J. Math. Soc. Japan
Vol. 47, No. 3, 1995

On Varea's conjecture

By Jian-Hua ZHOU

(Received Nov. 5, 1993)

1. Main result.

Unless stated otherwise, Lie algebras mentioned in this paper are always assumed to be finite dimensional over algebraically closed field F.

For $x \in L$, denote $C(x)$ the centralizer of x in L. L is called to be centralizer nilpotent (abbreviated c. n.) provided that the centralizer $C(x)$ is nilpotent for all nonzero $x \in L$. Such algebras have been studied by Benkart and Isaacs in [**1**].

For $x \in L$, denote $E_L(x)$ the Engle subalgebra of L determined by x, i.e., the Fitting null-component of ad x in L. L is called to be Engel subalgebraically anisotropic (abbreviated E. a.) if every proper Engel subalgebra $E_L(x)$ of L has no any ad-nilpotent element of L. E. a. Lie algebras have been studied by Varea in [**2**].

In many special cases, [**2**] has proved that E. a. Lie algebras are c. n.. It is conjected that only E. a. simple Lie algebras are $sl(2, F)$, $W_p(F)$ and $sl(3, F)/F \cdot 1$, where char $F = p > 0$. Under this conjecture, [**2**] has proved that every E. a. Lie algebra is c. n..

The aim of this short paper is to prove Varea's conjecture. In fact, the way used here is more direct than that of [**2**]. We will prove the following

THEOREM. *Let L be a Lie algebra over an algebraically closed field F. Then L is E. a. if and only if L is c.n..*

2. The proof of Theorem.

Sufficiency can be follow upon application of Theorem 2.5 of [**1**]. The main effort in the following will be to prove the necessity. Some preliminaries will be needed.

Let U be an abelian subalgebra of L. Then L can be decomposed into direct sum of weight spaces L_λ. That is, $L = \sum_\lambda L_\lambda$, where L_λ is the largest subspace of L on which ad $u - \lambda(u)$ is nilpotent for all $u \in U$. Let $K_\lambda = \{\eta \in L \mid \text{ad}\, u(\eta) = \lambda(u)\eta$, for all $u \in U\}$, $K = \sum_\lambda K_\lambda$. Then K is a subalgebra of L, for $[K_\lambda, K_\mu] < K_{\lambda+\mu}$.

Motivated by [3], we have

LEMMA 1. *Let $D \in \mathrm{Der}\, L$. If there exists an integer n such that $D^n(K)=0$, then D is nilpotent on L.*

PROOF. For an eigenvalue λ of D, let L^λ be the largest subspace of L on which $D-\lambda$ is nilpotent. Then $L=\sum_\lambda L^\lambda$ and $[L^\lambda, L^\mu] \subseteqq L^{\lambda+\mu}$. Since $D^n(K)=0$, $K \subseteqq L^0$ and hence $U \subseteqq K_0 \subseteqq K \subseteqq L^0$. So, $[U, L^\lambda] \subseteqq [L^0, L^\lambda] \subseteqq L^\lambda$, that is, L^λ is a U-submodule of L.

Suppose that $L^\lambda \neq 0$. Since U is abelian, there exists $\eta \in L^\lambda$ such that η is a common eigenvector of $\mathrm{ad}\, U$. Clearly, $\eta \in K$. So, $L^0 \cap L^\lambda \supseteqq K \cap L^\lambda \neq 0$ and therefore $\lambda=0$. The result follows.

It is clear that the subalgebra of a c.n. algebra is yet c.n.. However, it is difficult to prove that the subalgebra of an E.a. algebra is E.a.. By Lemma 1, we can prove the following

LEMMA 2. *Let L be E.a., then K is E.a. for every abelian subalgebra U of L.*

PROOF. Let $x \in K$ such that $\mathrm{ad}_K x$ is not nilpotent. Then $\mathrm{ad}_L x$ is certainly not nilpotent. Since L is E.a. and $E_K(x) \subseteqq E_L(x)$, $E_K(x)$ has no nonzero ad-nilpotent element of L. By Lemma 1, $E_K(x)$ has no nonzero ad-nilpotent element of K. So, K is E.a..

The following lemma is due to Benkart and Isaacs [1].

LEMMA 3. *Suppose that G is a c.n. Lie algebra over F. Let $U \subseteqq G$ be a nilpotent subalgebra with $\dim U \geqq 2$. Then $\mathrm{ad}\, u$ is nilpotent on L for all $u \in U$.*

PROOF OF THEOREM. On the contrary, suppose that L is a Lie algebra such that

(1) L is E.a. but not c.n.;

(2) L has the lowest dimension with respect to property (1).

The aim in the following is to prove that the centralizer $C(x)$ of x for all $0 \neq x \in L$ is nilpotent. And this contradicts with above hypotheses on L.

Let $0 \neq x \in L$. If $C(x)=Fx$, then $C(x)$ is nilpotent clearly. So, in the following, we consider only the case of $C(x) \neq Fx$. There are two posibilities:

Case 1: For all $y \in C(x) \setminus Fx$, $U=\mathrm{Span}\{x, y\}$ is toral on L, i.e., every element in U is ad-semisimple on L.

In this case, $C(x)$ is toral on L. For F is algebraically closed, $C(x)$ is abelian. Of course, $C(x)$ is nilpotent.

Case 2: There exists $y \in C(x) \setminus Fx$ such that $U=\mathrm{Span}\{x, y\}$ is not toral on L.

In this case, we decompose L into weight spaces L_λ relative to U. That is, $L=\sum_\lambda L_\lambda$, where $L_\lambda=\{x\in L|$ there exists an integer n such that $(\mathrm{ad}\,\eta-\lambda(\eta))^n(x)=0$ for all $\eta\in U\}$. Let $K_\lambda=\{x\in L|\mathrm{ad}\,\eta(x)=\lambda(\eta)x$ for all $\eta\in U\}$ and $K=\sum_\lambda K_\lambda$. Then K is a subalgebra of L. Since U is not toral, $\dim K<\dim L$. By Lemma 2, K is E. a.. By the hypotheses on the dimension of L, K is c. n.. Then it follows from Lemma 3, $\mathrm{ad}\,u$ is nilpotent for all $u\in U$. In particular, $\mathrm{ad}\,x$ is nilpotent.

In this case, if there exists $z\in C(x)\setminus Fx$ such that $W=\mathrm{Span}\{x, z\}$ is toral on L, then $x\in Z(L)$, the center of L. Therefore, $Z(L)\neq 0$. It follows from Proposition 2.1 of [2] that L is nilpotent. It is a contradiction. Thus, in the case 2, every element in $C(x)$ is nilpotent on L. In particular, $C(x)$ is nilpotent.

Above discussion shows that, in any cases, L must be c. n.. This contradicts with the hypotheses on L.

References

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Jian-Hua ZHOU
Department of Mathematics
Southeast University
210018 Nanjing
P. R. China