus \mathcal{S}''(1)$ . Recall that spinor bundles have no intermediate cohomology. Let us look at the long cohomology sequences associated with the exact sequences:

$$0 \rightarrow \mathcal{F}_T(-1 + t) \rightarrow \mathcal{F}_T(t) \rightarrow \mathcal{F}_S(t) \rightarrow 0, \quad t \in \mathbf{Z}.$$

Firstly we deduce that  $H^1(\mathcal{F}_T(-\tau + t))$  surjects onto  $H^1(\mathcal{F}_T(t))$ , for every fixed  $t$  and every  $\tau \geq 0$ ; Serre Duality and Serre Vanishing imply that  $h^1(\mathcal{F}_T(t)) = 0, \forall t$ . The vanishing of  $h^i(\mathcal{F}_T(t))$  for  $2 \leq i \leq 4$  are dealt with similarly. We have proved that  $\mathcal{F}_T(t)$  has no intermediate cohomology, so that, by the analogue of Horrocks criterion in Fact 1.4.1,  $\mathcal{F}_T$  splits as a direct sum of line bundles and twists of spinor bundles on  $\mathcal{Q}^5$ . Since the rank of  $\mathcal{S}$  is four we see that either  $\mathcal{F}_T \simeq \mathcal{S}(j)$  or it splits completely as the direct sum of line bundles  $\mathcal{F}_T \simeq \bigoplus_{i=1}^4 \mathcal{O}_{\mathcal{Q}^5}(a_i)$ . Using the Castelnuovo-Mumford 0-regularity criterion for global generation (see [4], where it is proved for sheaves on  $\mathcal{Q}^4$ ; the case of any  $\mathcal{Q}^n$  is analogous) we see that  $\mathcal{F}_T$  is generated by global sections as soon as  $h^5(\mathcal{F}_T(-5)) = 0$  which follows from the cohomology sequence associated with the sequence (12) twisted by  $-5$  once we observe that  $h^4(\mathcal{S}_{T, \mathcal{Q}^5}(-3)) = 3$  and  $h^5(\mathcal{S}_{T, \mathcal{Q}^5}(-3)) = 0$ . Recall that 1 is the smallest integer  $j$  for which the spinor bundles twisted by  $j$  are generated by global sections. It follows that, for the splitting type of  $\mathcal{F}_T$ , we have either  $j \geq 1$  or  $a_i \geq 0, \forall i$ .

We can compute the Chern classes of  $\mathcal{F}_T$  using (12), the invariants of  $T$  and Lemma 1.1.2. Comparing Chern polynomials we deduce that  $\mathcal{F}_T$  cannot split as the direct sum of line bundles, and that, once it is a twist of the spinor bundle  $\mathcal{S}, j = 1 : \mathcal{F}_T \simeq \mathcal{S}(1)$ .

We repeat the argument, replacing  $S$  with  $T$  and  $T$  with  $X$ , to see that either  $F \simeq \mathcal{S}'(1)$  or  $F \simeq \mathcal{S}''(1)$ .

We have proved that every fourfold on  $\mathcal{Q}^6$  with surface section of type 5) is as in Type C) and has the prescribed presentation for its ideal sheaf; we have also proved that every threefold on  $\mathcal{Q}^5$  with surface section of type 5) is of Type D) and has the prescribed presentation for its ideal sheaf. Conversely, since  $\mathcal{S}(1)$ ,  $\mathcal{S}'(1)$  and  $\mathcal{S}''(1)$  are globally generated, we use Fact 1.1.7 and our maximal list of varieties of degree  $d = 4$  to prove that the variety  $D_2(\phi)$  is as in C) or D), where  $\phi$  is a general element of  $H^0(S(1)^3)$  and  $S$  one of the three spinor bundles in question. To be precise, Fact 1.1.7 implies that, for a general  $\phi$  on  $\mathcal{Q}^6$ ,  $D_1(\phi)$  is either empty or has the expected codimension six and  $D_2(\phi)$  will be nonsingular outside  $D_1(\phi)$ . Porteous' formula, [2], II.4.2 gives  $[D_1(\phi)] = c_3(S(1))^2 - c_2(S(1))c_4(S(1)) = 0$ ; the Chern classes of the spinor bundles are listed in 1.4.1. It follows that  $D_1(\phi) = \emptyset$ .

Using facts (1.1.4) and (1.1.5) we conclude the proof for type D). To complete the proof for type C) we remark that  $\mathcal{S}'(1)$  distinguishes, via the choice of three general sections, a nonsingular, integral component, say  $\mathfrak{S}'$ , of  $\mathfrak{S}$ . The same is true for  $\mathcal{S}''(1)$  which defines another, distinct, nonsingular component  $\mathfrak{S}''$ . Since  $\mathfrak{S}$  is nonsingular,  $\mathfrak{S} = \mathfrak{S}' \sqcup \mathfrak{S}''$ .

The dimension of the two components, which are abstractly isomorphic, can be computed using Riemann-Roch and Fact 1.1.5. The fourfold C) cannot be the hyperplane section of a fivefold; see [20].

*The case of Type 10).*

Assume that  $X$  is a threefold on  $\mathcal{Q}^5$  whose general surface section,  $S$ , is of type 10). Since  $K_S = -L|_S$  and the natural map  $Pic(X) \rightarrow Pic(S)$  is injective by Lefschetz theorem on hyperplane sections, we have  $K_X = -2L$ . Looking at [20] for degree  $d = 6$  we see that either  $(X, L)$  is as in Type F) or  $X$  is a scroll over  $\mathcal{Q}^2$ ; the latter case is not possible by Proposition 3.4.4. We have  $K_X = -2L$ . Fact 1.1.7 yields an extension:

$$0 \rightarrow \mathcal{O}_{\mathcal{Q}^5} \rightarrow F \rightarrow \mathcal{I}_X(3) \rightarrow 0,$$

with  $F$  locally free of rank two. By [24], 2.1.5 one shows that  $F(-2)$  is stable. Using Lemma 1.1.2 we deduce that, for the Chern classes of  $F$ ,  $c_1(F(-2)) = -1$  and  $c_2(F(-2)) = 1$ . By Fact 1.4.2 we see that  $F(-2)$  is a Cayley bundle. Conversely, [25], Theorem 3.7 ensures that the general section of a normalized Cayley bundle twisted by  $\mathcal{O}_{\mathcal{Q}^5}(2)$  vanishes exactly along a variety of type F). As in [3], page 209, we see that our scrolls are parameterized by an open dense set,  $\mathfrak{U}$  of a projective bundle over the fine moduli space,  $\mathbf{P}^7 \setminus \mathcal{Q}^6$ , of these Cayley bundles. This space is clearly rational and it has dimension 20.

$\mathfrak{U}$  admits a natural morphism onto the Hilbert scheme  $\mathfrak{S}$  of our scrolls. This morphism is one to one. To conclude it is enough to observe that  $\mathfrak{S}$  is nonsingular by Fact 1.1.5, for then the morphism in question is an isomorphism by Zariski Main Theorem.

Note that, again by [20], this threefold is the hyperplane section of only one fourfold,  $\mathbf{P}^2 \times \mathbf{P}^2$  embedded via the Segre embedding; this latter can be projected smoothly to  $\mathbf{P}^7$  but, after this embedding, it does not lie on a smooth quadric  $\mathcal{Q}^6$  by Proposition 3.5.1.

The cases 11), 18), and  $Z_F^{10}$ .

These cases are analogous to the one of type 5). If the general surface section,  $S$ , of a threefold  $X$  on  $\mathcal{Q}^5$  is of type 11) then there is a morphism with connected fibers  $f : S \rightarrow \mathbf{P}^1$  all fibers of which are conics and  $K_S(1) = f^* \mathcal{O}_{\mathbf{P}^1}(1)$ , so that the former sheaf is generated by two global sections.

CLAIM.  $K_X(2)$  is generated by its global sections. Fix any point  $x \in X$ . Take any nonsingular hyperplane section  $S$  of  $X$  through  $x$ ; there are plenty of them since the dual variety  $\hat{X}$  does not contain hyperplanes. Kodaira Vanishing implies that  $H^0(X, K_X(2))$  surjects onto  $H^0(S, K_S(1))$  which in turn generates the stalk of  $K_S(1)$ . The claim follows.

A computation analogous to the one of type 5) allows us to conclude that if the threefold in question exists than it is of type G). To prove its existence, we use Fact 1.1.7 for a general morphism  $\phi : \mathcal{O}_{\mathcal{Q}^5}^2 \rightarrow \mathcal{O}_{\mathcal{Q}^5}(1)^3$ . This threefold cannot be the hyperplane section of any fourfold by [20].

If the type of the general surface section is 18) then we have a morphism  $f : S \rightarrow \mathcal{Q}^2$  which is the blowing up of  $\mathcal{Q}^2$  at 10 points and  $K_S(2) \simeq f^* \mathcal{O}_{\mathcal{Q}^2}(1)$  is generated by four global sections. We argue as above and get Type L).

If  $X$  is a threefold on  $\mathcal{Q}^5$  with general section  $S$  of type  $Z_F^{10}$  then the sectional genus  $g = 8$ . By looking at the list in [12], we see that  $X$  is a Del Pezzo fibration  $f : X \rightarrow Y$  over a curve  $Y$  and that  $X$  is not the hyperplane section of any fourfold. By looking at the proof of [12] Proposition 4.1 we see that the base of the fibration,  $Y$ , is a rational curve and that  $K_X(1) \simeq f^* \mathcal{O}_{\mathbf{P}^1}(1)$  is generated by two global sections. We argue as above and conclude that the type is N).  $\square$

### 3. Scrolls on quadrics.

#### 3.1. Statement of the main result.

In this section we classify scrolls as codimension two subvarieties of  $\mathcal{Q}^n$ , for  $n \geq 5$ . A *scroll*, here, is a nonsingular subvariety  $X \subseteq \mathcal{Q}^n \subseteq \mathbf{P}^{n+1}$  which admits a surjective morphism  $p : X \rightarrow Y$  to a lower dimensional variety  $Y$ , such that  $p$  has equidimensional fibers and the general scheme theoretic fiber is a linear subspace of  $\mathbf{P}^{n+1}$  of the appropriate dimension. The case  $\dim Y = 0$  is the theory of maximal dimensional linear spaces in quadrics, a well known subject; see [19]. From now on we assume  $\dim Y > 0$ .

By standard arguments, see [13] 2.7, we can assume, without loss of generality, that  $Y$  is nonsingular and that the polarized pair  $(X, L) \simeq (\mathbf{P}(\mathcal{E}), \xi_{\mathcal{E}})$ , where  $L$  is the restriction of the hyperplane bundle to  $X$  and  $\mathcal{E} := p_* L$  is an ample, rank  $\mu := \dim X - \dim Y + 1$ , locally free sheaf generated by its global sections.

REMARK 3.1.1. We assume that  $n \geq 5$  since surfaces on  $\mathcal{Q}^4$  which are scrolls over curves have been classified by Goldstein in [15]. They correspond to the surfaces of type 2), 3), 5), and 9) in [4].

In what follows the Types C), D), F) and I) below refer to Theorem 2.1.1; we say that a nonsingular threefold,  $X$ , on  $\mathcal{Q}^5$  is of Type O), if it has degree  $d = 12$  and it is a scroll over a minimal  $K3$  surface.

**THEOREM 3.1.2.** *The following is the complete list of nonsingular codimension two subvarieties of quadrics  $\mathcal{Q}^n$ ,  $n \geq 5$ , which are scrolls.*

Type C),  $n = 6$ ,  $d = 4$ , scroll over  $\mathbf{P}^1$  and over  $\mathbf{P}^3$ ;

Type D),  $n = 5$ ,  $d = 4$ , scroll over  $\mathbf{P}^1$ ;

Type F),  $n = 5$ ,  $d = 6$ , scroll over  $\mathbf{P}^2$ ;

Type L),  $n = 5$ ,  $d = 8$ , scroll over  $\mathcal{Q}^2$ ;

Type O),  $n = 5$ ,  $d = 12$ , scroll over a minimal K3 surface.

**PROOF.** The proof is the consequence of the lengthy analysis that constitutes the rest of the paper. Here we give the reader directions toward the various relevant statements.

By Fact 3.2.1 we need to deal only with the cases  $n = 5, 6$ .

Scrolls over curves are classified by Proposition 3.3.1. They correspond to types C) and D).

There are no fourfolds which are scrolls over a surface, by Proposition 3.5.1.

The only fourfold which is a scroll over a threefold is of type C), by Proposition 3.5.2.

Threefolds which are scrolls over a surface have degree  $d = 6, 8, 12$  by Proposition 3.4.3 and the base surfaces are as in Proposition 3.4.4. The classification in degrees  $d = 6, 8$  is complete; correspondingly we get types F) and L).

For an example and for the general properties of varieties of Type O) see Section 3.4.1.  $\square$

### 3.2. Preliminary facts.

The Barth-Larsen theorem implies that if  $X$  is a nonsingular codimension two subvariety of  $\mathcal{Q}^n$ , then the fundamental group  $\pi_1(X)$  is trivial for  $n \geq 6$ , and  $\text{Pic}(X) \simeq \mathbf{Z}$ , generated by the hyperplane bundle, for  $n \geq 7$ ; see [5]. Since  $\text{Pic}(X) \simeq \mathbf{Z}$  as soon as  $n \geq 7$ , we have:

**FACT 3.2.1.** There are no codimension two scrolls on  $\mathcal{Q}^n$  for  $n \geq 7$  and, for  $n = 6$ , any such is simply connected.

It is therefore enough to study threefolds on  $\mathcal{Q}^5$  which are scrolls over curves and surfaces and fourfolds on  $\mathcal{Q}^6$  which are scrolls over curves, surfaces, and threefolds.

Let us begin the analysis by fixing some notation. We start with a scroll of degree  $d$ ; let  $e_i := c_i(\mathcal{E})$ ,  $x_i := c_i(X)$  and  $y_i := c_i(Y)$ . Since  $p^*$  is injective it is harmless to denote  $p^*\alpha$  simply by  $\alpha$  while performing computations in the cohomology ring of  $X$ .

The tautological relation is

$$\sum_{i=0}^{\mu} (-1)^i L^{\mu-i} \cdot e_i = 0. \quad (13)$$

Finally, recall the usual exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow p^*(\mathcal{E}^\vee) \otimes L \rightarrow \mathcal{T}_X \rightarrow p^*\mathcal{T}_Y \rightarrow 0, \quad (14)$$

which is obtained by pasting together the relative Euler sequence [18], II.8.13 and the short exact sequence associated with the epimorphism  $dp : \mathcal{T}_X \rightarrow p^*\mathcal{T}_Y$ .

### 3.3. Scrolls over curves on $\mathcal{Q}^5$ and on $\mathcal{Q}^6$ .

The following is proved independently of Theorem 2.1.1.

**PROPOSITION 3.3.1.** *Let  $(X, L)$  be scroll over a nonsingular curve  $Y$ , on  $\mathcal{Q}^n$ . Then  $(X, L)$  is one of the following:*

$$(3.3.1.1) \quad n = 6, (\mathbf{P}_{\mathbf{P}^1}(\mathcal{E}), \xi_{\mathcal{E}}), \mathcal{E} := \mathcal{O}_{\mathbf{P}^1}(1)^4.$$

$$(3.3.1.2) \quad n = 5, (\mathbf{P}_{\mathbf{P}^1}(\mathcal{E}), \xi_{\mathcal{E}}), \mathcal{E} := \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^2;$$

*In particular, in both cases,  $d = 4$  and the embedding is projectively normal.*

**PROOF.** Let  $F \simeq \mathbf{P}^{n-3}$  be any fiber of the scroll. We cut (2) with  $F \cdot L^{n-5}$  and solve in  $d$ . We get  $d = 4$ , so that the structure of  $(X, L)$  is given by Theorem 8.10.1 of [6].

In both cases it is easy to write down explicit equations for the morphism associated with  $|\xi|$ ; we can check directly that  $\xi$  is very ample, that the image lies in a smooth quadric and that the embedding is projectively normal.  $\square$

**REMARK 3.3.2.** Case (3.3.1.1) above is the Segre embedding of  $\mathbf{P}^1 \times \mathbf{P}^3$ . It is a scroll over a curve if we look at the first projection. If we look at the second projection it is a scroll over  $\mathbf{P}^3$  with associated vector bundle  $\mathcal{O}_{\mathbf{P}^3}(1)^2$ . Case (3.3.1.2) is a general hyperplane section of (3.3.1.1); the natural morphism onto  $\mathbf{P}^3$  exhibits  $X$  as the blow up of  $\mathbf{P}^3$  along a line.

### 3.4. Threefolds on $\mathcal{Q}^5$ which are scrolls over surfaces.

**LEMMA 3.4.1.** *Let  $X \subseteq \mathcal{Q}^5$  be a codimension two scroll over a surface  $Y$ . Then either  $d = 8$  or we have:*

$$g - 1 = \frac{1}{8}d(d - 6),$$

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) = \frac{1}{144}(d^3 - 18d^2 + 96d),$$

$$e_1^2 = \frac{3}{2}d, \quad e_2 = \frac{d}{2},$$

$$K_Y \sim_n \frac{1}{6}(d - 12)e_1.$$

**PROOF.** We follow closely a procedure which can be found in [26]. By (14) we get:

$$x_1 = 2L - e_1 + y_1;$$

$$x_2 = 2Ly_1 - e_1y_1 + y_2;$$

$$x_3 = 2y_2L.$$

We plug the above equalities in (2) and (3) and get the following two equations:

$$\left(5 - \frac{d}{2}\right)L^2 + L \cdot e_1 - 3L \cdot y_1 + e_1^2 + y_1^2 - e_1 \cdot y_1 - y_2 = 0; \tag{15}$$

$$\frac{d^2}{2} - 2d + (d - 8)L^2 \cdot e_1 - (d - 8)L^2 \cdot y_1 - 4L \cdot e_1^2 - 4L \cdot y_1^2 - 2L \cdot y_2 + 8L \cdot e_1 \cdot y_1 = 0. \tag{16}$$

We cut (15) and the tautological relation with  $L$ ,  $e_1$  and  $y_1$  respectively. This way we get six relations which together with (16) and the relation  $L^3 = d$  give a system of eight linear equations in the variables:  $v := (L^3; L^2e_1; L^2y_1; Le_1^2; Le_1y_1; Ly_1^2; Le_2; Ly_2)$ . The matrix associated with the linear system is:

$$M := \begin{pmatrix} 5 - d/2 & 1 & -3 & 1 & -1 & 1 & 0 & -1 \\ 0 & 5 - d/2 & 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 5 - d/2 & 0 & 1 & -3 & 0 & 0 \\ d/2 - 2 & d - 8 & -d + 8 & -4 & 8 & -4 & 0 & -2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the linear system can be expressed as  $Mv^t = (0, 0, 0, 0, 0, 0, 0, d)$ .

Since  $\det M = 72 - 9d$ , the above system of equations has a unique solution if and only if  $d \neq 8$ .

Let us assume  $d \neq 8$ . Then the unique solution is:

$$\begin{aligned} & \{L^3; L^2e_1; L^2y_1; Le_1^2; Le_1y_1; Ly_1^2; Le_2; Ly_2\} \\ & = \left\{ d; \frac{3d}{2}; \frac{d}{4}(12 - d); \frac{3d}{2}; \frac{d}{4}(12 - d); \frac{d^3}{24} - d^2 + 6d; \frac{d}{2}; \frac{d^3}{24} - \frac{d^2}{2} + 2d \right\}. \end{aligned} \tag{17}$$

We can use (17) to compute the genus of a general curve section,  $C$ , of  $X$ . This genus equals the arithmetic genus of the line bundle  $e_1$  on  $Y$ ; we get

$$2(g - 1) = -e_1y_1 + e_1^2 = -Le_1y_1 + Le_1^2 = \frac{1}{4}d(d - 6).$$

An analogous computation gives

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = (1/144)(d^3 - 18d^2 + 96d) = \chi(\mathcal{O}_S), \tag{18}$$

where the first equality is a standard fact about projective bundles which can be proved using the Leray Spectral sequence and the last one follows from the fact that  $S$  is birationally equivalent to  $Y$ .

To prove that  $K_Y$  is numerically equivalent to a rational multiple of  $e_1$ , we use Hodge index theorem for the surface  $Y$ : by (17),  $K_Y^2e_1^2 = (K_Y \cdot e_1)^2$ , so that  $K_Y \sim_n qe_1$  for some rational number  $q$  which is straightforward to compute.  $\square$



LEMMA 3.4.2. *Let  $X$  be a threefold scroll over a surface on  $\mathcal{Q}^5$ . Then  $d \leq 42$ . Moreover, if a general curve section,  $C$ , is contained in another quadric hypersurface of  $\mathbf{P}^6$ , then  $d \leq 12$ .*

PROOF. By Lemma 3.4.1 we have  $g - 1 = (1/8)d(d - 6)$ .

Assume that a general curve section  $C$  is not contained in any surface, in the corresponding  $\mathcal{Q}^3$ , of degree strictly less than  $2 \cdot 7$ . Then Proposition 1.2.7 implies  $d \leq 42$ .

Assume  $C$  is contained in a surface  $\mathcal{S} \subseteq \mathcal{Q}^3$  of degree  $2 \cdot 6$ . Proposition 1.2.6 implies  $d \leq 27$ . The same argument repeated for surfaces of degrees  $2 \cdot 5$ ,  $2 \cdot 4$  and  $2 \cdot 3$  gives  $d \leq 18$  in all three cases.

Let us assume that  $C$  is contained in a surface of degree  $2 \cdot 2$  and that  $d > 8$ ; by Proposition 1.2.1,  $X$  is contained in another quadric hypersurface of  $\mathbf{P}^6$ . We now prove that, under the above assumptions on  $C$ ,  $d \leq 12$ . We plug  $\sigma = 2$  and the values of  $\chi(\mathcal{O}_S)$  and  $g - 1$ , from Lemma 3.4.1, in inequality (11); we get

$$-\frac{1}{288}d(d+6)(d-12) \geq -\frac{1}{12} \left[ \frac{1}{2}dL^2 - (K_X + 5L)^2 \right] D. \quad (19)$$

By Lemma 3.4.1 we have

$$K_X = -2L + \frac{1}{6}(d-6)e_1.$$

We now plug the above expression for  $K_X$  in (19) using the following relations  $e_2 = (d/2)f$ ,  $L^2D - Le_1D + e_2D = 0$ , where  $f$  is a fiber of the scroll. After simplifications the result is

$$-d(d+6)(d-12) \geq 12(d+6)Le_1D + d(d+6)(d-12)Df.$$

Since  $Le_1D \geq 0$  and  $Df \geq 0$  we get  $d \leq 12$ . Moreover, if  $d = 12$  then  $D$  must be empty. Finally if  $C$  were contained in a surface of degree  $2 \cdot 1$  then the same would be true for  $X$ , by Theorem 1.2.1. But then  $X$  would be a scroll on a quadric  $\hat{\mathcal{Q}}^4$  of  $\mathbf{P}^5$  with at most one singular point. Weil and Cartier divisors coincide on  $\hat{\mathcal{Q}}^4$  and  $\text{Pic}(\hat{\mathcal{Q}}^4) \simeq \mathbf{Z}$  by [18] II.6 Ex. 6.5. It would follow that  $X$  is a complete intersection, a contradiction.  $\square$

LEMMA 3.4.3. *Let  $X$  be a threefold scroll over a surface on  $\mathcal{Q}^5$ . Then  $d = 6, 8$  or  $12$ .*

PROOF. By Lemma 3.4.2,  $d \leq 42$ ; by Lemma 3.4.1, since the invariants there given must be integers we see that the only possibilities for the pairs  $(d, g)$  with  $d > 12$  are  $(18, 28)$ ,  $(24, 55)$ ,  $(30, 91)$ ,  $(36, 136)$  and  $(42, 190)$ . We prove that the cases  $d = 18, 24, 30, 36, 42$  cannot occur.

Let  $C \subseteq \mathcal{Q}^3 \subseteq \mathbf{P}^4$  be the general curve section of  $X$  and  $\Gamma \subseteq \mathcal{Q}^2 \subseteq \mathbf{P}^3$  be the general hyperplane section of  $C$ . We denote by  $h_C(i) := h^0(\mathcal{O}_{\mathbf{P}^4}(i)) - h^0(\mathcal{I}_{C, \mathbf{P}^4}(i))$  the Hilbert function of  $C \subseteq \mathbf{P}^4$  and by  $h_\Gamma(i) := h^0(\mathcal{O}_{\mathbf{P}^3}(i)) - h^0(\mathcal{I}_{\Gamma, \mathbf{P}^3}(i))$  the Hilbert function of  $\Gamma \subseteq \mathbf{P}^3$ . Clearly  $h_C(i) \leq h^0(\mathcal{O}_C(i))$ , for every  $i$ .

*The case (18, 28).* By Riemann-Roch and Serre Duality we have  $h^0(\mathcal{O}_C(i)) = 18i - 27$  for  $i \geq 4$ ; in particular  $h^0(\mathcal{O}_C(4)) = 45$ .  $C$  cannot be contained in another

quadric of  $\mathbf{P}^4$ , since otherwise, by Proposition 1.2.1 and Lemma 3.4.2,  $d \leq 12$ .  $C$  cannot be contained in an integral cubic of  $\mathbf{P}^4$ , otherwise, we would get that the genus would be maximal with respect to the bound prescribed by Proposition 1.2.5 and, since  $\varepsilon = 0$ ,  $C$  would be a complete intersection, forcing  $X$  to be one too in view of [18], III.9 Ex. 9.6. For the same reason  $C$  cannot be contained in an integral quartic of  $\mathbf{P}^4$ . It follows that there are no quartic hypersurfaces containing  $C$  except for the ones which are the union of  $\mathcal{Q}^3$  with another quadric; in particular  $h_C(4) = 55$ . We get  $55 = h_C(4) \leq h^0(\mathcal{O}_C(4)) = 45$ , a contradiction. The case  $d = 18$  cannot occur.

*The case (24, 55).* As in the previous case we deduce that  $C$  is contained in a unique quadric of  $\mathbf{P}^4$ ,  $C$  is not contained in any integral cubic or quartic of  $\mathbf{P}^4$ . This gives  $h_C(4) = 55$ . As before  $h^0(\mathcal{O}_C(5)) = 66$ . By [17] Lemma 3.1 we have  $h_C(5) \geq h_C(4) + h_r(5)$  and by [17] Lemma 3.4 we also have that  $h_r(5) \geq 16$ . It follows that  $55 + 16 \leq h_C(5) \leq h^0(\mathcal{O}_C(5)) = 66$ , a contradiction. The case  $d = 24$  cannot occur.

*The cases (30, 91), (36, 136) and (42, 190).* They are treated as the case  $d = 24$ . In the first case  $h^0(\mathcal{O}_C(7)) = 120$  and the only hypersurfaces of degree seven of  $\mathbf{P}^4$  which contain  $C$ , contain  $\mathcal{Q}^3$ , so that  $h_C(7) = 140$ , again a contradiction. In the second case  $h^0(\mathcal{O}_C(8)) = 153$  and the only hypersurfaces of degree eight of  $\mathbf{P}^4$  which contain  $C$ , contain  $\mathcal{Q}^3$ , so that  $h_C(8) = 289$ , again a contradiction. In the last case  $h^0(\mathcal{O}_C(10)) = 231$  and the only hypersurfaces of degree nine of  $\mathbf{P}^4$  which contain  $C$ , contain  $\mathcal{Q}^3$ , so that  $h_C(9) = 385$ . In particular  $h_C(10) > 385$ , by [17] Lemma 3.1, again a contradiction. □

The proof of the following is independent of Theorem 2.1.1.

**PROPOSITION 3.4.4.** *Let things be as in Lemma 3.4.3. If  $d = 6$  then  $Y \simeq \mathbf{P}^2$ ; if  $d = 8$  then  $Y \simeq \mathcal{Q}^2$ ; if  $d = 12$  then  $Y$  is a minimal K3 surface.*

**PROOF.** Let  $d = 6$ . By Lemma 3.4.1  $-K_Y$  is ample and  $K_Y^2 = 9$ ; by the classification of Del Pezzo surfaces we conclude that  $Y \simeq \mathbf{P}^2$ . Let  $d = 8$ . The proof of Proposition 3.5.1 gives  $Y \simeq \mathcal{Q}^2$ . Let  $d = 12$ . By Lemma 3.4.1  $K_Y$  is numerically trivial, so that  $Y$  is a minimal model. (18) prescribes  $\chi(\mathcal{O}_Y) = 2$ , so that, by the Enriques-Kodaira classification,  $Y$  is a K3 surface. □

**3.4.1. The case of Type O).**

The purpose of this section is twofold. First we give an example of a scroll of Type O), making the list of Theorem 3.1.2 effective. Then we collect information about the arbitrary variety of this type.

Let  $X$  be of Type O),  $\beta_{i,j} := h^i(\mathcal{I}_{X,\mathcal{Q}^5}(j))$  and  $\sigma_i := h^i(\mathcal{I}_{X,\mathcal{Q}^5}(-1) \otimes \mathcal{S})$ . The sheaves  $\Psi_i$  are defined in [1].

**THEOREM 3.4.5.** *Let  $\mathcal{S}^7 \xrightarrow{\phi} \Psi_3 \oplus \mathcal{O}_{\mathcal{Q}^5}^3$  be a general morphism. Then,  $\phi$  is injective,  $X := D_{27}(\phi)$  is a variety of Type O) and we have a resolution of the form*

$$0 \rightarrow \mathcal{S}^7 \xrightarrow{\phi} \Psi_3 \oplus \mathcal{O}_{\mathcal{Q}^5}^3 \rightarrow \mathcal{I}_{X,\mathcal{Q}^5}(3) \rightarrow 0.$$

We will prove this theorem after Proposition 3.4.8. First we determine some properties of the arbitrary variety of Type O).

**PROPOSITION 3.4.6.** *Let  $X \subseteq \mathcal{Q}^5$  be of Type O).*

*Then:*

$$g = 10; K_X L^2 = -6; K^2 L = -6; K_X^3 = 12; c_2(X)L = 24, c_1(\mathcal{E})^2 = 18, c_2(\mathcal{E}) = 6.$$

*The cohomology of  $\mathcal{O}_X(l)$ :*

$$\begin{aligned} h^1(\mathcal{O}_X(t)) &= 0, \forall t \in \mathbf{Z}; \\ h^2(\mathcal{O}_X(t)) &= 0, \forall t \in \mathbf{Z} \text{ except for } h^2(\mathcal{O}_X) = 1; \\ h^2(\mathcal{O}_X(t)) &= 0, \forall t \geq -1. \end{aligned}$$

*The following is the Beilinson-Kapranov  $E_1^{p,q}$  table for the sheaf  $\mathcal{I}_{X, \mathcal{Q}^5}(3)$ ; see [1] Theorem 5.6. A letter a on the left of a vector bundle B means B direct sum with itself a times.*

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 7\mathcal{S} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Psi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_{1,2}\Psi_1 & \beta_{1,3}\mathcal{O}_X \\ 0 & 0 & 0 & 0 & \beta_{0,2}\Psi_1 & \beta_{0,3}\mathcal{O}_X, \end{array}$$

where  $\beta_{0,2} - \beta_{1,2} = 0, \beta_{0,3} - \beta_{1,3} = 3$ .

Either  $\beta_{2,0} = 1$  and  $\beta_{0,3} = 7$  or  $\beta_{2,0} = 0$  and  $0 \leq \beta_{1,3} \leq 21$ .

If  $\beta_{2,0} = 1$  then  $\mathcal{I}_{X, \mathcal{Q}^5}(3)$  can be expressed as the cohomology of a monad of the form (see [24] for the definition of monads):

$$0 \rightarrow 7\mathcal{S} \xrightarrow{m_1} \Psi_3 \oplus \Psi_1 \oplus \mathcal{O}_{\mathcal{Q}^5}(1) \xrightarrow{n_1} 4\mathcal{O}_{\mathcal{Q}^5} \rightarrow 0 \tag{20}$$

If  $\beta_{2,0} = 0$  then  $\mathcal{I}_{X, \mathcal{Q}^5}(3)$  can be expressed as the cohomology of a monad of the form:

$$0 \rightarrow 7\mathcal{S} \xrightarrow{m_2} \Psi_3 \oplus 3\mathcal{O}_{\mathcal{Q}^5} \oplus \beta_{1,3}\mathcal{O}_{\mathcal{Q}^5} \xrightarrow{n_2} \beta_{1,3}\mathcal{O}_{\mathcal{Q}^5} \rightarrow 0. \tag{21}$$

**PROOF.** The first list of invariants can be read off from Lemma 3.4 when  $d = 12$ .

As to  $h^2(\mathcal{O}_X(t))$  we argue as follows. Via the projection formula and Leray Spectral Sequence,  $h^2(\mathcal{O}_X(t)) = 0, \forall t < 0$  and for the same reason  $h^2(\mathcal{O}_X) = 1$ . Since  $K_Y$  is trivial, Leray Spectral Sequence and Le Potier's Vanishing Theorem [27] give  $h^i(\mathcal{O}_X(1)) = h^i(\mathcal{O}_Y(\mathcal{E})) = h^i(\mathcal{O}_Y(K_Y \otimes \mathcal{E})) = 0, \forall i \geq 2$ . By Serre Duality and the fact that  $L_{|S}(K_S - mL_{|S}) = 6 - 12m$  we see that  $h^2(\mathcal{O}_S(m)) = 0, \forall m \geq 1$ ; we conclude for  $h^2$  by an easy induction using the sequences

$$0 \rightarrow \mathcal{O}_X(m-1) \rightarrow \mathcal{O}_X(m) \rightarrow \mathcal{O}_S(m) \rightarrow 0. \tag{22}$$

The vanishings of the  $h^3$ 's are obvious consequences of Serre Duality.

$h^1(\mathcal{O}_X(t)) = 0, \forall t < 0$  by Kodaira vanishing. For  $t = 0$  the vanishing follows from  $h^1(\mathcal{O}_Y) = 0$ . Since  $X \subseteq \mathbf{P}^6$  is linearly normal by a result of Fujita's (cf. [23], §4) and  $\chi(\mathcal{O}_X(1)) = 7$  by Riemann-Roch, we have  $h^1(\mathcal{O}_X(1)) = 0$ . To prove the remaining vanishings for  $h^1(\mathcal{O}_X(t))$  we argue by induction using the long cohomology sequences associated with the sequences (22), the analogue ones obtained by replacing  $X$  and  $S$  by

$S$  and  $C$  (a general curve section of  $S$ ) and observing that the linear systems  $|\mathcal{O}_C(t)|$  are non-special for  $t \geq 2$ .

The Beilinson-Kapranov table is obtained as follows. The vanishings  $\beta_{i,j} = 0$  for  $i = 1, 2, 3, 4, 5$  and  $j = -1, 0, 1, 2, 3$ , except for  $\beta_{3,0} = h^0(\mathcal{O}_X) = 1$ , are obtained by taking the cohomology of the exact sequences

$$0 \rightarrow \mathcal{I}_{X, \mathcal{Q}^5}(l) \rightarrow \mathcal{O}_{\mathcal{Q}^5}(l) \rightarrow \mathcal{O}_X(l) \rightarrow 0 \tag{23}$$

and by plugging the above values for the cohomology of  $\mathcal{O}_X(t)$ . For the same reason  $\beta_{i,j} = 0$  for  $i = 0, 1, 2, j = -1, 0$ .  $\beta_{0,1}$  is zero because  $X$  cannot be degenerate (see the proof of Lemma 3.4.2).  $\beta_{1,1} = 0$  since  $X \subseteq \mathbf{P}^6$  is linearly normal. The relations on the remaining  $\beta$ 's come from the shape of the Hilbert polynomial

$$\chi(\mathcal{I}_{X, \mathcal{Q}^5}(t)) = \frac{1}{60}t^5 + \frac{5}{24}t^4 - t^3 + \frac{19}{24}t^2 + \frac{59}{60}t - 1,$$

which vanishes for  $t = -1, 1, 2$ , and has value three for  $t = 3$ .

Because of how this spectral sequence works ( $E_\infty = E_6$  and  $E_\infty^{p,q} \simeq \{0\}$  for  $p + q \neq 0$ ), we see that  $\sigma_i = 0$ , for  $i = 0, 1, 2$ .

$\sigma_5 = 0$  by observing the cohomology of (23) tensored with  $\mathcal{S}$ . We use the same sequences, together with Riemann-Roch for  $\mathcal{S}$  and for  $\mathcal{S}|_X$  to get

$$\chi(\mathcal{S}(t)) = \frac{1}{15}t(t+1)(t+2)(t+3)(t+4),$$

and

$$\chi(\mathcal{S}|_X(t)) = 8t^3 - 6t^2 + 7;$$

it follows that  $-\sigma_3 + \sigma_4 = \chi(\mathcal{I}_{X, \mathcal{Q}^5} \otimes \mathcal{S}(-1)) = 7$ .

We now prove that  $\sigma_3 = 0$ . There is at most one nontrivial differential from  $\sigma_3 \mathcal{S}$ , namely the one that hits  $E_4^{-2,0}$ . On the other hand  $E_4^{-2,0} = E_2^{-2,0} = \text{Ker } d_1^{-2,0}$ . It is enough to show that the last group is trivial. We consider two cases. The former is when  $\beta_{0,2} = 0$ ; in this case  $\sigma_3$  is clearly zero. The latter is when  $\beta_{0,2} \neq 0$ . Then  $\beta_{0,2} = 1$  otherwise  $X$  would have  $d \leq 8$ , the degree of the intersection on  $\mathcal{Q}^5$  of two hypersurfaces of degree four.  $\beta_{0,3} = 7$  otherwise  $X$  would be a complete intersection on  $\mathbf{P}^6$  of type  $(2, 2, 3)$ , a contradiction. By Kapranov's explicit resolution of the diagonal on  $\mathcal{Q}^n \times \mathcal{Q}^n$ , see [1], we infer that  $d_1^{-2,0}$  coincides with the injection  $(\Psi_1 \simeq) \Omega^1(1)_{\mathbf{P}^6|_{\mathcal{Q}^5}} \rightarrow \mathcal{O}_{\mathcal{Q}^5}^7$ , obtained by restricting the Euler sequence of  $\mathbf{P}^6$  to  $\mathcal{Q}^5$ . It follows that  $\sigma_3 = 0$  and therefore  $\sigma_4 = 7$ .

If  $\beta_{2,0} \neq 0$ , we have seen above that  $\beta_{2,0} = 1, \beta_{3,0} = 7$  and that  $d_1^{-2,0}$  coincides with the injection  $(\Psi_1 \simeq) \Omega^1(1)_{\mathbf{P}^6|_{\mathcal{Q}^5}} \rightarrow \mathcal{O}_{\mathcal{Q}^5}^7$  whose cokernel is  $\mathcal{O}_{\mathcal{Q}^5}(1)$ . The statement associated with (20) follows from [1].

Similarly, we see that the statement associated with (21) holds when  $\beta_{2,0} = 0$ . Since the morphism  $n_2$  is trivial on  $(3 + \beta_{1,3})\mathcal{O}_{\mathcal{Q}^5}$ , the restriction  $\nu := n_{2|_{\Psi_3}}$  is surjective. Recall that the rank of  $\Psi_3$  is 26. If  $\beta_{1,3} > 21$ , then the kernel of the map  $\nu$  would be a locally free sheaf of rank  $r < 5$  with the fifth Chern class  $c_5 = c_5(\Psi_3) \neq 0$ , a contradiction. □

The following is essentially due to Peskine and Szpiro; see [23], §1.

LEMMA 3.4.7. *Let  $X$  be a codimension two nonsingular subvariety of a nonsingular variety  $Z$  of dimension  $n \leq 5$  and  $L_i, i = 1, 2$ , two line bundles on  $Z$  such that the sheaves  $\mathcal{I}_{X,Z}(L_i)$  are globally generated on  $Z$ . Let  $s_i \in H^0(\mathcal{I}_{X,Z}(L_i))$  be the choice of two general sections and  $V_i$  the two effective divisors associated with the  $s_i$ . Then  $V_1 \cap V_2 = X \cup Y$ , as schemes, where  $Y$  is nonsingular.*

We now give a family of examples of degree  $d = 12$  scrolls on  $\mathcal{Q}^5$ .

PROPOSITION 3.4.8. *Let  $\rho : \mathcal{C}^\vee \rightarrow \mathcal{O}_{\mathcal{Q}^5}(2)^3$  be a generic morphism. Then  $X := D_1(\rho)$  is a variety of Type O) such that  $\mathcal{I}_{X,\mathcal{Q}^5}(3)$  is generated by global sections;  $X$  is linked to a variety  $X'$  of Type F) via the complete intersection of two general elements of  $|\mathcal{I}_{X,\mathcal{Q}^5}(3)|$ . Conversely, if  $X \subseteq \mathcal{Q}^5$  is of Type O) and  $\mathcal{I}_{X,\mathcal{Q}^5}(3)$  is generated by global sections, then  $X = D_1(\rho)$  for some  $\rho$  as above and  $X$  is linked as above to a variety of Type F).*

PROOF. For facts about Cayley bundles, see Fact 1.4.2.  $\mathcal{C}(2)^3$  is generated by global sections and Fact 1.1.7 implies that  $X := D_1(\rho)$  is a codimension two nonsingular subvariety of  $\mathcal{Q}^5$  and that we have the following exact sequence

$$0 \rightarrow \mathcal{C}^\vee(-2) \xrightarrow{\rho} \mathcal{O}_{\mathcal{Q}^5}^3 \rightarrow \mathcal{I}_{X,\mathcal{Q}^5}(3) \rightarrow 0. \tag{24}$$

We compute the total Chern class of  $\mathcal{I}_{X,\mathcal{Q}^5}(3)$  via (24):  $1 + 3h + 6h^2 + 9h^3 + 9h^4$ .

We compare it with Lemma 1.1.2:

$$\gamma_1 = 3, \gamma_2 = 6 = \frac{1}{2}d, \gamma_3 = \frac{1}{2}(K_X + 2L)L^2 = 9, \gamma_4 = \frac{1}{2}(K_X + 2L)^2L = 9, \gamma_5 = \frac{1}{2}(K_X + 2L)^3 = 0,$$

where  $L$  denotes  $\mathcal{O}_{\mathcal{P}^6}(1)|_X$ . It follows that  $X$  has degree  $d = 12$ . By [6] Proposition 7.2.2,  $K_X + 2L$  is generated by global sections since  $(X, L)$  cannot be isomorphic to either  $(\mathcal{P}^3, \mathcal{O}_{\mathcal{P}^3}(1))$ ,  $(\mathcal{Q}^3, \mathcal{O}_{\mathcal{Q}^3}(1))$  or to a scroll over a curve since they all have degree  $d = 4$  by Proposition 3.3.1. The fact that  $\gamma_5 = 0$  implies that  $K_X + 2L$  cannot be big and the fact that  $\gamma_4 \neq 0$  implies that the Kodaira dimension  $\kappa(K_X + 2L) = 2$ , so that, by [6] Theorem 7.3.2,  $(X, L)$  is an adjunction theoretic scroll over a surface and, by [6] Proposition 14.1.3, it is actually a scroll in our sense. By Lemma 3.4.4,  $X$  is a degree  $d = 12$  scroll over a K3 surface. The linking part is proved using Lemma 3.4.7 to produce an  $X'$  of degree  $d' = 6$  and by observing that the mapping cone construction yields a resolution for  $\mathcal{I}_{X',\mathcal{Q}^5}(3)$  which coincides with the one of a variety of Type F).

The converse is proved in a similar way. □

PROOF OF THEOREM 3.4.5. For the varieties constructed in Proposition 3.4.8 we have, by (24), that  $\beta_{2,0} = 0$  and  $\beta_{0,3} = 3$ . By looking at the display of the monad (21) with these invariants we get the desired resolution for the ideal sheaves of these varieties. It also follows that, for the generic morphism  $\phi$ ,  $\phi$  is injective and  $X := D_{27}(\phi)$  is of Type O) as in the proof of Proposition 3.4.8. □

### 3.5. 4-folds which are scrolls on $\mathcal{Q}^6$ .

PROPOSITION 3.5.1. *There are no fourfolds scrolls over surfaces on  $\mathcal{Q}^6$ .*

PROOF. By contradiction, assume that  $X^4$  is such a scroll. Cutting (2) with a fiber  $F \simeq \mathbf{P}^2$ , we get  $d = 8$ . We take a general hyperplane section and obtain a scroll,  $X$ , on  $\mathcal{Q}^5$  so that the previous analysis applies. In fact, if a special fiber,  $F$ , were isomorphic to  $\mathbf{P}^2$ , then  $F|_F \simeq \mathcal{O}_{\mathbf{P}^2}(-1)$  so that we would have a contraction morphism  $\eta : X \rightarrow X'$  and the structural morphism  $p : X \rightarrow Y$  would factor through  $\eta$  violating the upper semicontinuity of the dimension of the fibers.

We solve the linear system contained in the proof of Lemma 3.4 for  $d = 8$  and we get that the solutions depend on one additional parameter  $t$ :

$$\begin{aligned} & \{L^3; L^2e_1; L^2y_1; Le_1^2; Le_1y_1; Ly_1^2; Le_2; Ly_2\} \\ & = \left\{ 8; 36 - \frac{9}{2}t; 24 - 3t; 36 - \frac{9}{2}t; 24 - 3t; 16 - 2t; 28 - \frac{9}{2}t; t \right\}. \end{aligned} \tag{25}$$

We observe that  $K_Y^2 = (4/9)e_1^2$  and that  $K_Y \cdot e_1 = -(2/3)e_1^2$ . Since  $e_1$  is ample, the Hodge Index Theorem implies that  $3K_Y \sim_n -2e_1$ . It follows that  $Y$  has to be a Del Pezzo surface. On such a  $Y$ , numerical and rational equivalence coincide and  $3K_Y$  is not divisible by 2 unless  $Y$  is a smooth  $\mathcal{Q}^2$ . In this case  $t = b_2 = 4$ . In particular  $\deg e_2 = 10$  and  $g = 4$ . By [21] this case cannot occur if  $\dim X \geq 4$ . This contradicts the existence of scrolls over surfaces in  $\mathcal{Q}^6$  of dimension four.  $\square$

PROPOSITION 3.5.2. *The only scroll over a threefold on  $\mathcal{Q}^6$  is  $\mathbf{P}^1 \times \mathbf{P}^3$  embedded with the Segre embedding.*

PROOF. The proof runs along the lines of Lemma 3.4.1. Using (14) we compute the Chern classes of  $X$ :

$$\begin{aligned} x_1 &= 2L - e_1 + y_1; \\ x_2 &= 2Ly_1 - e_1y_1 + y_2; \\ x_3 &= 2Ly_2 - e_1y_2 + y_3; \\ x_4 &= 2Ly_3. \end{aligned}$$

After having plugged these relations in (2), (3) and (4) we get, respectively:

$$\left(\frac{1}{2}d - 8\right)L^2 + L(4y_1 - 2e_1) - e_1^2 + e_1y_1 - y_1^2 + y_2 = 0, \tag{26}$$

$$\begin{aligned} & 8L^3 + L^2(4e_1 - 8y_1) + L(-2e_1y_1 + 4y_1^2 - 4y_2) \\ & + e_1^3 - e_1^2y_1 + e_1y_1^2 - e_1y_2 - y_1^3 + 2y_1y_2 - y_3 = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} & 6L^4 - 8L^3y_1 + L^2(4e_1^2 - 4e_1y_1 + 8y_1^2 - 8y_2) \\ & + L(-2e_1^3 + 2e_1y_1^2 - 2e_1y_2 - 4y_1^3 + 8y_1y_2 - 4y_3) = 0. \end{aligned} \tag{28}$$

We cut the tautological relation and (26) with the following classes:  $L^2, Le_1, Ly_1, e_1^2, y_1^2, e_1y_1, e_2$  and  $y_2$ ; we cut (27) with  $L, e_1$  and  $y_1$ . Considering also (28) we get a total of twenty linear equations in the seventeen variables:  $L^4, L^3e_1, L^3y_1, L^2e_1^2, L^2e_1y_1, L^2e_2,$

$L^2y_1^2, L^2y_2, Le_1^3, Le_1^2y_1, Le_1e_2, Le_1y_1^2, Le_1y_2, Le_2y_1, Ly_1^3, Ly_1y_2$  and  $Ly_3$ . We leave out, on purpose, the condition  $L^4 = d$ . We leave to the reader to check that the resulting linear system has a nontrivial solution only for  $d = 4$ . If  $d = 4$  we use Theorem 2.1.1 to conclude.  $\square$

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Mark Andrea de CATALDO  
Max-Planck-Institut für Mathematik,  
Gottfried-Claren-Strasse 26,  
53225 Bonn,  
Germany.  
e-mail: markan@mpim-bonn.mpg.de