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Some problems of minima concerning the oval.[*]

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§1. As is well known, the maximum area of those ovals which carry in every direction an assigned breadth has been found by use of the method of central symmetrization; this method transforms any oval into a central oval having greater area and the same breadth with the given oval in every direction. But, the method to make the area smaller is not yet known. We think this is the main difficulty in minimum problems of ovals under some conditions on the breadth.

However, some special cases in this direction were studied by Hayashi, Pál and others, as an analogue to the isoperimetric problem or as a solution of Kakeya's problem. The main inequalities which have been got are the following:

(1) $F \geqq \Delta^2/\sqrt{3}$ (Pál)[1];

(2) $2F > \Delta D$ (Kubota)[2];

(3) $4F \geqq \Delta L - b\Delta$ when $2\sqrt{3}\,\Delta \leqq L$,

where b is a positive root smaller than $2\Delta/\sqrt{3}$ of the equation

$2Lx^3-(L^2-\Delta^2)x^2-2L\Delta^2x+L^2\Delta^2=0$ (Yamanouchi)[3];

(4) $4F \geqq (L-2D)\sqrt{4LD-L^2}$, when $2D < L \leqq 3D$ (Kubota)[2];

(5) $2F \geqq (\pi-\sqrt{3})B^2$, when $B=D=\Delta=L/\pi$ (Lebesgue[4][5] and Blaschke)[6];

(6) $4F \geqq (L-2D)\sqrt{3}D$, when $3D \leqq L \leqq \pi D$ (Kubota)[7].

In these inequalities we denote by F the area, by L the perimeter, by D the diameter which is the length of the greatest chord and at the same time the length of the greatest breadth, and by Δ the length of the smallest breadth. The minimum ovals in the cases (1)~(5) are respectively the following:

(1) a regular triangle whose height is Δ when Δ is given;

(2) a triangle whose largest side is of length D and smallest height is Δ, when D and Δ are so given that $\Delta \leqq \sqrt{3}D/2$;
(3) an isosceles triangle whose heights corresponding to equal sides are Δ and whose perimeter is equal to L, when Δ and L are so given that $2\sqrt{3}\Delta \leqq L$;
(4) an isosceles triangle whose two equal sides are of length D and whose perimeter is equal to L, when D and L are so given that $2D < L \leqq 3D$;
(5) the Reuleaux triangle when the constant breadth B is given.

The equality of (6) does not occur unless $L=3D$ and the minimum oval is a regular triangle.

As can easily be seen by considering the above-mentioned inequalities, the problems of minimum figures for the following cases remain unsolved: (1) $\sqrt{3}D/2 < \Delta < D$, (2) $\pi\Delta < L < 2\sqrt{3}\Delta$, (3) $3D < L < \pi D$.

These problems, as Dr. Fujiwara said in his paper, lie beyond the scope of the elementary theory of maxima and minima in the infinitesimal calculus as well as the classical theory of calculus of variations.

The object of this note is to give a method which transforms any oval into an oval having smaller area and the same breadth in every direction by extending Lebesgue's treatment for the curves of constant breadth, and also to give a minimum figure for the cases $\sqrt{3}D/2 \leqq \Delta \leqq D$ and $\pi\Delta \leqq L \leqq 2\sqrt{3}\Delta$.[10]

§ 2. Let $\mathfrak{C}$ be an oval and $\mathfrak{C}'$ be a central symmetrization of $\mathfrak{C}$, in other words, $\mathfrak{C}'$ is a central convex curve which has the same breadth with $\mathfrak{C}$ in every direction; the oval, which is similar to $\mathfrak{C}'$ with the ratio 2 : 1, is called a breadth curve of $\mathfrak{C}$. The following properties of the breadth curve are to be mentioned.

PROPERTY 1. *The breadth curve of $\mathfrak{C}$ with its centre at O is an envelope of such a moving line t that the distance from O to t is equal to the breadth of a pair of supporting lines of $\mathfrak{C}$ parallel to t.*

PROPERTY 2. *Let O be a fixed point, and QQ' be a moving chord joining two supporting points of parallel supporting lines of $\mathfrak{C}$. Then, the locus of such a moving point P as $OP \,\underline{\underline{\parallel}}\, QQ'$ is a breadth curve of $\mathfrak{C}$ whose centre is O. Moreover a supporting line at P of the locus is parallel to a supporting line at Q of $\mathfrak{C}$.*

PROOF. Using the polar tangential coordinates with its origin at O, we write the equation of $\mathfrak{C}$

(1) $$P=P(\theta);$$

then the breadth curve of $\mathfrak{C}$ with its centre at O will be given by

(2) $$P=P(\theta)+P(\theta+\pi).$$

The envelope of $\mathfrak{C}$ will be represented parametrically by

$(3)_1$ $$Z=\{P'(\theta)-iP(\theta)\}e^{i\theta}$$

for any differentiable point θ of $P(\theta)$,

$(3)_2$ $$Z=\{\lambda P'_+(\theta)+(1-\lambda)P'_-(\theta)-iP(\theta)\}e^{i\theta} \qquad (0\leqq\lambda\leqq 1)$$

for any non-differentiable point θ of $P(\theta)$ (that is, a point at which $\mathfrak{C}$ has a rectilinear part parallel to the direction θ),

where Z is the Gaussian coordinate of a point on $\mathfrak{C}$, i the imaginary unit, and $P'_-(\theta)$ and $P'_+(\theta)$ are the left-side and right-side differential coefficients respectively.

The supporting lines at Q and Q' are parallel to each other. Hence, if we put

$$\overrightarrow{OQ}=\{\lambda_0 P'_+(\theta_0)+(1-\lambda_0)P'_-(\theta_0)-iP(\theta_0)\}e^{i\theta},$$

$$\overrightarrow{OQ'}=\{\lambda_1 P'_+(\theta_0+\pi)+(1-\lambda_1)P'_-(\theta_0+\pi)-iP(\theta_0+\pi)\}e^{i(\theta+\pi)},$$

we get

$$\overrightarrow{Q'Q}=\overrightarrow{OQ}-\overrightarrow{OQ'}=[P'_-(\theta_0)+P'_-(\theta_0+\pi)+\lambda_0\{P'_+(\theta_0)-P'_-(\theta_0)\}$$
$$+\lambda_1\{P'_+(\theta_0+\pi)-P'_-(\theta_0+\pi)\}-i\{P(\theta_0)+P(\theta_0+\pi)\}]e^{i\theta_0}.$$

Further, if we put

$$\lambda_0\{P'_+(\theta_0)-P'_-(\theta_0)\}+\lambda_1\{P'_+(\theta_0+\pi)-P'_-(\theta_0+\pi)\}$$
$$=\mu_0\{P'_+(\theta_0)+P'_+(\theta_0+\pi)-P'_-(\theta_0)-P'_-(\theta_0+\pi)\}$$

and $$P(\theta)+P(\theta+\pi)\equiv B(\theta),$$

then μ_0 is a value in the interval from λ_0 to λ_1, and

$$\overrightarrow{Q'Q}=\{\mu_0 B'_+(\theta_0)+(1-\mu_0)B'_-(\theta_0)-iB(\theta_0)\}e^{i\theta_0}$$
$$(0\leqq\mu_0\leqq 1).$$

Therefore, the locus of P such that $\overrightarrow{OP}=\overrightarrow{Q'Q}$ is given by

$$(4)_1 \qquad Z=\{B'(\theta)-iB(\theta)\}e^{i\theta}$$

for any differentiable point θ of $B(\theta)$,

$$(4)_2 \qquad Z=\{\mu B'_+(\theta)+(1-\mu)B'_-(\theta)-iB(\theta)\}e^{i\theta} \qquad (0\leqq\mu\leqq 1)$$

for any non-differentiable point θ of $B(\theta)$.

Comparing (3) with (4), we see that (4) is represented by (2) in the polar tangential equation. That is, the locus of P is the breadth curve (2). Thus, Property 2 is proved.

From this proof, we see

$$\{B'_+(\theta)-B'_-(\theta)\}e^{i\theta}=\{P'_+(\theta)-P'_-(\theta)\}e^{i\theta}-\{P'_+(\theta+\pi)-P'_-(\theta+\pi)\}e^{i(\theta+\pi)}.$$

Therefore, we get the following

PROPERTY 3. *Let $\mathfrak{B}$ be a breadth curve of $\mathfrak{C}$. If $\mathfrak{B}$ has a pair of parallel rectilinear parts of length l, then $\mathfrak{C}$ has two parallel rectilinear parts* (*one of them may shrink to a point*) *and the sum of their lengths is equal to l.*

If $\mathfrak{B}_0$ be a breadth curve whose centre is O, P_1 be a point on $\mathfrak{B}_0$ and $\mathfrak{B}_1$ be the translation of $\mathfrak{B}_0$ by $\overrightarrow{OP_1}$, then, since the breadth curve is central, $\mathfrak{B}_1$ passes through O. Further, let us denote by P_2 one of the intersection of $\mathfrak{B}_0$ and $\mathfrak{B}_1$, by P_2P_3 a chord of $\mathfrak{B}_0$ parallel to OP_1 and by P_4, P_5 and P_6 symmetrical points of P_1, P_2 and P_3 with respect to O respectively. Then P_4, P_5 and P_6 lie on $\mathfrak{B}_0$, and

$$\overrightarrow{P_5P_6}=\overrightarrow{P_3P_2}=\overrightarrow{OP_1}.$$

Therefore P_6 is another intersection of $\mathfrak{B}_0$ and $\mathfrak{B}_1$.

Similarly, if we denote by $\mathfrak{B}_2$ the translation of $\mathfrak{B}_0$ by $\overrightarrow{OP_2}$, then $\mathfrak{B}_2$ is carried by the translation $\overrightarrow{P_2P_1}$ into $\mathfrak{B}_1$; hence $\mathfrak{B}_2$ passes through each of O, P_1 and P_3. Thus we get the congruent relations

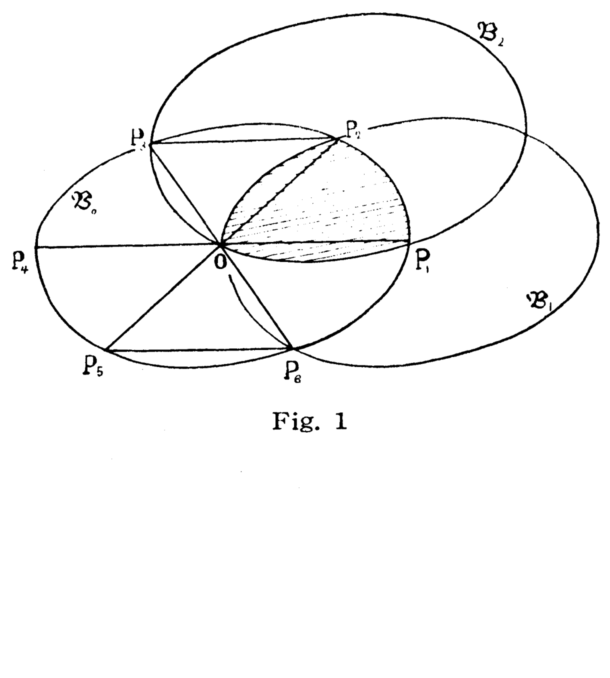

Fig. 1

(1) $$\text{sector } O\widehat{P_2P_3} = \text{sector } O\widehat{P_5P_6} = \text{sector } P_2\widehat{O}P_1,$$

(2) $$\text{sector } O\widehat{P_3P_4} = \text{sector } P_1\widehat{P_2O},$$

where the sector $O\widehat{P_2P_3}$ represents a domain enclosed by two radii OP_2, OP_3 and the minor arc $\widehat{P_2P_3}$ of $\mathfrak{B}_0$.

By (1), (2) and Property 1, the common part of three convex domains $\mathfrak{B}_0$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ (the shaded domain in Fig. 1), is an oval having the breadth assigned by $\mathfrak{B}_0$ in every direction. Similarly we get several ovals using $\mathfrak{B}_i$ and $\mathfrak{B}_{i+1}$ where $\mathfrak{B}_i$ denotes the translation of $\mathfrak{B}_0$ by $\overrightarrow{OP_i}$, but they are all congruent to each other. Therefore one of them will be called "the asymmetric oval" determined by a hexagon $P_1P_2P_3P_4P_5P_6$ and $\mathfrak{B}_0$, and $P_1P_2P_3P_4P_5P_6$ "the base hexagon".

From the construction of $P_1P_2P_3P_4P_5P_6$, we see that any convex polygon $P_1P_2P_3P_4P_5P_6$ inscribed in a breadth curve $\mathfrak{B}$ can be taken as the base hexagon when and only when it is symmetrical with respect to the centre of $\mathfrak{B}$ and

$$P_2P_3 = \frac{1}{2} P_1P_4 .$$

§ 3. We shall now proceed to prove the following

THEOREM 1. *Let $\mathfrak{B}$ be a breadth curve and $\mathfrak{S}$ be a hexagon of the largest area among all base hexagons inscribed in $\mathfrak{B}$. Then, the asymmetric oval determined by $\mathfrak{S}$ and $\mathfrak{B}$ is of the smallest area among all ovals with the breadth assigned by $\mathfrak{B}$ in every direction.*

PROOF. First we shall consider a case when the breadth curve $\mathfrak{B}$ has no angular point and denote by $\mathfrak{C}$ an oval whose breadth curve is $\mathfrak{B}$.

If we put the equations of $\mathfrak{B}$ as

$$x = X(s), \qquad y = Y(s),$$

where x and y are rectangular coordinates of a point on the curve and the parameter s is the curve length. Assume further that P_1, P_2 and P_3 are three vertices of a base hexagon corresponding to $s=s_1$, $s=s_2$ and $s=s_3$ and denote by θ_1, θ_2 and θ_3 three directions of supporting lines of the breadth curve at the respective points such that $\theta_1 \leqq \theta_2 \leqq \theta_3 \leqq \theta_1 + \pi$. Then, from the conditions of base hexagon, we get

$$X(s_1)=X(s_2)-X(s_3)\,, \qquad Y(s_1)=Y(s_2)-Y(s_3)\,,$$

whose Jacobian is

$$\frac{\partial(X(s_1),\,Y(s_1))}{\partial(s_2,\,s_3)}=\begin{vmatrix}\cos\theta_2 & -\cos\theta_3\\ \sin\theta_2 & -\sin\theta_3\end{vmatrix}=\sin(\theta_2-\theta_3)\,. \tag{1}$$

Thus we get the following two cases:

1) whatever the base hexagon may be chosen, Jacobian (1) is not equal to zero;
2) for a special base hexagon Jacobian (1) is equal to zero.

In the case 1), not only $\sin(\theta_2-\theta_3)\neq 0$ but also $\sin(\theta_3-\theta_1)\neq 0$, $\sin(\theta_1-\theta_2)\neq 0$, and s_2 and s_3 will be expressed by continuous functions of s_1. Therefore θ_1, θ_2 and θ_3 vary continuously when a base hexagon $\mathfrak{S}$, inscribed in $\mathfrak{B}$, moves continuously.

We construct supporting lines of $\mathfrak{C}$ parallel to those of $\mathfrak{B}$ at vertices of $\mathfrak{S}$ and denote by $\mathfrak{S}'$ the convex hexagon surrounded by these supporting lines of $\mathfrak{C}$. If we write the equation of $\mathfrak{C}$ as

$$P=P(\theta)\,,$$

then the rectangular coordinates of vertices of $\mathfrak{S}'$ will be given by

$$x=\{P(\theta_2)\cos\theta_3-P(\theta_3)\cos\theta_2\}/\sin(\theta_2-\theta_3)\,,$$

$$y=\{P(\theta_2)\sin\theta_3-P(\theta_3)\sin\theta_2\}/\sin(\theta_2-\theta_3)\,, \qquad \text{etc.}$$

Since $\sin(\theta_2-\theta_3)\neq 0$ etc., $\mathfrak{S}'$ is not a parallelogram and the vertices of $\mathfrak{S}'$ move continuously as $\mathfrak{S}$ rotates continuously.

If we denote by $A_1, A_2, A_3, A_4, A_5, A_6$ the vertices of $\mathfrak{S}'$ and by A_{25} the mid point of A_2A_5, then the oriented area of the triangle $A_1A_4A_{25}$ varies continuously as $\mathfrak{S}'$ moves continuously. Moreover, if $\mathfrak{S}'$ rotates by π around $\mathfrak{C}$, A_2 and A_5 interchange their positions and the sign of the area $A_1A_4A_{25}$ changes. Therefore, there is at least one hexagon $A_1A_2A_3A_4A_5A_6$ such that

$$\text{the area of } A_1A_4A_{25}=0\,.$$

Hence A_1, A_4 and A_{25} are collinear. In the hexagon thus obtained, $A_1A_2A_4A_5$ is a parallelogram since $A_1A_2 \perp\!\!\!\perp A_4A_5$. From $A_6A_1 \| A_4A_3$, $A_6A_5 \| A_2A_3$ we see that $\Delta A_6A_1A_5 \equiv \Delta A_3A_4A_2$. Therefore $A_1A_2A_3A_4A_5A_6$ is a central hexagon. If we denote by $P_1P_2P_3P_4P_5P_6$ the base hexagon corresponding to $A_1A_2A_3A_4A_5A_6$ in the construction of $\mathfrak{S}'$, then, by

Property 2, there are six supporting points $Q_1, Q_2, \cdots, Q_6$ such that Q_i $(i=1,2,\cdots,6)$ lies on A_iA_{i+1} and

$$\overrightarrow{Q_1Q_4}=\overrightarrow{P_1O}\,, \qquad \overrightarrow{Q_5Q_2}=\overrightarrow{OP_2}\,, \qquad \overrightarrow{Q_3Q_6}=\overrightarrow{OP_6}\,(=\overrightarrow{P_2P_1})\,.$$

Therefore

$$\overrightarrow{Q_1Q_4}+\overrightarrow{Q_5Q_2}+\overrightarrow{Q_3Q_6}=0\,. \tag{2}$$

Draw two lines A_1A_3', A_1A_5' parallel to Q_6Q_3, Q_1Q_4 and let them cut A_3A_4, A_4A_5 in A_3', A_5' respectively (see the left side in Fig. 2). Then we have[8]

$$\overrightarrow{A_3'A_1}=\overrightarrow{Q_3Q_6}\,, \tag{3}$$

$$\overrightarrow{A_1A_5'}=\overrightarrow{Q_1Q_4}\,, \tag{4}$$

and

$$\overrightarrow{A_3'A_1}+\overrightarrow{A_1A_5'}+\overrightarrow{A_5'A_3'}=0\,. \tag{5}$$

By (2)~(5) we get

$$\overrightarrow{Q_5Q_2}=\overrightarrow{A_5'A_3'}\,. \tag{6}$$

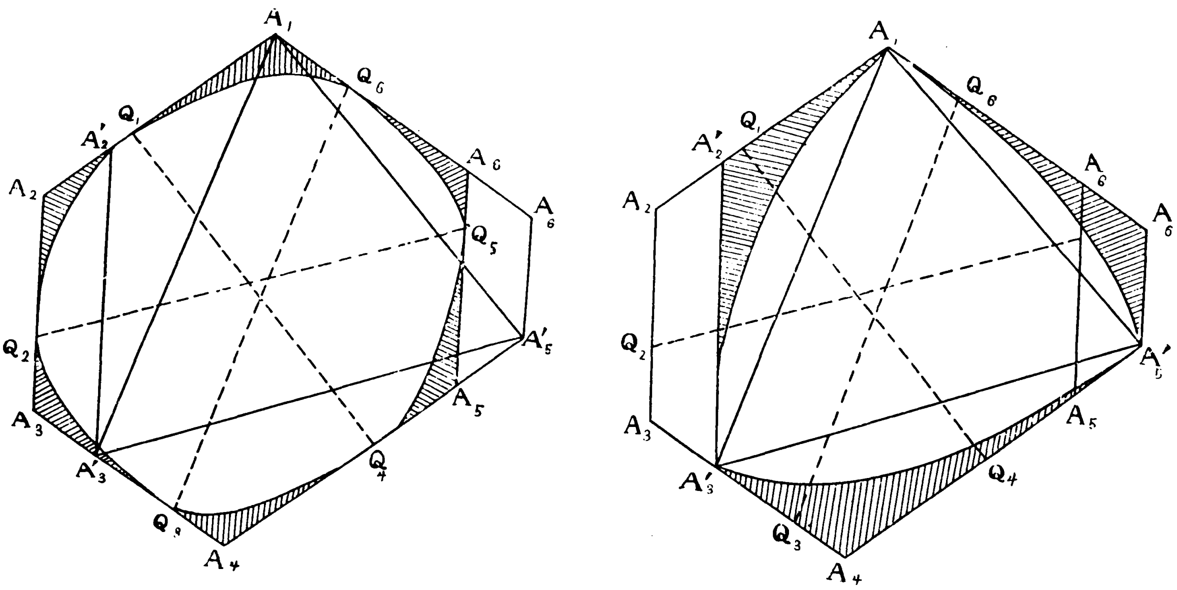

Fig. 2

Next draw $A_3'A_2'$ and $A_5'A_6'$ parallel to A_2A_3, and let them meet A_1A_2 and A_1A_6 in A_2', A_6' respectively. Then, by (6) two pairs of parallels $A_2'A_3'$, $A_6'A_5'$ and A_2A_3, A_6A_5 have the same breadth. Therefore, using $A_2A_3=A_6A_5$, we get

(7) $$\text{area } A_1A_2A_3A_4A_5A_6 \geqq \text{area } A_1A_2'A_3'A_4A_5'A_6'.$$

On the other hand, by (3), (4) and Property 2, the breadth curve of $\mathfrak{C}$ whose centre is A_1 passes through A_3' and A_5. Similarly A_5' and A_1 (or A_1 and A_3') lie on the breadth curve having its centre at A_3' (or A_5'). We denote by $(A_1\widehat{A_3'A_5'})$ the area bounded by two segments A_1A_3', A_1A_5' and the arc $\widehat{A_3'A_5'}$ of breadth curve and by $(A_1\widehat{Q_1Q_6})$ the area bounded by two segments A_1Q_1, A_1Q_6 and the arc $\widehat{Q_1Q_6}$ of $\mathfrak{C}$. Then, by the same method with Lebesgue's treatment for the curve of constant breadth we get

(8) $$\begin{cases} (A_1\widehat{Q_1Q_6})+(A_4\widehat{Q_4Q_3}) \leqq (A_4\widehat{A_5'A_3'}), \\ (A_3\widehat{Q_3Q_2})+(A_6\widehat{Q_6Q_5}) \leqq (A_6'\widehat{A_1A_5}), \\ (A_5\widehat{Q_5Q_4})+(A_2\widehat{Q_2Q_1}) \leqq (A_2'\widehat{A_3'A_1}). \end{cases}$$

Therefore by (7) and (8), the area of the domain bounded by three arcs $\widehat{A_1A_3'}$, $\widehat{A_3'A_5'}$ and $\widehat{A_5'A_1}$ is not greater than that of $\mathfrak{C}$ (the right side in Fig. 2). This domain, as is clear, is an asymmetric oval. Hence, the area of an asymmetric oval is not greater than the area of an arbitrarily taken oval.

In the case 2), we have $\sin(\theta_2-\theta_3)=0$ where θ_1, θ_2 and θ_3 satisfy $\theta_1 \leqq \theta_2 \leqq \theta_3 \leqq \theta_1+\pi$. Therefore $\theta_2=\theta_3$ or $\theta_2=\theta_3-\pi=\theta_1$. Accordingly, we shall consider the case $\theta_2=\theta_3$ only, without loss of generality.

In this case two supporting lines of $\mathfrak{B}$ at P_2 and P_3 coincide with each other. Therefore the supporting lines of $\mathfrak{B}$ at the vertices of the base hexagon $\mathfrak{S}$ form a parallelogram and at least one pair of opposite sides P_2P_3 and P_5P_6 of the base hexagon coincides with two rectilinear parts of $\mathfrak{B}$.

If we denote by $P_2'P_3'$ the rectilinear part on which the segment P_2P_3 lies, we have

(9) $$P_1O=P_2P_3 \leqq P_2'P_3'.$$

Since the segment $P_2'P_3'$ is a rectilinear part of the breadth curve, there exist two rectilinear parts Q_2Q_3, Q_5Q_6 (one of which may shrink to a point) on the oval $\mathfrak{C}$ such that

(10) $$\overrightarrow{Q_2Q_3}+\overrightarrow{Q_6Q_5}=\overrightarrow{P_2'P_3'}.$$

Draw two parallel supporting lines t, t' of $\mathfrak{E}$ parallel to a supporting line of $\mathfrak{B}$ at P_1; then, by Property 2, there are two supporting points Q_1, Q_4 on t, t' respectively such that

$$\overrightarrow{Q_1Q_4}=\overrightarrow{P_1O}\,. \tag{11}$$

By (9), (10), (11), we see that Q_2Q_3 and Q_6Q_5 are parallel arcs of $\mathfrak{E}$ parallel to Q_1Q_2 and

$$Q_1Q_4 \leqq Q_2Q_3+Q_5Q_6\,.$$

If we use the parallelogram[8)] formed by two pairs of parallels t, t' and Q_2Q_3, Q_5Q_6, then, as can be seen from Fig. 3, we get easily the same conclusion as in the case 1): The area of an asymmetric oval is not greater than the area of an arbirtrarily taken oval.

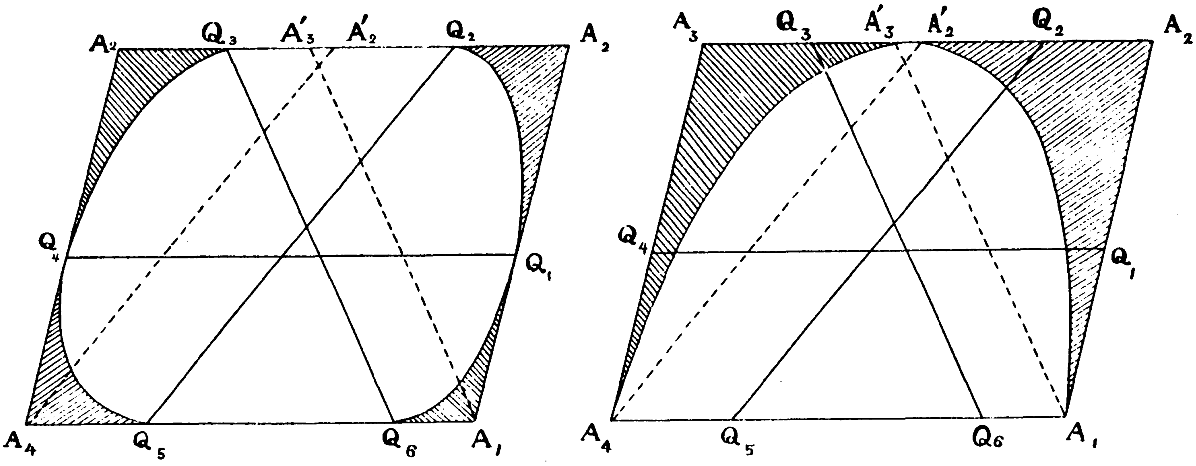

Fig. 3.

On the other hand, the area of an asymmetric oval is given by

$$\frac{1}{2}\begin{pmatrix}\text{the area of}\\ \text{the breadth curve}\end{pmatrix}-\frac{1}{3}\begin{pmatrix}\text{the area of}\\ \text{the base hexagon}\end{pmatrix}.$$

The first term being constant, the oval with minimum area is got when the base hexagon has maximum area. Thus, our theorem is proved when $\mathfrak{B}$ has no angular point.

Let us consider the case in which $\mathfrak{B}$ has angular points.

Let $\mathfrak{E}_\varepsilon$ be an outer ε-parallel curve of $\mathfrak{E}$, $\mathfrak{B}_\varepsilon$ a breadth curve of $\mathfrak{E}_\varepsilon$ and $\mathfrak{S}_\varepsilon$ a base hexagon of the largest area inscribed in $\mathfrak{B}_\varepsilon$. Then

$\mathfrak{B}_\varepsilon$ is an outer 2ε-parallel curve of $\mathfrak{B}$ and has no angular point. Therefore we get

$$(\text{the area of } \mathfrak{C}_\varepsilon) \geqq \frac{1}{2} (\text{the area of } \mathfrak{B}_\varepsilon) - \frac{1}{3} (\text{the area of } \mathfrak{S}_\varepsilon).$$

Let us denote by O the cocentre of $\mathfrak{B}$ and $\mathfrak{B}_\varepsilon$, by $P_1P_2, \cdots, P_6$ the vertices of $\mathfrak{S}_\varepsilon$, and by Q_i $(i=1,2,\cdots,6)$ the meet of OP_i with $\mathfrak{B}$, and assume that

$$OP_1/OQ_1 = \underset{i=1,2,\cdots,6}{\text{Max.}} OP_i/OQ_i.$$

If we construct the hexagon $Q_1Q_2'Q_3'Q_4Q_5'Q_6'$ similar and similarly situated to $\mathfrak{S}_\varepsilon$, then we get

$$OQ_i' \leqq OQ_i \qquad (i=2,3,5,6);$$

therefore $\mathfrak{B}$ enclose the hexagon $Q_1Q_2'Q_3'Q_4Q_5'Q_6'$. If we denote by $Q_1Q_2''Q_3''Q_4Q_5''Q_6''$ the base hexagon with the diagonal Q_1Q_4 inscribed in $\mathfrak{B}$, then

$$Q_2'Q_3' = \frac{1}{2} Q_1Q_4 = Q_2''Q_3'',$$

and therefore the breadth between $Q_2'Q_3'$ and $Q_6'Q_5'$ is not greater than the breadth between $Q_2''Q_3''$ and $Q_6''Q_5''$. Accordingly, the area of $Q_1Q_2'Q_3'$ $Q_4Q_5'Q_6'$ is not greater than the area of $Q_1Q_2''Q_3''Q_4Q_5''Q_6''$. Hence, if we denote by P any point on $\mathfrak{B}_\varepsilon$, by Q the meet of OP with $\mathfrak{B}$ and by λ_ε the maximum of $(OP:OQ)^2$, then we get

$$(\text{the area of } \mathfrak{S}_\varepsilon) \leqq (OP_1/OQ_1)^2 (\text{the area of } Q_1Q_2''Q_3''Q_4Q_5''Q_6'')$$
$$\leqq \lambda_\varepsilon (\text{the area of } \mathfrak{S})$$

where $\mathfrak{S}$ is the maximal base hexagon inscribed in $\mathfrak{B}$. Thus we get

$$(\text{the area of } \mathfrak{C}_\varepsilon) \geqq \frac{1}{2} (\text{the area of } \mathfrak{B}_\varepsilon) - \frac{1}{3} \lambda_\varepsilon (\text{the area of } \mathfrak{S}).$$

If ε tends to zero, then $\mathfrak{C}_\varepsilon$ and $\mathfrak{B}_\varepsilon$ converge to $\mathfrak{C}$ and $\mathfrak{B}$ respectively and λ_ε converges to 1. Therefore, the right side of the last inequality converges to the area of asymmetric oval determined by $\mathfrak{B}$ and $\mathfrak{S}$. Thus, we get the result that the asymmetric oval determined by $\mathfrak{B}$ and $\mathfrak{S}$ is of the smallest area among all ovals with the breadth assigned by $\mathfrak{B}$ in every direction.

§ 4. By virtue of Theorem 1, in order to find the minimum of all ovals with given D and Δ or perimeter L, it will be sufficient to consider the case of asymmetric ovals only, that is, the minimum problems reduce to the problems concerning central ovals $\mathfrak{B}$ and central hexagons $\mathfrak{S}$.

The following form of defining D and Δ is suitable for later purposes in the study of asymmetric ovals. D is the radius of the circumscribed circle about $\mathfrak{B}$ concentric with $\mathfrak{B}$ and Δ is the radius of the inscribed circle in $\mathfrak{B}$ concentric with $\mathfrak{B}$. By Crofton's theorem, the perimeter of $\mathfrak{B}$ is $2L$.

The following lemmas are simple but important.

LEMMA 1. *If we replace two minor arcs of the breadth curve by two parallel chords, we get another asymmetric oval, which has a smaller area and perimeter than the original oval, where the parallel chords are assumed not to meet the base hexagon and that they are symmetric with respect to the centre of the breadth curve.*

LEMMA 2. *Let us denote by $\mathfrak{B}$ and $P_1P_2P_3P_4P_5P_6$ the breadth curve and the base hexagon of the given asymmetric oval $\mathfrak{E}$. Draw four lines P_2P_2', P_3P_3', P_5P_5', P_6P_6' parallel to the supporting line of $\mathfrak{B}$ at P_1, and choose four points P_2', P_3', P_5' and P_6' respectively on them such that $P_2'P_3'$ lies on the opposite side of P_1P_4 with respect to P_2P_3 and $P_5'P_6'$ on the opposite side of P_1P_4 with respect to P_5P_6, that $\overrightarrow{P_2P_2'}=\overrightarrow{P_3P_3'}=\overrightarrow{P_5'P_5}=\overrightarrow{P_6'P_6}$, and that $P_2'P_3'$ meets or touches $\mathfrak{B}$. Then, an asymmetric oval, whose base hexagon is $P_1P_2'P_3'P_4P_5'P_6'$ and whose breadth curve is the minimum oval enclosing $P_2'P_3'P_5'P_6'$ and $\mathfrak{B}$, has a smaller area than $\mathfrak{E}$.*

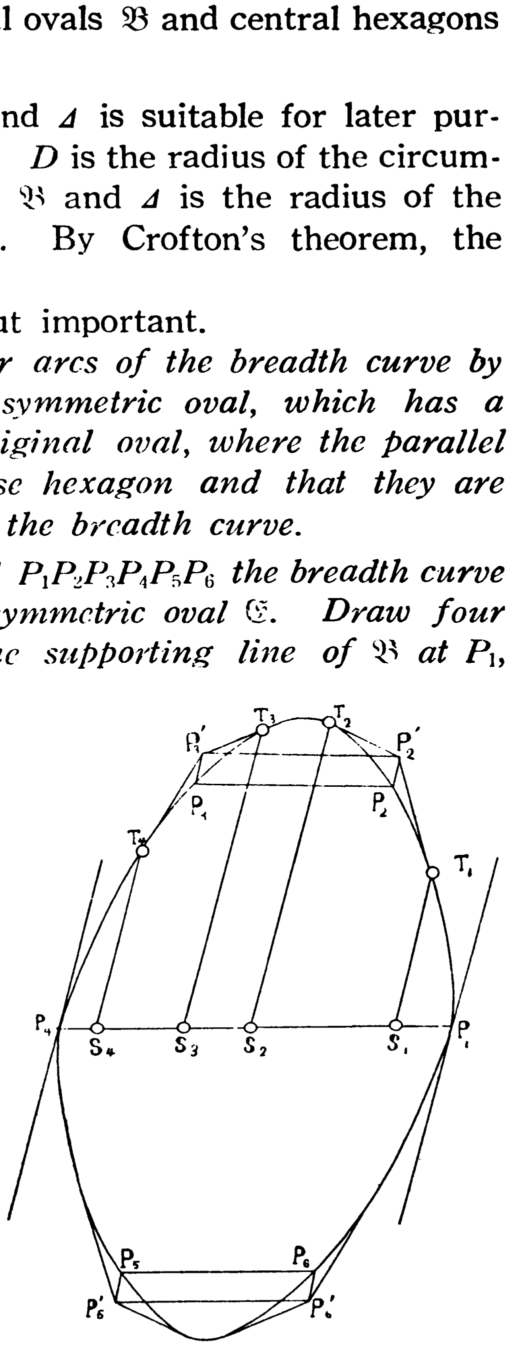

Fig. 4.

PROOF. Draw four tangent lines $P_2'T_1$, $P_2'T_2$; $P_3'T_3$, $P_3'T_4$ from P_2' and P_3' to $\mathfrak{B}$ and let them touch at T_1, T_2, T_3, T_4 respectively. We denote by $\mathfrak{B}'$ the minimum oval enclosing both

$P_2'P_3'P_5'P_6'$ and $\mathfrak{B}$, and by S_i $(i=1,2,3,4)$ the intersections of P_1P_4 with four parallels passing through A_i $(i=1,2,3,4)$ and parallel to P_2P_2'. It is clear that the hexagon $P_1P_2'P_3'P_4P_5'P_6'$ can be taken as a base hexagon. If we express the area of X by $|X|$ symbolically, then

$$\frac{1}{2}\{|\mathfrak{B}'|-|\mathfrak{B}|\}\leqq|\text{concave quadrangle } P_2T_1P_2'T_2|$$

$$+|\text{concave quadrangle } T_3P_3'T_4P_3|$$

$$=\frac{1}{2}S_1S_2\cdot P_2P_2'\sin\omega+\frac{1}{2}S_3S_4\cdot P_3P_3'\sin\omega \quad (\omega=\angle P_1B_4T_4)$$

$$=\frac{1}{2}(S_1S_2+S_3S_4)\cdot P_2P_2'\sin\omega$$

$$\leqq\frac{1}{2}P_1P_4\cdot P_2P_2'\sin\omega$$

$$=\frac{1}{3}\{|P_1P_2'P_3'P_4P_5'P_6'|-|P_1P_2P_3P_4P_5P_6|\}.$$

Therefore

$$\frac{1}{2}|\mathfrak{B}'|-\frac{1}{3}|P_1P_2'P_3'P_4P_5'P_6'|\leqq\frac{1}{2}|\mathfrak{B}|-\frac{1}{3}|P_1P_2P_3P_4P_5P_6|$$

Thus Lemma 2 is proved.

In Lemma 2, by making the breadth between $P_2'P_3'$ and $P_6'P_5'$ greater, we can obtain an asymmetric oval of a smaller area. But, in doing so, we must bear in mind that L, D and Δ become greater in general.

Let us consider the minimum area when D and Δ are so given that

$$D\geqq\Delta\geqq\sqrt{3}D/2.$$

Denote by $\mathfrak{C}$ an asymmetric oval satisfying given conditions and by $\mathfrak{K}$ the circle of radius Δ concentric with the breadth curve of $\mathfrak{C}$.

For our case there is at least a pair of points on the breadth curve whose distances from the centre are D. If such points are not the vertices of base hexagon, then, by applying[9)] the method of Lemma 2 to a pair of arcs on which a pair of points above mentioned lies, we get an asymmetric oval whose base hexagon has a pair of sides of length D and has a smaller area than the original oval.

If we denote by $P_1P_2\cdots P_6$ the new base hexagon whose sides P_2P_3 and P_5P_6 are of length D and by $\mathfrak{B}$ the minimum oval enclosing $P_1P_2\cdots P_6$ and $\mathfrak{K}$, then an asymmetric oval determined by $P_1P_2\cdots P_6$ and $\mathfrak{B}$ has smaller area than $\mathfrak{C}$, and D and $\varDelta$ are the same with those of $\mathfrak{C}$.

Transform the hexagon $P_1P_2\cdots P_6$ into $P_1P_2'P_3'P_4P_5'P_6'$ by the symmetrization with respect to the perpendicular bisector of P_1P_4, and denote by $\mathfrak{B}'$ the minimum oval enclosing $P_1P_2'P_3'P_4P_5'P_6'$ and $\mathfrak{K}$. Then we see that

$$|\mathfrak{B}|\geqq|\mathfrak{B}'|, \qquad |P_1P_2P_3P_4P_5P_6|=|P_1P_2'P_3'P_4P_5'P_6'|,$$

hence

$$\frac{1}{2}|\mathfrak{B}'|-\frac{1}{3}|P_1P_2'P_3'P_4P_5'P_6'|\leqq\frac{1}{2}|\mathfrak{B}|-\frac{1}{3}|P_1P_2P_3P_4P_5P_6|,$$

and that $P_1P_2'P_3'P_4P_5'P_6'$ satisfies the condition of base hexagon. Therefore an asymmetric oval determined by $\mathfrak{B}'$ and $P_1P_2'P_3'P_4P_5'P_6'$ has a smaller area than the original oval. Construct the regular hexagon $P_1QRP_4Q'R'$, whose sides are of length D, then QR meets or touches $\mathfrak{K}$, since $\varDelta\geqq\sqrt{3}D/2$.

By Lemma 2, we see that the asymmetric oval whose base hexagon is $P_1QRP_4Q'R'$ and whose breadth curve is a minimum oval enclosing $P_1QRP_4Q'R'$ and $\mathfrak{K}$ has the minimum area, when $D\geqq\varDelta\geqq\sqrt{3}D/2$. Thus we arrive at the following

THEOREM 2. *If D and $\varDelta$ of the oval are given so that*

$$D\geqq\varDelta\geqq\sqrt{3}D/2,$$

then the following inequality holds:

$$F\geqq3\varDelta\left\{\sqrt{D^2-\varDelta^2}+\varDelta\left(\sin^{-1}\frac{\varDelta}{D}-\frac{\pi}{3}\right)\right\}-\frac{\sqrt{3}}{2}D^2.$$

The equality occurs when and only when the oval is an asymmetric curve whose base hexagon is a regular hexagon of sides D and whose

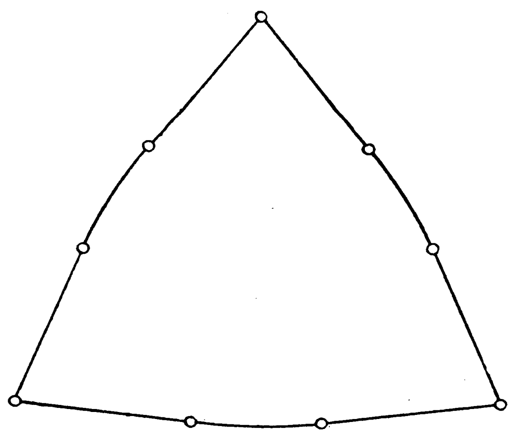

Fig. 5.

breadth curve is a minimum oval enclosing the base hexagon and a concentric circle with radius Δ.

This minimum figure is bounded by three circular arcs and six rectilinear parts, and resembles to the Reuleaux triangle as is seen in Fig. 5.

§ 5. Let us consider the minimum area when Δ and L are so given that

$$\pi\Delta \leqq L \leqq 2\sqrt{3}\Delta .$$

We have only to consider the asymmetric oval in this case too.

Denote by $\mathfrak{E}$ an asymmetric oval satisfying the given conditions, by $P_1P_2P_3P_4P_5P_6$ the base hexagon of $\mathfrak{E}$, by $\mathfrak{B}$ the breadth curve of $\mathfrak{E}$, and by $\mathfrak{K}$ the circle whose radius is Δ and concentric with $\mathfrak{B}$. If every side of the base hexagon has no common point with $\mathfrak{K}$, then the minimum distance of supporting lines of $\mathfrak{B}$ from the centre O is greater than Δ. Therefore at least one pair of sides of base hexagon meets or touches $\mathfrak{K}$. So we may assume that P_2P_3 and P_5P_6 meet or touch $\mathfrak{K}$.

Transform the hexagon $P_1P_2P_3P_4P_5P_6$ into $P_1P'_2P'_3P_4P'_5P'_6$ by the symmetrization with respect to the perpendicular bisector of P_1P_4, and denote by $\mathfrak{B}'$ the minimum oval enclosing $P_1P'_2P'_3P_4P'_5P'_6$ and $\mathfrak{K}$ and by $\mathfrak{E}'$ an asymmetric oval determined by $P_1P'_2P'_3P_4P'_5P'_6$ and $\mathfrak{B}'$. Then $\mathfrak{B}'$ is obtained by applying the method of Lemma 1 to the symmetrization of $\mathfrak{B}$ with respect to the perpendicular bisector of P_1P_4. Therefore we get

$$|\text{hexagon } P_1P'_2P'_3P_4P'_5P'_6| = |\text{hexagon } P_1P_2P_3P_4P_5P_6|,$$

$$|\mathfrak{B}'| \leqq |\mathfrak{B}|,$$

and hence

$$|\mathfrak{E}'| \leqq |\mathfrak{E}|. \tag{1}$$

Steiner's symmetrization and the method of Lemma 1 generally make the perimeter smaller; therefore, if we denote by (X) the perimeter of X symbolically, then we get

$$(\mathfrak{E}') \leqq (\mathfrak{E}), \tag{2}$$

and

$$\Delta\pi \leqq (\mathfrak{E}'), \tag{3}$$

for the length of the smallest breadth of $\mathfrak{E}'$ is $\varDelta$.

By virtue of the assumption concerning P_2P_3, we see that $P_2'P_3'$ and $P_5'P_6'$ meet or touch $\mathfrak{K}$. Then we get the following two cases:

1) every side of $P_1P_2'P_3'P_4P_5'P_6'$ meets or touches $\mathfrak{K}$;
2) two pairs of sides P_1P_2', P_4P_3' and $P_3'P_4$, $P_6'P_5'$ have no common point with $\mathfrak{K}$.

Let us consider the case 2). For this case P_1P_2', $P_3'P_4$, P_4P_5' and $P_6'P_1$ are rectilinear parts of $\mathfrak{B}'$. Denote by $P_2'A_1$ and A_2P_3' the other rectilinear parts of $\mathfrak{B}'$ passing through P_2' and P_3' respectively; then the oval $OP_2'\overset{\frown}{A_1A_2}P_3'O$ is an asymmetric oval determined by $P_1P_2'P_3'P_4P_5'P_6'$ and $\mathfrak{B}'$, that is $\mathfrak{E}'$.

Denote by M the middle point of arc $\overset{\frown}{A_1A_2}$, by X a moving point on $P_2'\overset{\frown}{A_1M}$ which is an arc of $\mathfrak{E}'$, and by Y the reflecting point of X with respect to OM. Then the distance from X to OY varies continuously as X moves continuously, and when X comes to P_2' it is greater than $\varDelta$. Accordingly there are two points A, A' on $P_2'\overset{\frown}{A_1A_2}P_3'$ such that they are symmetric with respect to OM and the distance from A to OA' is equal to $\varDelta$.

If we denote by $\mathfrak{E}''$ the convex domain surrounded by OA, OA' and the arc of $\mathfrak{E}'$ joining A, A', then $\mathfrak{E}''$ has a smaller area and perimeter than those of $\mathfrak{E}'$. Further we see that $\mathfrak{E}''$ is an asymmetric oval and every side of the base hexagon of $\mathfrak{E}''$ meets or touches $\mathfrak{K}$. Therefore, by substituting $\mathfrak{E}''$ for $\mathfrak{E}'$, we see that the case 2) reduces to the case 1).

Let us consider the case 1). In this case every side of $P_1P_2'P_3'P_4P_5'P_6'$ meets or touches $\mathfrak{K}$. Draw two pairs of tangents P_1Q, P_1Q' and $P_2'R, P_2'R'$, and let them touch $\mathfrak{K}$ at Q, Q' and R, R' respectively. If we put

$$\angle P_1OQ=\theta_0, \qquad \angle P_2'OR=\varphi_0, \qquad (\mathfrak{E}')=l,$$

then we have

$$|\mathfrak{E}'|=(\varDelta l-\varDelta^2\sec\theta_0\sqrt{4\sec^2\varphi_0-\sec^2\theta_0})/2, \tag{4}$$

$$l=2\varDelta\{2(\tan\varphi_0-\varphi_0)+\tan\theta_0-\theta_0+\pi/2\} \tag{5}$$

$$(\pi\varDelta\leqq l\leqq L\leqq 2\sqrt{3}\varDelta).$$

Now let us consider the maximum value of u where

$$u=\sec^2 x(4\sec^2 y-\sec^2 x), \tag{6}$$

when x and y are connected by

$$2(\tan y-y)+\tan x-x=(l-\pi\Delta)/(2\Delta) \tag{7}$$

$$(0\leqq x<\pi/2, \qquad 0\leqq y<\pi/2).$$

By (7), y is a continuous decreasing function of x, and

$$\frac{dy}{dx}=-\frac{1}{2}\cot^2 y\tan^2 x,$$

$$\frac{du}{dx}=4\sec^3 x\tan x\operatorname{cosec} y(\sec^2 y+\tan^2 y+\tan x\tan y)\sin(y-x).$$

On the other hand, if we put $y=x$ in (7), we get

$$\tan x-x=(l-\pi\Delta)/(6\Delta).$$

This equation has only one root in the interval $(0, \pi/2)$. Denote this root by θ_1 and the value of x corresponding to $y=0$ by α; then we see

$$y-x>0 \quad \text{when} \quad 0\leqq x<\theta_1,$$

$$y-x<0 \quad \text{when} \quad \theta_1<x<\alpha,$$

and therefore

$$\frac{du}{dx}=0 \quad \text{when} \quad x=0,$$

$$\frac{du}{dx}>0 \quad \text{when} \quad 0<x<\theta_1,$$

$$\frac{du}{dx}<0 \quad \text{when} \quad \theta_1<x<\alpha.$$

Consequently a maximum of u is got when $x=\theta_1$ and hence $x=y=\theta_1$. So we have

$$\sec\theta_0\sqrt{4\sec^2\varphi_0-\sec^2\theta_0}\leqq\sqrt{3}\sec^2\theta_1,$$

$$|\mathfrak{C}'|\geqq\Delta(l-\sqrt{3}\Delta\sec^2\theta_1)/2, \tag{8}$$

$$\tan\theta_1-\theta_1=(l-\pi\Delta)/6\Delta. \tag{9}$$

When θ_1 and l vary under the condition (9), we have

$$\frac{dl}{d\theta_1}=6\Delta\tan^2\theta_1\geqq 0\,,$$

$$\frac{d}{d\theta_1}(l-\sqrt{3}\Delta\sec^2\theta_1)=-2\sqrt{3}\Delta\tan\theta_1\left\{\left(\tan\theta_1-\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}\right\}\leqq 0$$

in the interval $0\leqq\theta_1<\pi/2$. Therefore, by making l greater, we can obtain a smaller value of $(l-\sqrt{3}\Delta\sec^2\theta_1)$.

If we denote by θ the root of

$$\tan x-x=(L-\pi\Delta)/6\Delta$$

in the interval $(0,\pi/2)$, then we get

$$0\leqq\theta_1\leqq\theta\leqq\pi/6\,,$$

since $\pi\Delta\leqq l\leqq L\leqq 2\sqrt{3}\Delta$. Therefore

(10) $$|\mathfrak{E}|\geqq|\mathfrak{E}'|\geqq(\Delta L-\sqrt{3}\Delta^2\sec^2\theta)/2\,.$$

Thus we arrive at the following

THEOREM 3. *If Δ and L of the oval are so given that*

$$\pi\Delta\leqq L\leqq 2\sqrt{3}\Delta\,,$$

then the following inequality holds:

$$2F\geqq\Delta L-\sqrt{3}\Delta^2\sec^2\theta\,,$$

where θ is the root of $\tan\theta-\theta=(L-\pi\Delta)/(6\Delta)$ *in the interval* $0\leqq\theta\leqq\pi/6$; *the equality occurs when and only when the oval is an asymmetric curve whose base hexagon is a regular hexagon of sides $\Delta\sec\theta$ and whose breadth curve is a minimum oval enclosing the base hexagon and the concentric circle with radius Δ.*[10]

Notes

(1) J. Pál: Ein Minimumproblem für Ovale. Math. Ann. 83 (1921). Cf. throughout this note as a reference the excellent report by T. Bonnesen u. W. Fenchel, Theorie der konvexen Körper. (Ergebn. d. Math. III 1.) Berlin (1934).

(2) T. Kubota: Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen. Sci. Rep. Tôhoku Univ. 12 (1923).

(3) M. Yamanouchi: Notes on closed convex figures. Proc. Phys.-Math. Soc. Japan 14 (1932).

(4) Lebesgue: Sur le problème des isopérimètres et sur les domaines de largeur constante. Bull. Soc. Math. France C. R. (1914).
(5) H. Lebesgue: Sur quelques questions de minimum, relatives aux courbes orbiformes, et sur leurs rapports avec le calcul des variations. J. Math. Pures Appl. **4** (1921).
(6) W. Blaschke: Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts. Math. Ann. **76** (1915).
(7) T. Kubota: Eine Ungleichheit für Eilinien. Math. Z. **20** (1924).
(8) In 1917 (?) Mr. K. Yanagihara proved the following theorem at the ordinary meeting of Mathematical Institute in Tohoku University: Let E_0, E_1, E_2 be three congruent ovals which are situated in a homogetic positions and touch with each other. If we construct a ring of congruent ovals $E_3, E_4, \cdots$ around E_0, so that E_i, E_{i-1}, E_0 are located in homogetic positions and touch with each other, then E_6 touches E_1, that is, E_7 is the very oval E_1. If we denote the internal common tangent of E_0, E_i by t_i, then, by placing E_1 in a suitable position, we can make the three pairs of opposite sides of the convex hexagon formed by $t_1, t_2, \cdots, t_6$ have respectively equal lengths. Cf. "Sūri Zasso" (Miscellaneous notes in mathematics) 2, Tokyo-Butsuri-Gakko-Zasshi (Journal of Physics School) **26** (1917). The hexagon $A_1A_2\cdots A_6$ in Fig. 3 is applicable to the hexagon in the above mentioned Yanagihara's theorem, and the parallelogram in Fig. 4 is a special case of it.
(9) The determination of the arc to which Lemma 2 is applied, owes to Prof. Hombu. Our previous proof was as follows: If every side of $P_1P_2\cdots P_6$ is not of length D, then by applying the method of Lemma 2, or by repetitions of it to every side of base hexagon, if necessary, we can make two sides of the base hexagon be of length D. But in doing so the area of the asymmetric oval becomes smaller.
(10) We have considered the case when D and Δ or L and Δ are given, and not the case when D and L are so given that $3D \leqq L \leqq \pi D$. For this case, by a property of the convex polygon in Favard's paper (Ann. École Norm. 46, 1929), we get

$$2F \geqq D^2\{[\pi/2\theta]\sin 2\theta + \sin(2\theta[\pi/2\theta]) - \sqrt{3}\} \quad (\geqq DL\cos\theta - \sqrt{3}D^2),$$

where [ ] is Gauss's notation and θ is the root of $[\pi/2\theta]\sin\theta + \cos(\theta[\pi/2\theta]) = L/(2D)$ in the interval $(0, \pi/6)$. The equality occurs when and only when the oval is an asymmetric curve whose base hexagon is a regular hexagon of sides D and whose breadth curve is a regular $6n$-polygon inscribed in a circle with radius D. This inequality differs in some respects from Theorem 2 or 3, for the equality does not occur unless

$$L = 6nD\sin(\pi/6n), \qquad n = 1, 2, 3, \cdots.$$

(*) Read at the annual meeting of the Math. Soc. of Japan held in June 2, 1951.

Added in proof by D. Hemmi.

After we wrote this paper I received D. Ohmann's paper (Math. Z. **55**, 1952. 347-352) and M. Sholander's (Trans. Amer. Math. Soc. **73**, 1952. 139-173). The former does not touch the case $3D < L < \pi D$ and the latter gives the partial results and conjectures a property of the solution. The proofs of Sholander's conjecture and the inequality in Notes (10) may be seen in Bull. Yamagata Univ. (Natural science) **2** (1953) 157-170 and **3** (1953) 1-11.