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Expansion growth of smooth codimension-one foliations

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0. Introduction.

The entropy of foliations is defined by Ghys, Langevin and Walczak ([**G-L-W**]) as follows. Let $\mathcal{F}$ be a codimension q foliation of class C^0 on a compact manifold M. Fixing a finite foliation cover $\mathcal{U}$ of $(M, \mathcal{F})$, we obtain the holonomy pseudogroup $\mathcal{H}$ of local homeomorphisms of $\boldsymbol{R}^q$ induced by $\mathcal{U}$. We define an integer $s_n(\varepsilon)$ $(n \in \boldsymbol{N}, \varepsilon>0)$ to be the maximum cardinality of (n, ε)-separating sets with respect to the holonomy pseudogroup $\mathcal{H}$. Then $s_n(\varepsilon)$ is monotone increasing on n and monotone decreasing on ε. The *entropy* $h(\mathcal{F}, \mathcal{U})$ of the foliation $\mathcal{F}$ is defined by the following formula:

$$h(\mathcal{F}, \mathcal{U}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon).$$

When we fix a sufficiently small positive real number ε, we notice that the monotone increasing map $s_n(\varepsilon)$ with respect to n represents the degree of the expansion of the foliation. In [**E1**], we considered the growth type of $s_n(\varepsilon)$ defined in the growth type set which is an extension of the usual growth type set (cf. [**H-H2**]) and we proved that the growth type of $s_n(\varepsilon)$ depends only on $(M, \mathcal{F})$. Therefore it becomes a topological invariant for foliations. We call it the *expansion growth* of $(M, \mathcal{F})$. By computing the expansion growth of several typical codimension 1 foliation of class C^0, we showed that the expansion growth of codimension 1 foliation of class C^0 takes uncountably many values.

In this paper, we compute the expansion growth of codimension 1 foliations of class C^2. The main result of this paper is the following.

THEOREM. *Let $\mathcal{F}$ be a transversely oriented codimension* 1 *foliation of class C^2 on a compact manifold M. Let K be an $\mathcal{F}$-saturated set.*

(1) *If $\bar{K}$ has a resilient leaf, then $\eta(K)=[e^n]$.*

(2) *If $\bar{K}$ has no resilient leaf and* level$(K)<\infty$, *then $\eta(K)=[n^{\mathrm{level}(K)}]$.*

(3) *Otherwise, $\eta(K)=[1, n, n^2, \cdots]$.*

Here $\eta(K)$ means the expansion growth of $(M, \mathcal{F})$ on K and the notation $[\cdot]$ means the growth type defined in section 1 and level(K) means supremum of the level of leaves contained in K. Contrary to the case of foliations of class C^0, our result says that the expansion growth of codimension 1 foliation of class C^2 takes only countably many values. While there exists a codimension 1 foliation of class C^2 containing a leaf whose growth type is fractional ([**C-C2**], [**He**], [**T2**]), we remark that the expansion growth is a typical growth type except for one case. As a corollary, we can easily deduce that the positivity of the entropy of codimension 1 foliation of class C^2 is equivalent to the existence of a resilient leaf, which was proved by Ghys, Langevin and Walczak ([**G-L-W**]).

In sections 1 and 2, we review the growth type set which is an extension of the usual growth type set and the expansion growth of foliations defined as an element of this growth type set. In section 3, we compute the expansion growth of codimension 1 foliations of class C^2.

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1. Growth.

In this section, we review the growth of an increasing sequence of increasing functions.

Let $\mathcal{G}$ be the set of non-negative increasing functions on $\boldsymbol{N}$:

$$\mathcal{G} = \{g : \boldsymbol{N} \to [0, \infty) ; g(n) \leqq g(n+1) \text{ for all } n \in \boldsymbol{N}\}.$$

Let $\tilde{\mathcal{G}}$ be the set of increasing sequences in $\mathcal{G}$:

$$\tilde{\mathcal{G}} = \{(g_j)_{j \in \boldsymbol{N}} \subset \mathcal{G} ; g_j(n) \leqq g_{j+1}(n) \text{ for all } j \in \boldsymbol{N} \text{ and } n \in \boldsymbol{N}\}.$$

We regard $\mathcal{G}$ as a subset of $\tilde{\mathcal{G}}$ by the map

$$\mathcal{G} \ni g \mapsto (g, g, g, \cdots) \in \tilde{\mathcal{G}}.$$

We define the growth type of an element of $\tilde{\mathcal{G}}$. We define a preorder $\leqq$ in $\tilde{\mathcal{G}}$ as follows. For $(g_j)_{j \in N}, (h_k)_{k \in N} \in \tilde{\mathcal{G}}$,

$$(g_j)_{j \in N} \leqq (h_k)_{k \in N} \Longleftrightarrow \exists B \in \boldsymbol{N}, \forall j \in \boldsymbol{N}, \exists k \in \boldsymbol{N}, \exists A > 0 \text{ such that } g_j(n) \leqq A h_k(Bn) \text{ for any } n \in \boldsymbol{N}.$$

The preorder $\leqq$ induces an equivalence relation $\cong$ by

$$(g_j)_{j \in N} \cong (h_k)_{k \in N} \Longleftrightarrow (g_j)_{j \in N} \leqq (h_k)_{k \in N} \text{ and } (g_j)_{j \in N} \geqq (h_k)_{k \in N}.$$

We define $\tilde{\mathcal{E}}$ to be the set of equivalence classes in $\tilde{\mathcal{G}}$:

$$\tilde{\mathcal{E}} = \tilde{\mathcal{G}} / \cong .$$

The equivalence class of $(g_j)_{j\in N}\in\tilde{\mathcal{G}}$ is written by $[g_j]_{j\in N}\in\tilde{\mathcal{E}}$ and is called the *growth type* of $(g_j)_{j\in N}$. Thus $\tilde{\mathcal{E}}$ is the set of all the growth types of increasing sequences of increasing functions and has the partial order $\leqq$ induced by the preorder $\lesssim$. The equivalence class of $g\in\mathcal{G}\subset\tilde{\mathcal{G}}$ is simply written by $[g]$. Let $\mathcal{E}$ be the set of such growth types:

$$\mathcal{E}=\{[g]\,;\, g\in\mathcal{G}\}\subset\tilde{\mathcal{E}}\,.$$

Then $\mathcal{E}$ is essentially equal to the partial ordered set of all the growth types of monotone increasing functions in the usual sense (cf. [**H-H2**]) and $\tilde{\mathcal{E}}$ can be considered as an extension of it.

The following relation is easy to be seen:

$$\begin{aligned}[0]\lneqq[1]\lneqq[n]\lneqq[n^2]\lneqq\cdots\lneqq[1,\,n,\,n^2,\,\cdots]\\ \lneqq[2^n]=[3^n]\lneqq[1,\,2^n,\,3^n,\,\cdots]\,.\end{aligned}$$

Here [0] (resp. [1]) is the growth of the constant function whose value is 0 (resp. 1). We say $[e^n]\in\mathcal{E}$ the *exactly exponential growth.* For $k\in\boldsymbol{N}\cup\{0\}$, we say $[n^k]\in\mathcal{E}$ the *exactly polynomial growth of degree k.* We say $[g_j]_{j\in N}\in\tilde{\mathcal{E}}$ to be *quasi-exponential* if

$$\lim_{j\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log g_j(n)>0\,.$$

Next we define the finite sum and the finite product of elements of $\tilde{\mathcal{E}}$. For $[g_j]_{j\in N}$, $[h_k]_{k\in N}\in\tilde{\mathcal{E}}$, we put

$$[g_j]_{j\in N}+[h_k]_{k\in N}=[g_j+h_j]_{j\in N}\,,$$

$$[g_j]_{j\in N}\cdot[h_k]_{k\in N}=[g_j\cdot h_j]_{j\in N}\,.$$

These definitions are clearly well-defined. The following relations are easy to be seen:

$$[n^k]+[n^l]=[n^{\max(k,l)}]\,.$$

$$[n^k]\cdot[n^l]=[n^{k+l}]\,.$$

$$[n^k]+[e^n]=[e^n]\,.$$

$$[n^k]\cdot[e^n]=[e^n]\,.$$

2. Expansion growth of foliations.

In this section, we define the expansion growth of a foliation on a compact manifold. Let $\mathcal{F}$ be a codimension q foliation of class C^0 on a compact $(p+q)$-dimensional manifold M.

Let $\mathcal{U}=\{(U_i,\,\varphi_i)\}_{i=1}^{A}$ be a *good foliation cover* of $(M,\,\mathcal{F})$. That is, it satisfies the following conditions.

(1) $\{U_i\}_{i=1}^{A}$ is an open covering of M.

(2) φ_i is a homeomorphism of U_i to $B_1^p(o)\times B_1^q(o_i)$, where $o=(0, \cdots, 0)\in \boldsymbol{R}^p$, $o_i=(3i, \cdots, 3i)\in \boldsymbol{R}^q$ and $B_1^p(z)=\{x\in \boldsymbol{R}^p ; |x-z|<1\}\subset \boldsymbol{R}^p$.

(3) If $U_i\cap U_{i'}\neq\emptyset$, then there exists a homeomorphism $\phi_{ii'}: \Phi_{i'}(U_i\cap U_{i'})\rightarrow \Phi_i(U_i\cap U_{i'})$ such that $\Phi_i=\phi_{ii'}\circ\Phi_{i'}$ on $U_i\cap U_{i'}$, where $\Phi_i=\mathrm{pr}\circ\varphi_i$ and $\mathrm{pr}: \boldsymbol{R}^p\times \boldsymbol{R}^q \rightarrow \boldsymbol{R}^q$ is the projection to the second factor.

We put $B_i=B_1^q(o_i)\subset \boldsymbol{R}^q$ and $T_i=\varphi_i^{-1}(\{o\}\times B_i)\subset M$. Put $T=\bigcup_{i=1}^{A} T_i\subset M$ and $B=\bigcup_{i=1}^{A} B_i\subset \boldsymbol{R}^q$. We remark that $\Phi_i|_{T_i}$ is a homeomorphism of T_i to B_i. We define a map $\iota: B\rightarrow T$ by $\iota(x)=(\Phi_i|_{T_i})^{-1}(x)\in T_i\subset T$ for $x\in B_i\subset B$.

We define a pseudogroup of local homeomorphisms of $B\subset \boldsymbol{R}^q$ induced by a foliation cover $\mathcal{U}$. Put $\mathcal{H}_1=\{id_B\}\cup\{\phi_{ii'}\}_{i,i'=1}^{A}$. Then we define $\mathcal{H}_n$ ($n\in \boldsymbol{N}$) as follows:

$$\mathcal{H}_n=\{f_n\circ\cdots\circ f_1 ; f_l\in\mathcal{H}_1\}.$$

Here the composition map $f_2\circ f_1$ is defined on $\mathrm{domain}(f_1)\cap f_1^{-1}(\mathrm{domain}(f_2))$. Put $\mathcal{H}=\bigcup_{n\in \boldsymbol{N}}\mathcal{H}_n$. We call $\mathcal{H}$ a *pseudogroup* of local homeomorphisms of B induced by a foliation cover $\mathcal{U}$.

Let x, y be points of B and let n be a natural number. We define a number $D_n^{\mathcal{H}_1}(x, y)$ as follows:

$$D_n^{\mathcal{H}_1}(x, y)=\max\{|f(x)-f(y)| ; f\in\mathcal{H}_n,\ x, y\in\mathrm{domain}(f)\}.$$

Let ε be a positive number. Two points x and y of B are said to be *$(n, \varepsilon, \mathcal{H}_1)$-separated* if $D_n^{\mathcal{H}_1}(x, y)\geqq\varepsilon$. Otherwise x and y are said to be *$(n, \varepsilon, \mathcal{H}_1)$-close*. Let K' be a subset of $B\subset \boldsymbol{R}^q$. A subset $S\subseteqq B$ is said to be an *$(n, \varepsilon, \mathcal{H}_1, K')$-separating set* if S is a subset of K' and for any $x, y\in S$, x and y are $(n, \varepsilon, \mathcal{H}_1)$-separated. A subset $R\subseteqq B$ is said to be an *$(n, \varepsilon, \mathcal{H}_1, K')$-spanning set* if for any $x\in K'$, there exists $y\in R$ such that x and y are $(n, \varepsilon, \mathcal{H}_1)$-close. Put

$$s_n^{\mathcal{H}_1}(\varepsilon, K')=\max\{\# S ; S \text{ is an } (n, \varepsilon, \mathcal{H}_1, K')\text{-separating set}\},$$

$$r_n^{\mathcal{H}_1}(\varepsilon, K')=\min\{\# R ; R \text{ is an } (n, \varepsilon, \mathcal{H}_1, K')\text{-spanning set}\}.$$

Let $(\varepsilon_j)_{j\in \boldsymbol{N}}$ be a monotone decreasing sequence of positive numbers which converges to 0. We can easily notice that $(s_n^{\mathcal{H}_1}(\varepsilon_j, K'))_{j\in \boldsymbol{N}}$ and $(r_n^{\mathcal{H}_1}(\varepsilon_j, K'))_{j\in \boldsymbol{N}}$ are elements of $\tilde{\mathcal{I}}$. We recall the following.

THEOREM 2.1 ([**E1**]). *Let $\mathcal{F}$ be a codimension q foliation class C^0 on a compact $(p+q)$-dimensional manifold M and let K be a subset of M. Let $\mathcal{U}$ be a good foliation cover of $(M, \mathcal{F})$ and let $(\varepsilon_j)_{j\in \boldsymbol{N}}$ be a monotone decreasing sequence of positive numbers which converges to* 0. *Put $\Phi(K)=\bigcup_{i=1}^{A}\Phi_i(K\cap U_i)\subseteqq B$. Then*

$$[s_n^{\mathcal{H}_1}(\varepsilon_j, \Phi(K))]_{j\in \boldsymbol{N}}=[r_n^{\mathcal{H}_1}(\varepsilon_j, \Phi(K))]_{j\in \boldsymbol{N}}\in\tilde{\mathcal{E}}$$

and this growth type is independent of the choice of $\mathcal{U}$ *and* $(\varepsilon_j)_{j\in N}$. ■

By Theorem 2.1, $[s_n^{\mathcal{H}_1}(\varepsilon_j, \Phi(K))]_{j\in N}=[r_n^{\mathcal{H}_1}(\varepsilon_j, \Phi(K))]_{j\in N}\in\tilde{\mathcal{E}}$ depends only on $(M, \mathcal{F})$ and $K\subseteq M$ and it is a topological invariant for foliations on compact manifolds.

DEFINITION 2.2. We call the above growth type the *expansion growth* of $(M, \mathcal{F})$ on K and denote it by

$$\eta(K, \mathcal{F})(\text{or simply } \eta(K)) \in \tilde{\mathcal{E}}.$$

Ghys, Langevin and Walczak defined the entropy $h(\mathcal{F}, \mathcal{U})$ of a foliation $\mathcal{F}$ to be

$$h(\mathcal{F}, \mathcal{U}) = \lim_{\varepsilon\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log s_n^{\mathcal{H}_1}(\varepsilon, B).$$

From the definition, we can easily see that the entropy of a foliation $\mathcal{F}$ on a compact manifold M is not zero if and only if $\eta(M, \mathcal{F})\in\tilde{\mathcal{E}}$ has quasi-exponential growth.

EXAMPLE.

(1) If $(M, \mathcal{F})$ is a bundle foliation, then $\eta(M, \mathcal{F})=[1]$.

(2) If $(M, \mathcal{F})$ is a Reeb foliation, then $\eta(M, \mathcal{F})=[n]$.

(3) If $(T^m, \mathcal{F})$ is a linear foliation on the m-dimensional torus, then $\eta(T^m, \mathcal{F})=[1]$.

(4) If $(T^2, \mathcal{F})$ is a Denjoy foliation, then $\eta(T^2, \mathcal{F})=[n]$.

Finally we describe several properties of the expansion growth.

LEMMA 2.3 ([**E**]). *Let* K *and* K' *be subsets of* M. *Then*

$$\eta(\bar{K}) = \eta(K).$$

$$\eta(K\cup K') = \eta(K)+\eta(K')$$

and if $K\subseteq K'$ *then*

$$\eta(K) \leqq \eta(K').$$ ■

3. Expansion growth of smooth codimension-one foliations.

In this section, we restrict ourselves to a transversely oriented codimension 1 foliation $\mathcal{F}$ of class C^2 on a compact manifold M except for Lemma 3.3. The following theorem is the main theorem of this paper.

THEOREM 3.1. *Let* $\mathcal{F}$ *be a transversely oriented codimension* 1 *foliation of class* C^2 *on a compact manifold* M. *Let* K *be an* $\mathcal{F}$*-saturated set.*

(1) *If* $\bar{K}$ *has a resilient leaf, then* $\eta(K)=[e^n]$.

(2) *If* $\bar{K}$ *has no resilient leaf and* level$(K)<\infty$, *then* $\eta(K)=[n^{\text{level}(K)}]$.

(3) *Otherwise,* $\eta(K)=[1, n, n^2, \cdots]$.

Here we define the *level* of a leaf L as follows:

$$\text{level}(L)=\sup\{l\,;\,\bar{L}_0 \subsetneqq \cdots \subsetneqq \bar{L}_l=\bar{L} \text{ such that } L_k \text{ is a leaf of } \mathcal{F}\}.$$

We define the level of an $\mathcal{F}$-saturated set K as follows:

$$\text{level}(K)=\sup\{\text{level}(L)\,;\,L \text{ is a leaf contained in } K\}.$$

For the proof of Theorem 3.1, take an 1-dimensional foliation $\mathcal{G}$ transverse to $\mathcal{F}$. Let $\mathcal{U}$ be a good foliation cover of $(M, \mathcal{F})$. We may assume that $\mathcal{U}=\{(U_i, \varphi_i)\}_{i=1}^{A}$ is a bidistinguished foliation cover of $(\mathcal{F}, \mathcal{G})$. We use the notations which we defined in section 2. Put $T_i=\varphi_i^{-1}(\{o\}\times B_i)$. We note that $\mathcal{U}$ can be taken so that for $i\neq i'$, $\bar{T}_i\cap\bar{T}_{i'}=\emptyset$. Put $T=\bigcup_{i=1}^{A} T_i\subset M$. We identify $T\subset M$ with $B\subset \boldsymbol{R}$ by the map ι^{-1}. Let $\mathcal{H}_1$, $\mathcal{H}_n$ and $\mathcal{H}$ be as in section 2. We may assume that each element of $\mathcal{H}$ is orientation-preserving.

We define the *growth* of a leaf L as follows:

$$gr(L)=[\#\mathcal{H}_n(y)]\in\mathcal{E}$$

where $y\in L\cap T$.

We review the theory of the level of leaves developed by Cantwell and Conlon ([**C-C1**]).

PROPOSITION 3.2 ([**C-C1**]). *Let* $\mathcal{F}$ *be a transversely oriented codimension* 1 *foliation of class* C^2 *on a compact manifold* M.

(1) *If* Y *is a local minimal set, then* $\text{level}(Y)<\infty$.

(2) *If* L *is a totally proper leaf, then* $\bar{L}$ *consists of finitely many proper leaves and* $gr(L)=[n^{\text{level}(L)}]$.

(3) *For a non-negative integer* l, *the union of leaves whose level is at most* l *is compact.*

(4) *The union of leaves whose level is finite is dense in* M.

(5) *Each leaf whose level is infinite has no proper side.* ■

We recall the following result for codimension 1 foliations of class C^0 which was proved in [**E1**].

PROPOSITION 3.3 ([**E1**]). *Let* $\mathcal{F}$ *be a transversely oriented codimension* 1 *foliation of class* C^0 *on a compact manifold* M.

(1) $\eta(M)\leqq[e^n]$.

(2) *If* L *is a resilient leaf, then* $\eta(L)=[e^n]$.

(3) *If* L *is a totally proper leaf, then* $\eta(L)=gr(L)$.

(4) *If* M *is a minimal set without holonomy, then* $\eta(M)=[1]$. ■

By (3) of Proposition 3.3 and (2) of Proposition 3.2, we have the following lemma.

LEMMA 3.4. *If L is a totally proper leaf, then* $\eta(L)=[n^{\mathrm{level}(L)}]$. ■

The key step for the proof of Theorem 3.1 is to prove the following.

PROPOSITION 3.5. *Let Y be an open local minimal set without holonomy such that $Y\neq M$. Then*

$$\eta(Y)=\eta(\delta Y)+[n]\cdot gr(\delta Y).$$

where δY is the union of border leaves of Y (which consists of finitely many leaves) and

$$gr(\delta Y)=\sum_{L \text{ is a leaf contained in } \delta Y} gr(L).$$

By this proposition, we can easily deduce the following lemma.

LEMMA 3.6. *Let Y be an open local minimal set.*

(1) *If $\bar{Y}$ has a resilient leaf, then* $\eta(Y)=[e^n]$.

(2) *If $\bar{Y}$ has no resilient leaf, then* $\eta(Y)=[n^{\mathrm{level}(Y)}]$.

PROOF. (1) Suppose that $\bar{Y}$ has a resilient leaf. Then by (1) and (2) of Proposition 3.3, we have $\eta(Y)=[e^n]$.

(2) We consider the case where $\bar{Y}$ has no resilient leaf. We remark that Y is an open local minimal set without holonomy. Moreover by the theorem of Sacksteder ([**S**]), $\bar{Y}$ has no exceptional local minimal set. So by (2) of Proposition 3.2, $\bar{Y}-Y$ consists of finitely many totally proper leaves. So by $\mathrm{level}(Y)<\infty$ and $\mathrm{level}(\bar{Y}-Y)=\mathrm{level}(Y)-1$, we have $gr(\delta Y)=[n^{\mathrm{level}(Y)-1}]$ and $\eta(\delta Y)=[n^{\mathrm{level}(Y)-1}]$ by Lemma 3.4. Hence if $Y\neq M$ then by Proposition 3.5.

$$\eta(Y)=\eta(\delta Y)+[n]\cdot gr(\delta Y)=[n^{\mathrm{level}(Y)-1}]+[n]\cdot[n^{\mathrm{level}(Y)-1}]=[n^{\mathrm{level}(Y)}].$$

If $Y=M$ then obviously $\mathrm{level}(Y)=0$. So by (4) of Proposition 3.3, we have

$$\eta(Y)=[1]=[n^{\mathrm{level}(Y)}].$$ ■

We prepare some arguments to show Proposition 3.5. We fix a nuclear-arm decomposition of Y (cf. [**D**]):

$$Y=X\cup K_1\cup\cdots\cup K_s,$$

where X is a nuclear and K_m ($m=1,\cdots,s$) is an arm. We may assume that $\mathcal{G}|_{K_m}$ is a product foliation and that $\mathcal{F}|_{K_m}$ is a foliated bundle. Moreover we may assume that ∂T does not intersect any K_m.

Let $\{I_i\}_{i\in \boldsymbol{N}}$ be the set of all components of $Y\cap T$. Identifying T and B in the section 2 by $\iota: B\to T$, we put $I_i=(a_i, b_i)$ ($i\in \boldsymbol{N}$). By taking sufficiently large nuclear, we may assume that, if I_i is contained in some arm K_m and $I_i\cap\mathrm{domain}(f)\neq\emptyset$ for some $f\in\mathcal{H}_1$, then $I_i\subset\mathrm{domain}(f)$. By choosing the indices i carefully, we may assume that there exists a map $\kappa:\boldsymbol{N}\to\boldsymbol{N}$ such that $i\leqq\kappa(n)$

if and only if there exists $f\in\mathcal{H}_n$ and $x\in I_i\cap\mathrm{domain}(f)$ such that $f(x)\in X$. We can easily see that $[\kappa(n)]=gr(\partial Y)$. Put

$$\mathcal{G}_1=\{f|_{\mathrm{domain}(f)\cap I_i}\,;\,f\in\mathcal{H}_1,\ i\in\boldsymbol{N}\},$$

$$\mathcal{G}_n=\{f_n\circ\cdots\circ f_1\,;\,f_k\in\mathcal{G}_1\}\quad(n\in\boldsymbol{N}),$$

and

$$\mathcal{G}=\bigcup_{n\in\boldsymbol{N}}\mathcal{G}_n.$$

We remark that for any $f\in\mathcal{H}$ or for any $f\in\mathcal{G}$, f can be extended to the C^2-diffeomorphism of $\overline{\mathrm{domain}(f)}$ to $\overline{\mathrm{range}(f)}$.

For each $m\in\{1,\cdots,s\}$, fix a component I_{i_m} of $K_m\cap T$. For any component I of $K_m\cap T$, fix a map $f_I\in\mathcal{G}$ satisfying the following conditions:

(1) $f_I(I_{i_m})=I$.

(2) When we decompose $f_I=g_n\circ\cdots\circ g_1$ ($g_k\in\mathcal{G}_1$, $n\in\boldsymbol{N}$), $g_k\circ\cdots\circ g_1(I_{i_m})$ ($k=0,\cdots,n$) are distinct components of $K_m\cap T$.

Since $\mathcal{F}|_Y$ is without holonomy, $\pi_1(Y)$ acts freely on each leaf J of $\mathcal{G}|_Y$. Since Y is an open local minimal set, the subgroup G_J of $\mathrm{Homeo}_+(J)$ induced by the action of $\pi_1(Y)$ has the minimal set J. So there exists a homeomorphism h_J of J to $\boldsymbol{R}$ (resp. S^1) such that the subgroup $G_{\boldsymbol{R}}$ (resp. G_{S^1}) of $\mathrm{Homeo}_+(\boldsymbol{R})$ (resp. $\mathrm{Homeo}_+(S^1)$) induced by h_J and G_J is a subgroup of translations of $\boldsymbol{R}$ (resp. rotations of S^1). We identify the group of translations of $\boldsymbol{R}$ with $\boldsymbol{R}$. By the above consideration, there exists a homeomorphism h_i ($i\in\boldsymbol{N}$) of I_i into $\boldsymbol{R}$ satisfying the following condition. For any $f\in\mathcal{G}$, there exists $\alpha_f\in G_{\boldsymbol{R}}$ such that

$$h(f(x))=h(x)+\alpha_f\quad\text{for any } x\in\mathrm{domain}(f)$$

where $h(x)=h_i(x)$ if $x\in I_i$. Moreover we may suppose that $\alpha_{f_I}=0$ for any component I of $(K_1\cup\cdots\cup K_s)\cap T$. We remark that if I_i is contained in some arm then $h_i(I_i)=\boldsymbol{R}$.

LEMMA 3.7. *There exists a large real number α such that*

$$\alpha>|\alpha_f|$$

for any $f\in\mathcal{G}_1$.

PROOF. Fix $m\in\{1,\cdots,s\}$. Let f be an element of $\mathcal{G}_1$ in the arm K_m. That is, domain(f) is a component I of $K_m\cap T$ and range(f) is a component I' of $K_m\cap T$. We consider the map $f_{I'}^{-1}\circ f\circ f_I$ of I_{i_m}. Here we can decompose

$$f_{I'}^{-1}\circ f\circ f_I=g_n\circ\cdots\circ g_1\quad(g_k\in\mathcal{G}_1,\ n\in\boldsymbol{N}).$$

Here $g_k\circ\cdots\circ g_1(I_{i_m})$ ($k=0,\cdots,n$) are components of $K_m\cap T$. We remark that any three components of these components do not coincide.

Put $f_k=g_k\circ\cdots\circ g_1$ ($k=0,\cdots,n$). Let x, y be points of $\bar{I}_{i_m}$. Then

$$\begin{aligned}\frac{f_n'(x)}{f_n'(y)} &= \prod_{k=1}^{n} \frac{g_k'(f_{k-1}(x))}{g_k'(f_{k-1}(y))}\\ &= \prod_{k=1}^{n}\left(1+\frac{g_k'(f_{k-1}(x))-g_k'(f_{k-1}(y))}{g_k'(f_{k-1}(y))}\right)\\ &= \prod_{k=1}^{n}\left(1+\frac{g_k''(z_k)(f_{k-1}(x)-f_{k-1}(y))}{g_k'(f_{k-1}(y))}\right)\\ &\leqq \prod_{k=1}^{n}(1+C\,|f_{k-1}(x)-f_{k-1}(y)|)\\ &\leqq e^{C\sum_{k=1}^{n}|f_{k-1}(x)-f_{k-1}(y)|}\\ &\leqq e^{4AC}\end{aligned}$$

where z_k is a point of $\overline{f_{k-1}(I_{i_m})}$ determined by the mean value theorem and

$$C=\frac{\max\{|g''(z)|\,;\,g\in\mathcal{H}_1,\ z\in\overline{\mathrm{domain}(g)}\}}{\min\{g'(z)\,;\,g\in\mathcal{H}_1,\ z\in\overline{\mathrm{domain}(g)}\}}$$

and

$$\sum_{k=1}^{n}|f_{k-1}(x)-f_{k-1}(y)|\leqq 2\sum_{i=1}^{A}|T_i|=4A.$$

So

$$f_n'(x)\leqq e^{4AC}f_n'(y).$$

Since f_n is a diffeomorphism of I_{i_m}, there exists $y\in I_{i_m}$ such that $f_n'(y)=1$. Therefore

$$f_n'(x)\leqq e^{4AC}$$

for any $x\in\bar{I}_{i_m}$. Hence we have

$$f_n\left(\frac{a_{i_m}+b_{i_m}}{2}\right)\in[a_{i_m}+c,\ b_{i_m}-c]\subset(a_{i_m},\ b_{i_m})=I_{i_m}$$

where

$$c=\frac{b_{i_m}-a_{i_m}}{2e^{4AC}}.$$

So

$$\begin{aligned}|\alpha_{f_{I'}^{-1}\circ f\circ f_I}|=|\alpha_{f_n}|&=\left|h_{i_m}\left(f_n\left(\frac{a_{i_m}+b_{i_m}}{2}\right)\right)-h_{i_m}\left(\frac{a_{i_m}+b_{i_m}}{2}\right)\right|\\ &\leqq h_{i_m}(b_{i_m}-c)-h_{i_m}(a_{i_m}+c).\end{aligned}$$

By $\alpha_{f_I}=\alpha_{f_{I'}}=0$, we have

$$\alpha_f=\alpha_{f_{I'}^{-1}\circ f\circ f_I}.$$

Since c does not depend on the choice of f, there exists a large real number α_m such that

$$\alpha_m > |\alpha_f|$$

for any $f\in\mathcal{G}_1$ in the arm K_m.

The cardinality of elements of $\mathcal{G}_1$ which is not in any arms are finite. So there exists a large real number α such that

$$\alpha > |\alpha_f|$$

for any $f\in\mathcal{G}_1$. ■

PROOF OF PROPOSITION 3.5. First we show that

$$\eta(Y) \geqq \eta(\delta Y)+[n]gr(\delta Y).$$

Obviously $\eta(Y)\geqq\eta(\delta Y)$. Fix a leaf $F\subseteqq\delta Y$. We show that $\eta(Y)\geqq[n]gr(F)$. We may assume that the negative side of F is contained in Y and $b_1\in\partial I_1$ is a point of F. For each $n\in\boldsymbol{N}$ and for each $b\in\mathcal{H}_n(b_1)-\mathcal{H}_{n-1}(b_1)$, we fix $f_b\in\mathcal{H}_n$ such that $b=f_b(b_1)$. Since ∂T does not intersect any arms, there exists $c\in I_1$ such that f_b is defined on $[c, b_1]$ for any $b\in\mathcal{H}(b_1)$. There exists a loop γ based on b_1 contained in F such that the holonomy map f_γ of I_1 induced by γ is a contraction to b_1. Here we may assume that f_γ is defined on $[c, b_1]$. Take a large natural number N such that $f_\gamma\in\mathcal{H}_N$.

Take a positive real number $\delta<1$ such that $\delta<|c-f_\gamma(c)|$ and δ is sufficiently small for $\mathcal{U}$. For $n\in\boldsymbol{N}$, put

$$S_n = \{f_b(f_\gamma^l(c))\,;\, b\in\mathcal{H}_n(b_1),\ l=1, \cdots, n\}.$$

Then S_n is an $(A(N+1)n, \delta, \mathcal{H}_1, Y\cap T)$-separating set. For we take any two points x, y of S_n. We may assume that $x\leqq y$ and $x=f_b(f_\gamma^l(c))$. We apply the argument of Lemma 2.5 and Theorem 3.3 in [**E1**] along $f_\gamma^{-1}\circ f_b^{-1}\in\mathcal{H}_{(N+1)n}$. Then there exists $f\in\mathcal{H}_{A(N+1)n}$ such that $x, y\in\text{domain}(f)$ and $|f(x)-f(y)|\geqq\delta$. So x and y are $(A(N+1)n, \delta, \mathcal{H}_1)$-separated. It follows that

$$s^{\mathcal{H}_1}_{A(N+1)n}(\delta, Y\cap T) \geqq \#S_n \geqq n\cdot\#\mathcal{H}_n(b_1).$$

Therefore

$$\eta(Y) \geqq [n]gr(F).$$

Hence

$$\eta(Y) \geqq \eta(\delta Y)+[n]gr(\delta Y).$$

Next we show that

$$\eta(Y) \leqq \eta(\delta Y)+[n]gr(\delta Y).$$

Fix a positive real number ε. There exists a large integer n_0 such that $|I_i|<\varepsilon$ for any $i>\kappa(n_0)$. We take a positive real number δ such that for any $i\leqq\kappa(n_0)$ and for any $x, y\in I_i$, if $|x-y|\geqq\varepsilon$ then $|h_i(x)-h_i(y)|\geqq\delta$. We take points $z_1, \cdots, z_N\in\boldsymbol{R}$ satisfying the following conditions.

$$\begin{cases} z_1 \leqq h_i\left(a_i+\frac{\varepsilon}{2}\right) & \text{if } |I_i| \geqq \varepsilon . \\ z_N \geqq h_i\left(b_i-\frac{\varepsilon}{2}\right) & \text{if } |I_i| \geqq \varepsilon . \\ 0 \leqq z_{k+1}-z_k < \delta . \\ z_N-z_1 \geqq \alpha . \\ h_i(I_i)\cap\{z_1, \cdots, z_N\} \neq \emptyset & \text{for any } i \in \boldsymbol{N}. \end{cases}$$

Fix a positive integer n. Put

$$R_n = \bigcup_{i=1}^{\kappa(n+n_0)} h_i^{-1}(\{z_k+l\alpha\,;\, k=1, \cdots, N,\ l=-n, \cdots, n\}).$$

Let R'_n be an $(n, \varepsilon, \mathcal{H}_1, \delta Y\cap T)$-spanning set with the minimum cardinality.

We will show that $R_n\cup R'_n$ is an $(n, 2\varepsilon, \mathcal{H}_1, Y\cap T)$-spanning set. Take any point x of $Y\cap T$. Let I_i be a component of $Y\cap T$ containing x.

First we consider the case where $i\leqq\kappa(n+n_0)$. Let $y\in R_n$ be a point which gives the minimum value of $|x-y|$. We remark that y is a point of I_i. We may assume that $x\leqq y$. We show that $D_n^{\mathcal{H}_1}(x, y)<\varepsilon$. Suppose $D_n^{\mathcal{H}_1}(x, y)\geqq\varepsilon$. Then there exists $f\in\mathcal{G}_n$ such that $|f(x)-f(y)|\geqq\varepsilon$. Let I_j be a component of $Y\cap T$ containing $f(x)$ and $f(y)$. By

$$|I_j| > |f(x)-f(y)| \geqq \varepsilon,$$

we have $j\leqq\kappa(n_0)$. Then

$$(f(x), f(y))\cap\left(a_j+\frac{\varepsilon}{2}, b_j-\frac{\varepsilon}{2}\right) \neq \emptyset$$

and

$$|h_j(f(x))-h_j(f(y))| \geqq \delta .$$

So

$$(h_j(f(x)), h_j(f(y)))\cap(z_1, z_N) \neq \emptyset .$$

On the other hand, $h_j(f(x))=h_i(x)+\alpha_f$ and $h_j(f(y))=h_i(y)+\alpha_f$. So $|h_i(x)-h_i(y)|\geqq\delta$. By $f\in\mathcal{G}_n$ and Lemma 3.7, we have $|\alpha_f|\leqq n\alpha$. So

$$(h_i(x), h_i(y))\cap(z_1-n\alpha, z_N+n\alpha) \neq \emptyset .$$

Therefore there exists a point

$$z \in \{z_k+l\alpha\,;\, k=1, \cdots, N,\ l=-n, \cdots, n\}\cap(h_i(x), h_i(y)) .$$

So

$$h_i^{-1}(z) \in R_n\cap(x, y) .$$

This contradicts the choice of y. Hence $D_n^{\mathcal{H}_1}(x, y)<\varepsilon$.

Next we consider the case where $i>\kappa(n+n_0)$. Obviously, $|x-b_i|<\varepsilon$. By $b_i\in\delta Y\cap T$, there exists $y\in R_n'$ such that $D_n^{\mathcal{H}_1}(b_i, y)<\varepsilon$. We show that $D_n^{\mathcal{H}_1}(x, y)<2\varepsilon$. Given any $f\in\mathcal{H}_n$ such that $x, y\in$domain(f). By $f\in\mathcal{H}_n$ and the choice of $\kappa(n)$, $f(x)$ is contained in the arm K_m containing x. So f is defined on $[a_i, b_i]$ and $(f(a_i), f(b_i))=I_j$ for some $j>\kappa(n_0)$. So

$$|f(x)-f(b_i)| < |f(a_i)-f(b_i)| < \varepsilon .$$

By $D_n^{\mathcal{H}_1}(b_i, y)<\varepsilon$, we have $|f(b_i)-f(y)|<\varepsilon$. Therefore $|f(x)-f(y)|<2\varepsilon$. So $D_n^{\mathcal{H}_1}(x, y)<2\varepsilon$.

By the above two results, $R_n\cup R_n'$ is an $(n, 2\varepsilon, \mathcal{H}_1, Y\cap T)$-spanning set. Hence

$$r_n^{\mathcal{H}_1}(2\varepsilon, Y\cap T)\leqq \#R_n+\#R_n'\leqq r_n^{\mathcal{H}_1}(\varepsilon, \delta Y\cap T)+N(2n+1)\cdot\kappa(n+n_0).$$

Then we can take a large positive real number C such that

$$r_n^{\mathcal{H}_1}(2\varepsilon, Y\cap T)\leqq r_n^{\mathcal{H}_1}(\varepsilon, \delta Y\cap T)+Cn\cdot\kappa(2n)$$

for any $n\in\boldsymbol{N}$. So

$$\eta(Y)\leqq\eta(\delta Y)+[n]gr(\delta Y).$$

This completes the proof of Proposition 3.5. ■

PROOF OF THEOREM 3.1. If $\bar{K}$ has a resilient leaf then by (1) and (2) of Proposition 3.3, we have $\eta(K)=[e^n]$.

We consider the case where $\bar{K}$ has no resilient leaf. We remark that $\bar{K}$ contains no open local minimal set with holonomy. Moreover by the theorem of Sacksteder, $\bar{K}$ contains no exceptional local minimal set. These facts imply that each local minimal set contained in $\bar{K}$ is a totally proper leaf or an open local minimal set without holonomy whose closure has no resilient leaf. By Proposition 3.2, we can take a set $\{L_j\}_{j\in\boldsymbol{N}}$ of leaves contained in $\bar{K}$ satisfying the following conditions.

(1) L_j is a totally proper leaf or a leaf contained in some open local minimal set Y without holonomy whose closure has no resilient leaf.

(2) $\bigcup_{j\in\boldsymbol{N}}L_j$ is dense in $\bar{K}$.

(3) $\sup\{\mathrm{level}(L_j)\,;\,j\in\boldsymbol{N}\}=\mathrm{level}(K)$.

(4) Any border leaves of components of $M-\bar{K}$ are contained in $\{L_j\}_{j\in\boldsymbol{N}}$.

By Lemma 3.4 and Lemma 3.6, we have $\eta(\bar{L}_j)=[n^{\mathrm{level}(L_j)}]$. Put $K_m=\bigcup_{j=1}^m\bar{L}_j$. Then

$$\begin{aligned}\eta(K_m)&=\eta(\bar{L}_1)+\cdots+\eta(\bar{L}_m)\\&=[n^{\mathrm{level}(L_1)}]+\cdots+[n^{\mathrm{level}(L_m)}]=[n^{\mathrm{level}(K_m)}].\end{aligned}$$

We will show that

$$\eta(\bar{K}) = [n^{\mathrm{level}(K_1)}, n^{\mathrm{level}(K_2)}, n^{\mathrm{level}(K_3)}, \cdots].$$

Fix a positive real number $\varepsilon<1$. Then we can take a large natural number m such that for any component I of $T-K_m$ if $|I|\geqq\varepsilon$ or $\partial I\cap\partial T\neq\emptyset$ then I is a component of $T-\bar{K}$.

Let n be a natural number. Let R_n be an $(n, \varepsilon, \mathcal{H}_1, K_m\cap T)$-spanning set with the minimum cardinality. We show that R_n is an $(n, 2\varepsilon, \mathcal{H}_1, \bar{K}\cap T)$-spanning set. Take any point $x\in\bar{K}\cap T$. If $x\in K_m\cap T$ then there exists $y\in R_n$ such that $D_n^{\mathcal{H}_1}(x, y)<\varepsilon$. We consider the case where $x\in(\bar{K}-K_m)\cap T$. Let Y be a component of $M-K_m$ containing x. By the choice of K_m, Y is a foliated bundle and $(Y\cup\delta Y)\cap\partial T=\emptyset$ and $|I|<\varepsilon$ for any component I of $Y\cap T$. So there exists $z\in\delta Y\cap T$ such that x and z are contained in some T_i. Since z is a point of $K_m\cap T_i$, there exists $y\in R_n\cap T_i$ such that $D_n^{\mathcal{H}_1}(z, y)<\varepsilon$. We show that $D_n^{\mathcal{H}_1}(x, y)<2\varepsilon$. Take any $f\in\mathcal{H}_n$ such that $x, y\in\mathrm{domain}(f)$. By $(Y\cup\delta Y)\cap\partial T=\emptyset$, we have $z\in\mathrm{domain}(f)$. By $|I|<\varepsilon$ for any component I of $Y\cap T$, we have $|f(x)-f(z)|<\varepsilon$. By $D_n^{\mathcal{H}_1}(z, y)<\varepsilon$, we have $|f(z)-f(y)|<\varepsilon$. So $|f(x)-f(y)|<2\varepsilon$. Therefore $D_n^{\mathcal{H}_1}(x, y)<2\varepsilon$. Hence R_n is an $(n, 2\varepsilon, \mathcal{H}_1, \bar{K}\cap T)$-spanning set. It follows that

$$r_n^{\mathcal{H}_1}(2\varepsilon, \bar{K}\cap T)\leqq \# R_n = r_n^{\mathcal{H}_1}(\varepsilon, K_m\cap T).$$

On the other hand,

$$\eta(K_m) = [n^{\mathrm{level}(K_m)}].$$

So there exists a large number C such that

$$r_n^{\mathcal{H}_1}(\varepsilon, K_m\cap T)\leqq Cn^{\mathrm{level}(K_m)}$$

for any $n\in\boldsymbol{N}$. Hence

$$r_n^{\mathcal{H}_1}(2\varepsilon, \bar{K}\cap T)\leqq r_n^{\mathcal{H}_1}(\varepsilon, K_m\cap T)\leqq Cn^{\mathrm{level}(K_m)}$$

for any $n\in\boldsymbol{N}$. This implies that

$$\eta(\bar{K})\leqq [n^{\mathrm{level}(K_1)}, n^{\mathrm{level}(K_2)}, n^{\mathrm{level}(K_3)}, \cdots].$$

Next we show the converse inequality. Obviously for any $m\in\boldsymbol{N}$,

$$\eta(\bar{K})\geqq\eta(K_m) = [n^{\mathrm{level}(K_m)}].$$

So there exists positive numbers δ, C such that

$$C\cdot s_n^{\mathcal{H}_1}(\delta, \bar{K}\cap T)\geqq n^{\mathrm{level}(K_m)}$$

for any $n\in\boldsymbol{N}$. This implies that

$$\eta(\bar{K})\geqq [n^{\mathrm{level}(K_1)}, n^{\mathrm{level}(K_2)}, n^{\mathrm{level}(K_3)}, \cdots].$$

Finally if level$(K)<\infty$ then

$$\eta(K)=\eta(\bar{K})=[n^{\mathrm{level}(K)}]$$

and otherwise

$$\eta(K)=[1, n, n^2, \cdots]. \qquad \blacksquare$$

By Theorem 3.1, we can easily deduce the following conclusion which is proved by Ghys, Langevin and Walczak.

COROLLARY 3.8 ([**G-L-W**]). *Let $\mathcal{F}$ be a transversely oriented codimension* 1 *foliation of class C^2 on a compact manifold M. Then the entropy of $\mathcal{F}$ is not zero if and only if $\mathcal{F}$ has a resilient leaf.* ■

If $\mathcal{F}$ is a real-analytic codimension 1 foliation then level$(M)<\infty$ by [**C-C3**]. So we have the following corollary.

COROLLARY 3.9. *Let $\mathcal{F}$ be a transversely oriented codimension* 1 *foliation of class C^ω on a compact manifold M. Let K be an $\mathcal{F}$-saturated set.*

(1) *If $\bar{K}$ has a resilient leaf, then $\eta(K)=[e^n]$.*

(2) *Otherwise, $\eta(K)=[n^{\mathrm{level}(K)}]$.* ■

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