J. Math. Soc. Japan
Vol. 49, No. 2, 1997

On polynomials which determine holomorphic mappings

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(Received Apr. 28, 1995)

§ 1. Introduction.

We say that two meromorphic functions f and g on $\boldsymbol{C}$ share the value a if the zeros of $f-a$ and $g-a$ ($1/f$ and $1/g$ if $a=\infty$) are the same. In [**N**], R. Nevanlinna showed the following two results:

THEOREM A. *If two distinct nonconstant meromorphic functions on* $\boldsymbol{C}$ *share four values by counting multiplicities, then one is a Möbius transformation of the other.*

THEOREM B. *If two nonconstant meromorphic functions on* $\boldsymbol{C}$ *share five values, then they are identical.*

These results are interesting from the viewpoint of determining meromorphic or holomorphic functions but it is troublesome to check four or five pairs of values of meromorphic or holomorphic functions. Also there are results in [**F1**] and [**F2**] which show the uniqueness of holomorphic mappings into complex projective spaces.

Recently, H.-X. Yi proved the following:

THEOREM C. *Let* n *and* m *be two positive integers such that* n *and* m *have no common factor and* $n>2m+4$. *Let* a *and* b *be two nonzero constants such that the algebraic equation* $P(w)=w^n+aw^{n-m}+b=0$ *has no multiple roots. If two nonconstant entire functions* f *and* g *satisfy* $P(g)=\alpha P(f)$ *for some entire function* α *without zeros, then* $f=g$.

In this, it is enough for determining holomorphic functions to check only one pair of holomorphic functions. So, the author asks the following two questions:

QUESTION 1. *Do there exist polynomials* P_n *of variables* $z_1, \cdots, z_n$ *with the property*:

(P) *if two algebraically nondegenerate holomorphic mappings* f *and* g *of* $\boldsymbol{C}$ *into* $\boldsymbol{C}^n$ *satisfy* $P_n(g)=\alpha P_n(f)$ *for some entire function* α *without zeros, then* $f=g$.

QUESTION 2. *Do there exist homogeneous polynomials* H_n *of variables* $w_0, \cdots, w_n$ *with the following property*:

(H) *if two algebraically nondegenerate holomorphic mappings f and g of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ with reduced representations $\tilde{f}$ and $\tilde{g}$ respectively satisfy $H_n(\tilde{g})=\alpha H_n(\tilde{f})$ for some entire function α without zeros, then $f=g$.*

Though the explanation of the terminologies in the above statements is left to the next section, we see that Yi's result shows the existence of P_1. In this paper, we shall show the existence of H_n, which makes the existence of P_n trivial.

It is assumed that the reader is familier with the fundamental concepts of the value distribution theory (or Nevanlinna theory) of meromorphic functions (see [**H**]).

§ 2. Basic results in the value distribution theory.

We begin to explain terminologies. For a holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$, its representation is a holomorphic mapping $\tilde{f}=(f_0, \cdots, f_n)$ of $\boldsymbol{C}$ into $\boldsymbol{C}^{n+1}$ such that $\tilde{f}(\boldsymbol{C})\neq\{\mathbf{o}\}$ and $f(z)=(f_0(z):\cdots:f_n(z))$ for each $z\in\boldsymbol{C}-\tilde{f}^{-1}(\mathbf{o})$, where $(w_0:\cdots:w_n)$ is a homogeneous coordinate system of $\boldsymbol{P}^n(\boldsymbol{C})$. If $\tilde{f}^{-1}(\mathbf{o})=\emptyset$, we say that $\tilde{f}$ is reduced.

DEFINITION 1. Let f be a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{C}^n$. If there exists no nonzero polynomial P of variables $z_1, \cdots, z_n$ such that $P(f)\equiv 0$, then it is said that f is algebraically nondegenerate.

DEFINITION 2. Let f be a holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ with a representation $\tilde{f}$.

(i) If there exists no nonzero homogeneous polynomial H of variables $w_0, \cdots, w_n$ such that $H(\tilde{f})\equiv 0$, then it is said that f is algebraically nondegenerate.

(ii) If there exists no nonzero linear homogeneous polynomial L of variables $w_0, \cdots, w_n$ such that $L(\tilde{f})\equiv 0$, then it is said that f is linearly nondegenerate.

REMARK. For holomorphic mappings into $\boldsymbol{C}$ or $\boldsymbol{P}^1(\boldsymbol{C})$, algebraic nondegeneracy coincides with nonconstantness.

For a meromorphic function φ and a positive integer p, $N^p(r, \varphi)$ represents the truncated counting function of $N(r, \varphi)$ by p as $\overline{N}(r, \varphi)$, i.e.,

$$N^p(r, \varphi)=\int_{r_0}^{r}\frac{n^p(t, \varphi)}{t}dt \quad (r>r_0),$$

where $n^p(t, \varphi)$ is the sum of minimum of p and multiplicity of pole of φ at each point z in $\{z; |z|<t\}$ and r_0 is a fixed positive number. The following is fundamental:

$$N^p(r, \varphi)\leqq N(r, \varphi)\leqq T(r, \varphi)+O(1). \tag{2.1}$$

Furthermore, for a holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ and a hyperplane H in $\boldsymbol{P}^n(\boldsymbol{C})$ represented by the equation of homogeneous coordinates $a_0w_0+\cdots+a_nw_n=0$, we use $T_f(r)$, $N_{f,H}(r)$, $N^p_{f,H}(r)$ and $S_f(r)$. We explain these. Let $\tilde{f}=(f_0, \cdots, f_n)$ be a reduced representation of f. Then $T_f(r)$ and $N_{f,H}(r)$ are given for $r>r_0$ by

$$T_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log\|\tilde{f}(re^{i\theta})\|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log\|\tilde{f}(r_0e^{i\theta})\|d\theta$$

and

$$N_{f,H}(r)=N(r, 1/F)$$

for $r>r_0$ if $F:=a_0f_0+\cdots+a_nf_n\not\equiv 0$, where $\|\cdot\|$ is the L^2-norm in $\boldsymbol{C}^{n+1}$. We also define

$$N^p_{f,H}(r)=N^p(r, 1/F).$$

The correspondence to (2.1) is

$$N^p_{f,H}(r)\leqq N_{f,H}(r)\leqq T_f(r)+O(1). \tag{2.2}$$

Finally, $S_f(r)$ represents quantities such that

$$S_f(r)=o(T_f(r)) \quad (r\to\infty,\ r\notin E),$$

where E is an exceptional set of (r_0, ∞) with finite linear measure.

THEOREM D ([**C**], [**Sh**], [**St**]). *Let f be a linearly nondegenerate holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ and $H_1, \cdots, H_q$ hyperplanes in general position in $\boldsymbol{P}^n(\boldsymbol{C})$. Then*

$$(q-n-1)T_f(r)\leqq\sum_{j=1}^{q}N^n_{f,H_j}(r)+S_f(r).$$

Also, we will need the following

THEOREM E ([**C**], [**M**]). *Let f be a nonconstant meromorphic function on $\boldsymbol{C}$ and $a_1, \cdots, a_q$ distinct complex numbers. If all the zeros of $f-a_j$ have the multiplicity at least m_j, where m_j are arbitrarily fixed positive integers* $(1\leqq j\leqq q)$, *then*

$$\sum_{j=1}^{q}\left(1-\frac{1}{m_j}\right)\leqq 2.$$

PROOF. In the situation Theorem E, the inequality of Theorem D becomes

$$(q-2)T_f(r)\leqq\sum_{j=1}^{q}N^1(r, 1/(f-a_j))+S_f(r).$$

By the assumption of multiplicity of zeros of $f-a_j$ and (2.2), we have $N^1(r, 1/(f-a_j))\leqq N(r, 1/(f-a_j))/m_j\leqq T_f(r)/m_j+O(1)$. Hence, we get

$$(q-2)T_f(r) \leqq \sum_{j=1}^{q} \frac{1}{m_j} T_f(r) + S_f(r).$$

This induces the inequality required.

§ 3. Existence of H_1.

THEOREM 1. *Let p and d be two positive integers with $d>2p+8$ and $p \geqq 2$ which have no common factors. Then $H_1(w_0, w_1)=w_0^d+w_0^p w_1^{d-p}+w_1^d$ has the property (H).*

We prove a more precise result:

THEOREM 2. *Let H_1 be the homogeneous polynomial as in Theorem 1. Let f and g be algebraically nondegenerate holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ with reduced representations $\tilde{f}=(f_0, f_1)$ and $\tilde{g}=(g_0, g_1)$ respectively. If*

$$H_1(g_0, g_1) = \alpha H_1(f_0, f_1) \tag{3.1}$$

holds for some entire function α without zeros, then

$$g_0 = \beta f_0 \quad \text{and} \quad g_1 = \beta f_1,$$

where β is an entire function such that $\beta^d=\alpha$.

PROOF. Consider the holomorphic mapping F of $\boldsymbol{C}$ into $\boldsymbol{P}^2(\boldsymbol{C})$ with the reduced representation $\tilde{F}=(g_0^d, (g_0^p+g_1^p)g_1^{d-p}, \alpha f_0^d)$. Since there exist positive constants C_1 and C_2 such that the inequalities

$$1+|w^{d-p}+w^d|^2 \geqq C_1(1+|w^d|^2) \geqq C_2(1+|w|^2)^d$$

hold for all $w \in \boldsymbol{C}$, we get $\|\tilde{F}\| \geqq \sqrt{C_2}\|\tilde{g}\|^d$ and

$$\begin{aligned}\|\tilde{F}\|^2 &\geqq \frac{1}{2}\{|g_0^d+(g_0^p+g_1^p)g_1^{d-p}-\alpha f_0^d|^2+|\alpha f_0^d|^2\} \\ &= \frac{1}{2}\{|\alpha(f_0^p+f_1^p)f_1^{d-p}|^2+|\alpha f_0^d|^2\} \\ &\geqq \frac{C_2}{2}(|\alpha|\cdot\|\tilde{f}\|^d)^2.\end{aligned}$$

These induce

$$\log\|\tilde{F}\| \geqq \frac{d}{2}(\log\|\tilde{f}\|+\log\|\tilde{g}\|)+\frac{1}{2}\log|\alpha|+O(1)$$

and

$$T_F(r) \geqq \frac{d}{2}(T_f(r)+T_g(r))+O(1). \tag{3.2}$$

Also, by considering the hyperplanes $H_j: w_0+a_jw_1=0$ $(1\leqq j\leqq d)$ in $\boldsymbol{P}^1(\boldsymbol{C})$, where a_j are defined by the factorization $w_0^d+w_0^pw_1^{d-p}+w_1^d=\prod_{j=1}^d(w_0+a_jw_1)$, we have, by Theorem D and (2.2),

$$\begin{aligned}(d-2)T_g(r) &\leqq \sum_{j=1}^d N_{g,H_j}(r)+S_g(r)\\ &=\sum_{j=1}^d N\Big(r, \frac{1}{g_0+a_jg_1}\Big)+S_g(r)\\ &=\sum_{j=1}^d N\Big(r, \frac{1}{f_0+a_jf_1}\Big)+S_g(r) \leqq dT_f(r)+S_g(r) \qquad (3.3)\end{aligned}$$

and, similarly,

$$(d-2)T_f(r)\leqq dT_g(r)+S_f(r). \qquad (3.4)$$

Assume that F is linearly nondegenerate and consider hyperplanes

$$\tilde{H}_1: w_0=0,\quad \tilde{H}_2: w_1=0,\quad \tilde{H}_3: w_2=0$$

and

$$\tilde{H}_4: w_0+w_1-w_2=0$$

in $\boldsymbol{P}^2(\boldsymbol{C})$. Then by Theorem D, we have

$$\begin{aligned}T_F(r) &\leqq \sum_{j=1}^4 N_{F,\tilde{H}_j}^2(r)+S_F(r)\\ &=N^2\Big(r, \frac{1}{g_0^d}\Big)+N^2\Big(r, \frac{1}{(g_0^p+g_1^p)g_1^{d-p}}\Big)\\ &\quad +N^2\Big(r, \frac{1}{\alpha f_0^d}\Big)+N^2\Big(r, \frac{1}{\alpha(f_0^p+f_1^p)f_1^{d-p}}\Big)+S_F(r)\\ &\leqq 2\Big(N\Big(r, \frac{1}{g_0}\Big)+N\Big(r, \frac{1}{g_1}\Big)+N\Big(r, \frac{1}{f_0}\Big)+N\Big(r, \frac{1}{f_1}\Big)\Big)\\ &\quad +N\Big(r, \frac{1}{g_0^p+g_1^p}\Big)+N\Big(r, \frac{1}{f_0^p+f_1^p}\Big)+S_F(r)\\ &\leqq (p+4)(T_f(r)+T_g(r))+S_F(r). \qquad (3.5)\end{aligned}$$

It follows from (3.2), (3.3), (3.4) and (3.5) that $S_F(r)$ is $S_f(r)$ and also $S_g(r)$, and that $d/2\leqq p+4$, which contradicts to $d>2p+8$. Hence we conclude that F is linearly degenerate. So there exist constants c_0, c_1 and c_2 such that $(c_0, c_1, c_2)\neq(0, 0, 0)$ and that

$$c_0g_0^d+c_1(g_0^p+g_1^p)g_1^{d-p}+c_2\alpha f_0^d=0. \qquad (3.6)$$

Because g is nonconstant, $c_2\neq 0$.

First, we assume that $c_1=0$. In this case, noting $c_2\neq 0$, we have

$$\frac{c_0}{c_2} g_0{}^d = -\alpha f_0{}^d. \tag{3.7}$$

By adding this to (3.1) on each side, we have also

$$\left(1+\frac{c_0}{c_2}\right) g_0{}^d + (g_0{}^p + g_1{}^p) g_1{}^{d-p} = \alpha (f_0{}^p + f_1{}^p) f_1{}^{d-p}. \tag{3.8}$$

Furthermore, if $1+c_0/c_2 \neq 0$, then consider the nonconstant holomorphic mapping $h=((1+c_0/c_2)g_0{}^d : (g_0{}^p+g_1{}^p)g_1{}^{d-p})$ of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ and hyperplanes

$$\hat{H}_1 : w_0 = 0, \quad \hat{H}_2 : w_1 = 0 \quad \text{and} \quad \hat{H}_3 : w_0 + w_1 = 0$$

in $\boldsymbol{P}^1(\boldsymbol{C})$. By Theorem D, we have

$$\begin{aligned} dT_h(r) + O(1) &= dT_g(r) \\ &\leqq N^1\left(r, \frac{1}{g_0{}^d}\right) + N^1\left(r, \frac{1}{(g_0{}^p+g_1{}^p)g_1{}^{d-p}}\right) \\ &\quad + N^1\left(r, \frac{1}{(f_0{}^p+f_1{}^p)f_1{}^{d-p}}\right) + S_g(r) \\ &\leqq N\left(r, \frac{1}{g_0}\right) + N\left(r, \frac{1}{g_0{}^p+g_1{}^p}\right) + N\left(r, \frac{1}{g_1}\right) \\ &\quad + N\left(r, \frac{1}{f_0{}^p+f_1{}^p}\right) + N\left(r, \frac{1}{f_1}\right) + S_g(r) \\ &\leqq (1+p)T_f(r) + (2+p)T_g(r) + S_g(r). \end{aligned}$$

Hence we get $d \leqq (d/(d-2))(p+1)+(p+2)$ by (3.4). This and $d > 2p+8$ induce

$$d < \frac{d}{d-2}\left(1+\frac{d-8}{2}\right) + 2 + \frac{d-8}{2} = d - 4 - \frac{4}{d-2} < d,$$

which is a contradiction. Hence $1+c_0/c_2=0$. So we get two identities

$$(g_0{}^p + g_1{}^p) g_1{}^{d-p} = \alpha (f_0{}^p + f_1{}^p) f_1{}^{d-p} \tag{3.9}$$

and

$$g_0{}^d = \alpha f_0{}^d, \tag{3.10}$$

from (3.7) and (3.8). Put $\varphi = f_1/f_0$ and $\psi = g_1/g_0$, then it follows that from above two identities that $\varphi^{d-p}+\varphi^d = \psi^{d-p}+\psi^d$. In the deformation

$$\frac{(\varphi/\psi)^d - 1}{(\varphi/\psi)^{d-p} - 1} = -\frac{1}{\psi^p}$$

of this, the multiplicity of each zero of $\varphi/\psi - a$ is a multiple of p, where a $(\neq 1)$ is any dth root of 1 or $(d-p)$th root of 1. We have used the assumption that d and $d-p$ are relatively prime. If φ/ψ is nonconstant, then Theorem E claims that $(d+(d-p)-2)(1-1/p) \leqq 2$. However, by the assumptions $d > 2p+8$ and $p \geqq 2$, we see that the left hand side is greater than 2, which is a contradiction. Hence φ/ψ is constant. Since (3.10) gives $g_0=\alpha_0 f_0$, we further get $g_1=\alpha_1 f_1$, where $\alpha_0{}^d=\alpha$ and α_1 is a non-zero constant multiple of α_0. Substituting these into (3.1) gives

$$(\alpha-\alpha_0{}^p\alpha_1{}^{d-p})f_0{}^p f_1{}^{d-p}+(\alpha-\alpha_1{}^d)f_1{}^d=0.$$

Since f is nonconstant, we get $\alpha=\alpha_0{}^p\alpha_1{}^{d-p}$ and $\alpha=\alpha_1{}^d$. These and $\alpha=\alpha_0{}^d$ induce that $\alpha_0=\alpha_1$, and so $f=g$.

Secondly, we consider the case $c_0=0$. We may assume that $c_1=1$ without loss of generality. Then (3.6) becomes

$$(g_0{}^p+g_1{}^p)g_1{}^{d-p}+c_2\alpha f_0{}^d=0.$$

From this we see that the multiplicity of each zero of g_1 and g_0-ag_1 is a multiple of d, where a is any pth root of -1. By Theorem E, we get $(1+p)(1-1/d)\leqq 2$. However, the left hand side is greater than 2, which is a contradiction. So $c_0=0$ is impossible.

Finally, we consider the case that any $c_j\neq 0$ $(j=0, 1, 2)$. Consider the holomorphic mapping $(c_0g_0{}^d : c_1(g_0{}^p+g_1{}^p)g_1{}^{d-p})$ of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$. By Theorem D, we have

$$\begin{aligned} dT_g(r) &\leqq N^1\Big(r, \frac{1}{g_0{}^d}\Big)+N^1\Big(r, \frac{1}{(g_0{}^p+g_1{}^p)g_1{}^{d-p}}\Big)+N^1\Big(r, \frac{1}{f_0{}^d}\Big)+S_g(r) \\ &\leqq N\Big(r, \frac{1}{g_0}\Big)+N\Big(r, \frac{1}{g_0{}^p+g_1{}^p}\Big)+N\Big(r, \frac{1}{g_1}\Big)+N\Big(r, \frac{1}{f_0}\Big)+S_g(r) \\ &\leqq (2+p)T_g(r)+\frac{d}{d-2}T_g(r)+S_g(r). \end{aligned}$$

Hence we get $d\leqq 2+p+d/(d-2)<p+4$. This contradicts to $d>2p+8$.

After all, if (3.1) holds, then

$$g_0=\beta f_0 \quad \text{and} \quad g_1=\beta f_1,$$

where $\beta=\alpha_0=\alpha_1$ with $\beta^d=\alpha$. Q. E. D.

§4. Existence of H_n.

First, we prove the following lemma:

LEMMA 1. *Let H_1 be the homogeneous polynomial as in Theorem 1. Let f and g be algebraically nondegenerate holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^1(\boldsymbol{C})$ with reduced representations $\tilde{f}=(f_0, f_1)$ and $\tilde{g}=(g_0, g_1)$, respectively. If*

$$H_1(g_0, g_1)=h^d H_1(f_0, f_1) \tag{4.1}$$

holds for some meromorphic function h, then h is an entire function without zeros.

PROOF. We can represent $h=A/B$, where A and B are entire functions without common zeros. Then (4.1) can be replaced by

$$B^d(g_0{}^d+g_0{}^p g_1{}^{d-p}+g_1{}^d) = A^d(f_0{}^d+f_0{}^p f_1{}^{d-p}+f_1{}^d). \tag{4.2}$$

As in the proof of Theorem 2, let consider the holomorphic mapping F of $\boldsymbol{C}$ into $\boldsymbol{P}^2(\boldsymbol{C})$ with the representation $\tilde{F}=(B^d g_0{}^d,\ B^d(g_0{}^p+g_1{}^p)g_1{}^{d-p},\ A^d f_0{}^d)$ which can be proved to be reduced. Then, we have

$$T_F(r) \geqq \frac{d}{2}(T_f(r)+T_g(r)+N(r,\ 1/A)+N(r,\ 1/B))+O(1). \tag{4.3}$$

Assume that F is linearly nondegenerate and consider hyperplanes

$$H_1: w_0=0,\quad H_2: w_1=0,\quad H_3: w_2=0$$

and

$$H_4: w_0+w_1-w_2=0$$

in $\boldsymbol{P}^2(\boldsymbol{C})$. Then by Theorem D, we have

$$\begin{aligned}
T_F(r) &\leqq \sum_{j=1}^{4} N_{F,H_j}^2(r)+S_F(r)\\
&= N^2\Big(r,\ \frac{1}{B^d g_0{}^d}\Big)+N^2\Big(r,\ \frac{1}{B^d(g_0{}^p+g_1{}^p)g_1{}^{d-p}}\Big)\\
&\quad +N^2\Big(r,\ \frac{1}{A^d f_0{}^p}\Big)+N^2\Big(r,\ \frac{1}{A^d(f_0{}^p+f_1{}^p)f_1{}^{d-p}}\Big)+S_F(r)\\
&\leqq 2(N(r,\ 1/g_0)+N(r,\ 1/g_1)+N(r,\ 1/f_0)+N(r,\ 1/f_1))\\
&\quad +4(N(r,\ 1/A)+N(r,\ 1/B))\\
&\quad +N\Big(r,\ \frac{1}{g_0{}^p+g_1{}^p}\Big)+N\Big(r,\ \frac{1}{f_0{}^p+f_1{}^p}\Big)+S_F(r)\\
&\leqq (p+4)(T_f(r)+T_g(r))+4(N(r,\ 1/A)+N(r,\ 1/B))+S_F(r)\\
&\leqq (p+4)(T_f(r)+T_g(r)+N(r,\ 1/A)+N(r,\ 1/B))+S_F(r).
\end{aligned} \tag{4.4}$$

It follows from (4.4) that $S_F(r)$ satisfies

$$S_F(r) = o(T_f(r)+T_g(r)+N(r,\ 1/A)+N(r,\ 1/B)) \quad (r\to\infty,\ r\notin E),$$

where E is a subset of $(r_0,\ \infty)$ with finite linear measure. Hence, we get by (4.3) and (4.4) $d/2\leqq p+4$, which contradicts to $d>2p+8$. Therefore we conclude that F is linearly degenerate. So there exist constants c_0, c_1 and c_2 such that $(c_0,\ c_1,\ c_2)\neq(0,\ 0,\ 0)$ and that

$$B^d(c_0 g_0{}^d+c_1(g_0{}^p+g_1{}^p)g_1{}^{d-p})+c_2 A^d f_0{}^d = 0. \tag{4.5}$$

Since g is nonconstant, $c_2 \neq 0$.

First, let z_0 be a point such that $B(z_0)=0$. Since A and B have no common zeros, we have $f_0=0$ at z_0 by (4.5) and $f_0{}^d+f_0{}^p f_1{}^{d-p}+f_1{}^d=0$ at z_0 by (4.2). However, these imply $f_0(z_0)=f_1(z_0)=0$, which is impossible. Therefore B has no zeros.

Secondly, let z_0 be a point such that $A(z_0)=0$. Then, we have $g_0{}^d+g_0{}^p g_1{}^{d-p}+g_1{}^d=0$ and $c_0 g_0{}^d+c_1(g_0{}^p g_1{}^{d-p}+g_1{}^d)=0$ at z_0. Since $g_0{}^d$ and $g_0{}^p g_1{}^{d-p}+g_1{}^d$ have no common zeros, we get $c_0=c_1 \neq 0$. However, we can see from (4.2), (4.5) and $c_0=c_1$ that f is algebraically degenerate. Therefore A has no zeros. Q. E. D.

THEOREM 3. *Homogeneous polynomials H_n $(n \geqq 2)$ of $w_0, \cdots, w_n$ with degree d^n inductively defined by*

$$H_n(w_0, \cdots, w_n)=H_1(H_{n-1}(w_0, \cdots, w_{n-1}), w_0^{d^{n-1}-1} w_n)$$

have the property (H).

We prove more precisely

THEOREM 4. *Let f and g be algebraically nondegenerate holomorphic mappings of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$ with representations $\tilde{f}=(f_0, \cdots, f_n)$ and $\tilde{g}=(g_0, \cdots, g_n)$ respectively. If*

$$H_n(g_0, \cdots, g_n)=\alpha H_n(f_0, \cdots, f_n) \tag{4.6}$$

holds for some entire function α without zeros, then

$$g_j=\beta f_j \quad (0 \leqq j \leqq n),$$

where β is an entire function such that $\beta^{d^n}=\alpha$.

PROOF. We proceed the proof by induction on n.

For $n=1$, let A and B be entire functions such that $\tilde{f}/A$ and $\tilde{g}/B$ are reduced. Then, (4.6) changes into the form

$$B^d H_1\left(\frac{g_0}{B}, \frac{g_1}{B}\right)=\alpha A^d H_1\left(\frac{f_0}{A}, \frac{f_1}{A}\right).$$

Lemma 1 says that A/B is an entire function without zeros. Hence, we can use Theorem 1 and obtain

$$g_0/B=\beta_1 f_0/A \quad \text{and} \quad g_1/B=\beta_1 f_1/A,$$

where β_1 is an entire function such that $\beta_1{}^d=\alpha(A/B)^d$. Put $\beta=\beta_1 B/A$, then we get the conclusion for $n=1$.

Assume that the result is true for $n-1$ and consider the case for n. Since we can rewrite the identity (4.6) into the form

$$H_1(H_{n-1}(g_0, \cdots, g_{n-1}), g_0^{d^{n-1}-1}g_n) = \alpha H_1(H_{n-1}(f_0, \cdots, f_{n-1}), f_0^{d^{n-1}-1}f_n),$$

it follows from the above result for $n=1$ that

$$H_{n-1}(g_0, \cdots, g_{n-1}) = \beta_1 H_{n-1}(f_0, \cdots, f_{n-1}) \tag{4.7}$$

and

$$g_0^{d^{n-1}-1}g_n = \beta_1 f_0^{d^{n-1}-1}f_n, \tag{4.8}$$

where β_1 is an entire function such that $\beta_1{}^d=\alpha$. By the assumption of induction and (4.7), we have

$$g_j = \beta f_j \quad (0 \leqq j \leqq n-1), \tag{4.9}$$

where β is an entire function such that $\beta^{d^{n-1}}=\beta_1$. By using (4.9) and $\beta^{d^{n-1}}=\beta_1$, we obtain from (4.8)

$$g_n = \beta_1(f_0/g_0)^{d^{n-1}-1}f_n = \beta_1(1/\beta)^{d^{n-1}-1}f_n = \beta_1\beta(1/\beta)^{d^{n-1}}f_n = \beta f_n.$$

Also, we have $\beta^{d^n}=(\beta^{d^{n-1}})^d=\beta_1{}^d=\alpha$ by $\beta^{d^{n-1}}=\beta_1$ and $\beta_1{}^d=\alpha$. Q. E. D.

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