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# Geometry of weakly symmetric spaces

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## 1. Introduction.

Weakly symmetric spaces have been introduced by A. Selberg [**21**] in 1956. His motivation was to generalize the Poisson summation formula to what is now known as the Selberg trace formula. These homogeneous spaces have the property that the differential operators which are invariant under the action of the full isometry group form a commutative algebra, that is, these spaces are commutative. Whereas much work has been done on the harmonic analysis of the commutative and weakly symmetric spaces, in particular on $SL(2, \boldsymbol{R})$, the geometry of the weakly symmetric spaces has to our knowledge not been studied thoroughly. Only a few properties are given in [**3**]. The reason might be that Selberg's definition of weakly symmetric spaces appears to be rather abstract and only a few examples are known. The intention of this note is to point out that there is a nice geometrical characterizing property hidden in Selberg's definition leading to the construction of a whole list of new examples, and to stimulate further research on the Riemannian geometry of this class of spaces.

A Riemannian manifold $M$ is said to be a *weakly symmetric space* if there exists a subgroup $G$ of the isometry group $I(M)$ of $M$ acting transitively on $M$ and an isometry $f$ of $M$ with $f^2 \in G$ and $fGf^{-1}=G$, such that for any two points $p, q \in M$ there exists an isometry $g \in G$ with $g(p)=f(q)$ and $g(q)=f(p)$. By taking $G=I(M)$ and $f=id_M$ it can be seen easily that any Riemannian globally symmetric space is weakly symmetric. The crucial observation for our studies is the following

GEOMETRICAL CHARACTERIZATION OF WEAKLY SYMMETRIC SPACES. *A Riemannian manifold $M$ is weakly symmetric if and only if for any two points $p$, $q$ in $M$ there exists an isometry of $M$ mapping $p$ to $q$ and $q$ to $p$.*

The proof is elementary. First, suppose $M$ is weakly symmetric and $p$, $q$ are any two points in $M$. Let $g \in G$ be an isometry of $M$ with $g(p)=f(q)$ and $g(q)=f(p)$. Then $f^{-1}g$ is an isometry of $M$ mapping $p$ to $q$ and $q$ to $p$. To see the converse, put $G=I(M)$ and $f=id_M$.

The paper is organized as follows. In Section 2 we present a short collection of results showing that weakly symmetric spaces have nice geometrical and analytical properties. Most of these results have been proved in [3] and [21], but Proposition 1 and the short proof of Proposition 3 are new. Section 3 contains preliminary material concerning Hermitian structures, quaternionic Hermitian structures and Cayley Hermitian structures on real vector spaces. Such a structure arises naturally on each tangent space of a projective or hyperbolic space over $\boldsymbol{C}$, $\boldsymbol{H}$ or $Cay$. In Section 4 we classify all submanifolds (which are necessarily complete and totally geodesic) in two-point homogeneous spaces for which the reflection in it is a global isometry of the ambient space. The method is to reduce this geometrical problem to the algebraic one of determining all curvature-invariant linear subspaces of the tangent space at an arbitrary point for which the orthogonal complement is also curvature-invariant. This will be useful in Section 5 where we derive by geometrical arguments new examples of weakly symmetric spaces. All these examples are homogeneous hypersurfaces in the projective or hyperbolic spaces over $\boldsymbol{C}$, $\boldsymbol{H}$ or $Cay$. As an application of this we shall classify in Section 6 all simply connected weakly symmetric spaces in dimensions three and four.

## 2. Preliminaries.

Let $M_1$ and $M_2$ be Riemannian manifolds. If $f_1$ is an isometry of $M_1$ and $f_2$ is an isometry of $M_2$, then the map $(p_1, p_2) \mapsto (f_1(p_1), f_2(p_2))$ is an isometry of the Riemannian product $M_1 \times M_2$. Conversely, let $M$ be a reducible weakly symmetric space. Without loss of generality we may suppose that there is no flat factor. So, we may write $M = M_1 \times \cdots \times M_k$, where $M_i$ is irreducible and non-flat. If no two factors are isometric to each other, then it follows at once that each isometry of $M$ is a product isometry and hence, each $M_i$ is weakly symmetric. Now suppose there are isometric factors, for example let $M = M_1 \times M_1$. Then an isometry of $M$ is either a product isometry $(f_1, f_2)$, or it is of the form $\sigma \circ (f_1, f_2)$, where $\sigma(p, q) = (q, p)$ for all $p, q \in M$. Let $p, q \in M_1$ be two distinct points and $r \in M_1$ arbitrary. As $M = M_1 \times M_1$ is weakly symmetric, there exists an isometry $f$ of $M$ interchanging $(p, r)$ and $(q, r)$. Then $f$ is a product isometry, since the other possibility leads immediately to a contradiction. Hence, we have

PROPOSITION 1. *The Riemannian product of two weakly symmetric spaces is a weakly symmetric space, and conversely, the flat factor and each irreducible factor of a reducible weakly symmetric space are weakly symmetric.*

The following result is a trivial consequence of the geometrical characterization of weakly symmetric spaces [3].

PROPOSITION 2. *A Riemannian manifold $M$ is a weakly symmetric space if and only if for every maximal geodesic $\gamma$ in $M$ and any point $m$ of $\gamma$ there exists an isometry of $M$ which is a non-trivial involution on $\gamma$ with $m$ as fixed point.*

REMARK. Manifolds having the latter property have been introduced by Szabó [**22**] as *ray symmetric spaces.*

Now suppose that $G_m(r)$ is a geodesic hypersphere of radius $r$ and with center $m$ in a weakly symmetric space $M$. Let $\gamma: \boldsymbol{R} \rightarrow M$ be a geodesic in $M$ with $\gamma(0)=m$ and $\gamma(r) \in G_m(r)$. According to Proposition 2, there exists an isometry $f$ of $M$ with $f(m)=m$ and $f(\gamma(r))=\gamma(-r)$. Using the Weingarten equation for hypersurfaces, it follows that the shape operators of $G_m(r)$ at $\gamma(r)$ and $\gamma(-r)$ are conjugate to each other. Hence, we have proved

PROPOSITION 3. *The principal curvatures (counted with multiplicities) of any geodesic hypersphere in a weakly symmetric space are the same at antipodal points.*

See also [**3**] for an alternative proof.

An immediate consequence of Proposition 3 is that the mean curvature of any geodesic hypersphere in a weakly symmetric space is the same at antipodal points. The latter property characterizes the *spaces with volume-preserving geodesic symmetries* (also known as *D'Atri spaces*), see [**12**], [**19**] and [**24**] for further references. So we have

PROPOSITION 4. *Any weakly symmetric space is a space with volume-preserving geodesic symmetries.*

As another consequence of Proposition 3 we get (see [**3**] for details)

PROPOSITION 5. *The Jacobi operator on any weakly symmetric space has constant eigenvalues along geodesics.*

REMARK. Manifolds whose Jacobi operator has this property have been introduced by the authors in [**6**] as $\mathfrak{C}$*-spaces.* See [**6**] and [**3**] for further results and references.

We recall a remarkable result which is due to Selberg [**21**].

PROPOSITION 6. *On any weakly symmetric space the algebra of all isometry-invariant differential operators is commutative.*

Such spaces are known as *commutative spaces* or *Gelfand spaces.*

REMARK. In general, none of the geometrical properties stated in Propositions 3, 4 and 5 characterizes weakly symmetric spaces. In fact, any generalized Heisenberg group has the properties stated in these propositions, but only few of them are weakly symmetric spaces (see [5] and Proposition 7 below). On the other hand, we do not know of any example of a commutative space which is not a weakly symmetric space. So, as already stated by Selberg [21], we do not know if commutativity implies weak symmetry.

Finally we mention the following result which is proved in [2].

PROPOSITION 7. *Every (maximal) geodesic in a weakly symmetric space is the orbit of a one-parameter group of isometries.*

Riemannian manifolds whose geodesics have this property are known as *Riemannian g.o. spaces* [20]. Note that Proposition 5 is an immediate consequence of Proposition 7.

## 3. Vector spaces with special structures.

Let $V$ be a real vector space equipped with an inner product $\langle\, ,\rangle$. Dimensions are always considered as real dimensions.

### a) Vector spaces with a Hermitian structure.

A *Hermitian structure* on $V$ is an endomorphism $J$ of $V$ such that $J^2=-id_V$ and $\langle Ju, Jw\rangle=\langle u, w\rangle$ for all $u, w\in V$. When $V$ is equipped with such a Hermitian structure $J$, then the dimension of $V$ is even, say equal to $2n$. Further, a linear subspace $W$ of $V$ is said to be *complex* if $JW\subset W$, and $W$ is said to be *totally real* if $JW\perp W$. We omit the elementary proof of the following lemma.

LEMMA 1. *Let $V_1, \cdots, V_k$ be mutually orthogonal complex subspaces of $V$ with $V=V_1\oplus\cdots\oplus V_k$ and $\dim V_1=2$. Further, let $u\in V_1$ and $w=w_1+\cdots+w_k\in V$, $w_i\in V_i$ be non-zero and orthogonal. Then there exist $n$-dimensional totally real subspaces $U$, $W$ of $V$ such that $u\in U$, $w_i\in W$, $U\perp W$ (and hence also $V=U\oplus W$) and $U_i:=U\cap V_i$, $W_i:=W\cap V_i$ are totally real subspaces of $V_i$ with $V_i=U_i\oplus W_i$ for $i=1, \cdots, k$.*

### b) Vector spaces with a quaternionic Hermitian structure.

A *quaternionic Hermitian structure* on $V$ is a three-dimensional real vector space $\mathfrak{J}$ of linear transformations on $V$ for which there exists a basis $J_1$, $J_2$, $J_3$ consisting of Hermitian structures on $V$ such that $J_iJ_{i+1}=J_{i+2}$ for all $i\in\{1, 2, 3\}$ (index modulo 3). Any such basis of $\mathfrak{J}$ is called a *canonical basis*. When $V$ is equipped with a quaternionic Hermitian structure $\mathfrak{J}$, then the dimension of $V$ is a multiple of 4, say $4n$. We endow $\mathfrak{J}$ with the standard inner product for endomorphisms on inner product spaces. Then any canonical basis of $\mathfrak{J}$ is an orthogonal basis with elements of length $\sqrt{4n}$. Further, let $W$ be a linear subspace of $V$. Then $W$ is called a *quaternionic* subspace of $V$ if $JW\subset W$ for all $J\in\mathfrak{J}$; $W$ is said to be a *totally complex* subspace of $V$ if there exists a one-dimensional linear subspace $\mathfrak{J}_1$ of $\mathfrak{J}$ such that $JW\subset W$ for all $J\in\mathfrak{J}_1$ and $JW\perp W$ for all $J\in\mathfrak{J}_1^{\perp}$; and $W$ is said to be a *totally real* subspace of $V$ if $JW\perp W$ for all $J\in\mathfrak{J}$. Finally, $w_1, \cdots, w_n\in V$ are said to form a *quaternionic orthonormal basis* of $V$ if $w_1, J_1w_1, J_2w_1, J_3w_1, \cdots, w_n, J_1w_n, J_2w_n, J_3w_n$ is an orthonormal basis of $V$ for some (and hence for any) canonical basis $J_1, J_2, J_3$ of $\mathfrak{J}$.

LEMMA 2. *Let $V_1, \cdots, V_k$ be mutually orthogonal quaternionic subspaces of $V$ with $V=V_1\oplus\cdots\oplus V_k$ and $\dim V_1=4$. Further, let $u\in V_1$ and $w=w_1+\cdots+w_k\in V$, $w_i\in V_i$, be non-zero and orthogonal. Then there exist $2n$-dimensional totally complex subspaces $U$, $W$ of $V$ such that $u\in U$, $w_i\in W$, $U\perp W$ (and hence also $V=U\oplus W$) and $U_i:=U\cap V_i$, $W_i:=W\cap V_i$ are totally complex subspaces of $V_i$ with $V_i=U_i\oplus W_i$ for $i=1, \cdots, k$.*

PROOF. As $V_1$ is a four-dimensional quaternionic subspace of $V$ and $u, w_1\in V_1$, there exists a Hermitian structure $J\in\mathfrak{J}$ such that $Ju$ is perpendicular to $w_1$. We may choose a quaternionic orthonormal basis $\tilde{w}_1, \cdots, \tilde{w}_n$ of $V$ such that $\tilde{w}_i\in V_i$ for $i=1, \cdots, k$, $\tilde{w}_i\in V_j$ for some $j\in\{2, \cdots, k\}$ $(i=k+1, \cdots, n)$, $\langle u, \tilde{w}_1\rangle=0=\langle u, J\tilde{w}_1\rangle$ and $w_i\in\boldsymbol{R}\tilde{w}_i$ $(i=1, \cdots, k)$. Then $\tilde{w}_1, J\tilde{w}_1, \cdots, \tilde{w}_n, J\tilde{w}_n$ span a $2n$-dimensional totally complex subspace $W$ of $V$ with $w_i\in W$. The orthogonal complement $U=W^{\perp}$ is a $2n$-dimensional totally complex subspace of $V$ containing $u$. Both $U$ and $W$ have the stated properties. □

**c) Vector spaces with a Cayley Hermitian structure.**

A *Cayley Hermitian structure* on $V$ is a seven-dimensional real vector space $\mathfrak{J}$ of linear transformations on $V$ for which there exists a basis $J_1, \cdots, J_7$ of Hermitian structures on $V$ satisfying the following composition rules:

| | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ | $J_6$ | $J_7$ |
|---|---|---|---|---|---|---|---|
| $J_1$ | $-I$ | $J_4$ | $J_7$ | $-J_2$ | $J_6$ | $-J_5$ | $-J_3$ |
| $J_2$ | $-J_4$ | $-I$ | $J_5$ | $J_1$ | $-J_3$ | $J_7$ | $-J_6$ |
| $J_3$ | $-J_7$ | $-J_5$ | $-I$ | $J_6$ | $J_2$ | $-J_4$ | $J_1$ |
| $J_4$ | $J_2$ | $-J_1$ | $-J_6$ | $-I$ | $J_7$ | $J_3$ | $-J_5$ |
| $J_5$ | $-J_6$ | $J_3$ | $-J_2$ | $-J_7$ | $-I$ | $J_1$ | $J_4$ |
| $J_6$ | $J_5$ | $-J_7$ | $J_4$ | $-J_3$ | $-J_1$ | $-I$ | $J_2$ |
| $J_7$ | $J_3$ | $J_6$ | $-J_1$ | $J_5$ | $-J_4$ | $-J_2$ | $-I$ . |

The table has to be read according to the rule $J_1J_2=J_4$. Any such basis of $\mathfrak{J}$ is called a *canonical basis.* Note that for each $i\in\{1, \cdots, 7\}$ the subspace $\mathfrak{J}_3$ of $\mathfrak{J}$ spanned by $J_i, J_{i+1}, J_{i+3}$ (index modulo 7) is a quaternionic Hermitian structure on $V$ with $J_i$, $J_{i+1}$, $J_{i+3}$ as a canonical basis. When $V$ is equipped with a Cayley Hermitian structure $\mathfrak{J}$, then the dimension of $V$ is a multiple of 8, say $8n$. We endow $\mathfrak{J}$ with the standard inner product for endomorphisms on inner product spaces. Then any canonical basis of $\mathfrak{J}$ is an orthogonal basis with elements of length $\sqrt{8n}$. Further, let $W$ be a linear subspace of $V$. Then $W$ is said to be a *Cayley* subspace of $V$ if $JW\subset W$ for all $J\in\mathfrak{J}$; $W$ is said to be a *totally quaternionic* subspace of $V$ if there exists a three-dimensional linear subspace $\mathfrak{J}_3$ of $\mathfrak{J}$ defining a quaternionic Hermitian structure on $V$ such that $JW\subset W$ for all $J\in\mathfrak{J}_3$ and $JW\perp W$ for all $J\in\mathfrak{J}_3^{\perp}$; $W$ is said to be a *totally complex* subspace of $V$ if there exists a one-dimensional linear subspace $\mathfrak{J}_1$ of $\mathfrak{J}$ such that $JW\subset W$ for all $J\in\mathfrak{J}_1$ and $JW\perp W$ for all $J\in\mathfrak{J}_1^{\perp}$; and $W$ is said to be a *totally real* subspace of $V$ if $JW\perp W$ for all $J\in\mathfrak{J}$. The vectors $w_1, \cdots, w_n\in V$ are said to form a *Cayley orthonormal basis* of $V$ if $w_1, J_1w_1, \cdots, J_7w_1, \cdots, w_n, \cdots, J_7w_n$ is an orthonormal basis of $V$ for some (and hence for any) canonical basis $J_1, \cdots, J_7$ of $\mathfrak{J}$.

LEMMA 3. *Let $V_1, \cdots, V_k$ be mutually orthogonal Cayley subspaces of $V$ with $V=V_1\oplus\cdots\oplus V_k$ and* $\dim V_1=8$. *Further, let $u\in V_1$ and $w=w_1+\cdots+w_k\in V$, $w_i\in V_i$, be non-zero and orthogonal. Then there exist $4n$-dimensional totally quaternionic subspaces $U$, $W$ of $V$ such that $u\in U$, $w_i\in W$, $U\perp W$ (and hence also $V=U\oplus W$) and $U_i:=U\cap V_i$, $W_i:=W\cap V_i$ are totally quaternionic subspaces of $V_i$ with $V_i=U_i\oplus W_i$ for $i=1, \cdots, k$.*

PROOF. We may choose a Cayley orthonormal basis $\tilde{w}_1, \cdots, \tilde{w}_n$ of $V$ such that $\tilde{w}_i\in V_i$ for $i=1, \cdots, k$, $\tilde{w}_i\in V_j$ for some $j\in\{2, \cdots, k\}$ $(i=k+1, \cdots, n)$, $\langle u, \tilde{w}_1\rangle=0$ and $w_i\in\boldsymbol{R}\tilde{w}_i$ $(i=1, \cdots, k)$. Let $J_3$ be a Hermitian structure in $\mathfrak{J}$ such that $J_3\tilde{w}_1\in\boldsymbol{R}u$. Choose any Hermitian structure $J_1$ in $\mathfrak{J}$ orthogonal to $J_3$. $J_1$ and $J_3$ uniquely determine a Hermitian structure $J_7$ in $\mathfrak{J}$. Finally, choose any Hermitian structure $J_2$ in $\mathfrak{J}$ orthogonal to $J_1$, $J_3$, $J_7$. This then determines uniquely a canonical basis $J_1, \cdots, J_7$ of $\mathfrak{J}$. Put $\mathfrak{J}_3:=span\{J_1, J_2, J_4\}$ and $W:=span\{\tilde{w}_i, J\tilde{w}_i \mid i=1, \cdots, n; J\in\mathfrak{J}_3\}$. Clearly, $W$ is invariant by $\mathfrak{J}_3$. Also $JW\perp W$ for all $J\in\mathfrak{J}_3^{\perp}$, which can be verified easily by using the above composition rules. Similarly it follows that $U=W^{\perp}$ is a $4n$-dimensional totally quaternionic subspace of $V$ containing $u$. $U$ and $W$ have the stated properties. □

## 4. Isometric reflections on two-point homogeneous spaces.

In this section we classify all submanifolds in two-point homogeneous spaces for which the reflection of the ambient space with respect to the submanifold is an isometry. Submanifolds are always supposed to be connected. First we recall that a two-point homogeneous space is a connected Riemannian manifold $M$ with the property that for any two pairs of points $(p_1, q_1), (p_2, q_2)\in M\times M$

with $d(p_1, q_1)=d(p_2, q_2)$ there exists an isometry of $M$ mapping $p_1$ to $p_2$ and $q_1$ to $q_2$. Here, $d$ denotes the distance function on $M$ induced from the Riemannian metric. The two-point homogeneous spaces are known to be precisely the following spaces with the standard metrics (see [25], [23]):

- Euclidean spaces, denoted by $\boldsymbol{E}^n$;
- spheres, denoted by $S^n$;
- projective spaces over $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ and $Cay$, denoted by $\boldsymbol{R}P^n$, $\boldsymbol{C}P^n$, $\boldsymbol{H}P^n$ and $CayP^n$, respectively;
- hyperbolic spaces over $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ and $Cay$, denoted by $\boldsymbol{R}H^n$, $\boldsymbol{C}H^n$, $\boldsymbol{H}H^n$ and $CayH^n$, respectively.

The Cayley projective and hyperbolic spaces are defined only for $n\leqq 2$; in all other cases $n\geqq 1$ may be chosen arbitrarily. At each point in $\boldsymbol{C}P^n$ or $\boldsymbol{C}H^n$ the tangent space is equipped in a natural way with a Hermitian structure induced from the Kähler structure on the manifold. The quaternionic Kähler structure on $\boldsymbol{H}P^n$ or $\boldsymbol{H}H^n$ induces canonically a quaternionic Hermitian structure on each of its tangent spaces. And the Cayley structure on $CayP^2$ or $CayH^2$ determines at each tangent space a Cayley Hermitian structure.

We shall need the classification of all connected complete totally geodesic submanifolds in two-point homogeneous spaces. For the reader's convenience we list here those which are maximal in the sense that they are not contained in a higher-dimensional proper totally geodesic submanifold (see [26] for details):

| ambient space | submanifold |
|---|---|
| $\boldsymbol{E}^n$ | $\boldsymbol{E}^{n-1}$ |
| $S^n$ | $S^{n-1}$ |
| $\boldsymbol{R}P^n$ | $\boldsymbol{R}P^{n-1}$ |
| $\boldsymbol{C}P^n$ | $\boldsymbol{R}P^n$, $\boldsymbol{C}P^{n-1}$ |
| $\boldsymbol{H}P^n$ | $\boldsymbol{C}P^n$, $\boldsymbol{H}P^{n-1}$ |
| $CayP^2$ | $\boldsymbol{H}P^2$, $CayP^1$ |
| $\boldsymbol{R}H^n$ | $\boldsymbol{R}H^{n-1}$ |
| $\boldsymbol{C}H^n$ | $\boldsymbol{R}H^n$, $\boldsymbol{C}H^{n-1}$ |
| $\boldsymbol{H}H^n$ | $\boldsymbol{C}H^n$, $\boldsymbol{H}H^{n-1}$ |
| $CayH^2$ | $\boldsymbol{H}H^2$, $CayH^1$ . |

From this list one may easily deduce the full classification of complete connected totally geodesic submanifolds in two-point homogeneous spaces.

Let $M$ be a totally geodesic submanifold of a Riemannian manifold $N$ and $R^N$ the Riemannian curvature tensor of $N$. From the Codazzi equation it follows immediately that at each point $p\in M$ the tangent space $T_pM$ is *curvature-invariant*, that is, $R^N(u, v)w\in T_pM$ for all $u, v, w\in T_pM$. If $N$ is a

Riemannian globally symmetric space then also the converse holds, that is, if $W$ is a curvature-invariant linear subspace of $T_pN$ at some point $p\in N$, then there exists a unique connected complete totally geodesic submanifold $M$ in $N$ with $p\in M$ and $T_pM=W$ (see e.g. [**13**], p. 224, where this statement is formulated in terms of Lie triple systems). Using the above table of totally geodesic submanifolds, we derive a complete list of curvature-invariant linear subspaces in the tangent space at an arbitrary point of a two-point homogeneous space:

- for $\boldsymbol{E}^n$, $S^n$, $\boldsymbol{RP}^n$, $\boldsymbol{RH}^n$: any linear subspace;
- for $\boldsymbol{CP}^n$, $\boldsymbol{CH}^n$: totally real subspaces, complex subspaces;
- for $\boldsymbol{HP}^n$, $\boldsymbol{HH}^n$: totally real subspaces, totally complex subspaces, quaternionic subspaces, linear subspaces of a four-dimensional quaternionic subspace;
- for $CayP^2$, $CayH^2$: totally real subspaces, totally complex subspaces, totally quaternionic subspaces, Cayley subspaces, linear subspaces of an eight-dimensional Cayley subspace.

Note that we used here the fact that $CayP^1$ is isometric to $S^8$, $CayH^1$ to $\boldsymbol{RH}^8$, $\boldsymbol{HP}^1$ to $S^4$ and $\boldsymbol{HH}^1$ to $\boldsymbol{RH}^4$.

THEOREM 1. *The reflection in the following complete totally geodesic submanifolds are global isometries*:

| *ambient space* | *submanifold* |
|---|---|
| $\boldsymbol{E}^n$ | $\boldsymbol{E}^k$ |
| $S^n$ | $S^k$ |
| $\boldsymbol{RP}^n$ | $\boldsymbol{RP}^k$ |
| $\boldsymbol{CP}^n$ | $\boldsymbol{RP}^n$, $\boldsymbol{CP}^k$ |
| $\boldsymbol{HP}^n$ | $\boldsymbol{CP}^n$, $\boldsymbol{HP}^k$ |
| $CayP^2$ | $\boldsymbol{HP}^2$, $CayP^1$, $\{p\}$ |
| $\boldsymbol{RH}^n$ | $\boldsymbol{RH}^k$ |
| $\boldsymbol{CH}^n$ | $\boldsymbol{RH}^n$, $\boldsymbol{CH}^k$ |
| $\boldsymbol{HH}^n$ | $\boldsymbol{CH}^n$, $\boldsymbol{HH}^k$ |
| $CayH^2$ | $\boldsymbol{HH}^2$, $CayH^1$, $\{p\}$. |

*Here $k\in\{0, \cdots, n-1\}$, $n\geqq 2$, and for $k=0$ the submanifold consists of a single point $\{p\}$. Any submanifold of a two-point homogeneous space $N$ for which the reflection in it is a global isometry of $N$ is congruent to one of the above models.*

PROOF. First of all we remark that the reflection in the above submanifolds can always be defined locally. If this local reflection then turns out to be an isometry, it can be uniquely extended to a global isometry of the ambient space, for the ambient space (except $\boldsymbol{RP}^n$) is a simply connected complete analytic

Riemannian manifold (see e.g., [**15**], p. 256). In case of $\boldsymbol{R}P^n$ one argues that the global reflection of $S^n$ in $S^k$ commutes with the antipodal map on $S^n$ and hence descends to a global isometry of $\boldsymbol{R}P^n$ which is in fact the reflection in $\boldsymbol{R}P^k$.

Every connected component of the fixed point set of an isometry of a Riemannian manifold $N$ is a totally geodesic submanifold. Thus, in order that the reflection in a submanifold $M$ of $N$ is an isometry, $M$ is necessarily a totally geodesic submanifold. If $N$ is in addition a locally symmetric space, a local reflection in a totally geodesic submanifold $M$ is an isometry if and only if at each point $p \in M$ there exists a totally geodesic submanifold $\tilde{N}$ of $N$ with $p \in \tilde{N}$ and $T_p\tilde{N}=(T_pM)^\perp$ (see [**10**]). Note that the fact that a reflection in a submanifold of any of these spaces is a global isometry implies that the submanifold is complete. So it remains to find the connected complete totally geodesic submanifolds in two-point homogeneous spaces having this "totally geodesic normal space" property. According to the preceding remarks, we just have to find the curvature-invariant subspaces for which the orthogonal complement is also curvature-invariant. A straightforward case-by-case argument then yields the assertion. □

## 5. Examples.

In the original paper by Selberg [**21**] an explicit example of a weakly symmetric space which is not a Riemannian globally symmetric space is presented, namely the Lie group $SL(2, \boldsymbol{R})$ equipped with a suitable left-invariant Riemannian metric. This has been the only non-trivial weakly symmetric space known to us before we started this work. We will now combine the (pointwise) algebraic statements of Section 3 and the (global) geometric results of Section 4 to derive further examples of weakly symmetric spaces.

THEOREM 2. *Each of the following hypersurfaces, endowed with the induced Riemannian metric of the ambient space, is a weakly symmetric space for* $n \geqq 2$:

| *ambient space* | *hypersurface* |
|---|---|
| $\boldsymbol{C}P^n$ | *tube around* $\{p\}$, $\boldsymbol{C}P^1, \cdots$, *or* $\boldsymbol{C}P^{n-1}$ |
| $\boldsymbol{H}P^n$ | *tube around* $\{p\}$, $\boldsymbol{H}P^1, \cdots$, *or* $\boldsymbol{H}P^{n-1}$ |
| $CayP^2$ | *tube around* $\{p\}$ *or* $CayP^1$ |
| $\boldsymbol{C}H^n$ | *horosphere; tube around* $\{p\}$, $\boldsymbol{C}H^1, \cdots$, *or* $\boldsymbol{C}H^{n-1}$ |
| $\boldsymbol{H}H^n$ | *horosphere; tube around* $\{p\}$, $\boldsymbol{H}H^1, \cdots$, *or* $\boldsymbol{H}H^{n-1}$ |
| $CayH^2$ | *horosphere; tube around* $\{p\}$ *or* $CayH^1$. |

REMARKS. 1. We want to point out that each of these hypersurfaces has the geometrical features mentioned in Propositions 2–7.

2. In the compact case we consider only tubes for which the radius $r$ is less than the distance from the core to its focal set, more precisely, $r \in (0, \pi/\sqrt{c})$, where $c$ is the maximal value of the sectional curvature of the ambient space. In the non-compact case $r$ may be an arbitrary positive real number. In all these cases the tube is an embedded connected complete hypersurface in the ambient space.

3. Every geodesic hypersphere and every tube around a connected complete totally geodesic submanifold in $\boldsymbol{E}^n$, $S^n$, $\boldsymbol{R}P^n$ or $\boldsymbol{R}H^n$ is a Riemannian globally symmetric space. The same holds for horospheres in $\boldsymbol{R}H^n$. This known result follows also in an elegant way by using reflections with respect to geodesics orthogonal to the hypersurface and arguments analogous to those given in the proof of Theorem 2.

4. The focal set of $\boldsymbol{C}P^k$, $0 \leq k \leq n-1$, in $\boldsymbol{C}P^n$ is at distance $\pi/\sqrt{c}$ (for $c$ see Remark 2) from $\boldsymbol{C}P^k$ and congruent to $\boldsymbol{C}P^{n-k-1}$. Therefore every tube around $\boldsymbol{C}P^k$ is isometric to a suitable tube around $\boldsymbol{C}P^{n-k-1}$. Analogously, every tube around $\boldsymbol{H}P^k$ is congruent to a suitable tube around $\boldsymbol{H}P^{n-k-1}$, and every geodesic hypersphere in $CayP^2$ is congruent to a suitable tube around $CayP^1$.

5. The universal covering space of Selberg's example of $SL(2, \boldsymbol{R})$ is isometric to the universal covering space of a tube of some suitable radius around $\boldsymbol{C}H^1$ in $\boldsymbol{C}H^2$ (see Section 6).

6. As a consequence of the above theorem we have that *any geodesic hypersphere in a two-point homogeneous space is a weakly symmetric space.*

a) An unpublished result by S. Helgason and L. Vanhecke says that any geodesic hypersphere in a two-point homogeneous space is a commutative space. By means of Proposition 6 we obtain an alternative proof.

b) According to W. Ziller [**27**], a geodesic hypersphere in a compact two-point homogeneous space is a naturally reductive Riemannian homogeneous space if and only if the ambient space is not $CayP^2$. Therefore, geodesic hyperspheres in $CayP^2$ provide examples of weakly symmetric spaces which are not naturally reductive.

c) Two-point homogeneous spaces can in fact be characterized by the property that all geodesic hyperspheres are weakly symmetric spaces (this follows from Theorem 2, Proposition 4 and [**4**]).

7. Any generalized Heisenberg group which satisfies the $J^2$-condition (see [**11**]) is weakly symmetric since it is isometric to a horosphere in $\boldsymbol{C}H^n$, $\boldsymbol{H}H^n$ or $CayH^2$ (see also [**5**]).

PROOF OF THEOREM 2. Let $k \in \{0, \cdots, n-1\}$, $M$ a tube of radius $r \in (0, \pi/\sqrt{c})$ around $\boldsymbol{C}P^k$ (for $c$ see Remark 2) and $p$, $q$ any distinct points in

$M$. We denote by $2t$ the distance from $p$ to $q$ in $M$. Then there exists a geodesic $\gamma: \boldsymbol{R} \to M$ in $M$ with $\gamma(-t)=p$ and $\gamma(t)=q$. We put $m=\gamma(0)$ and denote by $\alpha: \boldsymbol{R} \to \boldsymbol{C}P^n$ the geodesic in $\boldsymbol{C}P^n$ with $\alpha(0) \in \boldsymbol{C}P^k$ and $\alpha(r)=m$. Let $V_1$ be the complex subspace of $T_m\boldsymbol{C}P^n$ determined by $\dot{\alpha}(r)$, the outward unit normal vector of $M$ at $m$. Further, by $V_2$ we denote the complex subspace of $T_m\boldsymbol{C}P^n$ obtained by parallel translation of $T_{\alpha(0)}\boldsymbol{C}P^k$ from $\alpha(0)$ to $m$ along $\alpha$. Finally, let $V_3$ be the orthogonal complement of $V_1 \oplus V_2$ in $T_m\boldsymbol{C}P^n$, which is also a complex subspace of $T_m\boldsymbol{C}P^n$. We now apply Lemma 1 with $u=\dot{\alpha}(r)$ and $w=\dot{\gamma}(0)$. According to this lemma, there exist $n$-dimensional totally real subspaces $U$, $W$ of $T_m\boldsymbol{C}P^n$ so that $\dot{\alpha}(r) \in U$, $\dot{\gamma}(0) \in W$, $T_m\boldsymbol{C}P^n=U \oplus W$ (orthogonal direct sum), and $U \cap V_2$ is a $k$-dimensional totally real subspace of $V_2$. Let $P$ be the connected complete totally geodesic submanifold of $\boldsymbol{C}P^n$ with $m \in P$ and $T_mP=U$. $P$ is congruent to $\boldsymbol{R}P^n$. By means of Theorem 1 the reflection $f$ of $\boldsymbol{C}P^n$ in $P$ is a global isometry of $\boldsymbol{C}P^n$. Furthermore, as $U \cap V_2$ is a $k$-dimensional totally real subspace of $V_2$ and $V_2$ is obtained by parallel translation of $T_{\alpha(0)}\boldsymbol{C}P^k$ from $\alpha(0)$ to $m$ along $\alpha$, the connected component of $P \cap \boldsymbol{C}P^k$ through $\alpha(0)$ is congruent to a totally geodesic $\boldsymbol{R}P^k$ in $\boldsymbol{C}P^k$. Therefore and as $\boldsymbol{C}P^k$ is totally geodesic in $\boldsymbol{C}P^n$, the reflection $f$ maps $\boldsymbol{C}P^k$ into itself. Since $f$ is an isometry of $\boldsymbol{C}P^n$ leaving $\boldsymbol{C}P^k$ invariant, it must map the tube $M$ about $\boldsymbol{C}P^k$ into itself. Thus, $f$ induces an isometry $g$ on $M$. As $\dot{\gamma}(0)$ is orthogoal to $T_mP$, $g$ reflects the geodesic $\gamma$ in $m=\gamma(0)$ and hence maps $p=\gamma(-t)$ to $q=\gamma(t)$ and $q$ to $p$. By the geometric characterization in the Introduction it now follows that $M$ is a weakly symmetric space.

The proof for $\boldsymbol{C}H^n$ is analogous, except that we may choose $r \in \boldsymbol{R}_+$.

The quaternionic case can be treated analogously; just replace totally real by totally complex, complex by quaternionic, $\boldsymbol{R}$ by $\boldsymbol{C}$, $\boldsymbol{C}$ by $\boldsymbol{H}$, and use Lemma 2 instead of Lemma 1.

Also the proof for $CayP^2$ and $CayH^2$ is analogous. Here one uses Lemma 3 instead of Lemma 1 and replaces totally real by totally quaternionic, complex by Cayley, $\boldsymbol{R}$ by $\boldsymbol{H}$, $\boldsymbol{C}$ by $Cay$ and $n$ by 2.

Next, let $M$ be a horosphere in $\boldsymbol{C}H^n$ and $p$, $q$ distinct points in $M$. We connect $p$ and $q$ by a geodesic $\gamma: \boldsymbol{R} \to M$ in $M$ with $p=\gamma(-t)$ and $q=\gamma(t)$, where $2t$ is the distance from $p$ to $q$ in $M$, and put $m=\gamma(0)$. Let $\alpha: \boldsymbol{R} \to \boldsymbol{C}H^n$ be the geodesic in $\boldsymbol{C}H^n$ parametrized by arc length such that $m=\alpha(0)$ and $M$ is the level set $F^{-1}(\{0\})$ of the Busemann function

$$F: \boldsymbol{C}H^n \to \boldsymbol{R}, \quad x \mapsto \lim_{t\to\infty}(d(x, \alpha(t))-t),$$

where $d$ denotes the distance function on $\boldsymbol{C}H^n$ induced by its Riemannian metric. In particular, $\dot{\alpha}(0)$ is a unit normal vector of $M$ at $m$. Denote by $V_1$ the complex subspace of $T_m\boldsymbol{C}H^n$ determined by $\dot{\alpha}(0)$ and by $V_2$ its orthogonal complement in

$T_m\boldsymbol{C}H^n$. We now apply Lemma 1 with $u=\dot{\alpha}(0)$ and $w=\dot{\gamma}(0)$ to obtain $n$-dimensional totally real subspaces $U$, $W$ of $T_m\boldsymbol{C}H^n$ with $\dot{\alpha}(0)\in U$, $\dot{\gamma}(0)\in W$ and $T_m\boldsymbol{C}H^n=U\oplus W$ (orthogonal direct sum). Let $P$ be the connected complete totally geodesic submanifold in $\boldsymbol{C}H^n$ with $m\in P$ and $T_mP=U$; $P$ is congruent to $\boldsymbol{R}H^n$. By means of Theorem 2 the reflection $f$ of $\boldsymbol{C}H^n$ in $P$ is a globally well-defined isometry of $\boldsymbol{C}H^n$. As $\dot{\alpha}(0)\in U=T_mP$, the entire geodesic $\alpha$ runs in $P$, and so $f$ leaves the Busemann function $F$ invariant, that is, $F\circ f=F$. This shows that $f$ maps $M$ into itself and hence induces an isometry $g$ of $M$. Finally, since $\dot{\gamma}(0)\in W\perp U=T_mP$, $g$ reflects $\gamma$ in $m=\gamma(0)$ and so maps $p=\gamma(-t)$ to $q=\gamma(t)$ and $q$ to $p$. Therefore, we see that $M$ is a weakly symmetric space.

The proof for a horosphere on $\boldsymbol{H}H^n$ or $CayH^2$ is analogous. □

Each of the model spaces mentioned in Theorem 2 is non-symmetric. For some of them this is known to some experts but we have not found an appropriate reference for a general argument. Since our proof of the non-symmetry involves lengthy case-by-case computations, we omit it here but indicate the lines along which it can be done. The shape operator and hence, via the contracted Gauss equation, the Ricci tensor of any of the model spaces can be computed in a standard way. It turns out that there are at most three distinct Ricci roots all of which are constant. The constancy of the Ricci roots follows also immediately from the fact that the model spaces are homogeneous hypersurfaces. The Einstein case occurs precisely for a geodesic hypersphere of radius $r$ in $\boldsymbol{H}P^n$ with $\tan^2(r\sqrt{c}/2)=2n$, the tube of radius $r$ around $\boldsymbol{H}P^{n-1}$ in $\boldsymbol{H}P^n$ with $\cot^2(r\sqrt{c}/2)=2n$ (this space is a geodesic hypersphere of the type mentioned just before), and a geodesic hypersphere of radius $r$ in $CayP^2$ with $\tan^2(r\sqrt{c}/2)=8/3$. These particular geodesic hyperspheres have been found by Jensen [**14**] and Bourguignon-Karcher [**9**] as providing examples of non-standard Einstein metrics on spheres. Because of this, the corresponding metrics cannot be symmetric (see for example [**27**]). Next, if the model space $M$ has two or three distinct Ricci roots, we may write $TM=D_0\oplus D_1$ or $TM=D_0\oplus D_1\oplus D_2$, respectively, where $D_0$, $D_1$ and $D_2$ denote the subbundles of $TM$ consisting of the eigenvectors corresponding to the distinct Ricci roots. If $\xi$ is a normal vector of $M$, then $J\xi$ is always an eigenvector of the Ricci tensor $S$ of $M$, where in the quaternionic and Cayley case $J$ is an arbitrary element of the quaternionic Kähler and Cayley structure of the ambient space, respectively. We may therefore assume that vectors of these type are contained in $D_0$. Then $D_1$ and $D_2$ are always invariant by the Kähler, quaternionic Kähler or Cayley structure of the ambient space. Now, if $M$ were a symmetric space, each bundle $D_i$ would be integrable. On the other hand, if $g$ denotes the Riemannian metric of $M$, $X$ is a non-zero vector in $D_1$ at some point $p\in M$, $J$ a Hermitian structure on the tangent space of the ambient space at $p$, and $\xi$ a non-zero normal vector of $M$ at $p$, then it follows from the Codazzi equation that $g([X, JX], J\xi)\neq 0$, whence $D_1$ is not integrable. This shows that also the non-Einstein model spaces are non-symmetric.

## 6. Classifications.

The following result will be useful for the classification problem and can be proved quickly by using the above Proposition 2 and Corollary 6.4 in [**15**], p. 256.

PROPOSITION 8. *The Riemannian universal covering space of a weakly symmetric space is also a weakly symmetric space.*

REMARK. Every simply connected complete Sasakian space form is isometric to the Riemannian unit sphere $S^n$, or a geodesic hypersphere in $\boldsymbol{C}P^n$ or $\boldsymbol{C}H^n$, or a horosphere in $\boldsymbol{C}H^n$, or the universal covering space of a tube around $\boldsymbol{C}H^{n-1}$ in $\boldsymbol{C}H^n$ (see [**1**]). Because the weak symmetry is preserved under homotheties of the metric, it follows from Theorem 2 and the above proposition that every simply connected complete Riemannian manifold which is homothetic to a Sasakian space form (a so-called $\alpha$-Sasakian space form) is weakly symmetric.

As an application of Theorem 2 we shall now classify all simply connected weakly symmetric spaces in dimensions three and four.

THEOREM 3. *A three-dimensional simply connected Riemannian manifold is a weakly symmetric space if and only if it is isometric to one of the following model spaces*:

(I) *a three-dimensional symmetric space*;
(II) *a geodesic hypersphere in $\boldsymbol{C}P^2$ or $\boldsymbol{C}H^2$*;
(III) *the universal covering space of a tube around $\boldsymbol{C}H^1$ in $\boldsymbol{C}H^2$*;
(IV) *a horosphere in $\boldsymbol{C}H^2$.*

REMARKS. 1. The spaces of type (II), (III) and (IV) are non-symmetric.

2. There are the following group theoretical characterizations of the spaces of type (II), (III) and (IV) (see [**17**] and [**7**] for details):

Type (II): The group $SU(2)$ with a left-invariant Riemannian metric $g$ such that the isotropy subgroup of $I^0(M, g)$ (=connected component of the isometry group of $(M, g)$) at any point is $SO(2)$. All these metrics depend on two arbitrary parameters.

Type (III): The universal covering group of $SL(2, \boldsymbol{R})$ with a left-invariant Riemannian metric $g$ such that the isotropy group of $I^0(M, g)$ at any point is $SO(2)$. The explicit form is $M=\boldsymbol{R}^3[t, x, y]$ with

$$ds^2=\frac{1}{|a+b|}dt^2+|a+b|e^{-2t}dx^2+(dy+\sqrt{2b}\,e^{-t}dx)^2,$$

where $a, b\in\boldsymbol{R}$ with $b>0$ and $a+b<0$.

Type (IV): The Heisenberg group of all matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

with any left-invariant Riemannian metric. Explicitly, $M=\boldsymbol{R}^3[x, y, z]$ with

$$ds^2=\frac{1}{2b}(dx^2+dz^2+(dy-xdz)^2),$$

where $b>0$ is arbitrary.

3. This list of spaces provides also the full classification for the following classes of Riemannian manifolds in the simply connected case: spaces with volume-preserving geodesic symmetries, commutative spaces, naturally reductive Riemannian homogeneous spaces, $\mathfrak{C}$-spaces (see [**6**] and [**17**] for details).

PROOF OF THEOREM 3. From Theorem 2 and Proposition 8 we see that any of the above model spaces is a weakly symmetric space. Conversely, suppose that $M$ is a simply connected weakly symmetric space. By means of Proposition 4 $M$ is a space with volume-preserving geodesic symmetries. O. Kowalski [**17**] proved that then $M$ is either a symmetric space or one of the Lie groups mentioned in Remark 2. The authors showed in [**7**] that all these Lie groups can be realized geometrically by the spaces listed in (II), (III) and (IV). □

THEOREM 4. *A four-dimensional simply connected Riemannian manifold is a weakly symmetric space if and only if it is either a symmetric space or a Riemannian product of a three-dimensional simply connected weakly symmetric space and* $\boldsymbol{R}$.

PROOF. The "if"-part of the assertion follows by means of Proposition 1. Conversely, suppose that $M$ is a four-dimensional simply connected weakly symmetric space. According to Proposition 7, $M$ is a g.o. space. It is proved in [**20**] that then $M$ is either a symmetric space or a Riemannian product of a Lie group as in the preceding Remark 2 and $\boldsymbol{R}$. Since these Lie groups are precisely the weakly symmetric spaces of type (II), (III) and (IV) in Theorem 3, the assertion eventually follows. □

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