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Ricci curvature, diameter and optimal volume bound

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§ 1. Motivation and main results.

It is well known that a complete Riemannian n-manifold M with the Ricci curvature $Ric(M)\geqq(n-1)k$ and the diameter $d(M)\leqq D$ has the volume bounded above by the volume $\tilde{v}_k(D)$ of a D-ball in the simply connected space form M_k^n with the constant sectional curvature k. In other words, if we rescale and normalize the metric so that $d(M)=\pi$ and consider the class $\boldsymbol{M}_k$ of all closed Riemannian n-manifold with $Ric(M)\geqq(n-1)k$ and $d(M)=\pi$, then the volume defines a function on $\boldsymbol{M}_k$:

$$vol: \boldsymbol{M}_k \longrightarrow \boldsymbol{R}^+$$

with the range in the interval $(0, \tilde{v}_k(\pi)]$. Note that the Myers theorem implies that k must be smaller than or equal to 1 since $d(M)=\pi$.

For $k=1$, the maximal diameter sphere theorem of Cheng [**Ch**] implies that $\boldsymbol{M}_k$ contains only one element, the n-sphere with its canonical metric *can*. Hence the range of *vol* on $\boldsymbol{M}_1$ contains the single value $\tilde{v}_1(\pi)$. To see that there is no positive lower bound on the function *vol* defineded on $\boldsymbol{M}_k$ for $k\leqq 0$, one can consider the flat tori: $S^1(\varepsilon)\times T^{n-1}$, $\varepsilon>0$ where $S^1(\varepsilon)$ is the circle with radius ε in $\boldsymbol{R}^2$ and T^{n-1} is a flat $(n-1)$-torus. For positive $k<1$, one can consider the suspension M_ε of an $(n-1)$-sphere, S_ε^{n-1}, in $\boldsymbol{R}^n$ with radius $\varepsilon<1$. Namely, $M_\varepsilon= S_\varepsilon^{n-1}\times_{\sin}[0, \pi]$. Note that M_ε is the n-sphere S^n. Then smooth the two singular points and rescale the metric to obtain a metric g_ε on M_ε with $d(g_\varepsilon)=\pi$ and $\min Ric(g_\varepsilon)\geqq 1-\eta_1(\varepsilon)$ and $vol(g_\varepsilon)\leqq\eta_2(\varepsilon)$ where the positive functions $\eta_1(\varepsilon)$ and $\eta_2(\varepsilon)$ approach zero as ε goes to zero. See also [**GP**] for a similar construction. This indicates that the lower bound of *vol* on $\boldsymbol{M}_k$ is also zero for positive $k<1$.

For the upper bound of *vol* on $\boldsymbol{M}_k$, one may ask if the upper bound $\tilde{v}_k(\pi)$ is obtainable by some Riemannian n-manifold in $\boldsymbol{M}_k$? The answer is yes only when $k=1/4$ or 1. They are obtained by $(\boldsymbol{RP}^n, 4can)$ and (S^n, can), respectively. Therefore it is natural to ask the following

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QUESTION. Is the upper bound $\tilde{v}_k(\pi)$ optimal for $k \neq 1/4$ or 1?

In other words, can one find an $M \in \boldsymbol{M}_k$ with the volume as close to $\tilde{v}_k(\pi)$ as possible for $k \neq 1/4$ or 1? If one asks the same question for the sectional curvature, then the answer is No! There does exist an upper bound smaller than $\tilde{v}_k(\pi)$ for $k \neq 1/4$ or 1. This was studied by the author in [**Wu1**] for $k>0$ and also independently by Grove and Petersen in [**GP**] for general k. For the Ricci curvature case, this is still unknown. However, for the subclass $\boldsymbol{M}_k^\Lambda = \{M \in \boldsymbol{M}_k \mid Ric(M) \leqq \Lambda\}$ of $\boldsymbol{M}_k$, the answer is again No! Specifically, we prove

THEOREM 1. *Given any $\Lambda \geqq k(n-1)$, there are a function $\Phi_\Lambda: (-\infty, 1) \to \boldsymbol{R}^+$ and a number $\delta_\Lambda > 0$ depending only on Λ and n with the properties*:

(1) *$\Phi_\Lambda(k) \leqq \tilde{v}_k(\pi)$ and equality holds only when $k=1/4$.*

(2) *$vol(M) \leqq \Phi_\Lambda(k)$ for any $M \in \boldsymbol{M}_k^\Lambda$.*

(3) *In case that $k=1/4$, then $M \in \boldsymbol{M}_k^\Lambda$ is diffeomorphic to RP^n, provided that $vol(M) \geqq \tilde{v}_k(\pi) - \delta_\Lambda$.*

Properties (1) and (2) together show that $\tilde{v}_k(\pi)$ is not an optimal upper bound for the volume function vol on $\boldsymbol{M}_k^\Lambda$ with $k \neq 1/4$ or 1. Theorem 1-(3) was also proved by Anderson [**A2**] for the case $k=1$. Note that if a Riemannian n-manifold M in $\boldsymbol{M}_k$ has the volume close to $\tilde{v}_k(\pi)$, then the relative volume comparison theorem implies that every r-ball in M has the volume close to $\tilde{v}_k(r)$ for $r \leqq \pi$. This motivates a notion to measure how the volume of a small ball is close to that of a small ball in the Euclidean space.

DEFINITION 1. For any $\theta \in (0, 1)$ we say that a complete Riemannian n-manifold M has θ-volume-radius, $vol-rad_\theta(M) \geqq r_0$ if for all $p \in M$ and $0 < r \leqq r_0$,

$$\frac{vol(B(p, r))}{\tilde{v}(r)} \geqq \theta$$

where $vol(B(p, r))$ denotes the volume of the r-ball, $B(p, r)$, in M around p and $\tilde{v}(r)$ is the volume of the r-ball in the Euclidean n-space.

This new invariant $vol-rad_\theta(M)$ will play an important role in our proof of Theorem 1. It can be estimated in terms of some familiar geometric invariants, e.g., curvature, diameter, volume and/or injectivity radius. This can be seen from the following examples.

EXAMPLE 1. Let M be a compact Riemannian n-manifold with the injectivity radius $i(M) \geqq i_0 > 0$. According to a result of Croke [**C**], we have $vol(B(p, r)) \geqq c_n \tilde{v}(r)$ for any $p \in M$ and $r \leqq i_0/2$ where the constant $c_n \in (0, 1)$ depends only on the dimension n. Thus, $vol-rad_{c_n}(M) \geqq i_0/2$.

EXAMPLE 2. Let M be a complete Riemannian n-manifold with the sectional curvature $K(M) \leqq k$ and $i(M) \geqq i_0 > 0$. The Rauch comparison theorem ([**CE**]) implies that for all $p \in M$ and $r \leqq i_0$, $vol(B(p, r)) \geqq \tilde{v}_k(r)$. Thus there is a continuous positive function $r(\theta): (0, 1) \to \boldsymbol{R}^+$ depending only on n and k with $\lim_{\theta\to 1} r(\theta)=0$ such that $vol-rad_\theta(M) \geqq r(\theta)$.

EXAMPLE 3. Let $\boldsymbol{M}$ denote the class of closed Riemannian n-manifolds M with $Ric(M) \geqq (n-1)k$, $d(M) \leqq D$ and $vol(M) \geqq v > 0$. The Bishop-Gromov volume comparison theorem ([**Gr2**]) implies that for any $M \in \boldsymbol{M}$,

$$\frac{vol(B(p, r))}{\tilde{v}_k(r)} \geqq \frac{vol(M)}{\tilde{v}_k(D)} \geqq \frac{v}{\tilde{v}_k(D)}$$

for all $p \in M$ and $r \leqq D$. Thus there are continuous functions $\theta: (0, \tilde{v}_k(D)) \to (0, 1)$ and $r: (0, 1) \to \boldsymbol{R}^+$ depending on k, D and n with $\lim_{v\to \tilde{v}_k(D)} \theta(v) = 1$ and $\lim_{\theta\to 1} r(\theta)=0$ such that for $M \in \boldsymbol{M}$ and $\theta \leqq \theta(v)$,

$$vol-rad_\theta(M) \geqq r(\theta) > 0\,.$$

EXAMPLE 4. In the Euclidean space $\boldsymbol{R}^3$, we consider the open surface M^2 of revolution $\{(x, e^{-x^2}\cos\theta, e^{-x^2}\sin\theta): (x, \theta) \in \boldsymbol{R} \times [0, 2\pi]\}$, with the induced metric. Hence one has $vol-rad_\theta(M^2)=0$ for any $\theta \in (0, 1)$. For an open complete Riemannian manifold M, if $vol-rad_\theta(M)>0$ for some θ, then the volume of M is not finite.

The technique we shall use to prove Theorem 1 is based on the Gromov-Hausdorff distance. It is now well known that the class $\boldsymbol{M}_k$ is precompact in the Gromov-Hausdorff topology for all k. In [**Gr**], Gromov showed that the Hausdorff convergence and the Lipschitz convergence are equivalent for the class of closed Riemannian n-manifolds M with $|K(M)| \leqq K$, $d(M) \leqq D$ and $vol(M) \geqq v > 0$. In [**P**], Peters extended this result to the $C^{1,\alpha}$ convergence. To prove Theorem 1, we need first establish a convergence result for the Ricci curvature and the θ-volume-radius.

THEOREM 2. *Given $\alpha \in (0, 1)$, there is a positive number $\theta^* < 1$ depending only on n and α such that the class $\boldsymbol{M}$ of closed Riemannian n-manifolds M which satisfy*

$$(*) \qquad |Ric(M)| \leqq (n-1)k, \qquad d(M) \leqq D, \qquad vol-rad_{\theta^*}(M) \geqq r_0 > 0$$

is $C^{1,\alpha}$-precompact. In particular, $\boldsymbol{M}$ contains at most finitely many diffeomorphism types.

REMARK. To prove this theorem, we shall employ a technique developed by Anderson in [**A2**]. For small $\theta \in (0, 1)$, the $C^{1,\alpha}$ conclusion in Theorem 2 is not true. This can be seen from the metrics g_i on $S^2 \times S^2$, constructed by Anderson [**A1**], which satisfy $1 \leqq Ric(g_i) \leqq 10$ and $vol(g_i) \geqq 1/10$, but which converge to an orbifold $C(RP^3)\#_\partial C(RP^3)$ in the Gromov-Hausdorff topology. Note that the Myers theorem implies that $d(g_i)\leqq\sqrt{3}\pi$. However, the class $\boldsymbol{M}$ may still contain only finitely many diffeomorphism types. See also [**Ga**], [**A2**] and [**AC**] for discussions under injectivity radius lower bound and/or the $L^{n/2}$ norm bound on the curvature tensor.

In view of the $C^{1,\alpha}$ convergence, the cone points in $C(RP^3)\#_\partial C(RP^3)$ motivate the following definition.

DEFINITION 2. Let (X, d) be an inner metric space. A point $p\in X$ is said to have the one-dimensional injectivity radius $inj_1(p)\geqq\varepsilon>0$ if there is a geodesic $\gamma: [-\varepsilon, \varepsilon]\to X$ with $\gamma(0)=p$ and $d(\gamma(s), \gamma(t))=|t-s|$, for all s, t in $[-\varepsilon, \varepsilon]$. If $inj_1(p)\geqq\varepsilon$ for all $p\in X$, we shall denote it by $inj_1(X)\geqq\varepsilon$.

The cone points do not have the one-dimensional injectivity radius $\geqq\varepsilon$ for any $\varepsilon>0$. It is also easy to check that the one-dimensional injectivity radius is upper semi-continuous with respect to the (pointed) Gromov-Hausdorff distance. Thus the metrics g_j on $S^2\times S^2$, constructed by Anderson, can not have a uniformly lower bound on $inj_1(g_j)$. This also shows that the class $\boldsymbol{M}(k, D, v)$ of all closed Riemannian n-manifold with $|Ric(M)|\leqq(n-1)k$, $d(M)\leqq D$ and $vol(M)\geqq v>0$ is not $C^{1,\alpha}$ precompact. However, for Riemannian 4-manifolds we do have a precompactness result about the one-dimensional injectivity radius.

THEOREM 3. *Let $\boldsymbol{M}(4)$ be the class of closed Riemannian 4-manifolds (M, g) which satisfy*

$$|Ric(M)|\leqq(n-1)k, \qquad d(M)\leqq D, \qquad vol(M)\geqq v>0, \qquad inj_1(M)\geqq\eta .$$

Then $\boldsymbol{M}(4)$ is $C^{1,\alpha}$-precompact for any $\alpha\in(0, 1)$.

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§ 2. Lipschitz convergence.

In this section we shall prove Theorems 2 and 3. Our method is a combination of the Gromov convergence theorem, the rescaling technique, and the splitting theorem of Cheeger and Gromoll. We note that some parts of the material here are inspired and modelled after Peters [**P**] and Anderson [**A2**] and the results and techniques in [**A2**] will be used freely in this paper. First, we recall the definition of the harmonic radius.

DEFINITION 3. A compact Riemannian n-manifold (M, g) is said to have a $C^{1,\alpha}$-harmonic (m, ε)-net, if there is a covering $\{B(x_k, \varepsilon)\}_{k=1}^{m}$ of M by ε-balls such that the balls $B(x_k, \varepsilon/2)$ are disjoint and each $B(x_k, 8\varepsilon)$ has a harmonic coordinate chart $\{u_j\}_{j=1}^{n}$ such that the metric tensor in these coordinates is $C^{1,\alpha}$ bounded, i.e., if $g_{ij}=g(\nabla u_i, \nabla u_j)$ on $B(x_k, 8\varepsilon)$, then

$$(+)\qquad \begin{cases} C^{-1}\delta_{ij} \leqq g_{ij} \leqq C\delta_{ij} & \text{(as bilinear form)} \\ \varepsilon^{1+\alpha}\|g_{ij}(y)\|_{C^{1,\alpha}} \leqq C \end{cases}$$

for some constants $\alpha\in(0, 1)$ and $C\gg 1$, where the norms are taken with respect to the coordinates $\{u_j\}$ on $B(x_k, 8\varepsilon)$. We shall say that the harmonic radius, $r_h(x)$, at a point $x\in M$ is at least ε if the ball $B(x, 8\varepsilon)$ has a coordinate chart $\{u_j\}_{j=1}^{n}$ with the property $(+)$.

REMARK. According to the Schauder estimates, any harmonic function u defined on the ball $B(x_k, 8\varepsilon)$ has $\|u\|_{C^{2,\alpha}}\leqq c(C, \varepsilon)\|u\|_{L^2}$. Similarly, we have a $C^{2,\alpha}$ bound for the transition functions.

A sequence of Riemannian n-manifolds (M_j, g_j) is said to converge, in the $C^{1,\alpha}$ topology, to a $C^{1,\alpha}$ Riemannian manifold (M, g) if M is a smooth manifold with a $C^{1,\alpha}$ metric tensor g and there are diffeomorphisms $f_j\colon M\to M_j$, for j sufficiently large, such that the pull back metric $f_j^*g_j$ converge to g in the $C^{1,\alpha}$ topology on M. Here, the $C^{1,\alpha}$ structures are defined with respect to some fixed $C^{1,\alpha}$ atlas on M, compactible with its smooth structure. The following fact about the harmonic radius is now well-known (cf. [**A2**]).

PROPOSITION 1. *The harmonic radius is continuous with respect to $C^{1,\alpha}$ convergence.*

The importance of the $C^{1,\alpha}$ harmonic (m, ε)-nets lies in the following

PROPOSITION 2. *A class of Riemannian n-manifolds, which have $C^{1,\alpha}$ harmonic (m, ε)-nets for some fixed m, ε and $C^{1,\alpha}$ Hölder constant $C\gg 1$ is precompact in the $C^{1,\delta}$ $(\delta<\alpha)$ topology.*

REMARK. As pointed out in [**A2**], Proposition 2 also holds locally for pointed complete Riemannian manifolds, provided that one works on compact subsets.

To prove theorem 2, we shall need a lemma of Anderson ([**A2**]) which can be rephrased as

THE GAP LEMMA. *There exists a $\theta^*\in(0, 1)$ depending only on the dimension n such that if (N, h) is a complete Ricci flat n-manifold with $vol-rad_{\theta^*}(N)=\infty$, then (N, h) is isometric to $(\boldsymbol{R}^n, can)$.*

Before we prove Theorem 2, we first show that there is a uniform lower bound for the harmonic radius of Riemannian n-manifolds with the property (*) in Theorem 2.

PROPOSITION 3. *Given $\alpha \in (0, 1)$ and $C \gg 1$, choose θ^* to be as in the Gap Lemma, there is a lower bound ε for the harmonic radius of Riemannian n-manifold M with the property:*

$$(*) \qquad |Ric(M)| \leqq (n-1)k, \qquad d(M) \leqq D, \qquad vol-rad_{\theta^*} \geqq r_0 > 0.$$

Namely, $r_h(x) \geqq \varepsilon$ for all $x \in M$.

PROOF. To prove this proposition, we shall follow the line of the proof of Lemma 2.1 in [**A2**]. We argue this proposition by contradiction. Suppose that Proposition 3 is not true. Then one can find a sequence of Riemannian n-manifolds (M_l, g_l) with the property (*); points $x_l \in M_l$ with $C^{1,\alpha}$ harmonic radius $r_h(x_l) = \varepsilon_l \to 0$ as $l \to \infty$ and the point x_l realizes the minimun value of the harmonic radius function r_h on M_l. Now we rescale the metrics so that $h_l = \varepsilon_l^{-2} g_l$. Thus, the ball $B(x_l, \varepsilon)$ is now a ball of radius 1 in the new metric h_l and in (M_l, h_l), x_l has harmonic radius $r_h(x_l) = 1$. Furthermore,

(1) $|Ric(M_l, h_l)| \leqq (n-1)\varepsilon_l^2 k$;

(2) $vol-rad_{\theta^*}(M_l, h_l) \geqq \varepsilon_l^{-1} r_0$, and

(3) h_l has the $C^{1,\alpha}$ harmonic radius $\geqq 1$.

Hence, by passing to a subsequence of pointed Riemannian n-manifolds, one can assume, by Proposition 2, that (M_l, h_l, x_l) converges, in the $C^{1,\delta}(\delta < \alpha)$ topology, to a $C^{1,\alpha}$ Riemannian n-manifold (N, h, x) with $x = \lim x_l$.

The Ricci curvature equation in harmonic coordinates $\{u_l^i\}_{i=1}^n$([**DK**]) is given by

$$(4) \qquad -2Ric(h_l)_{ij} = (h_l)^{rs} \frac{\partial^2 (h_l)_{ij}}{\partial u_l^r \partial u_l^s} + \cdots\cdots$$

where the dots indicate lower order terms involving at most one derivative of the metric. The equation (4) is a uniformly elliptic system of P.D.E. for which we have locally uniform $C^{1,\alpha}$ bounds on the coefficients $(h_l)^{rs}$, $C^{0,\alpha}$ bounds on the lower order terms and C^0 bounds on the left hand side $Ric(h_l)_{ij}$ by (1). Hence the elliptic regularity theory (cf. [**GT**] [**Mo**]) then gives a uniform $W^{2,p}$ bound on $\|h_l\|$ for all $p < \infty$.

The Sobolev embedding theorem implies that $W^{2,p}$ is compactly embedded in $C^{1,\beta}(\beta \in (0, 1))$ as long as $p > n/(1-\beta)$. Therefore by passing to a subsequence we obtain that h_l converges to h in the $C^{1,\beta}$ for all $\beta \in (0, 1)$. In particular, h_l converges to h in the $C^{1,\alpha}$ topology, not just in the $C^{1,\delta}(\delta < \alpha)$ topology. Hence, Proposition 1 implies that $r_h(x) = \liminf r_h(x_l) = 1$. On the other hand, since $|Ric(h_l)| \to 0$ in the C^0 topology and the harmonic coordinates $\{u_l^i\}$ of h_l

converge to a harmonic coordinates $\{u^i\}$ of h, h is a weak $C^{1,\alpha}$ solution to the Einstein equation:

$$Ric(h)_{ij}=0=-\frac{1}{2}h^{rs}\frac{\partial^2 h_{ij}}{\partial u^r \partial u^s}+\cdots\cdots$$

Thus the regularity theory implies that h is a smooth Ricci flat metric on N. Since all closed r-balls $\bar{B}(x, r)$ in N are compact, the Hopf-Rinow theorem implies that N is a complete Riemannian manifold.

Since the volume is continuous with respect to the $C^{1,\alpha}$ convergence, Property (2) then gives that $vol-rad_{\theta*}(N)=\infty$. Thus, (N, h) is a complete Ricci flat Riemannian n-manifold with $r_h(y)\geqq 1$ for all $y\in N$ and satisfies the volume condition in the Gap Lemma. Therefore, one has that (N, x, h) is isometric to $(\boldsymbol{R}^n, o, can)$. Since the usual coordinates of $\boldsymbol{R}^n$ are harmonic functions, the big constant $C\gg 1$ implies that $r_h(x)>1$. This contradicts the fact $r_h(x)=1$ and completes the proof of Proposition 3. □

We are now in a position to prove Theorem 2.

PROOF OF THEOREM 2. Choose θ^* to be as in Proposition 3. Suppose that Theorem 2 is not true, then one can find, by Propositions 2 and 3, a sequence of Riemannian n-manifolds (M_j, g_j) which satisfy the property (*) in Theorem 2 and have $C^{1,\alpha}$ harmonic (m_j, ε)-nets $\{B(x_{(j,l)}, \varepsilon)\}_{l=1}^{m_j}$ where ε is as in Proposition 3, and $m_j\to\infty$ as $j\to\infty$. Note that for each j, the balls $B(x_{(j,l)}, \varepsilon/2)$ are disjoint. We choose an l_j so that $B(x_{(j,l_j)}, \varepsilon/2)$ is the ball of smallest volume. By the Bishop-Gromov volume comparison theorem we have

$$m_j\leqq\frac{vol(M_j)}{vol(B(x_{(j,l_j)}, \varepsilon/2))}\leqq\frac{\tilde{v}_k(D)}{\tilde{v}_k(\varepsilon/2)}.$$

This is impossible since $m_j\to\infty$ as $j\to\infty$ and hence Theorem 2 holds. □

Next, we shall study the class $\boldsymbol{M}(4)$. Before doing so, we first prove a lemma about the fundamental group of an open manifold with nonnegative Ricci curvature.

LEMMA. *Let N be a complete Riemannian n-manifold with $Ric(N)\geqq 0$. If there is a point $p\in N$ with*

$$\lim\frac{vol(B(p, r))}{\tilde{v}(r)}=\theta>0,$$

then the fundamental group, $\pi_1(N)$, of N has order $\leqq 1/\theta$.

REMARK. The relative volume comparison theorem shows that $\lim_{r\to\infty} vol(B(p, r))/\tilde{v}(r)$ does exist. It is easy to check that for any other point $q\in N$, $\lim_{r\to\infty}vol(B(q, r))/\tilde{v}(r)$ is also equal to θ. In terms of the volume-radius, we have $vol-rad_\theta(N)=\infty$. This result was also proved by Anderson in [**A4**]. The author would like to thank the referee for pointing out this.

PROOF. Suppose that this lemma is not true, then one can find m distinct elements in $\pi_1(N)$, say $g_1=e, g_2, \cdots, g_m$, where m is the smallest integer greater than $1/\theta$. Consider the universal covering $\pi: \tilde{N}\to N$ with the pull back metric and hence $Ric(\tilde{N})\geqq 0$. Then $\pi_1(N)$ acts isometrically on $\tilde{N}$ as the deck transformations. Fix a point $\tilde{p}\in\tilde{N}$ with $\pi(\tilde{p})=p$ and let F be the Dirichlet fundamental domain associated to $\tilde{p}$ in $\tilde{N}$, i.e.,

$$F=\{y\in\tilde{N}\,|\,d(y, \tilde{p})\leqq d(y, g\tilde{p}) \quad \text{for all} \quad g\in\pi_1(N)\}.$$

Since $B(\tilde{p}, r)\cap F$ is mapped under the projection onto $B(p, r)$, $vol(B(\tilde{p}, r)\cap F)=vol(B(p, r))$. Let $s=\max\{d(p, g_j\tilde{p})\,|\,1\leqq j\leqq m\}$. Hence we obtain $B(g_j\tilde{p}, r)\subset B(\tilde{p}, r+s)$ for all $j=1, 2, \cdots, m$. Set $F_j(r)=g_j(B(\tilde{p}, r)\cap F)$, then $F_i(r)\cap F_j(r)$ has measure zero when $i\neq j$ and $\cup_{j=1}^m F_j(r)\subset B(\tilde{p}, r+s)$. Therefore, it follows that

$$mvol(B(p, r))\leqq vol(B(\tilde{p}, r+s)).$$

On the other hand, the Bishop volume comparison theorem gives

$$vol(B(\tilde{p}, r+s))\leqq\tilde{v}(r+s).$$

Thus,

$$m\leqq\frac{\tilde{v}(r+s)}{vol(B(p, r))}=\frac{\tilde{v}(r)}{vol(B(p, r))}\frac{\tilde{v}(r+s)}{\tilde{v}(r)}$$

for all $r>0$. Letting $r\to\infty$, we obtain $m\leqq 1/\theta$. This contradicts our choice of the integer m and hence the order of $\pi_1(N)$ is at most $1/\theta$. □

Next, we show that there is a uniform lower bound for the harmonic radius of Riemannian 4-manifolds in $\boldsymbol{M}(4)$.

PROPOSITION 4. *Let $\boldsymbol{M}(4)$ be as in Theorem* 3. *Then given any $\alpha\in(0, 1)$ and $C\gg 1$, there is an $\varepsilon>0$ such that the harmonic radius of $M\in\boldsymbol{M}(4)$ is at least ε, i.e., $r_h(x)\geqq\varepsilon$ for all $x\in M$.*

PROOF. We proceed as in the proof of Proposition 3. Suppose that Proposition 4 is not true, then one can find a subsequence of Riemannian 4-manifolds (M_j, g_j) in $\boldsymbol{M}(4)$ and points $x_j\in M_j$ with harmonic radius $r_h(x_j)=\varepsilon_j\to 0$ as $j\to\infty$ and the points x_j realize the minimum value of the harmonic radius function r_h on M_j. Once again, we rescale the metrics so that $h_j=\varepsilon_j^{-2}g_j$. Hence the ball $B(x_j, \varepsilon_j)$ is now a ball of radius 1 in the metric h_j. Moreover,

(1) $|Ric(M_j, h_j)|\leqq(n-1)\varepsilon_j^2 k$,

(2) h_j has the $C^{1,\alpha}$ harmonic radius $\geqq 1$.

(3) $inj_1(M_j, h_j)\geqq\varepsilon_j^{-1}\eta$ and

(4) by Example 3 in § 1, there is a $\theta=\theta(k, D, v, \eta, n)>0$ such that $vol-rad_\theta(M_j, h_j)\geqq\varepsilon_j^{-1}\eta$.

By passing to a subsequence of pointed Riemannian manifolds, one can assume that (M_j, x_j, h_j) converges, in the $C^{1,\delta}$ $(\delta<\alpha)$ topology, to a $C^{1,\alpha}$ Riemannian n-manifold (N, x, h) with $x=\lim x_j$. Now use the Ricci curvature equation in the harmonic coordinates and proceed the same argument as in the proof of Proposition 3 to obtain that (N, x, h) is actually a smooth complete Ricci flat Riemannian n-manifold and (M_j, x_j, h_j) also converges to (N, x, h) in the $C^{1,\alpha}$ topology. Thus, Proposition 1 implies that $r_h(x)=\liminf r_h(x_j)=1$.

To obtain a contradiction, it suffices to show that (N, h) is isometric to $(\boldsymbol{R}^4, can)$ and hence $r_h(x)>1$. According to Property (3) and the upper semicontinuity of the one-dimensional injectivity radius with respect to the pointed Hausdorff topology, we have $inj_1(x)=\infty$. Namely, there is a line $\gamma:\boldsymbol{R}\to N$ with $\gamma(0)=x$. The splitting theorem of Cheeger and Gromoll ([**CG**]) implies that N is isometric to a product space $N_1\times\boldsymbol{R}$ where N_1 is a codimension one totally geodesic submanifold of N. Thus, N_1 is a complete Ricci-flat 3-manifold. The Schouten-Struik theorem ([**SS**]) then gives that N_1 is flat. In turn, N is a flat 4-manifold.

Since the volume is continuous with respect to the $C^{1,\alpha}$ convergence, Property (4) implies that $vol-rad_\theta(N)=\infty$. The above lemma then asserts that $\pi_1(N)$ is a finite group and has order $\leqq 1/\theta$. That is, N is a flat manifold with a finite fundamental group. Therefore, N must be simply connected (cf. [**Mi**] Cor. 19.3). Thus the Cartan-Ambrose-Hicks theorem ([**CE**]) implies that (N, h) is isometric to $(\boldsymbol{R}^4, can)$ and this completes the proof of Proposition 4. □

Proposition 4 and the argument in the proof of Theorem 2 show that $\boldsymbol{M}(4)$ is a class of Riemannian 4-manifolds with $C^{1,\alpha}$ harmonic (m, ε)-nets for some fixed m, ε and $C^{1,\alpha}$ Hölder constant $C\gg 1$. Thus Proposition 2 implies that $\boldsymbol{M}(4)$ is $C^{1,\delta}$-precompact for any $\delta<\alpha$. Since $\alpha\in(0, 1)$ is arbitrary, $\boldsymbol{M}(4)$ is $C^{1,\alpha}$ precompact for any $\alpha\in(0, 1)$ and Theorem 3 holds. □

Our first application of Theorem 2 is a diameter pinching sphere theorem.

THEOREM 4. *Let θ^* be as in Theorem 2 for $C=10^n$ and $\alpha=1/2$. There exists a positive number $d^*<\pi$ depending only on the constants $k\geqq 1$, $r_0>0$ and n such that if M is a complete Riemannian n-manifold with $n-1\leqq Ric(M)\leqq(n-1)k$ and $vol-rad_{\theta^*}\geqq r_0$, then M is diffeomorphic to the unit sphere, S^n, in $\boldsymbol{R}^{n+1}$, provided that $d(M)\geqq d^*$.*

PROOF. Suppose that this is not true, then there is a sequence of Riemannian n-manifolds (M_j, g_j) with $n-1\leqq Ric(M_j)\leqq(n-1)k$, $vol-rad_{\theta_n}\geqq r_0$ and $d(M_j)\to\pi$ as $j\to\infty$ such that M_j is not diffeomorphic to S^n for all j. Since the class of Riemannian manifolds $\{(M_j, g_j)\}$ satisfies the conditions in Theorem 2, there is a subsequence (M_{j_i}, g_{j_i}) which converges to a $C^{1,\alpha}$ Riemannian n-manifold (X, h). It is easy to check that the Bishop-Gromov volume comparison theorem still holds for (X, h) and $d(X)=\pi$. Thus one can use Shiohama's proof ([**Sh**]) of the maximal diameter sphere theorem to conclude that $vol(X)=vol(S^n)$ and hence $vol(M_j)$ is close to $vol(S^n)$ for large j. Finally, Anderson's volume pinching sphere theorem [**A2**] implies that M_j is diffeomorphic to S^n for large j. This leads to a contradiction and completes the proof of Theorem 4. $\square$

REMARK. In general, a Riemannian n-manifold M with $Ric(M)\geqq n-1$ and $d(M)$ close to π is not necessarily homeomorphic to S^n; see [**A3**] and [**O**] for examples in dimensions $n\geqq 4$. For dimension 3, under an extra assumption on the lower bound of volume, the author ([**Wu2**]) shows that this is true.

§3. Volume functions Φ_Λ.

In this section we shall prove our Theorem 1. First, we recall the class $\boldsymbol{M}_k^\Lambda$. We normalize the metric of a compact Riemannian n-manifold M so that $d(M)=\pi$. Hence the Myers theorem implies that min $Ric(M)\leqq n-1$. Then we consider the class $\boldsymbol{M}_k^\Lambda$ of all closed Riemannian n-manifolds M with $(n-1)k\leqq Ric(M)\leqq \Lambda$ and $d(M)=\pi$. In what follows, we shall show that $vol(M)$ can not be arbitrarily close to $\tilde{v}_k(\pi)$ for $M\in \boldsymbol{M}_k^\Lambda$ with $k<1$ unless M is diffeomorphic to RP^n.

PROOF OF THEOREM 1. Suppose that there is a sequence of Riemannian n-manifolds (M_i, g_i) in $\boldsymbol{M}_k^\Lambda$, $k<1$, with $vol(M_i)\to\tilde{v}_k(\pi)$ as $i\to\infty$. Fix $C\gg 1$ and $\alpha\in(0, 1)$. Then there is an $r_0>0$ such that $vol-rad_{\theta*}(M_j, g_j)\geqq r_0$ when $vol(M_i)\geqq\theta^*\tilde{v}_k(\pi)$ where $\theta^*\in(0, 1)$ is as in Theorem 2. Then Theorem 2 implies that there is a subsequence, still denoted by (M_i, g_i), converging to a $C^{1,\alpha}$ Riemannian n-manifold (M, g). In particular, $d(M)=\pi$ and $vol(M)=\tilde{v}_k(\pi)$. Since M_i are diffeomorphic to M for large i, we may assume that the metrics g_i are defined on M.

Our next aim is to show that g is actually a smooth Einstein metric. To see this we shall follow closely the proof of the sphere theorem in [**A2**] and first show that $|Ric(g_i)-(n-1)kg_i|\to 0$ almost everywhere on M. Indeed, since $vol(B_{g_i}(x, \pi))/\tilde{v}_k(\pi)\to 1$ as $i\to\infty$ for all $x\in M$, for almost all unit vector $w\in T_{i,x}=T(M, g_i)_x$ the unit speed geodesics $\gamma_w(s)$ in the direction w are length minimizing for a distance $\pi-\varepsilon_i$ with $\varepsilon_i\to 0$ as $i\to\infty$. In other words, the set D_i of such w has measure $\geqq vol(S^{n-1})-\varepsilon_i$.

Let the point $\gamma_w(r)$ be inside the cut locus of x, and let $\theta_i(w, r)dw$ and $H_i(w, r)$ be the volume form and the mean curvature of the geodesic sphere

$S_x(r)=\{y\in(M, g_i)|d(x, y)=r\}$ at $\gamma_w(r)$ in (M, g_i), respectively. Then we have $H_i(w, r)=\theta_i'(w, r)/\theta_i(w, r)$ and the second variation formula ([**BZ**]) gives

$$\frac{\theta_i'(w, r)}{\theta_i(w, r)}=\sum_{j=1}^{n-1}\int_0^r |Y_j'(s)|^2-<R_i(Y_j(s), \gamma_w'(s))\gamma_w'(s), Y_j(s)>ds$$

where $Y_j(s)$ are Jacobi fields along γ_w with $Y_j(0)=0$ such that $\{Y_j(r)\}_{j=1}^{n-1}\cup\{\gamma_w'(r)\}$ forms an orthonormal basis for $TM_{\gamma_w(r)}$. Define a function $h_k: [0, r]\to\boldsymbol{R}$ by

$$h_k(s)=\begin{cases}\dfrac{\sin(\sqrt{k}s)}{\sqrt{k}}, & k>0\\ s, & k=0\\ \dfrac{\sinh(\sqrt{-k}s)}{\sqrt{-k}}, & k<0.\end{cases}$$

Let $X_j(s)=f(s)E_j(s)$ where $E_j(s)$ is the parallel vector field along $\gamma_w(s)$ with $E_j(r)=Y_j(r)$ and $f(s)=h_k(s)/h_k(r)$. It follows from the basic index lemma that

$$\begin{aligned}\frac{\theta_i'(w, r)}{\theta_i(w, r)}&=\sum_{j=1}^{n-1}\int_0^r |Y_j'(s)|^2-\langle R_i(Y_j(s), \gamma_w'(s))\gamma_w'(s), Y_j(s)\rangle ds\\ &\leqq\sum_{j=1}^{n-1}\int_0^r |X_j'(s)|^2-\langle R_i(X_j(s), \gamma_w'(s))\gamma_w'(s), X_j(s)\rangle ds\\ &=\int_0^r (n-1)|f'(s)|^2-f^2(s)\,Ric_i(\gamma_w'(s), \gamma_w'(s))ds\\ &=\int_0^r (n-1)[|f'(s)|^2-kf^2(s)]ds-\int_0^r [Ric_i(\gamma_w'(s), \gamma_w'(s))-(n-1)k]f^2(s)ds.\end{aligned}$$

An easy calculation yields

$$\begin{aligned}\frac{\theta_i'(w, r)}{\theta_i(w, r)}\leqq&\frac{d}{ds}[\ln h_k^{n-1}(s)]_{s=r}\\ &-\frac{1}{h_k^2(r)}\int_0^r [Ric_i(\gamma_w'(s), \gamma_w'(s))-(n-1)k]h_k^2(s)ds.\end{aligned}$$

This gives

$$\text{(1)}\qquad \frac{d}{ds}\left[\ln\frac{\theta_i(w, s)}{h_k^{n-1}(s)}\right]_{s=r}\leqq-\frac{1}{h_k^2(r)}\int_0^r [Ric_i(\gamma_w'(s), \gamma_w'(s))-(n-1)k]h_k^2(s)ds\leqq 0.$$

Since the volume form of geodesic spheres in (M, g_i) are close, on D_i, to the volume form $h_k^{n-1}(r)dw$ of the geodesic spheres in the simply connected space n-form M_k^n. It follows that for $w\in D_i$,

$$\text{(2)}\qquad \frac{d}{ds}\left[\ln\frac{\theta_i(w, s)}{h_k^{n-1}(s)}\right]_{s=r}\geqq-\delta_i$$

with $\delta_i\to 0$ as $i\to\infty$. Since $0\leqq|Ric(g_i)-(n-1)kg_i|\leqq\Lambda+|(n-1)k|$, inequalities

(1) and (2) imply that $|Ric(g_i)-(n-1)kg_i|\to 0$ a.e. on M and for any $p\geqq 1$,

$$\int_M |Ric(g_i)-(n-1)kg_i|^p dvol(g_i) \longrightarrow 0 .$$

Since g_i also converges to g in the $C^{1,\alpha}$ topology, g is a weak $C^{1,\alpha}\cap W^{2,p}$ solution of the Einstein equation in the harmonic coordinates $\{u_l\}_{l=1}^n$:

$$Ric(g)_{ij} = -\frac{1}{2}g^r s\frac{\partial_2 g_{ij}}{\partial u^r \partial u^s}+\cdots=(n-1)kg_{ij}.$$

The regularity theory of elliptic systems again implies that g is a smooth Einstein metric on M. On the other hand, the Hopf-Rinow theorem guarantees that g is complete. Since $vol(M, g)=\tilde{v}_k(\pi)$, we have $i(M, g)=d(M, g)=\pi$ and $K(M, g)\equiv 1$. From here there are several ways to obtain the results. We shall go with the Blaschke conjecture. We have that (M, g) is a compact Blaschke n-manifold with an Einstein metric g. By examining the proof of the Bishop volume comparison theorem ([**BC**]), one has the following three cases.

Case I. $k\leqq 0$. (M, g) has no conjugate points. Hence the Blaschke conjecture ([**B**] [**We**] [**Yg**]) implies that (M, g) must be isometric to (RP^n, can) up to a constant factor. This is impossible, since $Ric(RP^n, can)>0$.

Case II. $k>0$ and (M, g) has no conjugate points. Again, the Blaschke conjecture gives that (M, g) is isometric to $(RP^n, 4can)$ and $k=1/4$ since $d(M, g)=\pi$.

Case III. $k>0$ and (M, g) has conjugates points of index $(n-1)$. Namely, (M, g) is a Wiedersehen manifold. Thus the Blaschke conjecture implies, in this case, that (M, g) is isometric to (S^n, can) and $k=1$ since $d(M, g)=\pi$. This is again impossible since $k<1$.

These three cases together show that one can have a sequence of Riemannian n-manifolds (M_i, g_j) in $\boldsymbol{M}_k^A$, $k<1$, with $vol(M_i)\to\tilde{v}_k(\pi)$ only when $k=1/4$. In this case M_i is diffeomorphic to RP^n for large i. Thus the function Φ_A and the number δ_A do exist and this completes the proof of Theorem 1. □

REMARK. Our present proof does not yield estimates on Φ_A and δ_A. If one can find an explicit estimate for Φ_A, then one may be able to determine whether the upper bound $\tilde{v}_k(\pi)$ is optimal for the volume function vol on $\boldsymbol{M}_k$.

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