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Lagrangian submanifolds of C^n with conformal Maslov form and the Whitney sphere

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1. Introduction

Let C^n be the n-dimensional complex Euclidean space, $\langle\ ,\ \rangle$ the Euclidean metric and J the canonical complex structure on C^n. The Kaehler two form Ω is given by $\Omega(v,w)=\langle v,Jw\rangle$, for any vectors v, w in C^n. We say that an immersion $\psi: M\rightarrow C^n$ of an n-dimensional manifold M is *Lagrangian* if $\psi^*\Omega\equiv 0$.

The simplest examples of Lagrangian submanifolds of C^n are the totally geodesic ones, i.e. the Lagrangian subspaces of C^n. A second family of examples, known as the *Whitney spheres* [**17**], can be defined as a family of Lagrangian immersions of the unit sphere S^n, centered at the origin of R^{n+1}, in C^n given by

$$\psi(x_1,\dots,x_{n+1})=\frac{r}{1+x_{n+1}^2}(x_1,x_1x_{n+1},\dots,x_n,x_nx_{n+1})+A$$

where r is a positive number and A is a vector of C^n. We will refer to r and A as the radius and the center of the Whitney sphere. Up to dilatations of C^n all the Whitney spheres are congruent with the corresponding to $r=1$, $A=0$.

These examples have very interesting properties. From a topological point of view, it is well-known that the sphere cannot be embedded in C^n as a Lagrangian submanifold [**8**]. The Whitney spheres have the best possible behaviour, because they are embedded except at the poles of S^n where they have a double point.

From the point of view of the second fundamental form, they have the simplest behaviour, after the totally geodesic ones, because their second fundamental forms σ satisfy

$$\sigma(v,w)=\frac{n}{n+2}\{\langle v,w\rangle H+\langle Jv,H\rangle Jw+\langle Jw,H\rangle Jv\},$$

being H the mean curvature vector of the Whitney sphere. In some sense, these submanifolds play the role of the umbilical hypersurfaces of the Euclidean space R^{n+1}, in the family of Lagrangian submanifolds, and it seems natural that they could be characterized of several forms, in the same way that the Euclidean spheres. The first characterization of these spheres that we get is given in Theorem 2 and says that *the totally geodesic Lagrangian submanifolds and the Whitney spheres are the only*

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Lagrangian submanifolds of C^n *whose second fundamental forms have the above expression.* As a direct consequence of this result, we obtain for Lagrangian submanifolds of C^n an inequality involving the scalar curvature of M and $|H|^2$, and characterize the totally geodesic Lagrangian submanifolds and the Whitney spheres as the only ones that reach the equality (Corollary 3).

Following with the analogy between Lagrangian submanifolds of C^n and hypersurfaces of R^{n+1}, the Euclidean spheres are the basic examples of hypersurfaces with constant mean curvature. The Whitney spheres have not parallel mean curvature vector, property, which is generally taking, as a version on higher codimension of the notion of constant mean curvature, but their mean curvature vectors H satisfy that JH are conformal vector fields on the spheres. So, we will take this property as the Lagrangian version of the concept of hypersurfaces of constant mean curvature. As the dual form of JH is the Maslov form of our Lagrangian immersion, we will refer to these submanifolds as *Lagrangian submanifolds with conformal Maslov form.*

In paragraph 2, we study a natural family of this kind of immersions: those Lagrangian submanifolds with conformal Maslov form and such that JH is a principal direction of A_H. In Corollary 2, we classify locally this family, in terms of minimal Lagrangian submanifolds of the complex projective space CP^{n-1} and a 1-parameter family of plane curves, by a construction method, that generalizes the circle bundles on the minimal Lagrangian submanifolds of CP^{n-1} induced by the Hopf fibration $\Pi : S^{2n-1} \to CP^{n-1}$ (see Definition 1 and Proposition 3). In this contex, the Whitney sphere corresponds with the example constructs with the totally geodesic Lagrangian submanifold of CP^{n-1} and the simplest curve.

In paragraph 3, we use this local classification result to obtain remarkable consequences. The first one can be described as follows. A classic Hopf result says that the Euclidean sphere is the only compact surface of genus zero in R^3 with constant mean curvature. We obtain the Lagrangian version of this result, which is one of the most important global results in this paper:

The Whitney sphere is the only compact Lagrangian submanifold of C^n *with conformal Maslov form and null first Betti number.*

Another property of the Whitney spheres is related with their Ricci curvatures. It is known that there exist no Lagrangian compact submanifolds of C^n with positive Ricci curvature, and the Whitney spheres have positive Ricci curvatures everywhere except at two points where their Ricci curvatures vanish. In Corollary 5 we prove that *the Whitney sphere is the only compact Lagrangian submanifold of* C^n *with non-parallel conformal Maslov form such that* $\mathrm{Ric}(JH) \geq 0$.

The last important global result in this paper is given in Corollary 6. In it *we describe all the compact Lagrangian submanifolds of* C^n *with conformal Maslov form and whose first Betti number is one.*

In [**4**], I. Castro and the second author studied this family in the particular case of surfaces. In that paper the authors classified completely the family of compact

Lagrangian surfaces with conformal Maslov form, obtaining besides the Whitney sphere a two-parameter family of Lagrangian tori in $\boldsymbol{C}^2$.

Finally, when this paper was finished, the authors have known that in [**3**] our Corollary 3 has been proved by using another method.

2. The Gauss map

Following with the analogy between constant mean curvature hypersurfaces of $\boldsymbol{R}^{n+1}$ and Lagrangian submanifolds of $\boldsymbol{C}^n$ with conformal Maslov form, we are also going to characterize our family of submanifolds in terms of their Gauss maps.

Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an n-dimensional manifold M. The induced metric will be also denoted by $\langle\ ,\ \rangle$. Then, $\psi^* T\boldsymbol{C}^n = TM \oplus T^\perp M$, where $T^\perp M$ is the normal bundle of ψ. Let $\overline{\nabla}$ denotes the connection on $\psi^* T\boldsymbol{C}^n$ induced by the Levy-Civita connection of $\boldsymbol{C}^n$, and ∇, (respectively $\nabla^\perp$) denotes the Levi-Civita connection of the induced metric on M, (respectively the normal connection of ψ).

The most elementary properties of these kind of immersions are:

i) J defines a bundle-isomorphism from the tangent bundle to the normal bundle of ψ, such that

$$J \circ \nabla = \nabla^\perp \circ J$$

ii) The second fundamental form σ of ψ and the Weingarten endomorphism A_ξ associated to a normal vector ξ, are related by

$$\sigma(v, w) = JA_{Jv}w.$$

So, $\langle \sigma(v, w), Jz \rangle$ defines a symmetric trilineal form on TM.

iii) If H denotes the mean curvature vector of ψ, then from i), ii) and the Codazzi equation of ψ, we have that *JH is a closed vector field on M*.

Let $\mathscr{L} = \mathscr{L}(\boldsymbol{C}^n)$ be the space of Lagrangian n-planes in $(\boldsymbol{C}^n, \Omega)$. As usual, the Riemannian structure on $\mathscr{L}$ is defined by identifying $\mathscr{L}$ with the symmetric space $U(n)/O(n)$, where $U(n)$ (respectively $O(n)$) is the unitary group (respectively orthogonal group) of order n. $U(n)/O(n)$ is a symmetric subspace of the Grassmann manifold of n-planes in Euclidean $2n$-space: $O(2n)/O(n) \times O(n)$.

Let

$$\nu : M \to \mathscr{L} \equiv \frac{U(n)}{O(n)} \hookrightarrow \frac{O(2n)}{O(n) \times O(n)}$$

be the Gauss map of the Lagrangian immersion ψ. Following the ideas of Ruh and Vilms [**12**], the bundle $\nu^* T\mathscr{L}$ is isometric to the bundle of symmetric tensors of order two on $M : \mathscr{S}^2(TM)$, and via this identification, the tension field $\tau(\nu)$ of ν is identified with ∇JH. So, (see [**12**], Corollary 1), the Gauss map ν is harmonic if and only if H (or JH) is a parallel vector field.

The square of the determinant map

$$\det{}^2 : \frac{U(n)}{O(n)} \to S^1$$

defines a $SU(n)/SO(n)$-fiber bundle over the circle S^1. If α is the volume form on S^1 and

$$f = \det{}^2 \circ \nu : M \to S^1,$$

then $\omega = f^*(\alpha)$ is the Maslov form on M, and it is well-known (see [**10**]) that up to a constant, the Maslov form ω is the dual form of the vector field JH.

If $V = \ker(\det^2)_*$ is the vertical distribution of the fibration det^2, it is easy to see that the vertical component of the tension field $\tau(\nu)^V$ can be identify with the following cross section of $\mathscr{S}^2(TM) : \nabla JH - (\operatorname{div} JH/n)I$, where $\operatorname{div} JH$ is the divergence of JH and I is the identity. Now, as JH is a closed vector field, it is easy to check that $\nabla JH = (\operatorname{div} JH/n)I$ if and only if JH is a conformal vector field. So, we have proved the following result:

Proposition 1. *Let $\phi : M \to \mathbf{C}^n$ be a Lagrangian immersion. Then the Gauss map $\nu : M \to U(n)/O(n)$ is vertically harmonic, i.e. $\tau(\nu)^V \equiv 0$, if and only if JH is a conformal vector field.*

We remark that the above condition about the tension field of ν can be written in terms of the map $f : M \to S^1$. In fact, using again the ideas of Ruh and Vilms [**12**], we have that $d\nu_p(v) \equiv A_{Jv}$ for any vector v tangent to M at p, obtaining that $df_p(v) = 2n\langle H, Jv\rangle if(p)$. If locally, $f = e^{i\theta(t)}$ then, *JH is a conformal vector field* if and only if the Hessian of $\theta : \nabla^2\theta$ is proportional to the metric $\langle\ ,\ \rangle$.

3. Local study

In this section, we are going to study those Lagrangian submanifolds of $\mathbf{C}^n$ with conformal Maslov form such that JH is a principal direction of A_H. As many of the local results we will obtain are true if we consider instead of JH any closed and conformal vector field X, in this section we will deal with Lagrangian submanifolds of $\mathbf{C}^n$ which have a closed and conformal vector field X such that $\sigma(X, X) = \rho JX$ for some function ρ on the submanifold. Many results are known for manifolds which admit closed and conformal vector fields (see [**1**], [**11**], [**13**], [**14**], [**15**]). In the following result we summary some well known properties of these manifolds.

Lemma 1. *Let X be a (nontrivial) closed and conformal vector field on an n-dimensional Riemannian manifold $(M, \langle\ ,\ \rangle)$. Then*

1. *The gradients of $|X|^2$ and divergence of X are given by*

$$\nabla|X|^2 = \frac{2\operatorname{div} X}{n} X \quad |X|^2\nabla(\operatorname{div} X) = -\frac{n\operatorname{Ric}(X)}{(n-1)} X,$$

where Ric *denotes the Ricci curvature of M.*

2. *The curvature tensor R of M satisfies*

$$|X|^2 R(v,w)X = \frac{\mathrm{Ric}(X)}{(n-1)}\{\langle w, X\rangle v - \langle v, X\rangle w\}.$$

3. *If $n > 2$, X has nonvanishing divergence at its zeroes and they are isolated.*

4. *If $M' = \{p \in M / X_p \neq 0\}$, then the distribution*

$$p \in M' \to \mathscr{D}(p) = \{v \in T_pM / \langle v, X\rangle = 0\}$$

defines an umbilical foliation on $(M', \langle\ ,\ \rangle)$. In particular $|X|^2$ and $\operatorname{div} X$ *are constant on the connected leaves of $\mathscr{D}$.*

5. *If Δ^g is the Laplacian of the metric $g = |X|^{-2}\langle\ ,\ \rangle$, then*

$$\Delta^g \log|X| + \frac{\mathrm{Ric}(X)}{n-1} = 0.$$

6. (*Local description*) *X is a parallel vector field with respect to g, and so (M', g) is locally isometric to $(I \times N, dt^2 \times g')$, where I is an open interval in $\boldsymbol{R}$, for any $t \in I$, $t \times N$ is a leaf of the foliation $\mathscr{D}$, and $X = (\partial/\partial t, 0)$.*

Proof. First of all it is clear that a vector field X on M is closed and conformal if and only if

$$\nabla_v X = \frac{\operatorname{div} X}{n} v \tag{1}$$

for any vector v tangent to M. From (1), $\nabla|X|^2$ can be easily computed. Now using again (1) we obtain that the Hessian of $|X|^2$ is given by

$$\nabla^2|X|^2(v,w) = 2(\operatorname{div} X/n)^2\langle v, w\rangle + \frac{2}{n}\langle \nabla(\operatorname{div} X), v\rangle\langle X, w\rangle. \tag{2}$$

As $\nabla^2|X|^2$ and $\langle\ ,\ \rangle$ are symmetric tensors, the above expression says that

$$|X|^2\nabla(\operatorname{div} X) = \langle \nabla(\operatorname{div} X), X\rangle X.$$

Now derivating in (1) and using the above formula, we obtain

$$|X|^2 R(v,w)X = -\frac{\langle \nabla(\operatorname{div} X), X\rangle}{n}\{\langle w, X\rangle v - \langle v, X\rangle w\}$$

and so 1) and 2) follow.

In [**11**] (Proposition 4.1) it was proved that X has nonvanishing divergence at its zeroes when $n > 2$. If p is a zero of X (and hence $\operatorname{div} X(p) \neq 0$), from (2) it follows that the Hessian of $|X|^2$ at p is positive definite and so p is a isolated zero of X. This proves 3).

As X is a closed vector field, then $\mathscr{D}$ defines a foliation on M'. Also using (1), it is clear that the second fundamental form σ_0 of a leaf of $\mathscr{D}$ in M' is given by

$$\sigma_0(v,w) = -\frac{\operatorname{div} X}{n|X|^2}\langle v, w\rangle X, \tag{3}$$

and so the foliation is umbilical.

If Δ is the Laplacian of the metric $\langle\ ,\ \rangle$, them from (2) and 1) it is easy to get

$$|X|^2\Delta\log|X|^2 = \frac{2(n-2)(\operatorname{div}X)^2}{n^2} - \frac{2\operatorname{Ric}(X)}{n-1}.$$

Now, using that for any smooth function h on M (see [**2**], Theorem 1.159)

$$\Delta^g h = |X|^2\Delta h - (n-2)|X|^2\langle\nabla\log|X|, \nabla h\rangle,$$

the above formula joint with 1) prove 5).

Using the relation between the Levi-Civita connections of the conformal metrics g and $\langle\ ,\ \rangle$ (see [**2**], Theorem 1.159) it is clear, using 1) and (1), that X is a parallel vector field of (M', g), and so 6) is proved.

LEMMA 2. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an n-dimensional manifold M, and let X be a closed and conformal vector field* (*without zeroes*) *on M such that $\sigma(X,X) = \rho JX$ for some function ρ. Then:*

1. *For any point $p \in M$, the eigenvalues of A_{JX} on $\mathscr{D}(p)$ satisfy the quadratic equation*

$$\lambda^2 - \rho\lambda + \frac{\operatorname{Ric}(X)}{n-1} = 0.$$

In particular, there are at most two.

2. *If $v, w \in \mathscr{D}(p)$ are eigenvectors of A_{JX} corresponding to different eigenvalues, then $\sigma(v,w) = 0$.*
3. *The eigenvalues of A_{JX} are constant on the connected leaves of the foliation $\mathscr{D}$.*
4. *For any $v, w, z \in \mathscr{D}$, we have*

$$\langle(\nabla\sigma)(v,w,z), JX\rangle + \frac{\operatorname{div}X}{n}\langle\sigma(v,w), Jz\rangle = 0$$

where $\nabla\sigma$ denotes the covariant derivative of the second fundamental form σ.

PROOF. As X_p is an eigenvector of A_{JX}, then A_{JX} can be diagonalized on $\mathscr{D}(p)$. Now, from Lemma 1,2) and the Gauss equation of ψ it is easy to prove 1) and 2).

From Lemma 1,6), (M,g) is locally isometric to $(I \times N, dt^2 \times g')$, and up to that isometry, $|X|^2(t,x)$ is a function of t. Now Lemma 1,5) says that $\Delta^g\log|X|$ is also a function of t and so $\operatorname{Ric}(X)$ is constant on the connected leaves of $\mathscr{D}$.

On the other hand, taking derivative on $\sigma(X,X) = \rho JX$ and using (1) we get

$$\langle(\nabla\sigma)(v,X,X), Jw\rangle + \frac{2\operatorname{div}X}{n}\langle\sigma(v,X), Jw\rangle$$
$$= \langle\nabla\rho, v\rangle\langle X, w\rangle + \frac{\rho\operatorname{div}X}{n}\langle v, w\rangle.$$

Using the elementary properties of ψ and the Codazzi equation of ψ we obtain that all the terms on the above equation are symmetric on v and w, except possible

$\langle \nabla\rho, v\rangle\langle X, w\rangle$. So this one must be also symmetric and this means that

$$\nabla\rho = \frac{\langle \nabla\rho, X\rangle}{|X|^2} X.$$

In particular ρ is constant on the leaves of $\mathscr{D}$. Moreover, as $\mathrm{Ric}(X)$ is also constant on the leaves of $\mathscr{D}$, the other eigenvalues (solutions of the equation given in 1)) are also constant on the leaves of $\mathscr{D}$, which proves 3).

Finally we prove 4). Let

$$M' = \{p \in M / A_{JX} \text{ has two different eigenvalues on } \mathscr{D}(p)\}.$$

Then M' is an open subset of M, and if these eigenvalues are denoted by $\rho_i, i = 1,2$ and their multiplicities by $n_i, i = 1,2$, then $n\langle H, JX\rangle = \rho + n_1\rho_1 + n_2\rho_2$. From Lemma 2,1), $\rho_i, i = 1,2$ are continuous functions, and hence $n_i, i = 1,2$ are constant on each connected component of M'. So,

$$\mathscr{D}_i(p) = \{v \in \mathscr{D}_p / A_{JX}v = \rho_i v\}, \qquad i = 1,2$$

define distributions on the connected components of M' such that $\mathscr{D}(p) = \mathscr{D}_1(p) \oplus \mathscr{D}_2(p)$, for any $p \in M'$. If V, W are vector fields of $\mathscr{D}_i, i = 1,2$, then

$$\langle \sigma(V, W), JX\rangle = \rho_i\langle V, W\rangle.$$

Using 3) and taking derivative with respect to a vector field Z on $\mathscr{D}$ in the above expression, we get

$$\langle (\nabla\sigma)(Z, V, W), JX\rangle + \frac{\operatorname{div} X}{n}\langle \sigma(V, W), JZ\rangle = 0.$$

The property 4) on M' follows from this equation, by using the symmetry of σ and $\nabla\sigma$, 3) and the decomposition $\mathscr{D} = \mathscr{D}_1 \oplus \mathscr{D}_2$.

If M' is dense on M we finish. If not, $M - M'$ has nonempty interior and in it A_{JX} has only one eigenvalue. In this case, and using an easier version of the above reasoning, we prove 4) on $M - M'$, which finish the proof of Lemma 2.

Now, we are going to describe locally our family of Lagrangian submanifolds. We start with the easiest case, when the vector field X is parallel.

PROPOSITION 2. *Let $\psi : M \to C^n$ be a Lagrangian immersion of an n-dimensional manifold M. Then, there exists a parallel vector field X on M such that $\sigma(X, X) = \rho JX$ for certain function ρ if and only if locally M is the Riemannian product $M_1^{n_1} \times M_2^{n_2}$ with $n_1 \geq 1$, ψ is the product of the Lagrangian immersions $\psi_i : M_i^{n_i} \to C^{n_i}, i = 1,2$, being ψ_1 any regular curve of C if $n_1 = 1$, and being ψ_1 an spherical immersion (i.e. the image of ψ_1 lies in a hypersphere of C^{n_1}) if $n_1 \geq 2$. Moreover, in the first case, $X \equiv (\partial/\partial t, 0)$ being t the arc length parameter of ψ_1, and in the latter, $X \equiv (J(\psi_1 - B), 0)$ being B the center of the sphere where ψ_1 lies.*

PROOF. Assume that X is a parallel vector field such that $\sigma(X, X) = \rho JX$. If ρ is identically zero, from Lemma 2,1) and as $\mathrm{Ric}(X) = 0$, it follows that $A_{JX} \equiv 0$ and then

$\psi_*(X)$ defines a parallel vector field v in $\boldsymbol{C}^n$. So $\psi(M)$ lies in a hyperplane of $\boldsymbol{C}^n$ with normal vector Jv and we get a particular case of the case $n_1 = 1$ where the curve ψ_1 is a straight line.

If ρ is not identically zero, we consider the nonempty open subset

$$M' = \{p \in M/\rho(p) \neq 0\}.$$

Again, from Lemma 2,1), and as $\mathrm{Ric}(X) = 0$, it follows that A_{JX} has at most two different eigenvalues: ρ and 0 at the points of M'. So, if n_1 is the multiplicity of ρ as eigenvalue of A_{JX} we have that $n\langle H, JX\rangle = n_1\rho$, and hence n_1 is constant on each connected component of M'.

On the other hand, taking derivative in $\sigma(X, X) = \rho JX$ and using that X is parallel, we get

$$\begin{aligned}\langle \nabla\rho, X\rangle &= \sum_{i=1}^{n} \langle (\nabla\sigma)(e_i, X, X), Je_i\rangle\rangle \\ &= n\langle \nabla\langle H, JX\rangle, X\rangle = n_1\langle \nabla\rho, X\rangle.\end{aligned}$$

So, using Lemma 2,3), on M' we have that

$$(n_1 - 1)\nabla\rho = 0$$

and then on each connected component of M', either $n_1 = 1$ or $n_1 \geq 2$ and ρ is a non-null constant. As on the boundary of M' ρ is zero, if there exists a connected component of M' with $n_1 \geq 2$, then $M' = M$ and ρ is a non-null constant. In other case, if $n_1 = 1$ in all M', then it is also true in all M, by considering the case $A_{JX} \equiv 0$ as a particular case of the case $n_1 = 1$ with $\rho = 0$. So we have only two posibilities for A_{JX}:

$$\begin{pmatrix} \rho & & & & \\ & \ddots & & & \\ & & \rho & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad \begin{pmatrix} \rho & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

where ρ is respectively a non-null constant or a function.

In the first case, let $\mathscr{D}_i$, $i = 1, 2$ be the distributions on M defined by

$$\mathscr{D}_1(p) = \{v \in T_pM/A_{JX}v = \rho v\} \quad \mathscr{D}_2(p) = \{v \in T_pM/A_{JX}v = 0\}.$$

of dimensions n_1 and $n - n_1$ with $n_1 \geq 2$.

We are going to prove that $\mathscr{D}_i$, $i = 1, 2$ define totally geodesic foliations on M. To see it, it is sufficient to prove that if V is a vector field on $\mathscr{D}_2$ and W is any vector field on M, then $\nabla_W V$ is a vector field on $\mathscr{D}_2$. As V is a vector field on $\mathscr{D}_2$, $\sigma(X, V) = 0$, and then

$$(\nabla\sigma)(W, X, V) + \sigma(X, \nabla_W V) = 0.$$

So $\nabla_W V$ will be a vector field on $\mathscr{D}_2$ if and only if $(\nabla\sigma)(W, X, V) = 0$.

In other to prove that $(\nabla\sigma)(W, X, V) = 0$, we derivate $\sigma(X, X) = \rho JX$ with respect to V which is a vector field on $\mathscr{D}$. As X is parallel, and ρ is constant on the leaves of $\mathscr{D}$, we have $(\nabla\sigma)(V, X, X) = 0$. Now, if $W = \lambda X + W^{\mathscr{D}}$, where $W^{\mathscr{D}}$ is the component of W on $\mathscr{D}$, we obtain that $(\nabla\sigma)(W, X, V) = (\nabla\sigma)(W^{\mathscr{D}}, X, V) = \mu JX$ by using Lemma 2,4) and the fact that $\operatorname{div} X = 0$. Finally using the symmetries of $\nabla\sigma$ we obtain that

$$\mu|X|^2 = \langle(\nabla\sigma)(X, X, V), JW^{\mathscr{D}}\rangle = 0,$$

and so $(\nabla\sigma)(W, X, V) = 0$. Hence $\mathscr{D}_i$, $i = 1, 2$ define totally geodesic foliations on M.

As consequence, M is locally isometric to $M_1^{n_1} \times M_2^{n_2}$. Moreover, as the eigenvalues of A_{JX} are ρ and 0, from Lemma 2,2) it follows that $\sigma(V_1, V_2) = 0$ for V_i tangent to M_i, $i = 1, 2$. Now using a result of Moore [9] we conclude that ψ is the product of two Lagrangian immersions $\psi_i : M_i \to \boldsymbol{C}^{n_i}, i = 1, 2$. Finally, as X is a vector field tangent to M_1, if A^1_{JX} is the Weingarten endomorphism of ψ_1, then $A^1_{JX} = \rho I$ with ρ a non-null constant. Then, it is clear that $\psi_1 + (1/\rho)JX$ is a parallel vector field B in $\boldsymbol{C}^{n_1}$ along ψ_1, and then ψ_1 lies in a sphere centered at B.

In the second case, and as X is parallel, it is clear that $\mathscr{D}$ is a totally geodesic foliation. Moreover, as $\sigma(X, V) = 0$ for any vector field of $\mathscr{D}$, a similar reasoning to the above one concludes the proof.

The converse is straightforward.

Now, we will study the general case.

THEOREM 1. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an n-dimensional manifold M. Then X is a closed and nonparallel conformal vector field* (*without zeroes*) *on M such that $\sigma(X, X) = \rho JX$ for some function ρ if and only if the immersion is given by $\psi = FJX + A$, where $F : M \to \boldsymbol{C}^*$ is a non-real complex function of constant length and A a vector of $\boldsymbol{C}^n$.*

We start by proving the following result.

LEMMA 3. *If x is a point of M such that* $(\operatorname{div} X)(x) \neq 0$, *then A_{JX} has only one eigenvalue on $\mathscr{D}(x)$.*

PROOF. Recall that from Lemma 2,1), A_{JX} has at most two eigenvalues on $\mathscr{D}(x)$. Suppose that A_{JX} has two different eigenvalues ρ_1 and ρ_2 on $\mathscr{D}(x)$. Let N be a connected leaf of $\mathscr{D}$ with $x \in N$. From Lemma 2,3), ρ_1 and ρ_2 are the eigenvalues of A_{JX} on $\mathscr{D}(y) = T_yN$ at any point y of N. Let $\mathscr{F}_i, i = 1, 2$ be the distributions on N defined by

$$y \in N \to \mathscr{F}_i(y) = \{v \in T_yN / A_{JX}v = \rho_i v\}.$$

We are going to prove that $\mathscr{F}_i$ are totally geodesic foliations on N. If ∇^0 is the Levi-Civita connection of the induced metric on N, the Gauss formula of N in M joint with (3) says that

$$\nabla^0_V W = \nabla_V W + \frac{\operatorname{div} X}{n|X|^2}\langle V, W\rangle X \tag{4}$$

for any vector fields V and W tangent to N. If W is a vector field on $\mathscr{F}_i$, $i = 1,2$, then $\sigma(X, W) = \rho_i JW$. Derivating with respect to the vector field V tangent to M we obtain

$$(\nabla\sigma)(V, X, W) + \frac{\operatorname{div} X}{n}\sigma(V, W) + \sigma(X, \nabla_V W) = \rho_i J\nabla_V W.$$

Multiplying this equation by JZ with Z any vector field on N, and using Lemma 2,4), (4) and that $\sigma(X,X) = \rho JX$ we obtain

$$\sigma(X, \nabla_V^0 W) = \rho_i J\nabla_V^0 W$$

which proves that $\mathscr{F}_i, i = 1,2$ are totally geodesic foliations on N.

So, locally N is the Riemannian product $N_1^{n_1} \times N_2^{n_2}$ with $n_1 + n_2 = n - 1$, and $x = (p_0, q_0)$. If σ' denotes the second fundamental form of the immersion $\psi : N \to \boldsymbol{C}^n$, then $\sigma' = \sigma + \sigma_0$, and then Lemma 1,5) and Lemma 2,2) say that $\sigma'(v_1, v_2) = 0$ for vectors v_i tangent to $N_i, i = 1,2$. Now, using again a result proved in [**9**], we obtain that the immersion $\psi : N_1 \times N_2 \to \boldsymbol{C}^n$ is given by

$$\psi(p,q) = (\psi_1(p), \psi_2(q), a) \tag{5}$$

where ψ_i are immersions of N_i in Euclidean subspaces of $\boldsymbol{C}^n$, and a is a vector of $\boldsymbol{C}^n$. We can assume that N_i are connected, $i = 1,2$. For each $q \in N_2$, let

$$\psi_q : N_1 \to \boldsymbol{C}^n$$

be the immersion given by $\psi_q(p) = \psi(p,q)$. Then, $\mathscr{N}$ defined by

$$p \in N_1 \to \mathscr{N}(p) = \mathscr{F}_2(p,q) \oplus J\mathscr{F}_2(p,q)$$

defines a subbundle of the normal bundle of ψ_q, and it is straightforward to prove that $\mathscr{N}$ is a totally geodesic subbundle and parallel with respect to the normal connection of ψ_q. Using a very well known result of [**6**], there exists a subspace $\boldsymbol{C}^{n_1}(q)$ of $\boldsymbol{C}^n$ such that

$$\psi_q : N_1 \to \boldsymbol{C}^{n_1}(q) \subset \boldsymbol{C}^n.$$

But from (5), we have for any $q_1, q_2 \in N_2$ that

$$\psi_{q_1} = \psi_{q_2} + (0, \psi_2(q_1) - \psi_2(q_2), 0)$$

and then the subspaces $\{\boldsymbol{C}^{n_1}(q)/q \in N_2\}$ are parallel in $\boldsymbol{C}^n$. So, if $\boldsymbol{C}^{n_1}$ is the corresponding linear subspace and $V(p,q) = \operatorname{span}\{X(p,q), JX(p,q)\}$, then

$$\boldsymbol{C}^{n_1} = \mathscr{F}_1(p,q) \oplus J\mathscr{F}_1(p,q) \oplus V(p,q).$$

Changing the roles of N_1 and N_2, there exists a linear subspace $\boldsymbol{C}^{n_2}$ of $\boldsymbol{C}^n$ such that

$$\boldsymbol{C}^{n_2} = \mathscr{F}_2(p,q) \oplus J\mathscr{F}_2(p,q) \oplus V(p,q).$$

As the immersion is Lagrangian, we obtain that

$$\boldsymbol{C}^{n_1} \cap \boldsymbol{C}^{n_2} = V(p,q), \qquad \textit{for any } (p,q) \in N_1 \times N_2.$$

and then, $V(p,q)$ is independent of the point (p,q). Now, as X is a vector field in this complex line, we have that $X = fc + gJc$ where c is a vector in this line and f, g are functions on $N_1 \times N_2$. Derivating the last expression with respect to any vector v tangent to N at x, we obtain

$$\frac{\operatorname{div} X}{n} v = \langle \nabla f, v\rangle c + \langle \nabla g, v\rangle Jc$$

and as v is perpendicular to $\operatorname{span}\{X_x, JX_x\} \equiv \operatorname{span}\{c, Jc\}$ we get that $\operatorname{div} X(x) = 0$ which contradicts our assumption.

Proof of the Theorem. Suppose that $\psi = FJX + A$ with $F = a + ib$. Then $a^2 + b^2 = l \in \boldsymbol{R}^*$ and $\psi - A = aJX - bX$. Derivating this expression with respect to a vector v, taking tangent and normal components to ψ and using the elementary properties of ψ we obtain

$$-v = \langle \nabla b, v\rangle X + b\nabla_v X + aA_{JX}v$$

$$0 = -bA_{JX}v + \langle \nabla a, v\rangle X + a\nabla_v X.$$

From this equation it is easy to obtain

$$\nabla_v X = -(b/l)v$$

$$A_{JX}v = -(a/l)v + (1/l)\langle -a\nabla b + b\nabla a, v\rangle X.$$

So, X is a closed and conformal vector field such that $\sigma(X,X) = \rho JX$ with $\rho = (1/l)\langle -a\nabla b + b\nabla a, X\rangle - (a/l)$. Moreover, as b is no identically zero, then X is not a parallel vector field.

Conversely, suppose that X is a closed and conformal vector field such that $\sigma(X,X) = \rho JX$. As X is nonparallel, then

$$M' = \{p \in M/(\operatorname{div} X)(p) \neq 0\}$$

is a nonempty open subset in M. From Lemma 3 we know that A_{JX} has only one eigenvalue on $\mathscr{D}(p)$. So we have a fuction $\hat{\rho}$ on M' such that $\sigma(X, V) = \hat{\rho}JV$ for any vector field V on $\mathscr{D}$. Now we compute the gradient of $\hat{\rho}$. To do it we compute divergences in the equality $\rho X = A_{JX}X$, obtaining

$$\begin{aligned}\langle \nabla\rho, X\rangle + \rho \operatorname{div} X &= \operatorname{div}(A_{JX}X) \\ &= \sum_{i=1}^{n}\langle(\nabla\sigma)(e_i, X, X), Je_i\rangle + \sum_{i=1}^{n}\frac{2\operatorname{div} X}{n}\langle\sigma(e_i, X), Je_i\rangle \\ &= n\langle\nabla_X^\perp H, JX\rangle + 2\langle H, JX\rangle \operatorname{div} X.\end{aligned}$$

But $X\langle H, JX\rangle = \langle\nabla_X^\perp H, JX\rangle + (1/n)\langle H, JX\rangle \operatorname{div} X$, and then it follows that

$$\langle\nabla(n\langle H, JX\rangle - \rho), X\rangle = \rho \operatorname{div} X - \langle H, JX\rangle \operatorname{div} X. \tag{6}$$

As $n\langle H, JX\rangle = \rho + (n-1)\hat{\rho}$, we obtain that

$$n\langle \nabla\hat{\rho}, X\rangle = (\rho - \hat{\rho})\operatorname{div} X. \tag{7}$$

Let $h : M' \to R$ be the function defined by

$$h = \left(\frac{\operatorname{div} X}{n}\right)^2 + \hat{\rho}^2.$$

Then using Lemma 1,1), Lemma 2,1),3) and (7) its gradient satisfies

$$\nabla h = \frac{-2\operatorname{div} X}{n|X|^2}\left(\hat{\rho}^2 - \rho\hat{\rho} + \frac{\operatorname{Ric}(X)}{n-1}\right)X = 0,$$

and so h is constant on each connected component of M'.

If M' is dense in M, then h is constant on M. If not, $M - M'$ has nonempty interior, and in it X is parallel. So, by continuity, on $\operatorname{Int}(M - M')$, Proposition 2 says that A_{JX} on $\mathscr{D}$ is either $A_{JX} = \rho I$ or $A_{JX} = 0$, and then the function h is also defined on $\operatorname{Int}(M - M')$. In the second case, on the boundary of M', $h = 0$ which is impossible because h is a non-null constant on each connected component of M'. So it happens the first one, and hence $A_{JX} = \rho I$ on $\operatorname{Int}(M - M')$. As consequence, and using again Proposition 2, our function h, is also constant on each connected component of $\operatorname{Int}(M - M')$, which implies that h is constant on M, i.e. $h = \mu^2$ with $\mu > 0$.

Now, we consider the vectorial function

$$\Upsilon = \mu^2\psi - \frac{\operatorname{div} X}{n}X + \hat{\rho}JX : M \to C^n.$$

If v is any vector of $\mathscr{D}$, using Lemma 1,1), and (7) we have

$$\begin{aligned}\overline{\nabla}_v\Upsilon &= (\mu^2 - (\operatorname{div} X/n)^2)v - \frac{\operatorname{div} X}{n}\sigma(X, v) - \hat{\rho}A_{JX}v + \hat{\rho}\frac{\operatorname{div} X}{n}Jv \\ &= (\mu^2 - (\operatorname{div} X/n)^2 - \hat{\rho}^2)v = 0.\end{aligned}$$

Derivating with respect to X, and using again Lemma 1,1) and (7) we obtain

$$\overline{\nabla}_X\Upsilon = \left(\mu^2 + \frac{\operatorname{Ric}(X)}{n-1} - \left(\frac{\operatorname{div} X}{n}\right)^2 - \rho\hat{\rho}\right)X = (\mu^2 - h)X = 0.$$

So we obtain that Υ is a constant vector B in C^n. Now, if we define

$$F = (-1/\mu^2)(\hat{\rho} + i\operatorname{div} X/n) \quad \text{and} \quad A = B/\mu^2 \tag{8}$$

we finish the proof. Observe that F is not a real function because div$X$ is not identically zero.

Now, using Theorem 1 we will describe locally our family of Lagrangian immersions. To do it, we start introducing a family of examples.

Let CP^{n-1} be the $(n-1)$-dimensional complex projective space with the canonical Kaehler structure of constant holomorphic sectional curvature 4, and $\Pi : S^{2n-1} \to CP^{n-1}$

the Hopf fibration. If $\phi : N \to \boldsymbol{CP}^{n-1}$ is a Lagrangian immersion of an $(n-1)$-dimensional simply-connected manifold N, then it is well-known that ϕ has a horizontal lift (with respect to the Hopf fibration) to S^{2n-1}, and that, up to rotations on S^{2n-1}, it is unique. We will denote by $\tilde{\phi} : N \to S^{2n-1}$ this horizontal lift. Then, the horizontality means that $\langle \tilde{\phi}_*(v), J\tilde{\phi} \rangle = 0$ for any tangent vector v to N. We remark that only the Lagrangian immersions in $\boldsymbol{CP}^{n-1}$ have locally horizontal lifts.

DEFINITION 1. *Let $\gamma : I \to \boldsymbol{C}^*$ be a regular curve and $\phi : N \to \boldsymbol{CP}^{n-1}$ a Lagrangian immersion of an $(n-1)$-dimensional simply-connected manifold N. We define*

$$\gamma * \phi : I \times N \to \boldsymbol{C}^n$$

by $(\gamma * \phi)(t, x) = \gamma(t)\tilde{\phi}(x)$ where $\tilde{\phi}$ is a horizontal lift of ϕ.

We describe the properties of these kind of immersions without proof, because it is straightforward.

PROPOSITION 3. *Let $\gamma : I \to \boldsymbol{C}^*$ be a regular curve and $\phi : (N, g') \to \boldsymbol{CP}^{n-1}$ a Lagrangian isometric immersion of an $(n-1)$-dimensional simply-connected manifold N. Then*

1. *$\gamma * \phi : (I \times N, |\gamma'|^2 dt^2 \times |\gamma|^2 g') \to \boldsymbol{C}^n$ is a Lagrangian isometric immersion which is well defined up to rotations in $\boldsymbol{C}^n$.*
2. *$X = (|\gamma(t)|/|\gamma'(t)|)(\partial/\partial t, 0)$ is a closed and conformal vector field on $I \times N$ such that $\sigma(X, X) = \rho JX$ with $\rho = (|\gamma|/|\gamma'|^3)\langle \gamma'', J\gamma' \rangle$.*
3. *The second fundamental form σ of $\gamma * \phi$ satisfies*

$$\sigma((0, v), (0, w)) = \frac{\langle \gamma', J\gamma \rangle}{|\gamma||\gamma'|} g'(v, w)JX + \gamma * \sigma_\phi(v, w)^*$$

where σ_ϕ is the second fundamental form of ϕ, * denotes horizontal lift and v, w are vectors tangent to N.

4. *The mean curvature vector H of $\gamma * \phi$ is given by*

$$nH = a(t)JX + \frac{(n-1)}{|\gamma|^2} \gamma * H_\phi^*$$

where H_ϕ is the mean curvature vector of ϕ and

$$a = \frac{\langle \gamma'', J\gamma' \rangle}{|\gamma||\gamma'|^3} + (n-1)\frac{\langle \gamma', J\gamma \rangle}{|\gamma|^3|\gamma'|}.$$

5. *The divergence of X is given by*

$$\operatorname{div} X = n \frac{\langle \gamma(t), \gamma'(t) \rangle}{|\gamma(t)||\gamma'(t)|}.$$

So X is parallel if and only if γ is a reparametrization of a circle. In this case, $\gamma * \phi$ is the circle bundle over N induced, via ϕ, by the Hopf fibration $\Pi : S^{2n-1} \to \boldsymbol{CP}^{n-1}$.

COROLLARY 1. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an n-dimensional manifold M and X a closed and nonparallel conformal vector field (without zeroes) on M such that $\sigma(X,X) = \rho JX$ for some function ρ. Then locally ψ is congruent to one of the examples described in Proposition* 3.

PROOF. Let X be a closed and nonparallel conformal vector field (without zeroes) on M such that $\sigma(X,X) = \rho JX$. From Lemma 1,6), M is locally isometric to $(I \times N, |X|^2(dt^2 \times g'))$ where I is an open interval with $0 \in I$ and N is a simply-connected manifold. From Theorem 1, we have that $\psi = FJX + A$. We can take $A = 0$. Let $\psi_0 : N \to \boldsymbol{C}^n$ be the immersion given by $\psi_0(x) = \psi(0,x)$. Then $\psi_0(x) = F(0,x)JX_{(0,x)}$ and as X has constant length on the connected leaves of $\mathscr{D}$, we have $|\psi_0|$ is a constant l. Moreover, if v is any vector tangent to N, then

$$\langle(\psi_0)_*(v), J\psi_0\rangle = -F\langle\psi_*(0,v), X\rangle = 0$$

because $X = (\partial/\partial t, 0)$. Hence $(1/l)\psi_0$ defines an horizontal immersion of N in S^{2n-1}. We define $\phi = \Pi \circ (1/l)\psi_0$, being $\Pi : S^{2n-1} \to \boldsymbol{CP}^{n-1}$ the Hopf fibration. It is clear that ϕ is a Lagrangian isometric immersion of $(N, \mu^2 g')$ in $\boldsymbol{CP}^{n-1}$ and that $(1/l)\psi_0$ is the horizontal lift of ϕ, where $\mu^2 = (\operatorname{div} X/n)^2 + \hat{\rho}^2$. (see (8)).

Now, let x a point of N and consider the curve $\alpha : I \to \boldsymbol{C}^n$ defined by $\alpha(t) = \psi(t,x)$. As from Lemma 1,6), X is given by $(\partial/\partial t, 0)$, then $\alpha(t) = iF(t,x)\alpha'(t)$. Using (8), Lemma 1,1) and Lemma 2,3), we have that

$$\alpha'(t) = f(t)\alpha(t) \qquad \text{with} \qquad f(t) = \operatorname{div} X/n + i\hat{\rho} \tag{9}$$

being $\hat{\rho}$ the eigenvalue of A_{JX} on $\mathscr{D}$. Integrating the above equation, we get

$$\alpha(t) = e^{\int_0^t f(s)ds}\alpha(0),$$

which means that

$$\psi(t,x) = e^{\int_0^t f(s)ds}\psi(0,x).$$

As $\tilde{\phi}(x) = (1/l)\psi(0,x)$ and $|X|^2 = |\gamma'|^2$, defining

$$\gamma : I \to \boldsymbol{C}^* \quad \text{by} \quad \gamma(t) = le^{\int_0^t f(s)ds} \tag{10}$$

we obtain that $\gamma * \phi$ is congruent to our immersion ψ.

Now, we apply the above results to the case in which $X = JH$. In this case, as JH is always a closed vector field, we only have to assume that JH is a conformal vector field. As we indicated at the introduction, we will refer to these submanifolds as Lagrangian submanifolds with *conformal Maslov form*.

COROLLARY 2. *Let $u : \boldsymbol{R} \to \boldsymbol{R}$ be a solution of the O.D.E.*

$$u''(t) + \frac{2n^2}{(n+2)^2}e^{4u(t)} - n\lambda^2 e^{-2nu(t)} - \frac{n(n-2)\lambda}{n+2}e^{(2-n)u(t)} = 0,$$

$$e^{u(0)} = a, \quad u'(0) = 0, \quad \lambda = 0, 1, -1, \tag{11}$$

and $\phi : N \to \boldsymbol{CP}^{n-1}$ a minimal Lagrangian immersion of a simply connected $(n-1)$-dimensional manifold N. Then

$$\gamma_u * \phi : \boldsymbol{R} \times N \to \boldsymbol{C}^n$$

where $\gamma_u : \boldsymbol{R} \to \boldsymbol{C}$ is the curve

$$\gamma_u(t) = \frac{e^{u(t)}}{\mu} e^{-i \int_0^t \left(\frac{n}{n+2} e^{2u(s)} + \lambda e^{-nu(s)}\right) ds} \qquad \mu = \frac{n}{n+2} a^2 + \lambda a^{-n}$$

is a Lagrangian immersion with conformal Maslov form such that $JH \equiv (\partial/\partial t, 0)$ is a principal direction of A_H.

Conversely, any Lagrangian immersion of an n-dimensional manifold in $\boldsymbol{C}^n$ with nonparallel conformal Maslov form such that JH is a principal direction of A_H is, around each point where H does not vanish and up to dilatations of $\boldsymbol{C}^n$, congruent to one of the above examples.

Remark 1. *The equation* (11) *has two constant solutions when $\lambda = 1$ and $\lambda = -1$ given respectively by*

$$e^{u(t)} = \left(\frac{n+2}{2}\right)^{1/n+2} \qquad e^{u(t)} = \left(\frac{n+2}{n}\right)^{1/n+2}.$$

In the first case, the curve γ_u is a reparametrization of a circle of radius $(2/n+2)^{1/n+2}$ and $\gamma_u * \phi$ is the circle bundle over N induced, via ϕ, by the Hopf fibration. The latter is a degenerate case, because is the only example in Corollary 2 *with $\mu = 0$. It is not difficult to see that in this case $\hat{\rho} = 0$ and the corresponding immersion is not described in Proposition* 3, *but $\gamma_u * \phi$ corresponds with one of the examples described in Proposition* 2.

These examples are the only ones described in Corollary 2 *where H is parallel.*

Proof. In order to prove the properties of $\gamma_u * \phi$, we only have to check that the vector field X given in Proposition 3 is JH. As ϕ is a minimal immersion, then $H_\phi^* \equiv 0$, and then Proposition 3,3) says that $X = JH$ if and only if $a(t) = -n$. Now, from the definition of γ_u is straightforward to check that $a(t) = -n$.

Conversely, suppose that JH is a conformal vector field such that $A_H(JH) = -\rho JH$ (we use the same notation than in section 3). Then, from (6) and Lemma 1,1) it follows that

$$\nabla \rho = -(3 \operatorname{div} JH - \rho |H|^{-2} \operatorname{div} JH) JH,$$

and using again Lemma 1,1) we have

$$\nabla \left(\rho + \frac{3n}{n+2}|H|^2\right) = -(|H|^{-2} \operatorname{div} JH)(\rho + \frac{3n}{n+2}|H|^2).$$

From this equation and using one more time Lemma 1,1) we get

$$\nabla \left(\rho |H|^n + \frac{3n}{n+2}|H|^{n+2}\right) = 0,$$

and so this function is a constant that will be denoted by $(n-1)\lambda$. Hence, we have computed explicitely the function ρ obtaining

$$\rho = -\frac{3n}{n+2}|H|^2 + (n-1)\lambda|H|^{-n}$$

where λ is a constant. Also, as $-n|H|^2 = \rho + (n-1)\hat{\rho}$ we also obtain that

$$\hat{\rho} = -\frac{n}{n+2}|H|^2 - \lambda|H|^{-n}.$$

From these expressions, Lemma 1,5) and Lemma 2,1) we have that $|H|$ satisfies the following differential equation:

$$\Delta^g \log|H| + \frac{2n^2}{(n+2)^2}|H|^4 - n\lambda^2|H|^{-2n} - \frac{n(n-2)\lambda}{n+2}|H|^{2-n} = 0. \tag{12}$$

Following the proof of Corollary 1, our immersion ψ is locally congruent to $\gamma * \phi : I \times N \to \boldsymbol{C}^n$, where ϕ is a totally real immersion of a simply-connected manifold N in $\boldsymbol{CP}^{n-1}$ and $\gamma : I \to \boldsymbol{C}^*$ is given by

$$\gamma(t) = l e^{\int_0^t f(s)ds},$$

where $f = \operatorname{div} JH/n + i\hat{\rho}$. In this case, from Proposition 3,3) it follows that $H_\phi = 0$ and so ϕ is a minimal immersion.

As $|H|$ is constant on the leaves of $\mathscr{D}$, then for each point x at N

$$u(t) = \log|H|(t, x)$$

defines a real function on I, and from (12) it satisfies the following o.d.e.

$$u''(t) + \frac{2n^2}{(n+2)^2} e^{4u(t)} - n\lambda^2 e^{-2nu(t)} - \frac{n(n-2)\lambda}{n+2} e^{(2-n)u(t)} = 0.$$

Now, from Lemma 1,1) and the above expression of $\hat{\rho}$ it is easy to see that

$$\operatorname{div} JH(t,x) = nu'(t) \quad \hat{\rho}(t,x) = -\left(\frac{n}{n+2} e^{2u(t)} + \lambda e^{-nu(t)}\right)$$

and so the function f is given by

$$f(t) = u'(t) - i\left(\frac{n}{n+2} e^{2u(t)} + \lambda e^{-nu(t)}\right).$$

As the constant l given in Corollary 1 is $a\mu^{-1}$, we get the result. The only detail that we have to prove is that it is sufficient to take $\lambda = 0, 1, -1$. To see it, we suppose that $u(t)$ is a solution of equation (11) with $\lambda > 0$. Then it is easy to see that

$$v(t) = u(\lambda^{-2/n+2}t) - (1/n+2)\log\lambda$$

is a solution of equation (11) with $\lambda = 1$ and the function f_v defined in (9) is given by

$$f_v = \lambda^{-2/n+2} f_u.$$

So the curve γ_v is homothetic to γ_u. A similar reasoning can be used when $\lambda < 0$. This finishes the proof.

We are going to describe some of the curves γ_u given in Corollary 2.

LEMMA 4. *Up to dilatations of $\boldsymbol{C}^*$, the curves $\gamma_u : \boldsymbol{R} \to \boldsymbol{C}^*$ associated to the solutions of equation* (11) *with $\lambda = 0$ are linear reparametrizations of the curve*

$$\gamma(t) = \frac{1}{\cosh 2t}(\cosh t + i \sinh t).$$

PROOF. The general solution of equation (11) with $\lambda = 0$ is given by

$$e^{2u(t)} = \frac{a^2}{\cosh\left(\frac{2na^2}{n+2}t\right)}.$$

Now, using the expression of the curve γ_u given in Corollary 2, it is straightforward to see that

$$\gamma_u(t) = \frac{n+2}{an\cosh\left(\frac{2na^2}{n+2}t\right)}\left(\cosh\left(\frac{na^2}{n+2}t\right) + i\sinh\left(\frac{na^2}{n+2}t\right)\right)$$

and 1) follows.

REMARK 2. When $\lambda \neq 0$, the solutions of equation (11) are certain kind of hyperelliptic functions (elliptic functions when $n = 2$, see [**4**]), and it is not difficult to see that they are periodic. The corresponding curves γ_u are closed and embedded when $n = 2$ ([**4**]), but the authors do not know what happens with these curves when $n \geq 3$. They believe that the curves are also closed and embedded.

4. Compact case. The Whitney sphere

We start this section giving a local characterization of the Whitney sphere in terms of the second fundamental form.

THEOREM 2. *Let $\psi : M \to \boldsymbol{C}^n$ a Lagrangian immersion of an n-dimensional manifold M. The second fundamental form σ of ψ is given by*

$$\sigma(v, w) = \frac{n}{n+2}\{\langle v, w\rangle H + \langle Jv, H\rangle Jw + \langle Jw, H\rangle Jv\} \tag{13}$$

for any vectors v and w tangent to M if and only if either ψ is totally geodesic or $\psi(M)$ is an open set of the Whitney sphere.

PROOF. It is straightforward to check that the totally geodesic immersion and the Whitney sphere satisfied (13).

Conversely, taking in (13) $v = w$ with $|v| = 1$ and derivating with respect to an orthogonal vector z we have that

$$(\nabla\sigma)(z, v, v) = \frac{n}{n+2}(\nabla_z^\perp H + 2\langle\nabla_z^\perp H, Jv\rangle Jv).$$

Now using the Codazzi equation, the above equation says that

$$\nabla_z^\perp H = -\langle \nabla_z^\perp H, Jv\rangle Jv + \langle \nabla_v^\perp H, Jv\rangle Jz.$$

As z and v are perpendicular, the fact that JH is a closed vector field implies that $\langle \nabla_z^\perp H, Jv\rangle = 0$ and so

$$\nabla_z^\perp H = \langle \nabla_v^\perp H, Jv\rangle Jz$$

which means that JH is a conformal vector field on M.

If $H \equiv 0$, then from (13) the immersion ψ is totally geodesic. If H is nontrivial, Lemma 1,3) says that the zeroes of H are isolated when $n > 2$. If $n = 2$ it is a very well-known that on each complex neighbourhood of the surface a conformal vector field is a holomorphic vector field and its zeroes are also isolated. Hence M' defined by

$$M' = \{x \in M / H_x \neq 0\}$$

is a connected dense open subset of M. We will work in it.

From (13) it follows that

$$\sigma(JH, JH) = \frac{3n}{n+2}|H|^2 H, \qquad \sigma(v, w) = \frac{n}{n+2}\langle v, w\rangle H$$

for vectors v, w perpendicular to JH. So, from Lemma 2,1) it is clear that JH is nonparallel, and hence we obtain from Corollary 2 that our immersion is locally congruent to $\gamma_u * \phi$ where u is a solution of equation (11) with $\lambda = 0$, and, by using Proposition 3,3), ϕ is a totally geodesic.

Now, we describe the examples given in Corollary 2, when u is a solution of (11) with $\lambda = 0$ and ϕ is totally geodesic. Let $\phi : S^{n-1} \to \boldsymbol{CP}^{n-1}$ be the Lagrangian totally geodesic immersion of the sphere of radius 1. As the horizontal lift $\tilde{\phi} : S^{n-1} \to S^{2n-1}$ of ϕ is given by

$$\tilde{\phi}(v_1, \dots, v_n) = (v_1, 0, \dots, v_n, 0),$$

then using Lemma 4 we obtain that up to dilatations and translations of $\boldsymbol{C}^n$

$$\gamma_u * \phi : \boldsymbol{R} \times S^{n-1} \to \boldsymbol{C}^n$$

is given by

$$(\gamma_u * \phi)(t, v_1, \dots, v_n) = \left(\frac{1}{\cosh 2t}\right)(v_1 \cosh t, v_1 \sinh t, \dots, v_n \cosh t, v_n \sinh t).$$

If we identify $\boldsymbol{R}^n - \{0\}$ with $\boldsymbol{R} \times S^{n-1}$ via the conformal transformation $w \to (log|w|, w/|w|)$, then our Lagrangian immersion $\gamma_u * \phi$ defines a Lagrangian immersion $\psi : \boldsymbol{R}^n - \{0\} \to \boldsymbol{C}^n$ which can be written as

$$\psi(w) = \frac{1}{1+|w|^4}((1+|w|^2)w_1, (|w|^2 - 1)w_1, \dots, (1+|w|^2)w_n, (|w|^2 - 1)w_n),$$

being $w = (w_1, \dots, w_n)$.

It is clear that ψ extends regularly to 0 and ∞ and via the stereographic projection it defines a Lagrangian immersion, which will be also denoted by $\psi : S^n \to \boldsymbol{C}^n$ which is defined by

$$\psi(x_1, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}^2}(x_1, x_1 x_{n+1}, \dots, x_n, x_n x_{n+1})$$

which is the Whitney immersion.

So, following the above reasoning, if x is a point of M', then there exists a neighbourhood U of x such that $\psi(U)$ is contained in a Whitney sphere Σ. As M' is connected and dense in M, a standard argument proves that $\psi(M)$ is contained in Σ.

Corollary 3. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an n-dimensional manifold M. If τ is the scalar curvature of ψ, then*

$$\tau \leq \frac{n^2(n-1)}{n+2}|H|^2$$

and the equality holds if and only if either ψ is totally geodesic or $\psi(M)$ is the Whitney sphere.

Proof. We consider, on the unit tangent bundle to M, the inequality

$$\left|\sigma(v, v) - \frac{n}{n+2}(H + 2\langle H, Jv\rangle Jv)\right|^2 \geq 0. \tag{14}$$

Interpolating in (14), it is not difficult to get that the inequality in (14) implies that

$$\tau \leq \frac{n^2(n-1)}{n+2}|H|^2$$

and that the equality holds in the last inequality if and only if the equality holds in (14). But the equality in (14) means that the second fundamental form of ψ is given by (13). Now, our result follows from Theorem 2.

From now on, we are going to assume that our Lagrangian submanifold M is compact. We start proving the following result.

Theorem 3. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an n-dimensional compact manifold M with conformal Maslov form. If H has zeroes on M, then ψ is congruent to the Whitney sphere.*

Proof. It is clear that H does not vanish identically, because there are not compact minimal submanifolds in the Euclidean space. As JH is a closed and conformal vector field with at least a zero, then from [**13**] (Theorem 3), M is diffeomorph to a sphere.

We consider on M the vector field Z defined by

$$Z = J\sigma(JH, JH) - \frac{3n}{n+2}|H|^2 JH.$$

Using Lemma 1, it is easy to get that

$$\langle \nabla_v Z, w\rangle = \langle (\nabla\sigma)(JH, v, w), H\rangle + \frac{2\operatorname{div} JH}{n}\langle \sigma(v,w), H\rangle$$

$$- \frac{3\operatorname{div} JH}{n+2}\{|H|^2\langle v,w\rangle + 2\langle JH, v\rangle\langle JH, w\rangle\}.$$

From here it is clear that Z is a closed vector field and that $\operatorname{div} Z = 0$. So Z is a harmonic vector field on M. As there are not harmonic vector fields on a sphere, Z vanishes identically, and then we have that

$$\sigma(JH, JH) = \frac{3n}{n+2}|H|^2 H. \tag{15}$$

On the other hand, as JH is a closed vector field on a sphere, there exists a function $f : M \to \boldsymbol{R}$ such that $JH = \nabla f$. Now, the conformality of JH implies that the Hessian of f satisfies the following equation

$$\nabla^2 f = (\operatorname{div} JH/n)\langle\ ,\ \rangle.$$

From a result in [**15**] we get that M is conformally equivalent to a round sphere. As the umbilicity is invariant under conformal transformations (see [**5**], p. 18), and in a round sphere the umbilical hypersurfaces are spheres, we obtain that the leaves of the umbilical foliation $\mathscr{D}$ are spheres. Following the proof of Corollary 1, if N is one of the leaves of the foliation $\mathscr{D}$, then N admits a minimal Lagrangian immersion ϕ in $\boldsymbol{CP}^{n-1}$. When $n = 2$, then trivially ϕ is totally geodesic. If $n = 3$, using a result in [**18**], ϕ is totally geodesic, because ϕ is a minimal Lagrangian immersion of a surface of genus zero in $\boldsymbol{CP}^2$. If $n \geq 4$, Lemma 5 below says that the leaves of $\mathscr{D}$ have constant curvature and as they are spheres these constants are positives. So, using a result in [**7**] our minimal Lagrangian immersion ϕ is also totally geodesic.

In any case, ϕ is totally geodesic, and from Proposition 3,3),

$$\sigma(v,w) = \hat{\rho}\langle v,w\rangle H$$

for any v, w orthogonal to JH, and then (15) implies that $\hat{\rho} = (n/(n+2))|H|^2$. This fact joint with (15) imply that the second fundamental form σ of ψ is given by (13) at any point where H does not vanish. By continuity, σ is given by (13) on the whole M. Now our result follows from Theorem 2.

Lemma 5. *Let* $(M, \langle\ ,\ \rangle)$ *an n-dimensional* $(n \geq 4)$ *conformally flat Riemannian manifold admitting a closed a conformal vector field X. Then the leaves of the foliation* $\mathscr{D}$ *(see Lemma 1) with the induced metric have constant curvature.*

Proof. As M is conformally flat, the Weyl tensor of M vanishes, and then the curvature of M is given by (see [**2**], Definition 1.117)

$$(16)\qquad R(v,w,x,z) = -\frac{r}{(n-1)(n-2)} R_0(v,w,x,z) + \frac{1}{n-2}\{\langle v,z\rangle S(w,x) - \langle v,x\rangle S(w,z) + \langle w,x\rangle S(v,z) - \langle w,z\rangle S(v,x)\},$$

where S is the Ricci tensor, r is the scalar curvature of $(M,\langle\ ,\ \rangle)$ and

$$R_0(v,w,x,z) = \langle v,z\rangle\langle w,x\rangle - \langle v,x\rangle\langle w,z\rangle.$$

Taking $w = x = X, v, z$ perpendicular to X in the above expression and using Lemma 1,2) we obtain

$$S(v,z) = \frac{r - |X|^{-2}\operatorname{Ric}(X)}{n-1}\langle v,z\rangle.$$

Now, using this expression in (16) we obtain that

$$R(v,w,x,z) = \frac{r - 2|X|^{-2}\operatorname{Ric}(X)}{(n-1)(n-2)} R_0(v,w,x,z),$$

where v,w,x,z are perpendicular to X.

If N is a leaf of the foliation $\mathscr{D}$ and R' the Riemannian curvature of the induced metric on N, then the above expression, the Gauss equation of N in M and (3) imply that

$$R'(v,w,x,z) = \left\{\left(\frac{\operatorname{div} X}{n|X|}\right)^2 + \frac{r - 2|X|^{-2}\operatorname{Ric}(X)}{(n-1)(n-2)}\right\} R_0(v,w,x,z),$$

for any vectors v,w,x,z tangent to N. So, as $\dim N \geq 3$, N has constant curvature.

Corollary 4. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of a compact n-dimensional manifold M with conformal Maslov form. If the first Betti number of M vanishes, then ψ is congruent to the Whitney sphere.*

Remark 3. In [**4**], Corollaries 3 and 4 were proved when $n = 2$.

Proof. As JH is a closed vector field and the first Betti number of M is zero, then there exists a function $f : M \to \boldsymbol{R}$ such that $\nabla f = JH$. So H has zeroes at the critical points of f and the Corollary follows from Theorem 3.

Suppose that $\psi : M \to \boldsymbol{C}^n$ is a Lagrangian immersion of a compact n-dimensional manifold M, and that the Ricci curvature of M is positive: $\operatorname{Ric} > 0$. Then the Gauss equation of ψ says that

$$n\langle\sigma(v,v),H\rangle > \sum_{i=1}^{n} |\sigma(v,e_i)|^2 \geq 0.$$

In particular, $|H|^2 > 0$. As $\operatorname{Ric} > 0$, then $H^1_{dR}(M) = 0$, and then our closed vector field JH is the gradient of a function $f : M \to \boldsymbol{R}$. As M is compact, f has a critical point, which contradicts that $|H| > 0$. So *there exists no compact Lagrangian submanifolds of*

$\boldsymbol{C}^n$ *with* Ric > 0. If one consider the Whitney sphere, then it is easy to see from (13) that its Ricci curvature is given by

$$\mathrm{Ric}(v) = \frac{n^2}{(n+2)^2}(n|H|^2|v|^2 + (n-2)\langle H, Jv\rangle^2)$$

which proves that Ric is positive at any point except at the two zeroes of H, where Ric is zero. In the next result we characterize the Whitney sphere in terms of the Ricci curvature.

Corollary 5. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of a compact n-dimensional manifold M with conformal Maslov form. Then*

$$\mathrm{Ric}(JH) \geq 0$$

if and only if either ψ has parallel mean curvature vector or ψ is congruent to the Whitney sphere.

Remark 4. In [**16**] Urbano classified the Lagrangian compact submanifolds of $\boldsymbol{C}^n$ with parallel mean curvature vector, proving that they are product of Lagrangian compact submanifolds of complex Euclidean spaces which are minimal in certain hyperspheres.

Proof. It is clear that any compact Lagrangian submanifold with parallel mean curvature vector and the Whitney sphere satisfy that $\mathrm{Ric}(JH) \geq 0$.

Conversely, suppose that $\mathrm{Ric}(JH) \geq 0$. If H has a zero, Theorem 3 says that ψ is congruent to the Whitney sphere. In other case, H has not zeroes, and then the metric $g = |H|^{-2}\langle\ ,\ \rangle$ is defined on the whole M. Integrating the expression given in Lemma 1,5) with respect to the measure dv_g of the metric g we have

$$0 = \int_M \left(\Delta^g \log|H| + \frac{\mathrm{Ric}(JH)}{n-1}\right) dv_g.$$

As $\mathrm{Ric}(JH) \geq 0$, we have that $\mathrm{Ric}(JH) = 0$ and so Lemma 1,5) implies that $|H|$ is constant. Using Lemma 1,1) and (1) we obtain that H is parallel, and we finish the proof.

We finish this paper by proving the following global result.

Corollary 6. *Let $\psi : M \to \boldsymbol{C}^n$ be a Lagrangian immersion of an orientable compact n-dimensional manifold M with nonparallel conformal Maslov form such that the first Betti number of M is one. Then the universal covering $(\tilde{M}, \tilde{\psi})$ of (M, ψ) is congruent to $(\boldsymbol{R} \times N, \gamma_u * \phi)$, where u is a solution of* (11) *with $\lambda = 1, -1$ and $\phi : N \to \boldsymbol{CP}^{n-1}$ is a minimal Lagrangian immersion of a simply-connected complete manifold N.*

Remark 5. In [**4**] Lagrangian tori in $\boldsymbol{C}^2$ with conformal Maslov form and such that JH is not a principal direction of A_H have been constructed. Hence it is not true that, compact Lagrangian submanifolds with conformal Maslov form whose first Betti number is greater than one, have JH as a principal direction of A_H.

PROOF. From Theorem 3, H has no zeroes on M. So the vector field $|H|^{-n}JH$ is defined on the whole M. Using Lemma 1, it is easy to see that the above vector field is closed and that its divergence vanishes. So it is a harmonic vector field on M. Using a similar reasoning as in Theorem 3, we have that the vector field Z defined by $Z = J\sigma(JH, JH) - (3n/(n+2))|H|^2 JH$ is also a harmonic vector field, and as the first Betti number of M is one, then there exists a constant $(n-1)\lambda$ such that $Z = (n-1)\lambda|H|^{-n}JH$. This means that

$$\sigma(JH, JH) = \left(\frac{3n}{n+2}|H|^2 + (n-1)\lambda|H|^{-n}\right)H.$$

As consequence, if $\tilde{H}$ is the mean curvature vector of $\tilde{\psi}$, then $J\tilde{H}$ is also a conformal vector field without zeroes on $\tilde{M}$ and

$$\tilde{\sigma}(J\tilde{H}, J\tilde{H}) = \frac{3n}{n+2}(|\tilde{H}|^2 + (n-1)\lambda|\tilde{H}|^{-n})\tilde{H},$$

being $\tilde{\sigma}$ the second fundamental form of $\tilde{\psi}$. If the induced metric on $\tilde{M}$ by $\tilde{\psi}$ is also denoted by $\langle\ ,\ \rangle$, then the metric $g = |\tilde{H}|^{-2}\langle\ ,\ \rangle$ (see Lemma 1) is globally defined, and it is complete because it is the covering metric of the metric $|H|^{-2}\langle\ ,\ \rangle$ on M. As $\tilde{M}$ is simply-connected, there exists a function $h : \tilde{M} \to \boldsymbol{R}$ such that $\nabla^g h = J\tilde{H}$, where ∇^g denotes the gradient of the metric g. Hence, as by Lemma 1,6) $J\tilde{H}$ is a parallel vector field, the Hessian of h vanishes. So $(\tilde{M}, g)$ is isometric to $(\boldsymbol{R} \times N, dt^2 \times g')$, where g' is a complete metric. As $\tilde{M}$ is simply-connected, N is also simply-connected. Moreover, Corollary 2 says that $\tilde{\psi}$ is congruent to $\gamma_u * \phi$ where u is a solution of (11) with $\lambda = 0, 1, -1$ and ϕ is a minimal isometric immersion of the complete Riemannian manifold $(N, \mu^2 g')$ in $\boldsymbol{CP}^{n-1}$. To finish the proof, we will see that $\lambda = 0$ cannot be possible. As H has no zeroes, M is compact and $|H| = |\tilde{H}|$, we have that $|\tilde{H}|^2 \geq \varepsilon > 0$. Also and from Proposition 3 it follows that $|\tilde{H}|^2(t, x) = |\gamma_u(t)|^2$. If $\lambda = 0$, then Lemma 4 says that $|\gamma_u(t)|^2 = 1/\cosh 2t$ and hence $\lim_{t\to\infty}|\tilde{H}|^2(t, x) = 0$ which contradicts the above assertion about the length of $\tilde{H}$.

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